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CANONICAL IDEALS OF COHEN-MACAULAY PARTIALLY ORDERED SETS

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Introduction

Our dream is to revive the ideal theory in partially ordered sets from a viewpoint of commutative algebra.

Historically, the concept of ideals in commutative algebra was first studied by Dedekind, who considered the ring of algebraic integers in an algebraic number field.

Let S be a set and S^* the set whose elements are the various subsets of S. Then S^* turns out to be a lattice ordered by inclusion. On the other hand, we may regard S^* to be a commutative ring with identity if we define addition and multiplication in S^* as follows: $A + B := (A - B) \cup (B - A)$, $A \cdot B := A \cap B$. A subset I of the ring S^* is an ideal of S^* if and only if (i) $A \in I$, $B \in S^*$ and $B \subset A$ together imply $B \in I$ and (ii) $A \cup B \in I$ for any A, $B \in I$.

So, in Stone [Sto], a subset I of an arbitrary lattice L is called an ideal of L if (i) $\alpha \in I$, $\xi \in L$ and $\xi \leq \alpha$ together imply $\xi \in I$ and (ii) $\alpha \vee \beta \in I$ for any α, $\beta \in I$. Later, Frink [Fri] extended Stone's definition to partially ordered sets, abbreviated as posets. Here, ignoring the condition (ii) of Stone's definition, we call a subset I of an arbitrary poset Q a poset ideal of Q if $\alpha \in I$, $\beta \in Q$ and $\beta \leq \alpha$ together imply $\beta \in I$.

Recently, some remarkable works between commutative algebra and combinatorics have been accomplished ([Hoc_1], [Rei], [Sta_5], [Sta_9], [Sta_{13}]). One of the main topics in this area is the concept of Cohen-Macaulay posets, see [Bac], [Bjö], [H_3] and [Sta_8]. We now pay attention to poset ideals of Cohen-Macaulay posets to obtain certain ring-theoretical information.

Let $R = \bigoplus_{n\geq 0} R_n$ be an ASL (algebra with straightening laws [Eis]) domain on a Cohen-Macaulay poset Q over a field $R_0 = k$. Then by what means

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can we describe the canonical module K_R of R explicitly? Roughly speaking, as we can see in Stanley's paper [Sta_7], the canonical module K_R of a Cohen-Macaulay domain $R=\bigoplus_{n\geq 0} R_n$ is controlled by the numerical condition, i.e., the behavior of its Poincaré series $F(R,\lambda):=\sum_{n=0}^{\infty}(\dim_k R_n)\lambda^n$. In general, given a noetherian graded ring $R=\bigoplus_{n\geq 0} R_n$ defined over a field $R_0=k$, it is difficult to check whether R is Cohen-Macaulay and to calculate its Poincaré series. However, as soon as R turns out to be an ASL on a poset Q over a field k, the desired information can be obtained easily from the combinatorics of the poset Q.

In this paper, we introduce the concept of "canonical ideals" of Cohen-Macaulay posets (cf. (1.1)). If $R=\bigoplus_{n\geq 0} R_n$ is an ASL domain on a Cohen-Macaulay poset Q, which possesses a canonical ideal I, over a field $R_0=k$, then the canonical module K_R of R is isomorphic to the ideal $I\cdot R$ of R as graded R-modules up to shift in grading. This is a ring-theoretical background to define canonical ideals of Cohen-Macaulay posets.

Many interesting and important questions now occur. Among them, one of the fundamental problems is to classify all Cohen-Macaulay posets which possess canonical ideals. Our main result (3.12), in which the distributive lattices with canonical ideals are classified, is a starting point of this classification problem. We hope that the structure of Cohen-Macaulay posets with canonical ideals will be clear in our further study.

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§ 1. The what and why of canonical ideals

The purpose of this section is, first, to introduce the concept of canonical ideals of Cohen-Macaulay posets and, secondly, to state a ring-theoretical background of this notion.

To begin with, we summarize basic definitions and terminologies on combinatorics.

Every partially ordered set (*poset* for short) to be considered is finite, unless otherwise stated.

The *length* of a chain (totally ordered set) X is $\sharp(X) - 1$, where $\sharp(X)$ is the cardinality of X as a set.

The *rank* of a poset Q, denoted by $\operatorname{rank}(Q)$, is the supremum of lengths of chains contained in Q.

A poset Q is called *pure* if the length of any maximal chain of Q is equal to $\operatorname{rank}(Q)$.

The *height* (resp. *depth*) of an element α of a poset Q is the supremum of lengths of chains descending (resp. ascending) from α, and written as $\operatorname{height}_Q(\alpha)$ (resp. $\operatorname{depth}_Q(\alpha)$).

A *poset ideal* of a poset Q is a subset I such that $\alpha \in I$, $\beta \in Q$ and $\beta \leq \alpha$ together imply $\beta \in I$.

We say that a multichain $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$ of a poset Q *belongs to* a poset ideal I if $\alpha_i \in I$ for some i.

A *lattices* is a poset L any two of whose elements α and β have a greatest lower bound or "meet" denoted by $\alpha \wedge \beta$, and a least upper bound or "join" denoted by $\alpha \vee \beta$. A subposet P of a lattice L is called a *sublattice* of L if both $\alpha \wedge \beta$ and $\alpha \vee \beta$ in L are contained in P for all $\alpha, \beta \in P$.

Let $\mathbb{N}$ be the set of non-negative integers and $\mathbb{Z}$ the set of integers. A *weighted poset* (Q, ω) is a couple of a poset Q and a map ω, called a *weight* on Q, from Q to $\mathbb{N} - \{0\}$.

The weight of a multichain $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_p$ of a weighted poset (Q, ω) is defined to be $\sum_{1 \leq i \leq p} \omega(\alpha_i)$. For any non-negative integer n, let $c_n = c_n(Q, \omega)$ be the number of multichains of weight n. Thus in particular $c_0 = 1$. Then define the *Poincaré series* $F_{(Q,\omega)}(\lambda)$ of (Q, ω) to be the generating function

$$F_{(Q,\omega)}(\lambda) = \sum_{n=0}^{\infty} c_n \lambda^n \in \mathbb{Z}[[\lambda]]$$

of the sequence $\{c_n\}_{n \geq 0}$, which will turn out to be a rational function of the indeterminate λ.

Let I be a poset ideal of a poset Q and ω a weight on Q. For any positive integer n, write $c_n^I = c_n^I(Q, \omega)$ for the number of multichains of (Q, ω) of weight n which belong to I. The Poincaré series $F^I_{(Q,\omega)}(\lambda)$ of I in (Q, ω) is defined by

$$F^I_{(Q,\omega)}(\lambda) = \sum_{n=1}^{\infty} c_n^I \lambda^n \in \mathbb{Z}[[\lambda]].$$

Let Q be a poset and $A = k[X_\alpha; \alpha \in Q]$ the polynomial ring in $\sharp(Q)$-variables over a field k. Also, let I_Q be the ideal of A generated by all monomials of the form $X_\alpha X_\beta$ such that α and β in Q are incomparable. Set $k[Q]:= A/I_Q$, which is called the *Stanley-Reisner ring* of Q over k after the famous works [Sta_5] and [Rei].

A poset Q is called *Cohen-Macaulay* (resp. *Gorenstein*) over a field k if the Stanley-Reisner ring $k[Q]$ is Cohen-Macaulay (resp. Gorenstein).

Many interesting and important works of Cohen-Macaulay and Gorenstein posets are accomplished. Consult [Hoc_2] and [Sta_{11}] for further information.

We have now finished the preliminary steps for the definition of canonical ideals of Cohen-Macaulay posets.

