Higher Chow cycles on a family of Kummer surfaces

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Abstract. We construct a collection of families of higher Chow cycles of type $(2, 1)$ on a two-dimensional family of Kummer surfaces, and prove that for a very general member, they generate a subgroup of rank ≥ 18 in the indecomposable part of the higher Chow group. Construction of the cycles uses a finite group action on the family, and the proof of their linear independence uses Picard–Fuchs differential operators.

1 Introduction

In the celebrated paper [Blo86], Bloch defined the *higher Chow groups* $\mathrm{CH}^p(X, q)$ for a variety X, which are generalizations of the classical Chow groups. They are identified with the motivic cohomology when X is smooth. Higher Chow groups appear in many aspects of algebraic geometry and number theory and are related to many important problems. However, their structures are still mysterious for many varieties when the codimension p is greater than 1. In general, it is not easy to construct nontrivial higher Chow cycles (cf. [Fla92, Mil92, Spi99]).

We study $\mathrm{CH}^2(X, 1)$ for a certain type of Kummer surfaces X. A cycle $\xi \in \mathrm{CH}^2(X, 1)$ is called *indecomposable* if it is not contained in the image of the map

$$\mathbb{C}^\times \otimes_{\mathbb{Z}} \mathrm{Pic}(X) = \mathrm{CH}^1(X, 1) \otimes_{\mathbb{Z}} \mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^2(X, 1)$$

induced by the intersection product. The image of this map is called the *decomposable part*: this is an easily accessible part. The cokernel of this map is called the *indecomposable part* and is denoted by $\mathrm{CH}^2(X, 1)_{\mathrm{ind}}$. We are interested in the size of $\mathrm{CH}^2(X, 1)_{\mathrm{ind}}$. Explicit constructions of nontrivial elements of $\mathrm{CH}^2(X, 1)_{\mathrm{ind}}$ for $K3$ surfaces X were initiated by Müller–Stach [MS97]. Since then, further examples have been constructed (see [CDKL16, CL05, Col99, dAMS02, Ker13, Sas21]).

In this paper, we study a two-dimensional family of Kummer surfaces $\widetilde{\mathcal{X}} \to T$ associated with products of elliptic curves. We construct a collection of families of higher Chow cycles on $\widetilde{\mathcal{X}} \to T$. They generate a subgroup Ξ_t of $\mathrm{CH}^2(\widetilde{\mathcal{X}}_t, 1)_{\mathrm{ind}}$ for every $t \in T$. Our main result is the following.

Theorem 1.1 (*Theorem* 7.1) *For a very general*[1] $t \in T$, *we have* $\operatorname{rank} \Xi_t \geq 18$. *In particular,* $\operatorname{rank} \operatorname{CH}^2(\mathfrak{X}_t, 1)_{\mathrm{ind}} \geq 18$.

Our construction uses the realization of the Kummer surfaces as the desingularized double coverings of $\mathbb{P}^1 \times \mathbb{P}^1$, and our construction of higher Chow cycles uses special types of $(1,1)$-curves on $\mathbb{P}^1 \times \mathbb{P}^1$. This is similar to the construction of higher Chow cycles on abelian surfaces in [Sre14]. The fact that $\operatorname{CH}^2(\widetilde{\mathfrak{X}}_t, 1)_{\mathrm{ind}} \neq 0$ already follows from the result of Section 6 of [CDKL16] because our family is a base change of the family considered there. Our novelty here is the rank estimate $\operatorname{rank} \operatorname{CH}^2(\widetilde{\mathfrak{X}}_t, 1)_{\mathrm{ind}} \geq 18$. To produce many indecomposable cycles, we use a group action on the family as we will explain below.

A standard way to detect indecomposability is to use the *transcendental regulator map*

$$r: \operatorname{CH}^2(\widetilde{\mathfrak{X}}_t, 1) \longrightarrow H^3_{\mathcal{D}}(\widetilde{\mathfrak{X}}_t, \mathbb{Z}(2)) \simeq \frac{F^1 H^2(\widetilde{\mathfrak{X}}_t, \mathbb{C})^{\vee}}{H_2(\widetilde{\mathfrak{X}}_t, \mathbb{Z})} \longrightarrow\mkern-14mu\rightarrow \frac{H^{2,0}(\widetilde{\mathfrak{X}}_t)^{\vee}}{H_2(\widetilde{\mathfrak{X}}_t, \mathbb{Z})},$$

where the first map is the Beilinson regulator map and the last is the natural projection. It is known that the map r factors through $\operatorname{CH}^2(\widetilde{\mathfrak{X}}_t, 1)_{\mathrm{ind}}$. For a family $\{\xi_t\}_{t \in T}$ of higher Chow cycles on $\widetilde{\mathfrak{X}} \to T$, their transcendental regulators give rise to a certain type of normal function $\nu_{\mathrm{tr}}(\xi) = \{r(\xi_t)\}_{t \in T}$. If we take the pairing of $\nu_{\mathrm{tr}}(\xi)$ with a relative 2-form ω on $\widetilde{\mathfrak{X}} \to T$, this gives a multivalued holomorphic function on T.

Our construction begins with an initial cycle family $\xi_1 - \xi_0$ (see Section 4 for the definition). For this cycle family, the above multivalued function is given by the improper integral

$$(1) \qquad \mathcal{L}(a, b) = 2 \int_{\triangle} \frac{dx\,dy}{\sqrt{x(1-x)(1-ax)}\sqrt{y(1-y)(1-by)}},$$

where $\triangle = \{(x, y) \in \mathbb{R}^2 : 0 < y < x < 1\}$. The integral (1) is similar to the integral representation of Appell's hypergeometric functions (though the boundary $\partial\triangle$ is not necessarily contained in the branching locus of the integrand). What is important to us is that $\mathcal{L}$ satisfies the following system of inhomogeneous differential equations:

$$(2) \qquad \begin{cases} a(1-a)\dfrac{\partial^2 \mathcal{L}}{\partial a^2} + (1-2a)\dfrac{\partial \mathcal{L}}{\partial a} - \dfrac{1}{4}\mathcal{L} = \dfrac{2}{a-b}\left(\dfrac{\sqrt{1-b}}{\sqrt{1-a}} - 1\right), \\[4mm] b(1-b)\dfrac{\partial^2 \mathcal{L}}{\partial b^2} + (1-2b)\dfrac{\partial \mathcal{L}}{\partial b} - \dfrac{1}{4}\mathcal{L} = \dfrac{2}{a-b}\left(1 - \dfrac{\sqrt{1-a}}{\sqrt{1-b}}\right). \end{cases}$$

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[1] We use the word "very general" for the meaning that "outside of a countable union of proper(= not the whole space) analytic subsets."

We denote the differential operators appearing in the left-hand side of (2) by $\mathscr{D}_1$ and $\mathscr{D}_2$, respectively. Note that they are the Gauss hypergeometric differential operators of type $\left(\frac{1}{2}, \frac{1}{2}, 1\right)$. Since the periods of $\widetilde{\mathcal{X}} \to T$ are annihilated by $\mathscr{D} = (\mathscr{D}_1, \mathscr{D}_2)$, $\mathscr{D}$ is called a *Picard–Fuchs differential operator* on $\widetilde{\mathcal{X}} \to T$. From the differential equations (2), we see that the image of $(\xi_1)_t - (\xi_0)_t$ under the transcendental regulator is nontrivial for very general t.

The second part of our construction is to produce more cycles from $\xi_1 - \xi_0$ by using a group action on $\widetilde{\mathcal{X}} \to T$. For our purpose, automorphisms which act trivially on T are not sufficient. We consider a finite group $G_{\mathcal{X}}$ (isomorphic to a $\mathbb{Z}/2$-extension of $(\mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4)^2$) which acts on the total space $\widetilde{\mathcal{X}}$ and also on the base T. By letting $G_{\mathcal{X}}$ act on $\xi_1 - \xi_0$, we obtain the subgroup Ξ_t of $\mathrm{CH}^2(\widetilde{\mathcal{X}}_t, 1)_{\mathrm{ind}}$. To show that Ξ_t has rank ≥ 18, we consider the $G_{\mathcal{X}}$-action on the normal function of $\xi_1 - \xi_0$ and the differential equations (2) it satisfies. This is the second main calculation in this paper.

Inhomogeneous Picard–Fuchs differential equations arising from normal functions are studied recently (e.g., [dAMS08]). The differential equations (2) is a $\mathbb{Q}(0)$-extension of the exterior tensor product of the two Gauss hypergeometric differential equations. Hence, this is an example where the normal function can be expressed by a variant of hypergeometric functions. In [AO21], Asakura and Otsubo constructed a family of varieties and cycles whose normal functions are expressed by the generalized hypergeometric function ${}_3F_2$. Our cycles give an analogous example.

1.1 Outline of the paper

This paper is divided into two parts. Part 1 is devoted to the construction of the initial cycle family $\xi_1 - \xi_0$ and the computation of its normal function. In Section 2, we recall basic facts about higher Chow cycles and the regulator map. In Sections 3 and 4, we define the family of Kummer surfaces and higher Chow cycles. In Section 5, we compute the normal function of $\xi_1 - \xi_0$ and deduce the indecomposability. Part 2 is devoted to the construction and calculation of the group action. In Section 6, we construct the finite group action on the Kummer family. In Section 7, we construct more higher Chow cycles using the group action and calculate the images of their normal functions under the Picard–Fuchs differential operator. This calculation is based on the transformation formula proved in Section 8.

1.2 List of notations

We summarize the frequently used notations in this paper. Usually, $\widetilde{(-)}$ indicates a blow-up. The subscript 0 (e.g., T_0) means an initial space; when it is taken out (e.g., $T \to T_0$), this means taking an étale cover or an étale base change (e.g., $\widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}_0$).