(1.1) DEFINITION. Let Q be a Cohen-Macaulay poset of rank $d-1$ with a unique minimal element $-\infty$, and ω a weight on Q. Then a non-empty poset ideal I of Q is called a *canonical ideal* of the weighted poset (Q, ω) if the following conditions are satisfied:

(1.2) $F_{(Q,\omega)}(\lambda^{-1}) = (-1)^d \lambda^{-a} F^I_{(Q,\omega)}(\lambda)$ for some $a \in \mathbb{Z}$.

(1.3) The subposet $Q - I$ is Cohen-Macaulay with $\mathrm{rank}(Q - I) = d - 2$.

(1.4) EXAMPLE. a) First, consider the following Cohen-Macaulay poset

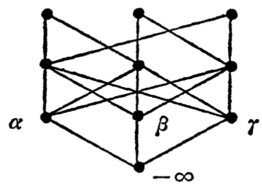
$Q =$ 

Fig. 1.

Let ω be the natural weight on Q, i.e., $\omega(x) = 1$ for any $x \in Q$. Then the Poincaré series of the weighted poset (Q, ω) is

$$F_{(Q,\omega)}(\lambda) = \frac{1 + 6\lambda + 9\lambda^2 + 2\lambda^3}{(1-\lambda)^4}.$$

Let $I_\alpha = \{-\infty, \alpha\}$. Since $Q - I_\alpha$ is Cohen-Macaulay and the Poincaré series of I_α in (Q, ω) is

$$F^{I_\alpha}_{(Q,\omega)}(\lambda) = \frac{2\lambda + 9\lambda^2 + 6\lambda^3 + \lambda^4}{(1-\lambda)^4},$$

the poset ideal I_α is a canonical ideal of (Q, ω). Of course, $I_\beta = \{-\infty, \beta\}$, $I_\gamma = \{-\infty, \gamma\}$ are also canonical ideals of (Q, ω). Thus a canonical ideal of a weighted poset is not necessarily unique even if it exists.

b) Secondly, let n be a positive integer and ω the weight on the Cohen-Macaulay poset

$Q =$

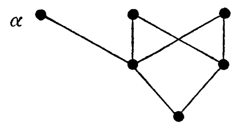


Fig. 2.

defined by $\omega(x) = 1$ if $x \neq \alpha$ and $\omega(\alpha) = n$. Then

$$F_{(Q,\omega)}(\lambda) = \frac{(1+\lambda)^2(1+\lambda+\lambda^2+\cdots+\lambda^{n-1})+\lambda^n}{(1-\lambda)^2(1-\lambda^n)}.$$

Hence, it can be checked that (Q, ω) has a canonical ideal if and only if $n = 1$.

c) Finally, let Q be the following Cohen-Macaulay poset

$Q =$

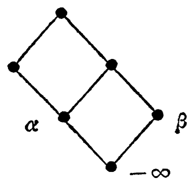


Fig. 3.

We write ω for the natural weight on Q and denote by ω' the weight on Q defined by $\omega'(x) = 1$ if $x \neq \alpha, \beta$ and $\omega'(\alpha) = \omega'(\beta) = 2$. Then $I = \{-\infty, \beta\}$ is a canonical ideal of (Q, ω), while $I = \{-\infty\}$ is a canonical ideal of (Q, ω').

A weighted poset (Q, ω) is called *numerically Gorenstein* over a field k if Q is Cohen-Macaulay over k and

$$F_{(Q,\omega)}(\lambda^{-1}) = (-1)^d \lambda^{-a} F_{(Q,\omega)}(\lambda)$$

for some $a \in \mathbb{Z}$, where $d = \operatorname{rank}(Q) + 1$.

If Q is a Cohen-Macaulay poset with a unique minimal element $-\infty$ and ω is a weight on Q, then the weighted poset (Q, ω) is numerically Gorenstein if and only if $I = \{-\infty\}$ is a canonical ideal of (Q, ω).

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(1.5) Proposition. *Let (Q, ω) be a Cohen-Macaulay weighted poset with a canonical ideal I. Then $(Q-I, \omega)$ is numerically Gorenstein.*

Proof. Since

$$F_{(Q,\omega)}(\lambda) = F^I_{(Q,\omega)}(\lambda) + F_{(Q-I,\omega)}(\lambda),$$

we have the equalities

$$\begin{aligned} F_{(Q-I,\omega)}(\lambda^{-1}) &= F_{(Q,\omega)}(\lambda^{-1}) - F^I_{(Q,\omega)}(\lambda^{-1}) \\ &= (-1)^d \lambda^{-a} F^I_{(Q,\omega)}(\lambda) - F^I_{(Q,\omega)}(\lambda^{-1}) \\ &= (-1)^{d-1}\lambda^{-a}[(-1)^d \lambda^a F^I_{(Q,\omega)}(\lambda^{-1}) - F^I_{(Q,\omega)}(\lambda)] \\ &= (-1)^{d-1}\lambda^{-a} F_{(Q-I,\omega)}(\lambda) \end{aligned}$$

for some $a \in \mathbb{Z}$ and $d = \operatorname{rank}(Q) + 1$. Hence $(Q-I, \omega)$ is numerically Gorenstein. Q.E.D.

Next, let us recall the definition and some basic results on algebras with straightening laws from [D-E-P] and [Eis].

Suppose that R is a commutative ring and Q, a subset of R, is a poset. A *monomial* is a product of the form $\alpha_1\alpha_2 \cdots \alpha_p$, where $\alpha_i \in Q$. A monomial $\alpha_1\alpha_2 \cdots \alpha_p$ is called *standard* if $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_p$. Now, let k be a field, R a k-algebra and Q a poset contained in R which generates R as a k-algebra. Then we call R an *algebra with straightening laws* (abbreviated as ASL) on Q over k if the following conditions are satisfied:

(ASL-1) The set of standard monomials is a basis of the algebra R as a vector space over k.

(ASL-2) If α and β in Q are incomparable (written as $\alpha \not\sim \beta$) and if

$$\alpha\beta = \sum_i r_i \gamma_{i1}\gamma_{i2} \cdots \gamma_{ip_i},$$

where $0 \ne r_i \in k$ and $\gamma_{i1} \le \gamma_{i2} \le \cdots$, is the unique expression, called the *straightening relation*, for $\alpha\beta$ in R as a linear combination of distinct standard monomials guaranteed by (ASL-1), then $\gamma_{i1} \le \alpha, \beta$ for every i.

Note that the right-hand side of the straightening relation in (ASL-2) is allowed to be the empty sum $(=0)$, but that, though 1 is a standard monomial, no $\gamma_{i1}\gamma_{i2} \cdots \gamma_{ip_i}$ can be 1.

It can be checked that the dimension of R as a k-algebra coincides with $\operatorname{rank}(Q) + 1$.

Let (Q, ω) be a weighted poset. An ASL R on Q over k is called an ASL on (Q, ω) if there is a grading $R = \bigoplus_{n \geq 0} R_n$ such that $R_0 = k$ and $\alpha \in R_{\omega(\alpha)}$ for every $\alpha \in Q$.

Since the Stanley-Reisner ring $k[Q]$ is the simplest ASL on Q over k, we also call $k[Q]$ the *discrete* ASL on Q over k. For any weight ω on Q, $k[Q]$ is an ASL on the weighted poset (Q, ω).

Let (Q, ω) be a weighted poset and R an ASL on (Q, ω) over k. Then R is Cohen-Macaulay (resp. Gorenstein) if the poset Q is Cohen-Macaulay (resp. Gorenstein) over k. This result is called a fundamental theorem in the theory of ASL, see [D-E-P]. Thanks to this fundamental theorem, we can obtain many information about any ASL on Q from the combinatorics of the poset Q.

The following lemma is quite essential in our work.

(1.6) LEMMA ([D-E-P]). *Let R be an ASL on a poset Q over a field k. If I is a poset ideal, then the set of standard monomials belonging to I is a basis of the ideal $I \cdot R$ of R as a vector space over k and the quotient ring $R/I \cdot R$ is an ASL on the subposet $Q - I$ over k.*

It is natural to ask why we present the concept of canonical ideals of Cohen-Macaulay posets. So, we now turn to the statement of a ring-theoretical background of canonical ideals of Cohen-Macaulay posets.

Let $R = \bigoplus_{n \geq 0} R_n$ be a noetherian graded ring defined over a field $R_0 = k$, and $M = \bigoplus_{n \in \mathbf{Z}} M_n$ a finitely generated graded R-module. The *Hilbert function* of M is defined by

$$H(M, n) = \dim_k M_n, \qquad \text{for } n \in \mathbb{Z}.$$

Thus in particular $H(M, n) = 0$ for $n \ll 0$. Define the Poincaré series of M to be

$$F_M(\lambda) = \sum_{n=-\infty}^{\infty} H(M, n)\lambda^n \in \mathbb{Z}[[\lambda]][\lambda^{-1}].$$

It is a consequence of the Hilbert syzygy theorem that $F_M(\lambda)$ is a rational function of λ.

The theory of canonical modules of noetherian graded rings is developed in [Sta$_7$] and [G-W]. We here summarize fundamental results from [H-K] and [Sta$_7$].

Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded ring defined over a field $R_0 = k$, and K_R the canonical module of R. Then the Poincaré series $F_{K_R}(\lambda)$ of K_R coincides with $(-1)^d F_R(\lambda^{-1})$, where $d = \dim(R)$.

If R is Gorenstein, then $F_R(\lambda^{-1}) = (-1)^d \lambda^{-a} F_R(\lambda)$ for some $a \in \mathbb{Z}$. Moreover, if R is a Cohen-Macaulay integral domain, then R is Gorenstein if and only if $F_R(\lambda^{-1}) = (-1)^d \lambda^{-a} F_R(\lambda)$ for some $a \in \mathbb{Z}$.

If R is a Cohen-Macaulay integral domain, then the canonical module K_R of R is isomorphic to a graded ideal I of R as graded R-modules up to shift in grading. In this case, if $I \neq R$, then R/I is Gorenstein and $\dim(R/I) = \dim(R) - 1$.

(1.7) LEMMA. *Let $R = \bigoplus_{n \geq 0} R_n$ be a Cohen-Macaulay graded domain defined over a field $R_0 = k$ with $\dim(R) = d$. Assume that I is a graded ideal of R which satisfies the following conditions:*

(i) *$F_R(\lambda^{-1}) = (-1)^d \lambda^{-a} F_I(\lambda)$ for some $a \in \mathbb{Z}$.*

(ii) *R/I is Cohen-Macaulay and $\dim(R/I) = d - 1$.*

Then the canonical module K_R of R is isomorphic to I as graded R-modules up to shift in grading.

Proof. Since

$$(*) \qquad 0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is an exact sequence of graded R-modules, we have the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \underline{\mathrm{Hom}}_R(R/I, K_R) \longrightarrow \underline{\mathrm{Hom}}_R(R, K_R) \longrightarrow \underline{\mathrm{Hom}}_R(I, K_R) \\ \longrightarrow \underline{\mathrm{Ext}}^1_R(R/I, K_R) \longrightarrow \underline{\mathrm{Ext}}^1_R(R, K_R) \longrightarrow \underline{\mathrm{Ext}}^1_R(I, K_R) \\ \longrightarrow \underline{\mathrm{Ext}}^2_R(R/I, K_R) \longrightarrow \underline{\mathrm{Ext}}^2_R(R, K_R) \longrightarrow \underline{\mathrm{Ext}}^2_R(I, K_R) \\ \longrightarrow \underline{\mathrm{Ext}}^3_R(R/I, K_R) \longrightarrow \cdots . \end{array}$$

On the other hand, $\underline{\mathrm{Hom}}_R(R/I, K_R) = 0$, $\underline{\mathrm{Ext}}^1_R(R/I, K_R) \simeq K_{R/I}$, $\underline{\mathrm{Hom}}_R(R, K_R) \simeq K_R$ and $\underline{\mathrm{Ext}}^1_R(R, K_R) = 0$ by (ii), see [G-W, (2.1.6)] and [G-W, (2.2.9)]. Thus we have the exact sequence

$$(**) \qquad 0 \longrightarrow K_R \longrightarrow \underline{\mathrm{Hom}}_R(I, K_R) \longrightarrow K_{R/I} \longrightarrow 0 .$$

By (i) we have the equality $F_{K_R}(\lambda)$ $(= (-1)^d F_R(\lambda^{-1})) = \lambda^{-a} F_I(\lambda)$, and by the same method as in the proof of (1.5) we can check $F_{K_{R/I}}(\lambda)$ $(= (-1)^{d-1} F_{R/I}(\lambda^{-1})) = \lambda^{-a} F_{R/I}(\lambda)$. Hence, thanks to (*) and (**), we obtain

$$F_{\underline{\mathrm{Hom}}_R(I, K_R(-a))}(\lambda) = F_R(\lambda) ,$$

where $K_R(-a)$ is a shift in grading of K_R. Thus there exists a degree preserving R-homomorphism $\varphi: I \to K_R(-a)$. Since R is an integral domain and $K_R(-a)$ is a fractional ideal of R, the map φ must be a multiplication by a homogeneous element of degree zero of the quotient field of R, hence φ is injective. Thus $I \simeq K_R(-a)$ since $F_I(\lambda) = \lambda^a F_{K_R}(\lambda) = F_{K_R(-a)}(\lambda)$.

Q.E.D.

The above result (1.7) is false if we drop the assumption that R is an integral domain. For example, let A be the polynomial ring $k[X_1, X_2, X_3, X_4, X_5]$ with the natural grading, i.e., $\deg(X_i) = 1$, R the quotient ring $A/(X_1X_2, X_3X_4)$ of A and I the ideal (X_1, X_2X_5) of R. Then R is reduced, $K_R \simeq X_5 \cdot R$, and $R/I \simeq R/(X_5)$ is Gorenstein, however, $I \not\simeq K_R$.

Let $R = \bigoplus_{n \geq 0} R_n$ be an ASL on a weighted poset (Q, ω) over a field $R_0 = k$. Then, by (ASL-1), the Poincaré series $F_R(\lambda)$ of R coincides with the Poincaré series $F_{(Q,\omega)}(\lambda)$ of (Q, ω). Moreover, if I is a poset ideal of Q, then $F_{I\cdot R}(\lambda) = F^I_{(Q,\omega)}(\lambda)$ by (1.6).

Hence, by virtue of a fundamental theorem of ASL and (1.6), we obtain the following result as a corollary to (1.7).

(1.8) Corollary. *Let (Q, ω) be a Cohen-Macaulay weighted poset which possesses a canonical ideal I, and $R = \bigoplus_{n \geq 0} R_n$ an ASL domain on (Q, ω). Then the canonical module K_R of R is isomorphic to the ideal $I \cdot R$ of R as graded R-modules up to shift in grading.*

This is the reason why we introduce the concept of canonical ideals of Cohen-Macaulay posets.

A weighted poset (Q, ω) is called *weakly Gorenstein* over a field k if Q is Cohen-Macaulay over k and there exists a Gorenstein ASL on (Q, ω) over k. If Q is Gorenstein over k, then (Q, ω) is weakly Gorenstein over k for any weight ω on Q. Also, a weakly Gorenstein weighted poset is automatically numerically Gorenstein. See [H_1], [H_2], [H_6], [H_7] and [Wat] for some results on Gorenstein posets.

A weighted poset (Q, ω) is called *integral* over a field k if there exists an ASL domain on (Q, ω) over k. Refer to [H_2], [H_4], [H_5], [H_8], [H-W] and [Wat] for some information on integral posets.