Notation	Explanation
S_0	$\mathbb{P}^1$ minus $0, 1, \infty$
T_0	An affine open subset of $S_0 \times S_0$
T	A finite étale cover of T_0
A_0	The coordinate ring of S_0
B_0	The coordinate ring of T_0
B	The coordinate ring of T
$\mathcal{Y}_0$ (resp. $\mathcal{Y}$)	A family of $\mathbb{P}^1 \times \mathbb{P}^1$ over T_0 (resp. T)
$\mathcal{X}_0$ (resp. $\mathcal{X}$)	a singular double cover of $\mathcal{Y}_0$ (resp. $\mathcal{Y}$)
$\mathcal{A}_0$ (resp. $\mathcal{A}$)	A family of products of elliptic curves over T_0 (resp. T)
Σ	A set of 4 sections of $\mathbb{P}^1 \times S_0$ and $\mathcal{E}$ over S_0
Σ^2	A set of 16 sections of $\mathcal{Y}, \mathcal{X}$ and $\mathcal{A}$ over T
$\widetilde{\mathcal{X}}_0$ (resp. $\widetilde{\mathcal{X}}$)	The family of Kummer surfaces over T_0 (resp. T)
$\widetilde{\mathcal{A}}_0$ (resp. $\widetilde{\mathcal{A}}$)	The blowing-up of $\mathcal{A}_0$ (resp. $\mathcal{A}$) along Σ^2
$\widetilde{\mathcal{Y}}_0$ (resp. $\widetilde{\mathcal{Y}}$)	The blowing-up of $\mathcal{Y}_0$ (resp. $\mathcal{Y}$) along Σ^2
U	An affine open subset of $\mathcal{Y}$
V, W	Affine open subsets of $\mathcal{X}$
$\mathcal{D}$	The family of diagonal lines on $\mathbb{P}^1 \times \mathbb{P}^1$ over T
$\mathcal{C}$	The pull-back of $\mathcal{D}$ by $\mathcal{X} \to \mathcal{Y}$
$\widetilde{\mathcal{C}}$	The strict transform of $\mathcal{C}$
$f(x), g(y)$	$f(x) = x(1-x)(1-ax),\ g(y) = y(1-y)(1-by) \in \mathcal{O}_{\mathcal{Y}}(U)$
G_{T_0}	A group ($\simeq \mathfrak{S}_3 \times \mathfrak{S}_3$) acting on T_0
G_T	A group ($\simeq \mathfrak{S}_4 \times \mathfrak{S}_4$) acting on T
$G_{\mathcal{Y}_0}$	A group ($\simeq \mathfrak{S}_4 \times \mathfrak{S}_4$) acting on $\mathcal{Y}_0$
$G_{\mathcal{Y}}$	A group ($\simeq (\mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4)^2$) acting on $\mathcal{Y}$
$G_{\mathcal{X}}$	A group ($\simeq (\mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4)^2 \times_{\mu_2} \mu_4$) acting on $\mathcal{X}$ and $\widetilde{\mathcal{X}}$
$\eta (= \eta_1 \eta_2)$	A 1-cocycle of $G_{\mathcal{Y}_0}$
ϕ_1, ϕ_2	A 1-cocycle of G_T
χ	A 1-cocycle of $G_{\mathcal{X}}$
Ψ_ρ	A $G_{\mathcal{X}}$-linearization of $\mathcal{O}_T^{\mathrm{an}}$ and $\mathcal{Q}$
Θ_ρ	A $G_{\mathcal{X}}$-linearization of $(\mathcal{O}_T^{\mathrm{an}})^{\oplus 2}$

2 Preliminaries

2.1 Higher Chow cycles of type (2,1)

For a smooth variety X over $\mathbb{C}$, let $\mathrm{CH}^p(X, q)$ be the higher Chow group defined by Bloch. In this paper, we treat the case $(p, q) = (2, 1)$. In this case, the following fact is well-known (see, e.g., [MS98, Corollary 5.3]).

Proposition 2.1 *The higher Chow group* $\mathrm{CH}^2(X, 1)$ *is isomorphic to the middle homology group of the following complex:*

$$K_2^{\mathrm{M}}(\mathbb{C}(X)) \xrightarrow{T} \bigoplus_{C \in X^{(1)}} \mathbb{C}(C)^\times \xrightarrow{\mathrm{div}} \bigoplus_{p \in X^{(2)}} \mathbb{Z} \cdot p.$$

Here, $X^{(r)}$ denotes the set of closed subvarieties of X of codimension r. The map T denotes the tame symbol map from the Milnor K_2-group of the function field $\mathbb{C}(X)$.

Therefore, a higher Chow cycle in $\mathrm{CH}^2(X,1)$ is represented by a formal sum

$$\text{(3)} \qquad \sum_j (C_j, f_j) \in \bigoplus_{C \in X^{(1)}} \mathbb{C}(C)^\times,$$

where C_j are prime divisors on X and $f_j \in \mathbb{C}(C_j)^\times$ are nonzero rational functions on them such that $\sum_j \mathrm{div}_{C_j}(f_j) = 0$ as codimension two cycles on X.

A *decomposable cycle* in $\mathrm{CH}^2(X,1)$ is an element of the image of the group homomorphism

$$\text{(4)} \qquad \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \Gamma(X, \mathcal{O}_X^\times) = \mathrm{CH}^1(X) \otimes_{\mathbb{Z}} \mathrm{CH}^1(X,1) \longrightarrow \mathrm{CH}^2(X,1)$$

induced by the intersection product. Let C be a prime divisor on X, and let $[C] \in \mathrm{Pic}(X)$ be the class corresponding to C. The image of $[C] \otimes \alpha$ under (4) is represented by $(C, \alpha|_C)$ in the presentation (3). The cokernel of (4) is denoted by $\mathrm{CH}^2(X,1)_{\mathrm{ind}}$. For $\xi \in \mathrm{CH}^2(X,1)$, ξ_{ind} denotes its image in $\mathrm{CH}^2(X,1)_{\mathrm{ind}}$. A cycle ξ is called *indecomposable* if $\xi_{\mathrm{ind}} \neq 0$.

2.2 The regulator map

By the canonical identification of $\mathrm{CH}^2(X,1)$ with the motivic cohomology $H^3_M(X, \mathbb{Z}(2))$, there exists the map

$$\text{(5)} \qquad \mathrm{reg} : \mathrm{CH}^2(X,1) \longrightarrow H^3_{\mathcal{D}}(X, \mathbb{Z}(2)) = \frac{H^2(X, \mathbb{C})}{F^2 H^2(X, \mathbb{C}) + H^2(X, \mathbb{Z}(2))}$$

called the *regulator map*. The target $H^3_{\mathcal{D}}(X, \mathbb{Z}(2))$ denotes the Deligne cohomology of X. This map can be regarded as the Abel–Jacobi map for $\mathrm{CH}^2(X,1)$. We recall an explicit formula for (5) following [Lev88, pp. 458–459].

Let X be a K3 surface over $\mathbb{C}$. By the Poincaré duality, the Deligne cohomology of X is isomorphic to the generalized complex torus

$$\text{(6)} \qquad H^3_{\mathcal{D}}(X, \mathbb{Z}(2)) \simeq \frac{(F^1 H^2(X, \mathbb{C}))^\vee}{H_2(X, \mathbb{Z})},$$

where $(F^1 H^2(X, \mathbb{C}))^\vee$ is the dual $\mathbb{C}$-vector space of $F^1 H^2(X, \mathbb{C})$ and we regard $H_2(X, \mathbb{Z})$ as a subgroup of $(F^1 H^2(X, \mathbb{C}))^\vee$ by the integration. Under the isomorphism (6), the image of the cycle ξ represented by a formal sum $\sum_j (C_j, f_j)$ under the regulator map is described as follows.

Let D_j be the normalization of the closed curve C_j on X. Let $\mu_j : D_j \to X$ denote the composition of $D_j \to C_j$ and $C_j \to X$. We will define a topological 1-chain γ_j on D_j. If f_j is constant, we define $\gamma_j = 0$. If f_j is not constant, we regard f_j as a finite morphism from D_j to $\mathbb{P}^1$. Then we define $\gamma_j = f_j^{-1}([\infty, 0])$, where $[\infty, 0]$ is a path on $\mathbb{P}^1$ from ∞ to 0 along the positive real axis. By the condition, $\sum_j \mathrm{div}_{C_j}(f_j) = 0$, $\gamma = \sum_j (\mu_j)_* \gamma_j$

is a topological 1-cycle on X. Since $H_1(X, \mathbb{Z}) = 0$, there exists a 2-chain Γ on X such that $\partial \Gamma = \gamma$. In this paper, γ and Γ are called the *1-cycle associated with ξ* and a *2-chain associated with ξ*, respectively. Then the image of ξ under the regulator map is represented by the pairing

$$\langle \mathrm{reg}(\xi), [\omega] \rangle = \int_\Gamma \omega + \sum_j \frac{1}{2\pi\sqrt{-1}} \int_{D_j - \gamma_j} \log(f_j) \mu_j^* \omega \quad ([\omega] \in F^1 H^2(X, \mathbb{C})).$$

Here, $\log(f_j)$ is the pull-back of the logarithmic function on $\mathbb{P}^1 - [\infty, 0]$ by f_j.

In this paper, we use the following variant of the regulator map.

Definition 2.2 The *transcendental regulator map* is the composite of the regulator map 5 and the projection induced by $H^{2,0}(X) \hookrightarrow F^1 H^2(X, \mathbb{C})$:

$$r : \mathrm{CH}^2(X, 1) \longrightarrow \frac{F^1 H^2(X, \mathbb{C})^\vee}{H_2(X, \mathbb{Z})} \longrightarrow\!\!\!\!\!\longrightarrow \frac{H^{2,0}(X)^\vee}{H_2(X, \mathbb{Z})}.$$

We denote this map by r.

By taking the pairing with a nonzero holomorphic 2-form ω on X, we have an isomorphism $H^{2,0}(X)^\vee / H_2(X, \mathbb{Z}) \simeq \mathbb{C} / \mathcal{P}(\omega)$, where $\mathcal{P}(\omega)$ is the subgroup of $\mathbb{C}$ defined by

$$\mathcal{P}(\omega) = \left\{ \int_\Gamma \omega \in \mathbb{C} : \Gamma \text{ is a topological 2-cycles on } X \right\},$$

i.e., $\mathcal{P}(\omega)$ is the set of periods of X with respect to ω. By the above formula, the image of $\xi \in \mathrm{CH}^2(X, 1)$ under the transcendental regulator map is

(7) $\langle r(\xi), [\omega] \rangle \equiv \int_\Gamma \omega \mod \mathcal{P}(\omega),$

where Γ is a 2-chain associated with ξ. If ξ is decomposable, $r(\xi) = 0$. This implies the following.

Proposition 2.3 *If $r(\xi) \neq 0$, we have $\xi_{\mathrm{ind}} \neq 0$. In other words, the transcendental regulator map factors through* $\mathrm{CH}^2(X, 1)_{\mathrm{ind}}$.

2.3 A relative setting

Since it is difficult to prove non-vanishingness of an element of $H^{2,0}(X)^\vee / H_2(X, \mathbb{Z})$, we use its relative version. Let $\pi : \mathcal{X} \to S$ be an algebraic family of $K3$ surfaces over a variety S. We define sheaves $\mathcal{P}$ and $\mathcal{Q}$ of abelian groups on S by

$$\mathcal{P} = \mathrm{Im}(R^2 \pi_* \underline{\mathbb{Z}}_\mathcal{X} \to \mathcal{H}om_{\mathcal{O}_S^{\mathrm{an}}}(\pi_* \Omega^2_{\mathcal{X}/S}, \mathcal{O}_S^{\mathrm{an}})),$$
$$\mathcal{Q} = \mathrm{Coker}(R^2 \pi_* \underline{\mathbb{Z}}_\mathcal{X} \to \mathcal{H}om_{\mathcal{O}_S^{\mathrm{an}}}(\pi_* \Omega^2_{\mathcal{X}/S}, \mathcal{O}_S^{\mathrm{an}})),$$

where $\underline{\mathbb{Z}}_\mathcal{X}$ is the constant sheaf on $\mathcal{X}$ and $\mathcal{O}_S^{\mathrm{an}}$ is the sheaf of holomorphic functions on S. Note that $\mathcal{P}$ is a local system on S.

By considering the pairing with a nonzero relative 2-form $\omega \in \Gamma(\mathcal{X}, \Omega^2_{\mathcal{X}/S})$, we have an isomorphism $\mathcal{H}om_{\mathcal{O}^{\mathrm{an}}_S}(\pi_* \Omega^2_{\mathcal{X}/S}, \mathcal{O}^{\mathrm{an}}_S) \simeq \mathcal{O}^{\mathrm{an}}_S$. Hence, $\mathcal{P}$ is isomorphic to the subsheaf $\mathcal{P}_\omega$ of $\mathcal{O}^{\mathrm{an}}_S$ generated by period functions with respect to ω and $\mathcal{Q}$ is isomorphic to $\mathcal{Q}_\omega = \mathcal{O}^{\mathrm{an}}_S / \mathcal{P}_\omega$. For a local section φ of $\mathcal{O}^{\mathrm{an}}_S$, its image in $\mathcal{Q}_\omega$ is denoted by $[\varphi]$.

For each $s \in S$, there exists the evaluation map

$$\mathrm{ev}_s : \Gamma(S, \mathcal{Q}) \to H^{2,0}(\mathcal{X}_s)^\vee / H_2(\mathcal{X}_s, \mathbb{Z}).$$

Under the isomorphisms $\mathcal{Q} \simeq \mathcal{Q}_\omega$ and $H^{2,0}(\mathcal{X}_s)^\vee / H_2(\mathcal{X}_s, \mathbb{Z}) \simeq \mathbb{C}/\mathcal{P}(\omega_s)$ induced by ω and ω_s, the evaluation map coincides with the map

$$\mathcal{Q}_\omega = \mathcal{O}^{\mathrm{an}}_S / \mathcal{P}_\omega \ni [f] \longmapsto f(s) \mod \mathcal{P}(\omega_s) \quad \in \mathbb{C}/\mathcal{P}(\omega_s),$$

where $f(s) \in \mathbb{C}$ denotes the value of the holomorphic function f at s. The following elementary lemma is crucial for the result.

Lemma 2.4 *For a nonzero element $v \in \Gamma(S, \mathcal{Q})$, we have $\mathrm{ev}_s(v) \neq 0$ for very general $s \in S$.*

Proof Since the question is local, we can shrink S in the sense of the classical topology. By fixing a relative 2-form $\omega \in \Gamma(\mathcal{X}, \Omega^2_{\mathcal{X}/S})$, we have the isomorphism $\mathcal{Q} \simeq \mathcal{O}^{\mathrm{an}}_S / \mathcal{P}_\omega$. We may assume that there exist a holomorphic function $\varphi \in \mathcal{O}^{\mathrm{an}}_S(S)$ such that $v = [\varphi]$ and a free basis $f_1, f_2, \ldots, f_r$ of $\mathcal{P}_\omega(S)$.

For each $\underline{c} = (c_i) \in \mathbb{Z}^r$, we define the holomorphic function $F_{\underline{c}}$ by

$$F_{\underline{c}} = \varphi - \sum_{i=1}^{r} c_i f_i.$$

Since $v = [\varphi]$ is nonzero in $\mathcal{Q}_\omega(S)$, $F_{\underline{c}}$ is a nonzero holomorphic function for each $\underline{c} \in \mathbb{Z}^r$. Consider the countable family $\{F_{\underline{c}}\}_{\underline{c} \in \mathbb{Z}^r}$ of the holomorphic functions. Then outside of the zeros of the functions in this family, $\varphi(s) \notin \mathcal{P}(\omega_s) = \langle f_1(s), f_2(s), \ldots, f_r(s) \rangle_{\mathbb{Z}}$. Hence $\mathrm{ev}_s(v) \neq 0$. ∎

The following corollary is also used in technical propositions.

Corollary 2.5 *If local sections v, v' of $\mathcal{Q}$ on an open subset U satisfy $\mathrm{ev}_s(v) = \mathrm{ev}_s(v')$ for any $s \in U$, we have $v = v'$.*

Finally, we consider the regulator map in the relative setting. Suppose that we have irreducible divisors $\mathcal{C}_j$ on $\mathcal{X}$ which are smooth over S and nonzero rational functions f_j on $\mathcal{C}_j$ whose zeros and poles are also smooth over S. Assume that they satisfy the condition $\sum_j \mathrm{div}_{(\mathcal{C}_j)_s}((f_j)_s) = 0$ for each $s \in S$. Then we have a family of higher Chow cycles $\xi = \{\xi_s\}_{s \in S}$ such that $\xi_s \in \mathrm{CH}^2(\mathcal{X}_s, 1)$ is represented by the formal sum $\sum_j((\mathcal{C}_j)_s, (f_j)_s)$. A family of higher Chow cycles constructed in this way is called an *algebraic family of higher Chow cycles* in this paper.

Remark 2.6 In the above situation, let ξ be the higher Chow cycle on the total space $\mathcal{X}$ defined by the formal sum $\sum_j(\mathcal{C}_j, f_j)$. For each $s \in S$, $\xi_s \in \mathrm{CH}^2(\mathcal{X}_s, 1)$ is the pullback of ξ by $\mathcal{X}_s \hookrightarrow \mathcal{X}$. Hence, we can regard an algebraic family of higher Chow cycle as a higher Chow cycle of the total space $\mathcal{X}$.

If we shrink S in the sense of the classical topology, there exists a C^∞-family of topological 2-chains $\{\Gamma_s\}_{s \in S}$ such that Γ_s is a 2-chain associated with ξ_s. If we fix a relative 2-form ω, the function

$$S \ni s \mapsto \int_{\Gamma_s} \omega_s \in \mathbb{C}$$

is holomorphic (cf. [CL05, Proposition 4.1]). Hence, we can define the element $\nu_{\mathrm{tr}}(\xi) \in \Gamma(S, \mathcal{Q})$ such that

$$\mathrm{ev}_s(\nu_{\mathrm{tr}}(\xi)) = r(\xi_s)$$

for every $s \in S$. This $\nu_{\mathrm{tr}}(\xi)$ can be regarded as a part of the normal function associated with ξ.

By combining Proposition 2.3 and Lemma 2.4, we have the following.

Proposition 2.7   *Let $\mathcal{X} \to S$ be an algebraic family of K3 surfaces, and let $\xi = \{\xi_s\}_{s \in S}$ be an algebraic family of higher Chow cycles. Suppose $\nu_{\mathrm{tr}}(\xi) \neq 0$. Then for very general $s \in S$, we have $(\xi_s)_{\mathrm{ind}} \neq 0$.*

3 The family of Kummer surfaces

In this section, we define the family of Kummer surfaces. This is the famous family, and our main purpose is to fix the notation. Let $S_0 = \mathbb{P}^1 - \{0, 1, \infty\}$ and $A_0 = \mathbb{C}\left[c, \frac{1}{c(c-1)}\right]$ be the coordinate ring of S_0. Let $\mathcal{E} \to S_0$ be the Legendre family of elliptic curves over S_0. The fiber of $\mathcal{E} \to S_0$ over $c \in S_0$ is the elliptic curve

$$\mathcal{E}_c : y^2 = x(1-x)(1-cx).$$

We have the natural S_0-morphism $\mathcal{E} \to \mathbb{P}^1 \times S_0$ defined by $(x, y) \mapsto x$. Let Σ denote the set of 2-torsion sections of $\mathcal{E} \to S_0$. The set Σ consists of four sections corresponding to $x = 0, 1, 1/c, \infty$. We use the same symbol Σ for its image under $\mathcal{E} \to \mathbb{P}^1 \times S_0$.

Next, we define the family of Kummer surfaces and related families. Let T_0 be the (Zariski) open set of $S_0 \times S_0$ defined by

$$T_0 = \left\{ (a, b) \in S_0 \times S_0 : a \neq b, 1-b, \frac{1}{b}, \frac{1}{1-b}, \frac{b-1}{b}, \frac{b}{b-1} \right\}$$

and B_0 be the coordinate ring of T_0. We will define the following families of varieties over T_0.

Let $\mathcal{Y}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \times T_0$. We use x (resp. y) for coordinates of the first (resp. second) $\mathbb{P}^1$ of $\mathcal{Y}_0$. We define the family of abelian surfaces $\mathcal{A}_0 \to T_0$ by the restriction of $\mathcal{E} \times \mathcal{E} \to S_0 \times S_0$ to T_0. The fiber of $\mathcal{A}_0 \to T_0$ over $(a, b) \in T_0$ corresponds to the product of elliptic curves $\mathcal{E}_a \times \mathcal{E}_b$. We have the natural T_0-morphism $\mathcal{A}_0 \to \mathcal{Y}_0$ induced by
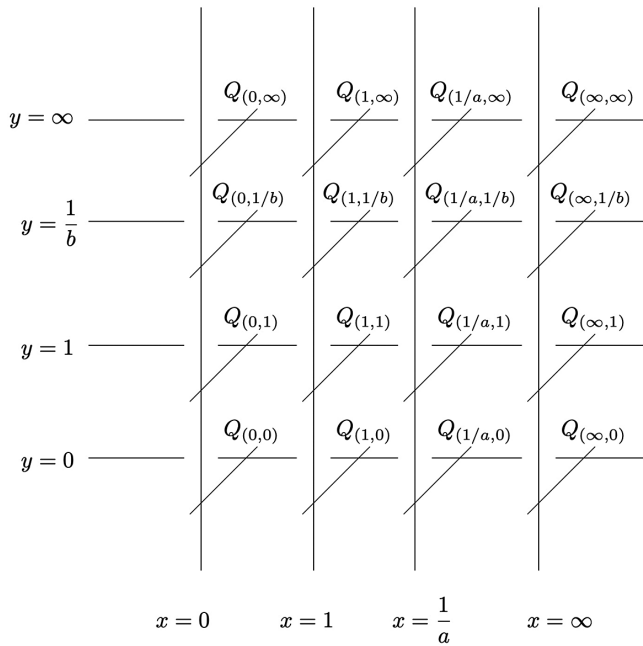
Figure 1: The exceptional divisors Q_σ on $\widetilde{\mathcal{X}}_0$.

the direct product of S_0-morphism $\mathcal{E} \to \mathbb{P}^1 \times S_0$. The map $\mathcal{A}_0 \to \mathcal{Y}_0$ is a 4:1 cover. Let $\iota : \mathcal{A}_0 \to \mathcal{A}_0$ be the map of taking inverses and we define $\mathcal{X}_0$ as the quotient of $\mathcal{A}_0$ by ι. Then the morphism $\mathcal{A}_0 \to \mathcal{Y}_0$ factors $\mathcal{X}_0$.