We close this section with the following

(1.9) Proposition. *Let (Q, ω) be a Cohen-Macaulay weighted poset which possesses a canonical ideal I. If (Q, ω) is integral then the weighted poset $(Q - I, \omega)$ is weakly Gorenstein.*

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§ 2. Edge-labelings of partially ordered sets

This section is a fundamental work which is indispensable for tho classification (3.12) of distributive lattices with canonical ideals. More systematic study related with this section will appear in [H_9].

Given a poset P, we write $P^\wedge$ for the poset obtained by adjoining a *new* pair of elements, written as $0^\wedge$ and $1^\wedge$, to P such that $0^\wedge < x < 1^\wedge$ for all $x \in P$. If we only require that $0^\wedge$ or $1^\wedge$ be adjoined, we write $P_{0^\wedge}$ or $P^{1^\wedge}$ respectively. We use the convention that $0^\wedge$ or $1^\wedge$ is *never* an element of P.

The symbol "$<\cdot$" denotes the covering relation, that is to say, $x <\cdot y$ means that $x < y$ and $x < z < y$ for no z. For any poset P, we write $\mathscr{C}(P)$ for its covering relation

$$\mathscr{C}(P) = \{(x, y) \in P \times P; x <\cdot y\}.$$

Thus, roughly speaking, $\mathscr{C}(P)$ is the set of "edges" in the Hasse diagram of the poset P.

An *edge-labeling* of P is a map $\delta: \mathscr{C}(P) \to \mathbb{N}$. Thus an edge-labeling corresponds to an assignment of non-negative integers to the edges of the Hasse diagram of P. The technique of edge-labelings originated in Stanley's work [Sta_3] and was developed by Björner [Bjö].

An edge-labeling δ of a poset P is called *positive* (resp. *non-zero*) if $\delta(x, y) > 0$ for any (resp. some) $x <\cdot y$ of P.

The edge-labeling which we are interested in is the following

(2.1) Definition. An edge-labeling δ of a poset P is called *path-free* if, for any two unrefinable chains

$$x = x_0 <\cdot x_1 <\cdot \cdots <\cdot x_n = y$$

and

$$x = y_0 <\cdot y_1 <\cdot \cdots <\cdot y_m = y$$

of P combining x with y, we have the equality

$$(***) \qquad \sum_{i=0}^{n-1} \delta(x_i, x_{i+1}) = \sum_{j=0}^{m-1} \delta(y_j, y_{j+1}).$$

Let P be an arbitrary poset. We denote by $\mathscr{D}(P^\wedge)$ the set of path-free edge-labelings of $P^\wedge$. Define the partial order in $\mathscr{D}(P^\wedge)$ as follows: $\delta \le \delta'$ if $\delta(x, y) \le \delta'(x, y)$ for every $x <\cdot y$ of $P^\wedge$. Also, let $\mathscr{D}_*(P^\wedge)$ (resp. $\mathscr{D}_+(P^\wedge)$) be the subposet of $\mathscr{D}(P^\wedge)$ which consists of all path-free positive (resp. non-zero) edge-labelings of $P^\wedge$. Let $\mathscr{M}_*(P^\wedge)$ (resp. $\mathscr{M}_+(P^\wedge)$) be the set of minimal elements of the poset $\mathscr{D}_*(P^\wedge)$ (resp. $\mathscr{D}_+(P^\wedge)$). Though we mainly consider finite posets only in this paper, we here study the infinite poset $\mathscr{D}(P^\wedge)$ exceptionally.

We make $\mathscr{D}(P^\wedge)$ an additive semigroup with identity by $(\delta + \delta')(x, y) := \delta(x, y) + \delta'(x, y)$. Note that if $\delta \leq \delta'$ then the edge-labeling $\delta' - \delta$, which is defined by $(\delta' - \delta)(x, y) := \delta'(x, y) - \delta(x, y)$, is contained in $\mathscr{D}(P^\wedge)$.

On the other hand, we naturally associate $\mathscr{D}(P^\wedge)$ (resp. $\mathscr{D}_*(P^\wedge)$) with the set of solutions in non-negative (resp. positive) integers to the system (***) of linear equations.

(2.2) EXAMPLE. The set $\mathscr{D}(P^\wedge)$ of

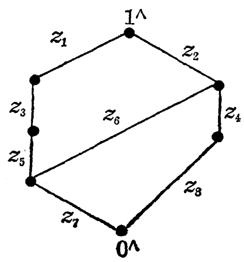
$P^\wedge =$ 

Fig. 4.

corresponds to the set of solutions in non-negative integers to the system

$$\begin{cases} z_1 + z_3 + z_5 = z_2 + z_6 \\ z_4 + z_8 = z_6 + z_7 \end{cases}$$

of linear equations.

For any poset ideal I, including $I = \varnothing$, of P, we denote by δ_I the path-free edge-labeling of $P^\wedge$ defined by

$$\delta_I(x, y) = \begin{cases} 1 & \text{if } x \in I \cup \{0^\wedge\} \text{ and } y \notin I \cup \{0^\wedge\} \\ 0 & \text{otherwise}. \end{cases} \tag{2.3}$$

(2.4) EXAMPLE. Consider the following poset

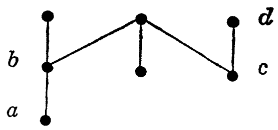
$P =$ 

Fig. 5.

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and the poset ideal $I = \{a, b, c, d\}$. Then the path-free edge-labeling δ_I of $P^\wedge$ looks like

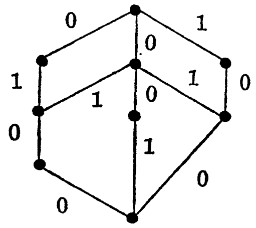

Fig. 6.

Now, it is natural to ask what the set $\mathscr{M}_+(P^\wedge)$ is.

(2.5) Proposition. *A path-free edge-labeling δ of $P^\wedge$ is contained in $\mathscr{M}_+(P^\wedge)$ if and only if $\delta = \delta_I$ for some poset ideal I of P.*

Proof. We easily see that $\delta_I \in \mathscr{M}_+(P^\wedge)$ for any poset ideal I of P. Conversely, let $\delta \in \mathscr{D}_+(P^\wedge)$ and I the poset ideal of P consisting of all elements x of P with the following property: For some (or equivalently, any) unrefinable chain

$$x = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_n = 1$$

of $P^{1\wedge}$, we have

$$\sum_{i=0}^{n-1} \delta(x_i, x_{i+1}) > 0 .$$

We claim $\delta_I \leq \delta$. Let $(x, y) \in \mathscr{C}(P^\wedge)$ with $\delta_I(x, y) = 1$ and

$$(0^\wedge \leq) \quad x <\cdot\, y = y_0 <\cdot\, y_1 <\cdot \cdots <\cdot\, y_m = 1^\wedge$$

one of the unrefinable chains of $P^\wedge$ combining x with $1^\wedge$. Then we have the inequailty

$$\delta(x, y) + \sum_{j=0}^{m-1} \delta(y_j, y_{j+1}) > 0$$

since $x \in I \cup \{0^\wedge\}$ and $\delta \in \mathscr{D}_+(P^\wedge)$. On the other hand, the equality $\sum_{0 \leq j \leq m-1} \delta(y_j, y_{j+1}) = 0$ holds since $y \notin I$, thus $\delta(x, y) > 0$ as desired. Q.E.D.

Remark. If I and I', $I \neq I'$, are poset ideals of P, then δ_I and $\delta_{I'}$ are incomparable in the poset $\mathscr{D}(P^\wedge)$.

We now study $\mathscr{D}_*(P^\wedge)$ and $\mathscr{M}_*(P^\wedge)$.