We define Σ^2 as the set of 2-torsion sections of $\mathcal{A}_0 \to T_0$. The set Σ^2 consists of 16 sections corresponding to $\Sigma \times \Sigma$. We use the same symbol Σ^2 for its images in $\mathcal{X}_0$ and $\mathcal{Y}_0$. Using the coordinates x and y, the set Σ^2 can be described as 16 points

$$(x, y) \in \{0, 1, 1/a, \infty\} \times \{0, 1, 1/b, \infty\}.$$

Let $\widetilde{\mathcal{A}}_0, \widetilde{\mathcal{X}}_0$, and $\widetilde{\mathcal{Y}}_0$ be the blowing-ups of $\mathcal{A}_0, \mathcal{X}_0$, and $\mathcal{Y}_0$ along Σ^2. The universality of the blowing-up induces the maps $\widetilde{\mathcal{A}}_0 \to \widetilde{\mathcal{X}}_0 \to \widetilde{\mathcal{Y}}_0$. Since $\widetilde{\mathcal{X}}_0 \to \mathcal{X}_0$ is the minimal resolution of singularities, $\widetilde{\mathcal{X}}_0 \to T_0$ is the family of Kummer surfaces. At $t = (a, b) \in T_0$, the fiber $\widetilde{\mathcal{X}}_t$ is the Kummer surface $\mathrm{Km}(\mathcal{E}_a \times \mathcal{E}_b)$ associated with $\mathcal{E}_a \times \mathcal{E}_b$. For $\sigma \in \Sigma^2$, Q_σ denotes the exceptional divisor over $\sigma \in \Sigma^2$. The configuration of Q_σ on $\widetilde{\mathcal{X}}_0$ is described in Figure 1.

Finally, we define local charts on $\mathcal{Y}_0$ and $\widetilde{\mathcal{X}}_0$. Let $U = \mathrm{Spec}\, B_0[x, y] \subset \mathcal{Y}_0$ be the affine open subset which is the complement of the divisors $x = \infty$ and $y = \infty$. The inverse image of U by $\widetilde{\mathcal{X}}_0 \to \mathcal{Y}_0$ is covered by two open affine subschemes $V = \mathrm{Spec}\, B_0[x, y, v]/(v^2 f(x) - g(y))$ and $W = \mathrm{Spec}\, B_0[x, y, w]/(w^2 g(y) - f(x))$, where $f(x) = x(1 - x)(1 - ax)$ and $g(y) = y(1 - y)(1 - by)$. These two subsets are glued by the relation $v = 1/w$.

4 Construction of initial higher Chow cycles

In this section, we construct families ξ_0, ξ_1, and ξ_∞ of higher Chow cycles on a base change $\widetilde{\mathcal{X}} \to T$ of $\widetilde{\mathcal{X}}_0 \to T_0$.

4.1 Construction of higher Chow cycles at fibers

First, we explain the construction at each fiber over $t = (a, b) \in T_0$. Let $\mathcal{D} \subset \mathbb{P}^1 \times \mathbb{P}^1 = (\mathcal{Y}_0)_t$ be the diagonal curve. Since $a \neq b$, $\mathcal{D}$ intersects with the branching locus of $(\mathcal{X}_0)_t \to (\mathcal{Y}_0)_t$ as Figure 2.

Let $\mathcal{C} \subset (\mathcal{X}_0)_t$ be the pull-back of $\mathcal{D}$ by $(\mathcal{X}_0)_t \to (\mathcal{Y}_0)_t$, and let $\widetilde{\mathcal{C}}$ be the strict transform of $\mathcal{C}$ by the blowing-up $(\widetilde{\mathcal{X}}_0)_t \to (\mathcal{X}_0)_t$. Then we see that $\widetilde{\mathcal{C}}$ is smooth and $\widetilde{\mathcal{C}} \to \mathcal{D}$ is a double covering ramified at two points. Hence, $\widetilde{\mathcal{C}}$ is isomorphic to $\mathbb{P}^1$. Furthermore, for each $\bullet \in \{0, 1, \infty\}$, $\widetilde{\mathcal{C}}$ intersects with $Q_{(\bullet,\bullet)}$ at two points $p_\bullet^+$ and $p_\bullet^-$ (cf. Figure 3).

Hence, we can find rational functions $\psi_0, \psi_1, \psi_\infty \in \mathbb{C}(\widetilde{\mathcal{C}})$ and $\varphi_\bullet \in \mathbb{C}(Q_{(\bullet,\bullet)})$ ($\bullet = 0, 1, \infty\}$) which satisfy the following relations:

$$
\begin{aligned}
\text{(8)} \qquad \operatorname{div}_{\widetilde{\mathcal{C}}}(\psi_0) &= p_0^- - p_0^+ = -\operatorname{div}_{Q_{(0,0)}}(\varphi_0), \\
\operatorname{div}_{\widetilde{\mathcal{C}}}(\psi_1) &= p_1^- - p_1^+ = -\operatorname{div}_{Q_{(1,1)}}(\varphi_1), \\
\operatorname{div}_{\widetilde{\mathcal{C}}}(\psi_\infty) &= p_\infty^- - p_\infty^+ = -\operatorname{div}_{Q_{(\infty,\infty)}}(\varphi_\infty).
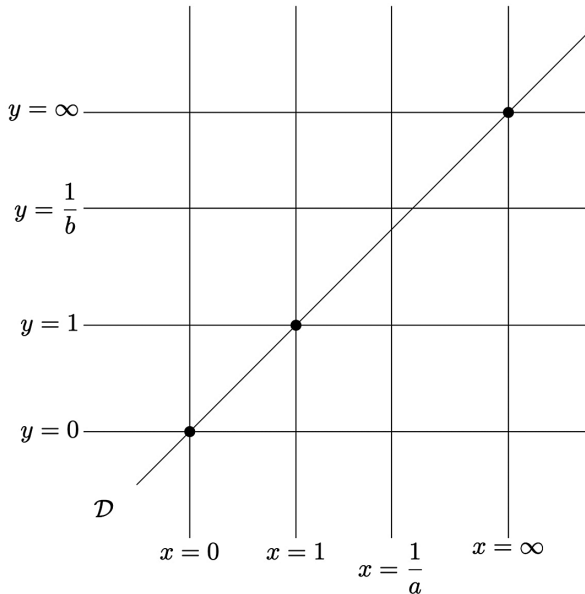\end{aligned}
$$

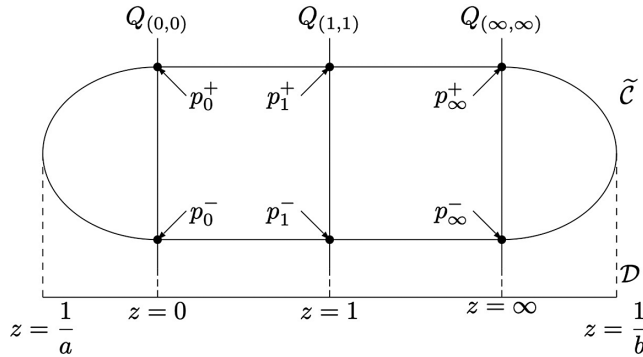Figure 2: The diagonal curve $\mathcal{D}$ and the branching locus.

Figure 3: The intersections of $\widetilde{\mathcal{C}}$, $Q_{(0,0)}$, $Q_{(1,1)}$, and $Q_{(\infty,\infty)}$.

We define $(\xi_0)_t, (\xi_1)_t, (\xi_\infty)_t \in \mathrm{CH}^2((\widetilde{\mathcal{X}}_0)_t, 1)$ by the formal sums

(9)
$$
\begin{aligned}
(\xi_0)_t &= (\widetilde{\mathcal{C}}, \psi_0) + (Q_{(0,0)}, \varphi_0), \\
(\xi_1)_t &= (\widetilde{\mathcal{C}}, \psi_1) + (Q_{(1,1)}, \varphi_1), \\
(\xi_\infty)_t &= (\widetilde{\mathcal{C}}, \psi_\infty) + (Q_{(\infty,\infty)}, \varphi_\infty).
\end{aligned}
$$

4.2 Construction of families of higher Chow cycles

To get families of higher Chow cycles, it is enough to construct rational functions $\varphi_\bullet$ and $\psi_\bullet$ on the family. However, the intersection points p_1^+ and p_1^- (resp. p_∞^+ and p_∞^-) interchange by the monodromy of T_0. Hence, it is impossible to construct such rational functions for $\bullet = 1, \infty$. Thus, it is necessary to take a finite étale base change of T_0 to define the families of cycles.

Let $B = B_0[\sqrt{a}, \sqrt{b}, \sqrt{1-a}, \sqrt{1-b}]$ and $T \to T_0$ be the finite étale cover corresponding to $B_0 \to B$. The base changes of $\mathcal{Y}_0, \mathcal{X}_0, \mathcal{A}_0, \widetilde{\mathcal{Y}}_0, \widetilde{\mathcal{X}}_0$, and $\widetilde{\mathcal{A}}_0$ by $T \to T_0$ are denoted by $\mathcal{Y}, \mathcal{X}, \mathcal{A}, \widetilde{\mathcal{Y}}, \widetilde{\mathcal{X}}$, and $\widetilde{\mathcal{A}}$, respectively.

Let $\mathcal{D} \subset \mathcal{Y}$ be the closed subscheme defined by the local equation $x = y$, let $\mathcal{C}$ be its pull-back by $\mathcal{X} \to \mathcal{Y}$, and let $\widetilde{\mathcal{C}}$ be its strict transform by the blowing up $\widetilde{\mathcal{X}} \to \mathcal{X}$. On the local chart V, $\widetilde{\mathcal{C}} \hookrightarrow \widetilde{\mathcal{X}}$ is described by the following ring homomorphism:

$$
B[x, y, v]/(v^2 f(x) - g(y)) \longrightarrow B[z, v]/(v^2(1 - az) - (1 - bz)); x, y, v \mapsto z, z, v.
$$

By this description, we see that $Q_{(0,0)}$ and $\widetilde{\mathcal{C}}$ intersect at $(x, y, v) = (0, 0, \pm 1)$, and $Q_{(1,1)}$ and $\widetilde{\mathcal{C}}$ intersect at $(x, y, v) = \left(1, 1, \pm\sqrt{1-b}/\sqrt{1-a}\right)$. By the local computation on another local chart containing $(\infty, \infty) \in \Sigma^2$, $Q_{(\infty,\infty)}$ and $\widetilde{\mathcal{C}}$ intersect at $(\xi, \eta, v') = \left(0, 0, \pm\sqrt{b}/\sqrt{a}\right)$, where local coordinates ξ, η, and v' are defined by $\xi = 1/x$, $\eta = 1/y$, and $v' = x^2 v/y^2$. Hence, we can define rational functions $\psi_\bullet \in \mathbb{C}(\widetilde{\mathcal{C}})$ and $\varphi_\bullet \in \mathbb{C}(Q_{(\bullet,\bullet)})$ ($\bullet \in \{0, 1, \infty\}$) by the following equations:

$$\psi_0 = (v + 1) \cdot (v - 1)^{-1}, \qquad\qquad \varphi_0 = (v - 1) \cdot (v + 1)^{-1},$$

$$\psi_1 = \left(v + \frac{\sqrt{1-b}}{\sqrt{1-a}}\right) \cdot \left(v - \frac{\sqrt{1-b}}{\sqrt{1-a}}\right)^{-1}, \quad \varphi_1 = \left(v - \frac{\sqrt{1-b}}{\sqrt{1-a}}\right) \cdot \left(v + \frac{\sqrt{1-b}}{\sqrt{1-a}}\right)^{-1},$$

$$\psi_\infty = \left(v' + \frac{\sqrt{b}}{\sqrt{a}}\right) \cdot \left(v' - \frac{\sqrt{b}}{\sqrt{a}}\right)^{-1}, \qquad \varphi_\infty = \left(v' - \frac{\sqrt{b}}{\sqrt{a}}\right) \cdot \left(v' + \frac{\sqrt{b}}{\sqrt{a}}\right)^{-1}.$$

They satisfy the relations in (8) and we define algebraic families of higher Chow cycles $\xi_0 = \{(\xi_0)_t\}_{t \in T}, \xi_1 = \{(\xi_1)_t\}_{t \in T}$ and $\xi_\infty = \{(\xi_\infty)_t\}_{t \in T}$ by the equations (9).