(2.6) PROPOSITION. $\sharp(\mathcal{M}_*(P^\wedge)) = 1$ *if and only if* P *is pure.*

Proof. To begin with, define $\delta^{[d]} \in \mathcal{D}_*(P^\wedge)$ to be

$$\delta^{[d]}(x, y) = \mathrm{depth}_{P^\wedge}(x) - \mathrm{depth}_{P^\wedge}(y) \tag{2.7}$$

for any $x <\cdot\, y$ of $P^\wedge$. We claim $\delta^{[d]} \in \mathcal{M}_*(P^\wedge)$. If

$$0^\wedge = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_r = 1^\wedge \qquad (r = \mathrm{rank}\,(P^\wedge))$$

is a maximal chain of $P^\wedge$, whose length is equal to rank$(P^\wedge)$, then $\delta^{[d]}(x_i, x_{i+1}) = 1$ for every i. If $\delta \in \mathcal{D}(P^\wedge)$ and $\delta < \delta^{[d]}$, then we have the inequality

$$\sum_{i=0}^{r-1} \delta(x_i, x_{i+1}) < \sum_{i=0}^{r-1} \delta^{[d]}(x_i, x_{i+1}),$$

thus $\delta(x_i, x_{i+1}) = 0$ for some i, $0 \le i < r$. Hence $\delta \notin \mathcal{D}_*(P^\wedge)$, and therefore $\delta^{[d]} \in \mathcal{M}_*(P^\wedge)$.

Now, the "if" part is easy. In fact, if P is pure then $\delta^{[d]}(x, y) = 1$ for any $x <\cdot\, y$ of $P^\wedge$. Obviously, $\delta^{[d]} \le \delta$ for any $\delta \in \mathcal{D}_*(P^\wedge)$. Hence $\mathcal{M}_*(P^\wedge) = \{\delta^{[d]}\}$.

To see why the "only if" part is true, assume that P is not pure. Then $\delta^{[d]}(\alpha, \beta) > 1$ for some $\alpha <\cdot\, \beta$ of $P^\wedge$. We consider the map $d'\colon P^\wedge \to \mathbb{N}$ defined by

$$d'(x) = \begin{cases} \mathrm{depth}_{P^\wedge}(x) + \delta^{[d]}(\alpha, \beta) - 1 & \text{if } \alpha \ne x \le \beta \\ \mathrm{depth}_{P^\wedge}(x) & \text{otherwise}. \end{cases}$$

By means of this map d', we define $\delta' \in \mathcal{D}_*(P^\wedge)$ to be $\delta'(x, y) := d'(x) - d'(y)$. Then $\delta'(\alpha, \beta) = 1$, thus $\delta^{[d]} \not\le \delta'$, hence $\sharp(\mathcal{M}_*(P^\wedge)) > 1$. Q.E.D.

From the above construction of $\delta' \in \mathcal{D}_*(P^\wedge)$, we immediately see the following

(2.8) COROLLARY. *For each covering relation* $x <\cdot\, y$ *of* $P^\wedge$ *there exists* $\delta \in \mathcal{M}_*(P^\wedge)$ *with* $\delta(x, y) = 1$.

(2.9) DEFINITION. Let P be an arbitrary poset and $\mathcal{I}$ a collection of poset ideals of P. Then $\mathcal{I}$ is called *basic* if the following conditions are satisfied:

(2.10) The empty set $\varnothing$ is contained in $\mathcal{I}$.

(2.11) If I and J are poset ideals such that $I \in \mathcal{I}$ and $J \subset I$, then $J \in \mathcal{I}$.

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(2.12) There exists $\delta_* \in \mathscr{D}(P^\wedge)$, called the *shifting* of $\mathscr{I}$, such that $\mathscr{M}_*(P^\wedge) = \{\delta_I + \delta_*; I \in \mathscr{I}\}$.

For which posets P does there exist a basic set $\mathscr{I}$?

(2.13) PROPOSITION. *Let P be an arbitrary poset. Then $\mathscr{I} = \{\varnothing\}$ is a basic set of P if and only if P is pure.*

Proof. Thanks to the proof of (2.6), if P is pure then $\mathscr{M}_*(P^\wedge) = \{\delta^{[d]}\}$, where $\delta^{[d]}$ is the edge-labeling (2.7). Let $\delta_* := \delta^{[d]} - \delta_\phi \in \mathscr{D}(P^\wedge)$. Then $\mathscr{I} = \{\varnothing\}$ is a basic set of P with the shifting δ_*.

On the other hand, if $\mathscr{I} = \{\varnothing\}$ is a basic set of P, then $\sharp(\mathscr{M}_*(P^\wedge)) = 1$, hence P is pure by (2.6). Q.E.D.

(2.14) LEMMA. *Let $\mathscr{I}$ be a basic set of a poset P and $\delta_* \in \mathscr{D}(P^\wedge)$ the shifting of $\mathscr{I}$. Then $\delta_*(x, y) \leq 1$ for each covering relation $x <\cdot\, y$ of $P^\wedge$. Moreover, $\delta_*(x, y) = 1$ if $0^\wedge < x <\cdot\, y$ $(\leq 1^\wedge)$.*

Proof. Thanks to (2.8), $\delta_*(x, y) \leq 1$ for each covering relation $x <\cdot\, y$ of $P^\wedge$. Also, since $\varnothing \in \mathscr{I}$, $\delta_\phi + \delta_*$ must be positive, hence $\delta_*(x, y) > 0$ if $0^\wedge < x <\cdot\, y$ $(\leq 1^\wedge)$. Q.E.D.

By the path-free property of the shifting δ_*, we obtain

(2.15) COROLLARY. *Assume that a poset P possesses a basic set $\mathscr{I}$. Then, for any element α of P, the interval*

$$[\alpha, 1^\wedge) := \{x \in P^{1^\wedge}; \alpha \leq x < 1^\wedge\}$$

of $P^{1^\wedge}$ is pure.

(2.16) LEMMA. *Assume that, for each element α of a poset P, the interval $[\alpha, 1^\wedge)$ of $P^{1^\wedge}$ is pure. Then, for any maximal chain of $P^\wedge$ of the form*

$$0^\wedge = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_{\mathrm{rank}(P^\wedge)} = 1^\wedge,$$

and for any $\delta \in \mathscr{M}_*(P^\wedge)$, we have $\delta(x_i, x_{i+1}) = 1$ for every $0 \leq i < \mathrm{rank}(P^\wedge)$.

Proof. Let $\delta \in \mathscr{M}_*(P^\wedge)$. We define the map $d_\delta \colon P^\wedge \to \mathbb{N}$ as follows. If $x \in P_{0^\wedge}$ and

$$x = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_n = 1^\wedge$$

is one of the unrefinable chains of $P^\wedge$ combining x with $1^\wedge$, then

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$$d_\delta(x) := \sum_{i=0}^{n-1} \delta(x_i, x_{i+1}), \tag{2.17}$$

and $d_\delta(1^\wedge) = 0$. Then $d_\delta(x) \geq \operatorname{depth}_{P^\wedge}(x)$ for every $x \in P^\wedge$. To obtain the conclusion, we have only to prove $d_\delta(0^\wedge) = \operatorname{rank}(P^\wedge)$.