5 Computation of the regulator

In this section, we compute the image of $(\xi_1)_t - (\xi_0)_t$ under the regulator map and prove its indecomposability. Our main result is the following.

Theorem 5.1 *For very general $t \in T$, $(\xi_1)_t - (\xi_0)_t$ is an indecomposable cycle.*

The main ingredients of the proof is the following two propositions.

Proposition 5.2 *Let ω be the relative 2-form $\frac{dx \wedge dy}{vf(x)}$ on the family $\widetilde{\mathfrak{X}} \to T$ and $\mathscr{D}_1, \mathscr{D}_2 : \mathcal{O}_T^{an} \to \mathcal{O}_T^{an}$ be the differential operators defined by*

$$\mathscr{D}_1 = a(1-a)\frac{\partial^2}{\partial a^2} + (1 - 2a)\frac{\partial}{\partial a} - \frac{1}{4},$$

$$\mathscr{D}_2 = b(1-b)\frac{\partial^2}{\partial b^2} + (1 - 2b)\frac{\partial}{\partial b} - \frac{1}{4}.$$

Let $\mathscr{D} = \begin{pmatrix} \mathscr{D}_1 \\ \mathscr{D}_2 \end{pmatrix} : \mathcal{O}_T^{an} \to (\mathcal{O}_T^{an})^{\oplus 2}$. Then for any local section f of $\mathcal{P}_\omega \subset \mathcal{O}_T^{an}$, we have $\mathscr{D}(f) = 0$. In particular, $\mathscr{D}$ factors the sheaf $\mathcal{Q}_\omega$. We use the same symbol $\mathscr{D}$ for the induced morphisms $\mathcal{Q}_\omega \to (\mathcal{O}_T^{an})^{\oplus 2}$ and $\mathcal{Q} \simeq \mathcal{Q}_\omega \to (\mathcal{O}_T^{an})^{\oplus 2}$.

Recall that $\mathcal{P}_\omega$ is the local system consisting of period functions with respect to ω and $\mathcal{Q}_\omega$ is the quotient of $\mathcal{O}_T^{an}$ by $\mathcal{P}_\omega$. The differential operator $\mathscr{D}$ is called a *Picard–Fuchs differential operator* because it annihilates all period functions with respect to ω.

By Proposition 2.7, to prove Theorem 5.1, it is enough to show $v_{tr}(\xi_1 - \xi_0)$ is nonzero. Then by the Proposition 5.2, it is enough to show $\mathscr{D}(v_{tr}(\xi_1 - \xi_0))$ is nonzero. To prove this, we will find an explicit multivalued function which represents $v_{tr}(\xi_1 - \xi_0)$.

For $(a, b) \in T_0$ such that $a, b \in \mathbb{R}_{<0}$, let $\mathcal{L}(a, b)$ be the improper integral

$$(10) \qquad \mathcal{L}(a, b) = 2 \int_\triangle \frac{dx\,dy}{\sqrt{x(1-x)(1-ax)}\sqrt{y(1-y)(1-by)}},$$

where $\triangle = \{(x, y) \in \mathbb{R}^2 : 0 < y < x < 1\}$. This integral converges and defines a local holomorphic function around (a, b). There exists a lift $\mathcal{L}$ of $\mathcal{L}(a, b)$ by $T \to T_0$ which satisfies the the following properties.

Proposition 5.3

(1) *For any $t \in T$, $\mathcal{L}$ can be analytically continued to an open neighborhood of t.*
(2) *The multivalued holomorphic function $\mathcal{L}$ represents $\nu_{\mathrm{tr}}(\xi_1 - \xi_0)$ under the isomorphism $\mathcal{Q}_\omega \simeq \mathcal{Q}$ induced by the relative 2-form ω in Proposition 5.2. In particular, for any $t \in T$, we have*

(11)
$$\langle r((\xi_1)_t - (\xi_0)_t), [\omega_t] \rangle \equiv \mathcal{L}(t) \mod \mathcal{P}(\omega_t),$$

where ω_t is the pull-back of ω at $\widetilde{\mathcal{X}}_t$.

(3) *The multivalued holomorphic function $\mathcal{L}$ satisfies the following system of the differential equations:*

(12)
$$\mathscr{D}(\mathcal{L}) = \frac{2}{a-b} \begin{pmatrix} \frac{\sqrt{1-b}}{\sqrt{1-a}} - 1 \\ 1 - \frac{\sqrt{1-a}}{\sqrt{1-b}} \end{pmatrix}.$$

In particular, $\mathscr{D}(\nu_{\mathrm{tr}}(\xi_1 - \xi_0))$ coincides with the right-hand side of (12).

Thus, we have $\nu_{\mathrm{tr}}(\xi_1 - \xi_0) \neq 0$ and this implies Theorem 5.1 by Proposition 2.7. We will prove Propositions 5.2 and 5.3 in this section.

5.1 The Picard–Fuchs differential operator

For $c \in \mathbb{C} - \mathbb{R}_{\geq 0}$, we consider the following improper integrals:

$$P_1(c) = \int_0^1 \frac{dx}{\sqrt{x(1-x)(1-cx)}}, \quad P_2(c) = \int_1^\infty \frac{dx}{\sqrt{x(1-x)(1-cx)}}.$$

As is well-known (see, e.g., [WW62, p. 253]), these integrals converge and give linearly independent solutions of the hypergeometric differential equation

(13)
$$c(1-c)\frac{d^2 P}{dc^2} + (1-2c)\frac{dP}{dc} - \frac{1}{4}P = 0.$$

For any $c \in S_0$, they can be analytically continued to a neighborhood of c and we use the same notation for the resulting multivalued functions.

Let θ be the relative 1-form $\frac{dx}{y}$ on the Legendre family of elliptic curves $\mathcal{E} \to S_0$. For $c \in \mathbb{C} - \mathbb{R}_{\geq 0}$, let γ_+, γ_- (resp. δ_+, δ_-) be lifts of paths $[0,1]$ and $[1,\infty]$ on $\mathbb{P}^1$ by $\mathcal{E}_c \to \mathbb{P}^1$. Let θ_c be the pull-back of θ at $\mathcal{E}_c$. Then we have

$$\int_{\gamma_+} \theta_c = -\int_{\gamma_-} \theta_c, \quad \int_{\delta_+} \theta_c = -\int_{\delta_-} \theta_c,$$

and they coincide with $\pm P_1(c)$ and $\pm P_2(c)$, respectively. Since $[\gamma_+] - [\gamma_-]$ and $[\delta_+] - [\delta_-]$ are generators of $H_1(\mathcal{E}_c, \mathbb{Z})$ (see, e.g., [CMP02, p. 10]), $2P_1$ and $2P_2$ are local basis of the period functions of $\mathcal{E} \to S_0$ with respect to θ. In particular, hypergeometric differential equation (13) is a Picard–Fuchs differential equation of $\mathcal{E} \to S_0$.

Next, we will find a Picard–Fuchs differential operator of $\widetilde{\mathcal{X}} \to T$. Recall that we have T-morphisms

$$\mathcal{A} \xleftarrow[\text{blowing-up}]{p} \widetilde{\mathcal{A}} \xrightarrow[\text{2:1 quotient}]{\pi} \widetilde{\mathcal{X}}.$$

We name the morphisms p and π as above. We have the relative 2-form $pr_1^*(\theta) \wedge pr_2^*(\theta)$ on $\mathcal{A} \to T$, where pr_i is the morphism $\mathcal{A} \to \mathcal{A}_0 \hookrightarrow \mathcal{E} \times \mathcal{E} \xrightarrow{pr_i} \mathcal{E}$. Then its pullback by p is stable under the covering transformation of π, so it descends to $\widetilde{\mathcal{X}}$. We denote the resulting 2-form on $\widetilde{\mathcal{X}}$ by ω. The relative 2-form ω is described as $\frac{dx \wedge dy}{v f(x)}$ and $\frac{dx \wedge dy}{w g(y)}$ on the local charts V and W. Then we have the following.

Proposition 5.4 *The four local holomorphic functions $2P_i(a)P_j(b)$ $(i, j \in \{1, 2\})$ are a local basis for the local system $\mathcal{P}_\omega \subset \mathcal{O}_T^{\mathrm{an}}$ generated by period functions of $\widetilde{\mathcal{X}} \to T$ with respect to ω.*

Proof   By the Künneth formula, we see that $4P_i(a)P_j(b)$ $(i, j \in \{1, 2\})$ is a local basis for the local system generated by period functions of $\mathcal{A} \to T$ with respect to $pr_1^*(\theta) \wedge pr_2^*(\theta)$. For $t \in T$, let ϕ be the morphism of Hodge structures defined by

$$\phi : H^2(\mathcal{A}_t) \xrightarrow{p^*} H^2(\widetilde{\mathcal{A}}_t) \xrightarrow{\pi_!} H^2(\widetilde{\mathcal{X}}_t),$$

where $\pi_!$ is the Gysin morphism induced by π. Since the mapping degree of π is 2, $\pi_! \circ \pi^* : H^2(\widetilde{\mathcal{X}}_t) \to H^2(\widetilde{\mathcal{X}}_t)$ equals multiplication by 2 (cf. [Voi02, Remark 7.29]). Since $\pi^*(\omega) = p^*(pr_1^*(\theta) \wedge pr_2^*(\theta))$, we have the relation

$$(14) \qquad\qquad \phi([(pr_1^*(\theta) \wedge pr_2^*(\theta))_t] = 2[\omega_t].$$

Let $\phi^\vee : H_2(\widetilde{\mathcal{X}}_t) \to H_2(\mathcal{A}_t)$ be the dual of ϕ. For any $[\Gamma] \in H_2(\widetilde{\mathcal{X}}_t, \mathbb{Z})$ and $[\Gamma'] \in H_2(\mathcal{A}_t, \mathbb{Z})$ such that $\phi^\vee([\Gamma]) = [\Gamma']$, we have

$$\int_\Gamma \omega_t = \frac{1}{2} \int_{\Gamma'} (pr_1^*(\theta) \wedge pr_2^*(\theta))_t$$

by (14). Since the right-hand side is a linear combination of $2P_i(a)P_j(b)$, we see that any period function of $\widetilde{\mathcal{X}} \to T$ with respect to ω is a linear combination of $2P_i(a)P_j(b)$. Furthermore, since ϕ is injective and its cokernel has no torsion (cf. [BHPV04], Chapter VIII, Proposition 5.1, and Corollary 5.6), $\phi^\vee$ is surjective. Thus, $2P_i(a)P_j(b)$ itself is a period function for $i, j \in \{1, 2\}$ and we have the result. ∎

Now we can prove Proposition 5.2.

Proof of Proposition 5.2   Let $f \in \mathcal{P}_\omega \subset \mathcal{O}_T^{\mathrm{an}}$ be any local section. Then f is a linear combination of $2P_i(a)P_j(b)$ by Proposition 5.4. Since $P_i(c)$ are solutions of (13), both of the differential operators $\mathscr{D}_1$ and $\mathscr{D}_2$ annihilate $2P_i(a)P_j(b)$. Hence, we have the result. ∎

5.2 Calculation of the regulator

For a while, we fix $t_0 \in T$ such that[2] $\sqrt{1-a}, \sqrt{1-b} \in \mathbb{R}_{>0}$. Our first goal is to compute the image of $(\xi_1)_t - (\xi_0)_t$ under the transcendental regulator map locally around t_0. For the computation, we construct a 2-chain and use (7). In the calculation, we replace a 2-chain associated with $(\xi_1)_t - (\xi_0)_t$ with another 2-chain to make the computation easier.

Let ξ be a higher Chow cycle on a $K3$ surface X, and let γ be the topological 1-cycle associated with ξ. A topological 2-chain Γ' is called a *2-chain associated with ξ in a weak sense* if there exists a finite family of curves $\{E_k\}_k$ on X and a topological 2-chain Γ_k on each E_k such that

$$(15) \qquad \partial\Gamma' = \gamma + \sum_k \partial\Gamma_k.$$

Let Γ be a 2-chain associated with ξ. If Γ' is a 2-chain associated with ξ in a weak sense, Γ and $\Gamma' + \sum_k \Gamma_k$ coincide up to topological 2-cycles. Since the pull-back of a holomorphic 2-form ω on E_k vanishes, we have

$$\int_{\Gamma'} \omega \equiv \int_{\Gamma} \omega \underset{(7)}{\equiv} \langle r(\xi), [\omega] \rangle \quad \mod \mathcal{P}(\omega).$$

Hence, we can also use Γ' for the computation of the transcendental regulator map.

We will construct a desired 2-chain. For t which is sufficiently close to t_0, let $\triangle_+$ and $\triangle_-$ be the images of the following maps:

$$(16) \qquad \begin{array}{ccc} \triangle = \{(x,y) \in \mathbb{R}^2 : 0 < y < x < 1\} & \longrightarrow & V(\subset \widetilde{\mathcal{X}}_t) \\ \cup & & \cup \\ (x,y) & \longmapsto & (x,y,v) = \left(x, y, \pm\dfrac{\sqrt{y(1-y)(1-by)}}{\sqrt{x(1-x)(1-ax)}}\right). \end{array}$$

Note that since t is sufficiently close to t_0, we may assume the function $\sqrt{1-ax}$ (resp. $\sqrt{1-by}$) do not ramify on $\triangle$ and we can fix the branch of it so that it takes a value 1 and $\sqrt{1-a}$ at $x = 0, 1$ (resp. 1 and $\sqrt{1-b}$ at $y = 0, 1$).

We define K_+ and K_- as the closures of $\triangle_+$ and $\triangle_-$, respectively. Let γ be a path $[0,1]$ on $\mathbb{P}^1$ reparameterized so that $\gamma(s) = s^2$ (resp. $1 - \gamma(s) = (1-s)^2$) on a neighborhood of 0 (resp. 1). Then if we replace x and y in the target in (16) by $\gamma(x)$ and $\gamma(y)$, respectively, (16) can be extended to a map from a compact-oriented manifold with corners, so K_+ and K_- are C^∞-chains.

Proposition 5.5 *For any t which is sufficiently close to t_0, we have*

$$\langle r((\xi_1)_t - (\xi_0)_t), [\omega_t] \rangle \equiv \int_{K_+} \omega_t - \int_{K_-} \omega_t \quad \mod \mathcal{P}(\omega_t).$$

To prove this, we should examine the boundaries of K_+ and K_-. Recall that we have the following morphisms.

$$\widetilde{\mathcal{X}}_t \xrightarrow{\text{2:1 quotient}} \widetilde{\mathcal{Y}}_t \xrightarrow{\text{blowing-up}} \mathcal{Y}_t = \mathbb{P}^1 \times \mathbb{P}^1$$

---

[2]Note that choosing a point on T is equivalent to choosing a point $(a,b) \in T_0$ and branches of $\sqrt{a}, \sqrt{b}, \sqrt{1-a}, \sqrt{1-b} \in \mathbb{C}$.
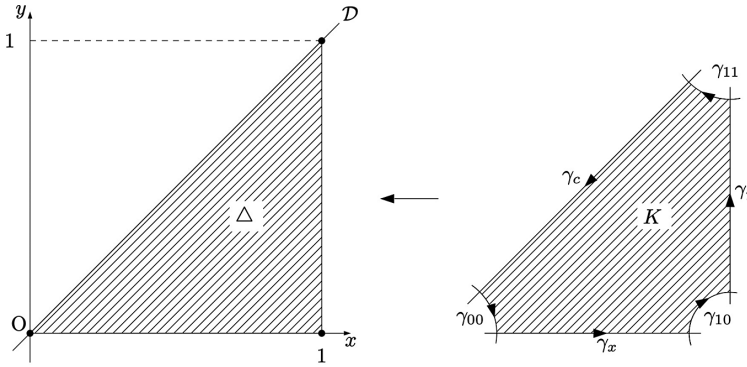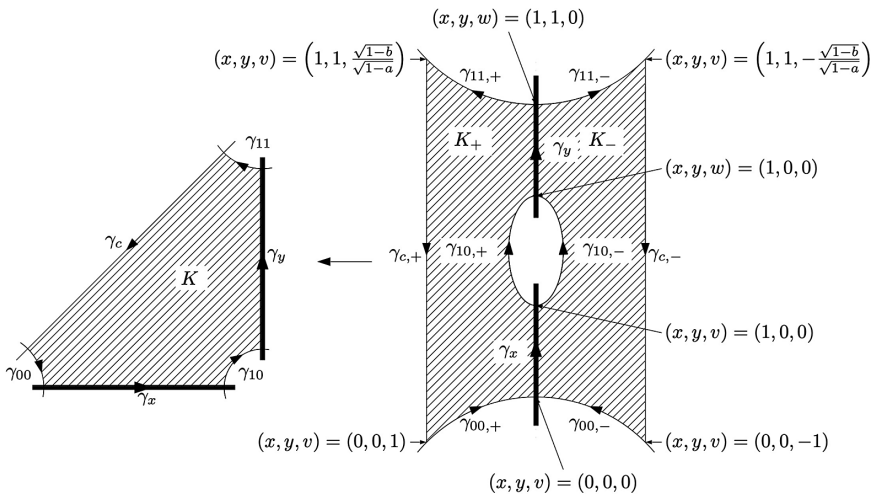
Figure 4: The 2-chain K and its boundary.

Figure 5: The 2-chains K_+ and K_- and their boundaries.

We regard $\triangle = \{(x, y) \in \mathbb{R}^2 : 0 < y < x < 1\}$ as a subset of $\mathcal{Y}_t = \mathbb{P}^1 \times \mathbb{P}^1$. Let K be the closure of the inverse image of $\triangle$ by $\widetilde{\mathcal{Y}}_t \to \mathcal{Y}_t$ (see Figure 4). We define paths $\gamma_c, \gamma_{11}, \gamma_y, \gamma_{10}, \gamma_x$, and γ_{00} on the boundary ∂K as in Figure 4.

Then $\widetilde{\mathcal{X}}_t \to \widetilde{\mathcal{Y}}_t$ induces the homeomorphism $K_+ \xrightarrow{\sim} K$ and $K_- \xrightarrow{\sim} K$ (see Figure 5). Here, the bold line in the figure denotes the branching locus of $\widetilde{\mathcal{X}}_t \to \widetilde{\mathcal{Y}}_t$. We define paths $\gamma_{c,\pm}, \gamma_{11,\pm}, \gamma_y, \gamma_{10,\pm}, \gamma_x$, and $\gamma_{00,\pm}$ on ∂K_+ and ∂K_- as in Figure 5. They satisfy the following properties:

(1) The paths $\gamma_{c,+}$ and $\gamma_{c,-}$ are the lifts of γ_c. Since γ_c is on the strict transformation of $\mathcal{D}$, they are on the curve $\widetilde{\mathcal{C}}$.

(2) Furthermore, since t is close to t_0, $\gamma_{c,+}$ (resp. $\gamma_{c,-}$) is a path from $(1, 1, \sqrt{1-b}/\sqrt{1-a})$ to $(0, 0, 1)$ (resp. $(1, 1, -\sqrt{1-b}/\sqrt{1-a})$ to $(0, 0, -1)$).

(3) The paths $\gamma_{00,+}, \gamma_{10,+}$, and $\gamma_{11,+}$ (resp. $\gamma_{00,-}, \gamma_{10,-}$, and $\gamma_{11,-}$) are the lifts of γ_{00}, γ_{10}, and γ_{11} and they are on the exceptional curves $Q_{(0,0)}, Q_{(1,0)}$, and $Q_{(1,1)}$, respectively.

(4) Since γ_x and γ_y on ∂K is contained in the branching locus of $\widetilde{\mathcal{X}}_t \to \widetilde{\mathcal{Y}}_t$, there exist the unique lifts of them and their lifts are contained in $\partial K_+ \cap \partial K_-$.

Then we can prove Proposition 5.5.