So, assume that $d_\delta(0^\wedge) > \operatorname{rank}(P^\wedge)$. Let $\mathfrak{A}$ be the subset of P consisting of all elements $x \in P$ with $\operatorname{depth}_{P^\wedge}(x) < d_\delta(x)$, and $\mathfrak{B} = P - \mathfrak{A}$. Since the interval $[\alpha, 1^\wedge)$ of $P^{1^\wedge}$ is pure for any $\alpha \in P$, the subset $\mathfrak{A}$ of P is a poset ideal of P. Also, if $x \in \mathfrak{A} \cup \{0^\wedge\}$, $y \in \mathfrak{B} \cup \{1^\wedge\}$ and $x <\cdot\, y \in \mathscr{C}(P^\wedge)$, then $d_\delta(x) - d_\delta(y) > 1$. We now define another map $d^\#\colon P^\wedge \to \mathbb{N}$ to be

$$d^\#(x) = \begin{cases} d_\delta(x) - 1 & \text{if } x \in \mathfrak{A} \cup \{0^\wedge\} \\ d_\delta(x) & \text{if } x \in \mathfrak{B} \cup \{1^\wedge\}, \end{cases}$$

and, by using this map $d^\#$, define $\delta^\# \in \mathscr{D}_*(P^\wedge)$ to be $\delta^\#(x, y) := d^\#(x) - d^\#(y)$. Then $\delta^\# < \delta$ in $\mathscr{D}_*(P^\wedge)$, which contradicts $\delta \in \mathscr{M}_*(P^\wedge)$. Q.E.D.

(2.18) PROPOSITION. *A poset P possesses a basic set if and only if the following conditions are satisfied:*

(2.19) *For any element α of P, the interval $[\alpha, 1^\wedge)$ of $P^{1^\wedge}$ is pure.*

(2.20) *The inequality* $\operatorname{rank}(P^\wedge) - \operatorname{depth}_{P^\wedge}(\beta) \leq 2$ *holds for any element $\beta \in P$ with $0^\wedge <\cdot\, \beta$ in $P^\wedge$.*

Proof. First, we shall prove the "only if" part. Thanks to (2.15), the condition (2.19) holds. Let δ_* be the shifting of a basic set $\mathscr{I}$ of P and $\delta^{[d]} \in \mathscr{M}_*(P^\wedge)$ the edge-labeling (2.7). Then $\delta^{[d]} = \delta_I + \delta_*$ for some $I \in \mathscr{I}$, hence

$$\begin{aligned} \operatorname{rank}(P^\wedge) - \operatorname{depth}_{P^\wedge}(\beta) &= \delta^{[d]}(0^\wedge, \beta) \\ &= \delta_I(0^\wedge, \beta) + \delta_*(0^\wedge, \beta) \leq 2 \end{aligned}$$

by (2.14) if $0^\wedge <\cdot\, \beta$ in $P^\wedge$.

Conversely, to prove the "if" part, let $\mathfrak{C}$ be the set of minimal elements of P and, for $i = 1, 2$,

$$\mathfrak{C}_i = \{x \in \mathfrak{C};\ \operatorname{rank}(P^\wedge) - \operatorname{depth}_{P^\wedge}(x) = i\}. \tag{2.21}$$

Let $\mathscr{I}$ be the set of poset ideals I of P with $I \cap \mathfrak{C}_1 = \varnothing$. Also, let $\delta_* \in \mathscr{D}(P^\wedge)$ be the edge-labeling defined by

$$\delta_*(x, y) = \begin{cases} 0 & \text{if } x = 0^\wedge \text{ and } y \in \mathfrak{C}_1 \\ 1 & \text{otherwise}. \end{cases}$$

We claim $\mathscr{I}$ is a basic set of P with the shifting δ_*.

Let $\delta \in \mathscr{M}_*(P^\wedge)$. Then $\delta - \delta_* \in \mathscr{D}_+(P^\wedge)$. Hence, thanks to (2.5), $\delta_I \leq \delta - \delta_*$ for some poset ideal I of P. If $x_1 \in \mathfrak{C}_1$ and

$$0^\wedge = x_0 <\cdot\, x_1 <\cdot \cdots <\cdot\, x_{\mathrm{rank}(P^\wedge)} = 1^\wedge$$

is a maximal chain of $P^\wedge$, then, by (2.16), $\delta(x_i, x_{i+1}) = 1$ for every i, $0 \leq i < \mathrm{rank}(P^\wedge)$. Thus $(\delta - \delta_*)(x_i, x_{i+1}) = 0$ if $i \geq 1$. So, $\delta_I(x_i, x_{i+1}) = 0$ if $i \geq 1$, hence $x_1 \notin I$, and therefore $I \in \mathscr{I}$. Now, $\delta_I + \delta_* \in \mathscr{D}_*(P^\wedge)$, $\delta \in \mathscr{M}_*(P^\wedge)$ and $\delta_I + \delta_* \leq \delta$ in $\mathscr{D}_*(P^\wedge)$ together imply $\delta_I + \delta_* = \delta$. Hence $\mathscr{M}_*(P^\wedge) \subset \{\delta_I + \delta_*; I \in \mathscr{I}\}$. On the other hand, $\delta_I + \delta_* \in \mathscr{D}_*(P^\wedge)$ if $I \in \mathscr{I}$. Thus, thanks to the remark after the proof of (2.5), $\mathscr{M}_*(P^\wedge) = \{\delta_I + \delta_*; I \in \mathscr{I}\}$.
Q.E.D.

(2.22) COROLLARY. *A basic set $\mathscr{I}$ of a poset P is unique if it exists.*

Proof. If P is pure, then $\sharp(\mathscr{M}_*(P^\wedge)) = 1$ by (2.6), hence $\mathscr{I} = \{\emptyset\}$ is a unique basic set of P.

Assume that P is not pure and that P satisfies the conditions (2.19) and (2.20). Let $\mathfrak{C}_i$ $(i = 1, 2)$ be the sets (2.21) and $\mathscr{I}$ a basic set consisting of all poset ideals I of P with $I \cap \mathfrak{C}_1 = \emptyset$. Let $\mathscr{I}'$ be another basic set of P and δ'_* the shifting of $\mathscr{I}'$. If a poset ideal $I' \in \mathscr{I}'$ contains an element $y \in \mathfrak{C}_1$, then $\delta_{I'}(0^\wedge, y) = 0$, thus $\delta'_*(0^\wedge, y) = 1$. Since $\delta'_*(x, y) = 1$ if $0^\wedge < x <\cdot\, y \leq 1^\wedge$ by the latter half of (2.14) and δ'_* is path-free, we have $\delta'_*(0^\wedge, x) = 2$ if $x \in \mathfrak{C}_2$, which contradicts the first half of (2.14). Thus $\mathscr{I}' \subset \mathscr{I}$. Since $\sharp(\mathscr{I}') = \sharp(\mathscr{I}) = \sharp(\mathscr{M}_*(P^\wedge))$, we have $\mathscr{I}' = \mathscr{I}$. Q.E.D.

So, from now on, we call *the* basic set $\mathscr{I}$ of a poset P.

§ 3. Which distributive lattices possess canonical ideals?

We now consider the problem of finding all distributive lattices which possess canonical ideals.

First, recall the Birkhoff's fundamental structure theorem [Bir, p. 59] for finite distributive lattices.

A lattice L is called *distributive* if the distributive laws

$$\alpha \wedge (\beta \vee \gamma) = (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$
$$\alpha \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$$

hold for all $\alpha, \beta, \gamma \in L$. A lattice L is distributive if and only if L contains neither

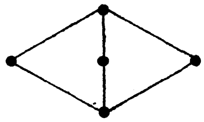

Fig. 7.

nor

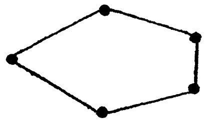

Fig. 8.

as a sublattice.

Let P be an arbitrary poset and $\mathcal{J}(P)$ the poset consisting of all poset ideals of P, ordered by inclusion. Then it can be checked immediately that $\mathcal{J}(P)$ is a distributive lattice. A classical fundamental structure theorem of Birkhoff guarantees the converse, that is to say, for any finite distributive lattice D, there exists a unique poset P such that $D = \mathcal{J}(P)$.

An element α of a lattice is called *join-irreducible* if $\alpha = \beta \vee \gamma$ implies $\alpha = \beta$ or $\alpha = \gamma$. Let D be a distributive lattice and P the subposet consisting of all join-irreducible elements of D. Then $D = \mathcal{J}(P)$. For example, if

$D =$

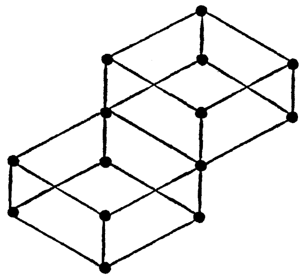

Fig. 9.

then

$P =$

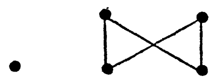

Fig. 10.

Next, let us consider the solutions in non-negative integers to a system of linear equations over $\mathbb{Z}$.

Let $\Phi = (a_{ij})_{\substack{1 \leqq i \leqq r \\ 1 \leqq j \leqq n}}$ be an $r \times n$ $\mathbb{Z}$-matrix and

$$E_\Phi := \left\{\beta = (\beta_1, \cdots, \beta_n) \in \mathbb{N}^n;\ \sum_{j=1}^{n} a_{ij}\beta_j = 0,\ 1 \leqq i \leqq r\right\}$$

the set of solutions in non-negative integers to the system of linear equations $\sum_{j=1}^{n} a_{ij}x_j = 0$ ($i = 1, 2, \cdots, r$) over $\mathbb{Z}$. Clearly, E_Φ is an additive semigroup with identity.

An element $\beta \in E_\Phi$ is called *fundamental* if $\beta = \gamma + \delta$ $(\gamma, \delta \in E_\Phi)$ implies $\gamma = \beta$ or $\delta = \beta$. We denote by FUND_Φ the set of non-zero fundamental elements of E_Φ. It is a consequence in classical invariant theory that there are only finitely many non-zero fundamental elements of E_Φ. Thus E_Φ is finitely generated as an additive semigroup, in other words, E_Φ is an affine semigroup. Consult [Sta$_2$], [Sta$_4$], [Sta$_6$] and [Sta$_{10}$] for further information.

Let k be a field and $k[X_1, X_2, \cdots, X_n]$ the polynomial ring in n-variables over k and $R_\Phi := k[E_\Phi]$ the affine semigroup ring

$$k[X^\beta, \beta \in E_\Phi] \qquad (\subset k[X_1, X_2, \cdots, X_n])$$

of E_Φ over k, where $X^\beta = X_1^{\beta_1} X_2^{\beta_2} \cdots X_n^{\beta_n}$ if $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$. By virtue of [Hoc$_1$], R_Φ is Cohen-Macaulay. Note that R_Φ is generated by $\{X^\beta; \beta \in \mathrm{FUND}_\Phi\}$ as a k-algebra. We call an element $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in E_\Phi$ *positive* if $\beta_i > 0$ for every $1 \leq i \leq n$. Let E_Φ^* be the set of positive elements of E_Φ and

$$k[E_\Phi^*] := k[X^\beta; \beta \in E_\Phi^*],$$

which is an ideal of R_Φ. Without loss of generality, we may assume that the set E_Φ^* is non-empty.

Assume, for the moment, that R_Φ is endowed a structure of a graded ring $\bigoplus_{n\geq 0} (R_\Phi)_n$ over $(R_\Phi)_0 = k$ such that each monomial X^β, $\beta \in E_\Phi$, is contained in $(R_\Phi)_n$ for some n $(= n_\beta) > 0$. Let $F_{R_\Phi}(\lambda)$ (resp. $F_{k[E_\Phi^*]}(\lambda)$) be the Poincaré series of the graded ring R_Φ (resp. the graded ideal $k[E_\Phi^*]$ of R_Φ). Then

(3.1) LEMMA. $F_{R_\Phi}(\lambda^{-1}) = (-1)^d F_{k[E_\Phi^*]}(\lambda)$, *where* $d = \dim(R_\Phi)$.

Proof. Consult [Sta$_2$, (23)] and [Sta$_2$, (26)]. Q.E.D.

(3.2) COROLLARY. *The canonical module* K_{R_Φ} *of* $R_\Phi = \bigoplus_{n\geq 0} (R_\Phi)_n$ *coincides with* $k[E_\Phi^*]$.

Proof. Consult [Sta$_7$, (6.7)]. Q.E.D.

(3.3) EXAMPLE. Let $\Phi = [1\ 1\ -2]$, so we study the solutions in non-negative integers to the linear equation $x + y - 2z = 0$ over $\mathbb{Z}$. Then $\mathrm{FUND}_\Phi = \{(2, 0, 1), (0, 2, 1), (1, 1, 1)\}$ and $R_\Phi = k[X^2Z, Y^2Z, XYZ]$. Then R_Φ is considered as a graded ring $\bigoplus_{n\geq 0} (R_\Phi)_n$ with $\deg(X^2Z) = p$, $\deg(Y^2Z) = q$ and $\deg(XYZ) = r$ if and only if $p + q = 2r$. Under the assumption

$p+q=2r$, we have

$$F_{R_\Phi}(\lambda)=\frac{1+\lambda^r}{(1-\lambda^p)(1-\lambda^q)}\,,\qquad F_{k[E_\Phi^*]}(\lambda)=\frac{\lambda^r(1+\lambda^r)}{(1-\lambda^p)(1-\lambda^q)}\,.$$

Thus $F_{R_\Phi}(\lambda^{-1})=F_{k[E_\Phi^*]}(\lambda)$.

Let $D=\mathscr{J}(P)$ be a distributive lattice and $\mathscr{D}(P^\wedge)$ the affine semigroup considered in the previous section. Also, let k be a field and $k[\mathscr{D}(P^\wedge)]$ the affime semigroup ring of $\mathscr{D}(P^\wedge)$ over k.

(3.4) LEMMA. *If we embed $D=\mathscr{J}(P)$ into $k[\mathscr{D}(P^\wedge)]$ by the injective map $\psi\colon D\ (=\mathscr{J}(P))\to k[\mathscr{D}(P^\wedge)]$ defined by*

$$\psi(I)=X^{\delta_I}\in k[\mathscr{D}(P^\wedge)]\qquad (I\in\mathscr{J}(P))\,,$$

then $k[\mathscr{D}(P^\wedge)]$ is an ASL on D over k.

Proof. By means of the map d_δ defined in (2.17), it is easy to see that the affine semigroup $\mathscr{D}(P^\wedge)$ is isomorphic to $\mathscr{S}(D)$ of [H_2, (3.2)]. Hence the conclusion follows from [H_2] immediately. Also, see [Gar]. Q.E.D.

Note that

$$\delta_I+\delta_J=\delta_{I\cap J}+\delta_{I\cup J}\tag{3.5}$$

for all $I, J\in\mathscr{J}(P)$, and that

$$k[\mathscr{D}(P^\wedge)]\simeq k[X_\alpha;\ \alpha\in D]/(X_\alpha X_\beta-X_{\alpha\wedge\beta}X_{\alpha\vee\beta};\ \alpha\not\sim\beta)\,.\tag{3.6}$$

Hence,

(3.7) COROLLARY. *Let $D=\mathscr{J}(P)$ be a distributive lattice and ω a weight on D. Then the k-algebra $k[\mathscr{D}(P^\wedge)]$ is an ASL on the weighted poset (D,ω), with respect to the embedding ψ, over k if and only if ω satisfies the equality*

$$\omega(\alpha)+\omega(\beta)=\omega(\alpha\wedge\beta)+\omega(\alpha\vee\beta)\tag{3.8}$$

for any $\alpha,\beta\in D$.

Before studying the k-algebra $k[\mathscr{D}(P^\wedge)]$ further, we recall the concept of "wonderful posets".