Proof of Proposition 5.5   It is enough to show that $K_+ - K_-$ is a 2-chain associated with $(\xi_1)_t - (\xi_0)_t$ in a weak sense. Note that $(\xi_1)_t - (\xi_0)_t$ is represented by the formal sum

$$\left(\widetilde{\mathcal{C}}, \psi_0^{-1}\psi_1\right) + \left(Q_{(0,0)}, \varphi_0^{-1}\right) + \left(Q_{(1,1)}, \varphi_1\right).$$

Let $\widetilde{\gamma}_c$ (resp. $\widetilde{\gamma}_{00}, \widetilde{\gamma}_{11}$) be the 1-chain on $\widetilde{\mathcal{C}}$ (resp. $Q_{(0,0)}, Q_{(1,1)}$) defined by the pull-back of $[\infty, 0]$ on $\mathbb{P}^1$ by the rational function $\psi_0^{-1}\psi_1$ (resp. $\varphi_0^{-1}, \varphi_1$). Then $\widetilde{\gamma}_c + \widetilde{\gamma}_{00} + \widetilde{\gamma}_{11}$ is the 1-cycle associated with ξ. By Figure 5, we have

$$\partial(K_+ - K_-) = (\gamma_{c,+} - \gamma_{c,-}) + (\gamma_{00,+} - \gamma_{00,-}) + (\gamma_{10,+} - \gamma_{10,-}) + (\gamma_{11,+} - \gamma_{11,-}).$$

Hence, we should show that the 1-chains $(\gamma_{c,+} - \gamma_{c,-} - \widetilde{\gamma}_c)$, $(\gamma_{00,+} - \gamma_{00,-} - \widetilde{\gamma}_{00})$, $(\gamma_{10,+} - \gamma_{10,-})$, and $(\gamma_{11,+} - \gamma_{11,-} - \widetilde{\gamma}_{11})$ are 1-boundaries on the curves $\widetilde{\mathcal{C}}, Q_{(0,0)}$, $Q_{(1,0)}$, and $Q_{(1,1)}$, respectively. Since $\widetilde{\mathcal{C}}, Q_{(0,0)}, Q_{(1,0)}$ and $Q_{(1,1)}$ are isomorphic to $\mathbb{P}^1$ and $H_1(\mathbb{P}^1) = 0$, it is enough to show that they are 1-cycles. By Figure 5, we have the following relations:

$$\partial(\gamma_{c,+} - \gamma_{c,-}) = \operatorname{div}_{\widetilde{\mathcal{C}}}(\psi_0^{-1}\psi_1) = \partial\widetilde{\gamma}_c, \quad \partial(\gamma_{00,+} - \gamma_{00,-}) = \operatorname{div}_{Q_{(0,0)}}(\varphi_0^{-1}) = \partial\widetilde{\gamma}_{00},$$

$$\partial(\gamma_{11,+} - \gamma_{11,-}) = \operatorname{div}_{Q_{(1,1)}}(\varphi_1) = \partial\widetilde{\gamma}_{11}, \quad \partial(\gamma_{10,+} - \gamma_{10,-}) = 0.$$

Hence, they are 1-cycles and we confirm that $K_+ - K_-$ is a 2-chain associated with $(\xi_1)_t - (\xi_0)_t$ in a weak sense. ∎

Let $\mathcal{L}$ be the local holomorphic function

$$\mathcal{L}(t) = \int_{K_+} \omega_t - \int_{K_-} \omega_t, \tag{17}$$

which is defined around t_0. Using the local description of ω_t, we see that

$$\int_{K_+} \omega_t = -\int_{K_-} \omega_t = \int_\triangle \frac{dxdy}{\sqrt{x(1-x)(1-ax)}\sqrt{y(1-y)(1-by)}}.$$

Hence, $\mathcal{L}(t)$ is a lift of $\mathcal{L}(a, b)$ in (10). Then we can prove Proposition 5.3.

Proof of Proposition 5.3   We will prove (1) and (2) simultaneously. Let $\{U_i\}_i$ be a good open cover of T such that for each i, there exists a local holomorphic function φ_i on U_i which represents $\nu_{\mathrm{tr}}(\xi_1 - \xi_0)|_{U_i}$. To prove (1) and (2), it is enough to show that $\mathcal{L}$ have an analytic continuation on each U_i and the resulting function coincides with φ_i up to an elements in $\mathcal{P}_\omega(U_i)$.

By Proposition 5.5, the equation (11) holds for any point t of an open neighborhood U_0 of t_0. Since the left-hand side of (11) corresponds to $\langle \mathrm{ev}_t(\nu_{\mathrm{tr}}(\xi_1 - \xi_0)), [\omega_t]\rangle$, we see that $[\mathcal{L}]$ corresponds to $\nu_{\mathrm{tr}}(\xi_1 - \xi_0)|_{U_0}$ by Corollary 2.5. Thus if $U_i \cap U_0 \neq \varnothing$, we

have $[\varphi_i]|_{U_i \cap U_0} = [\mathcal{L}]|_{U_i \cap U_0}$, so $\mathcal{L} - \varphi_i \in \mathcal{P}_\omega(U_i \cap U_0)$. Let p_i be the restriction of $\mathcal{L} - \varphi_i$ on $U_i \cap U_0$. Then we can extend p_i on U_i. The function $\varphi_i + p_i$ is a desired analytic continuation of $\mathcal{L}$ on U_i. For a general U_i, take a finite family $U_{i_1}, \ldots,$ $U_{i_n} = U_i$ such that $U_0 \cap U_{i_1} \neq \varnothing$ and $U_{i_{k-1}} \cap U_{i_k} \neq \varnothing$ for $k = 2, \ldots, n$ and repeat the above process.

Next, we will prove (3). We will compute $\mathcal{D}_1(\mathcal{L})$. It is enough to calculate it on a neighborhood of t_0. Hence, we may assume $\mathcal{L}$ is given by (17).

Let $H(a, x)$ be a local holomorphic function defined by $H(a, x) = -\frac{\sqrt{x(1-x)}}{2\sqrt{1-ax}^3}$. We can check that

$$\mathcal{D}_1\left(\frac{1}{\sqrt{x(1-x)(1-ax)}}\right) = \frac{\partial H(a, x)}{\partial x}.$$

Then by Stokes' theorem,[3] we have

$$\mathcal{D}_1\left(\frac{1}{2}\mathcal{L}\right) = \mathcal{D}_1\left(\int_{K_+} \frac{dx\,dy}{\sqrt{x(1-x)(1-ax)}\sqrt{y(1-y)(1-by)}}\right)$$

$$= \int_{K_+} d\left(\frac{H(a, x)\,dy}{\sqrt{y(1-y)(1-by)}}\right) = \int_{\partial K_+} \frac{H(a, x)\,dy}{\sqrt{y(1-y)(1-by)}},$$

and since the 1-form $\frac{H(a,x)\,dy}{\sqrt{y(1-y)(1-by)}}$ vanishes on ∂K_+ except $\gamma_{c,+}$, we have

$$= \frac{1}{2}\int_0^1 \frac{dz}{(1-bz)^{\frac{1}{2}}(1-az)^{\frac{3}{2}}} = \frac{1}{a-b}\int_1^{\frac{\sqrt{1-b}}{\sqrt{1-a}}} du = \frac{1}{a-b}\cdot\left(\frac{\sqrt{1-b}}{\sqrt{1-a}} - 1\right).$$

Here, we use the coordinate transform $u = \frac{\sqrt{1-bz}}{\sqrt{1-az}}$. We can compute $\mathcal{D}_2(\mathcal{L})$ similarly. ∎

6 The group action on the Kummer family

In this section, we construct a group action on $\widetilde{\mathfrak{X}} \to T$. To produce many higher Chow cycles, it is not enough to consider only automorphisms which are trivial on T. We will consider automorphisms of the following type.

Definition 6.1 Let $X \to S$ be a family of algebraic varieties. The *automorphism group* $\mathrm{Aut}(X \to S)$ *of* $X \to S$ consists of a pair (g, h) with $g \in \mathrm{Aut}(X)$ and $h \in \mathrm{Aut}(S)$ such that the following diagram commutes.

$$\begin{array}{ccc} X & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow{h} & S \end{array}$$

---

[3] To be more precise, we should take a reparametrization of K_+ using γ before Proposition 5.5.

We say that a group G acts on a family of varieties $X \to S$ if we have a group homomorphism $G \to \operatorname{Aut}(X \to S)$.

The main result of this section is as follows.

Proposition 6.2 *There exists a $G_{\mathcal{X}}$-action on $\widetilde{\mathcal{X}} \to T$, where $G_{\mathcal{X}}$ is a $\mathbb{Z}/2$-extension of $(\mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4)^2$.*

We use the following conventions for group actions.

(1) In this paper, we consider *left* group actions unless specified otherwise.
(2) Suppose a group G acts on a set M. The group G *stabilizes* a subset M' of M if we have $g \cdot m \in M'$ for any $g \in G$ and $m \in M'$.
(3) For a $\mathbb{C}$-scheme X, $\operatorname{Aut}(X)$ denotes the $\mathbb{C}$-automorphism group of X. For $\rho \in \operatorname{Aut}(X)$, $\rho^{\sharp}$ denotes the morphism of sheaves $\mathcal{O}_X \to \rho_* \mathcal{O}_X$.
(4) For $n \in \mathbb{Z}_{>1}$, μ_n denotes the group of nth roots of unity.
(5) For $n \in \mathbb{Z}_{\geq 1}$, $\mathfrak{S}_n$ denotes the symmetric group of degree n. For a set M, $\mathfrak{S}(M)$ denotes the symmetric group of M. The sign character is denoted by $\operatorname{sgn} : \mathfrak{S}(M) \to \mu_2$.

Before the construction of the $G_{\mathcal{X}}$-action on $\widetilde{\mathcal{X}} \to T$, we recall some formal properties of finite group actions on schemes in Section 6.1.

6.1 Generalities

A scheme with a group action (S, H, φ) is a triplet consisting of a $\mathbb{C}$-scheme S, a group H and a group homomorphism $\varphi : H \to \operatorname{Aut}(S)$. We usually omit φ from the notation and write (S, H). A morphism $(X, G) \to (S, H)$ between two such objects is a pair of a morphism $X \to S$ of varieties and a group homomorphism $G \to H$ satisfying the usual compatibility condition.

In Section 6.2, we use the following fiber product construction. Suppose we have two morphisms $(\pi, \psi) : (X, G) \to (S, H)$ and $(f, \varphi) : (S', H') \to (S, H)$. Recall that the fiber product of G and H' over H is defined by

$$G \times_H H' = \{(g, h) \in G \times H' : \psi(g) = \varphi(h)\}.$$

Then we define the fiber product of (X, G) and (S', H') over (S, H) by

(18) $$(X \times_S S', G \times_H H').$$

This is indeed the fiber product in the category of schemes with group actions.

Let (X, G) be a scheme with a group action. Then we have the natural right G-action on $\Gamma(X, \mathcal{O}_X^\times)$. A *1-cocycle* on $\Gamma(X, \mathcal{O}_X^\times)$ is a map $\chi : G \to \Gamma(X, \mathcal{O}_X^\times)$ satisfying

$$\chi(gh) = h^{\sharp}(\chi(g)) \cdot \chi(h)$$

for any $g, h \in G$. Here, $h^{\sharp} : \mathcal{O}_X \to h_* \mathcal{O}_X$ is the natural morphism induced by $h : X \to X$.