A poset Q is called *wonderful* (or locally semimodular) if the following condition holds in the poset $Q^\wedge$: If $y_1, y_2<z$ are covers of an element x, then there is an element $y\le z$ which is a cover of both y_1 and y_2.

(3.9) LEMMA. *A wonderful poset is Cohen-Macaulay over an arbitrary field.*

For the proof, consult [D-E-P, (8.1)] and [Bjö, (6.1)]. Also, see [H_3, (4.4)].

Without difficulty, we can check that every distributive lattice is wonderful.

(3.10) LEMMA ([D-E-P, Lemma 8.2]). *Let Q be a wonderful poset and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ a collection of minimal elements of Q. Define the poset ideal I of Q to be*

$$I = \{x \in Q;\ x \ngeq \alpha_i \text{ for all } i\}.$$

Then the subposet $Q - I$ is wonderful.

Now, our main result is

(3.11) THEOREM. *Let $D = \mathscr{J}(P)$ be a distributive lattice and ω a weight on D satisfying the condition* (3.8). *Then a poset ideal $\mathscr{I}$ of $\mathscr{J}(P)$ is a canonical ideal of the weighted poset (D, ω) if and only if $\mathscr{I}$ is the basic set of P.*

Proof. First, to prove the "if" part, assume that a poset ideal $\mathscr{I}$ of $\mathscr{J}(P)$ is a basic set of P with the shifting $\delta_* \in \mathscr{D}(P^\wedge)$. Let $\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]$ be the ideal of $k[\mathscr{D}(P^\wedge)]$ generated by $\{X^{\delta_I}; I \in \mathscr{I}\}$. Since

$$\mathscr{M}_*(P^\wedge) = \{\delta_I + \delta_*; I \in \mathscr{I}\},$$

we have

$$k[\mathscr{D}_*(P^\wedge)] = X^{\delta_*}(\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]).$$

Hence, if we consider $k[\mathscr{D}(P^\wedge)]$ to be a graded ring $\bigoplus_{n\geq 0}(k[\mathscr{D}(P^\wedge)])_n$ over $(k[\mathscr{D}(P^\wedge)])_0 = k$ with $\deg(X^{\delta_I}) = \omega(\alpha)$, $\alpha(\in D) = I(\in \mathscr{J}(P))$, for any $\alpha \in D$, then

$$F_{k[\mathscr{D}_*(P^\wedge)]}(\lambda) = \lambda^{-a} F_{\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]}(\lambda),$$

where $a = -\deg(X^{\delta_*})$. Thus, since $k[\mathscr{D}(P^\wedge)]$ is an ASL on (D, ω) over k,

$$(-1)^d F_{(D,\omega)}(\lambda^{-1}) = \lambda^{-a} F^{\mathscr{I}}_{(D,\omega)}(\lambda)$$

by (3.1), where $d = \operatorname{rank}(D) + 1$.

The remains of our work is to prove the subposet $D - \mathscr{I}$ is Cohen-Macaulay with $\operatorname{rank}(D - \mathscr{I}) = \operatorname{rank}(D) - 1$. Let $\mathfrak{C}_i$ $(i = 1, 2)$ be the subsets (2.21) of P. Then, by the latter half of the proof of (2.18), we obtain

$$\mathscr{I} = \{\alpha \in D(=\mathscr{J}(P));\ \alpha \not\geq \{x\} \in \mathscr{J}(P) \text{ for all } x \in \mathfrak{C}_1\}.$$

Hence, we have $\operatorname{rank}(D - \mathscr{I}) = \operatorname{rank}(D) - 1$ and, thanks to (3.10), $D - \mathscr{I}$ is wonderful.

Now, we shall prove the "only if" part. Since $\mathscr{I}$ is a canonical ideal of (D, ω), the ideal $\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]$ of the ASL $k[\mathscr{D}(P^\wedge)]$ on the weighted poset (D, ω) over k is isomorphic to the canonical module $K_{k[\mathscr{D}(P^\wedge)]} = k[\mathscr{D}_*(P^\wedge)]$ of $k[\mathscr{D}(P^\wedge)]$ as graded $k[\mathscr{D}(P^\wedge)]$-modules up to shift in grading. Let

$$b = \min\{n \in \mathbb{N};\ (k[\mathscr{D}_*(P^\wedge)])_n \neq 0\}$$

and

$$c = \min\{n \in \mathbb{N};\ (\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)])_n \neq 0\}.$$

Also, let f be a homogeneous element of degree $b - c$ of the quotient field of $k[\mathscr{D}(P^\wedge)]$ such that an isomorphism, up to shift in grading, from $\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]$ to $k[\mathscr{D}_*(P^\wedge)]$ is obtained by the multiplication of f. Let $X^{\delta_{I'}}$ $(I' \in \mathscr{I})$ be a monomial with $\deg(X^{\delta_{I'}}) = c$. Also, let

$$\mathscr{N} = \{\delta \in \mathscr{M}_*(P^\wedge);\ \deg(X^\delta) = b\}.$$

If $f\cdot X^{\delta_{I'}}$ is the linear combination

$$f\cdot X^{\delta_{I'}} = \sum_{\delta\in\mathscr{N}} c_\delta X^\delta \qquad (c_\delta \in k),$$

then, since $\delta_{I'} < \delta$ in $\mathscr{D}(P^\wedge)$ for any $\delta \in \mathscr{N}$, we obtain

$$f = \sum_{\delta\in\mathscr{N}} c_\delta X^{\delta-\delta_{I'}} \in k[\mathscr{D}(P^\wedge)].$$

Take $\delta' \in \mathscr{N}$ with $c_{\delta'} \neq 0$. Since $f\cdot X^{\delta_I} \in k[\mathscr{D}_*(P^\wedge)]$, $X^{\delta'-\delta_{I'}}\cdot X^{\delta_I}$ must be contained in $k[\mathscr{D}_*(P^\wedge)]$ for any $I \in \mathscr{I}$. Thus

$$X^{\delta'-\delta_{I'}}(\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]) \subset k[\mathscr{D}_*(P^\wedge)].$$

On the other hand, the Poincaré series of the ideal $X^{\delta'-\delta_{I'}}(\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)])$ coincides with that of $k[\mathscr{D}_*(P^\wedge)]$. Hence

$$X^{\delta'-\delta_{I'}}(\mathscr{I}\cdot k[\mathscr{D}(P^\wedge)]) = k[\mathscr{D}_*(P^\wedge)],$$

thus

$$\mathscr{M}_*(P^\wedge) = \{\delta_I + (\delta' - \delta_{I'});\ I \in \mathscr{I}\}.$$

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So, $\mathscr{I}$ is the basic set of P with the shifting $\delta_* = \delta' - \delta_{I'}$. Q.E.D.

(3.12) COROLLARY. *Let D be a distributive lattice, P the subposet consisting of all join-irreducible elements of D and ω a weight on D satisfying the condition* (3.8). *Then the weighted poset (D, ω) possesses a canonical ideal if and only if the poset P satisfies the conditions* (2.19) *and* (2.20).

Also, if P satisfies (2.19) *and* (2.20), *then the poset ideal*

$$\mathscr{I} = \left\{ \alpha \in D;\ \begin{array}{l} \alpha \ngeq \beta \text{ for any join-irreducible} \\ \text{element } \beta \text{ of } D \text{ with} \\ \operatorname{rank}(P^\wedge) - \operatorname{depth}_{P^\wedge}(\beta) = 1 \end{array} \right\}$$

of D is a canonical ideal of the weighted poset (D, ω).

Moreover, a canonical ideal $\mathscr{I}$ of (D, ω) is unique if it exists.

We should remark that the above corollary (3.12) is a somewhat surprising generalization of Stanley's famous result [Sta_{12}, Cor. 4.5.17 (b)].

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