Let $\mathscr{L}$ be a G-linearized line bundle on X. The G-linearization is the same as a collection $(\Phi_g : g^* \mathscr{L} \xrightarrow{\sim} \mathscr{L})_{g \in G}$ of isomorphisms satisfying the relation $\Phi_h \circ h^*(\Phi_g) = \Phi_{gh}$ for $g, h \in G$. We can construct a 1-cocycle as follows. Let s be a global section of

$\mathcal{L}$ such that $\mathrm{div}(s)$ is stable under the G-action. Then there exists the unique 1-cocycle $\chi : G \to \Gamma(X, \mathcal{O}_X^\times)$ such that

$$\Phi_g(g^*(s)) = \chi(g)^{-1} \cdot s \quad (g \in G).$$

Finally, we prove liftability of a group action by a cyclic covers. For a line bundle $\mathcal{L}$ and a section $s \in \Gamma(X, \mathcal{L}^{\otimes(-m)})$, the cyclic covering $Y \to X$ associated with $(\mathcal{L}, s)$ is defined as the relative spectrum of the $\mathcal{O}_X$-algebra $\bigoplus_{i=0}^{m-1} \mathcal{L}^{\otimes i}$ (cf. [BHPV04, p. 54]). The branching locus of $Y \to X$ coincides with $\mathrm{div}(s)$.

Proposition 6.3 *Let $\pi : Y \to X$ be the degree m cyclic covering associated with $(\mathcal{L}, s)$. Suppose a group G acts on X and a G-linearization $(\Phi_g)_{g \in G}$ of $\mathcal{L}$ are given. Assume that there exists a 1-cocycle $\chi : G \to \Gamma(X, \mathcal{O}_X^\times)$ such that*

$$\Phi_g^{\otimes(-m)}(g^*(s)) = \chi(g)^{-m} \cdot s$$

for any $g \in G$. Then there exists a G-action on Y such that π is G-equivariant.

Proof   For $g \in G$, we define an automorphism $\tilde{g} : Y \to Y$ as follows.

1. Let Y_1 be the cyclic covering associated with $(g^*\mathcal{L}, g^*(s))$. Then Y_1 is the fiber product of $Y \to X$ and $X \xrightarrow{g} X$. Since g is an isomorphism, $Y_1 \to Y$ is an isomorphism as well.
2. Let Y_2 be the cyclic covering associated with $(\mathcal{L}, \chi(g)^{-m} \cdot s)$. Then we have the isomorphism $Y_2 \simeq Y_1$ over X induced by Φ_g.
3. We have the $\mathcal{O}_X$-module automorphism on $\mathcal{L}$ defined by the multiplication of $\chi(g)^{-1}$. This automorphism induces an isomorphism $Y \simeq Y_2$ over X.

Composing these isomorphisms, we get an automorphism $\tilde{g} \in \mathrm{Aut}(Y)$.

(19)
$$\begin{array}{ccccccc}
Y & \xrightarrow[\sim]{(3)} & Y_2 & \xrightarrow[\sim]{(2)} & Y_1 & \xrightarrow[\sim]{(1)} & Y \\
\downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\
X & = & X & = & X & \xrightarrow[\sim]{g} & X
\end{array}$$

We can show that $G \to \mathrm{Aut}(Y); g \mapsto \tilde{g}$ is a group homomorphism by the condition on $(\Phi_g)_{g \in G}$ and the property of the 1-cocycles. Hence, we can construct a G-action on Y and π is G-equivariant by the construction. ∎

6.2 Construction of the groups and their actions

In this section, we will construct a group action on $\widetilde{\mathcal{X}} \to T$ by four steps.

1. We construct a $\mathfrak{S}(\Sigma)$-action on $\mathbb{P}^1 \times S_0 \to S_0$. By taking its direct product, we have a $G_{\mathcal{Y}_0}(= \mathfrak{S}(\Sigma)^2)$-action on $\mathcal{Y}_0 \to T_0$ which stabilizes Σ^2.
2. By taking the fiber product of the $G_{\mathcal{Y}_0}$-action on $\mathcal{Y}_0 \to T_0$ and a $G_T(= \mathfrak{S}_4^2)$-action on $T \to T_0$, we have a $G_{\mathcal{Y}}$-action on $\mathcal{Y} \to T$.
3. By considering a μ_2-extension $G_{\mathcal{X}}$ of $G_{\mathcal{Y}}$, we can construct a $G_{\mathcal{X}}$-action on $\mathcal{X} \to T$ by applying Proposition 6.3.
4. We lift the $G_{\mathcal{X}}$-action on $\mathcal{X} \to T$ to $\widetilde{\mathcal{X}} \to T$.

Throughout in this section, we identify $\mathrm{Aut}(S_0) = \mathfrak{S}(\{0, 1, \infty\})(\simeq \mathfrak{S}_3)$.

Proposition 6.4 *There exists a $\mathfrak{S}_4$-action on $\mathbb{P}^1 \times S_0 \to S_0$ satisfying the following properties.*

(1) *The natural map $\mathfrak{S}_4 \to \mathrm{Aut}(S_0) = \mathfrak{S}(\{0,1,\infty\})$ is surjective. For $\rho \in \mathfrak{S}_4$, its image in $\mathfrak{S}(\{0,1,\infty\})$ is denoted by $\underline{\rho}$.*

(2) *For $\rho \in \mathfrak{S}_4$ and $\sigma \in \Sigma$, we have $\underline{\rho} \circ \sigma \circ \underline{\rho}^{-1} \in \Sigma$. Moreover, the following map induces an isomorphism of groups.*

$$
\begin{array}{ccc}
\mathfrak{S}_4 & \longrightarrow & \mathfrak{S}(\Sigma) = \mathfrak{S}(\{0,1,1/c,\infty\}) \\
\cup & & \cup \\
\rho & \longmapsto & \left(\sigma \mapsto \underline{\rho} \circ \sigma \circ \underline{\rho}^{-1}\right)
\end{array}
$$
(20)

Since another group which is isomorphic to $\mathfrak{S}_4$ appears elsewhere, hereafter we denote this group by $\mathfrak{S}(\Sigma)$.

Proof Consider the following contravariant functor from the category of varieties to the category of sets.

$$
X \longmapsto \left\{ (Y \to X; p_1, p_2, p_3, p_4) : \begin{array}{l} Y \to X \text{ is a } \mathbb{P}^1\text{-bundle.} \\ p_1, \cdots, p_4 \text{ are its sections such that} \\ p_i(X) \cap p_j(X) = \varnothing \text{ for any } i \neq j. \end{array} \right\} / (X\text{-isom.})
$$

For each $(Y \to X; p_1, p_2, p_3, p_4)$, by considering the cross-ratio of $p_1(x)$, $p_2(x)$, $p_3(x)$, $p_4(x) \in \mathbb{P}^1$ at $x \in X$, we have the morphism $X \to S_0$. Then we can check that S_0 represents this functor and $(\mathbb{P}^1 \times S_0 \to S_0; 0, 1, \infty, 1/c)$ is the universal element. For each $\rho \in \mathfrak{S}_4$, ρ induces a natural automorphism on this functor. Combining with the fact that an automorphism on $\mathbb{P}^1$ which stabilizes four points is identity, this induces a $\mathfrak{S}_4$-action on $\mathbb{P}^1 \times S_0 \to S_0$. The property (2) is clear from this construction. The property (1) follows from the property of the cross-ratio. ∎

We need an explicit description of the $\mathfrak{S}(\Sigma)$-action on $\mathbb{P}^1 \times S_0$ for Step 3. We summarize them on Table 1. In the table, for each $\rho \in \mathfrak{S}(\Sigma)$, the image of c under $\underline{\rho}^\sharp : \mathcal{O}_{S_0} \to \underline{\rho}_* \mathcal{O}_{S_0}$ and the image of the local coordinate z on $\mathbb{P}^1$ under $\rho^\sharp : \mathcal{O}_{\mathbb{P}^1 \times S_0} \to \rho_* \mathcal{O}_{\mathbb{P}^1 \times S_0}$ are given.

From Proposition 6.4, we have a morphism $(\mathbb{P}^1 \times S_0, \mathfrak{S}(\Sigma)) \to (S_0, \mathfrak{S}(\{0,1,\infty\}))$. By taking the direct product, we have the morphism

(21)

$$
(\mathbb{P}^1 \times S_0 \times \mathbb{P}^1 \times S_0, \mathfrak{S}(\Sigma) \times \mathfrak{S}(\Sigma)) \longrightarrow (S_0 \times S_0, \mathfrak{S}(\{0,1,\infty\}) \times \mathfrak{S}(\{0,1,\infty\})).
$$

We denote the group $\mathfrak{S}(\Sigma) \times \mathfrak{S}(\Sigma)$ (resp. $\mathfrak{S}(\{0,1,\infty\}) \times \mathfrak{S}(\{0,1,\infty\})$) by $G_{\mathcal{Y}_0}$ (resp. G_{T_0}). Since $T_0 \subset S_0 \times S_0$ is stable under the G_{T_0}-action, (21) induces a morphism $(\mathcal{Y}_0, G_{\mathcal{Y}_0}) \to (T_0, G_{T_0})$. Furthermore, by the property (2) of Proposition 6.4, we see that $\Sigma^2 \subset \mathcal{Y}_0$ is stable under the $G_{\mathcal{Y}_0}$-action on $\mathcal{Y}_0$. We finish Step 1.

Next, we will construct a group action on T. Recall that T is a finite étale cover of T_0 corresponding to $B_0 \to B_0[\sqrt{a}, \sqrt{b}, \sqrt{1-a}, \sqrt{1-b}]$.

ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$
id	c	z

ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$
$(0\,1)$	$\frac{c}{c-1}$	$1-z$	$(0\,1/c)$	$1-c$	$\frac{1-cz}{1-c}$	$(0\,\infty)$	$\frac{1}{c}$	$\frac{1}{z}$
$(1/c\,\infty)$	$\frac{c}{c-1}$	$\frac{(1-c)z}{1-cz}$	$(1\,\infty)$	$1-c$	$\frac{z}{z-1}$	$(1\,1/c)$	$\frac{1}{c}$	cz

ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$
$(0\,1)(1/c\,\infty)$	c	$\frac{1-z}{1-cz}$	$(0\,1/c)(1\,\infty)$	c	$\frac{1-cz}{c(1-z)}$	$(0\,\infty)(1\,1/c)$	c	$\frac{1}{cz}$

ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$
$(0\,1\,1/c)$	$\frac{1}{1-c}$	$1-cz$	$(0\,1/c\,1)$	$\frac{c-1}{c}$	$\frac{c(1-z)}{c-1}$
$(0\,\infty\,1)$	$\frac{1}{1-c}$	$\frac{z-1}{z}$	$(0\,1\,\infty)$	$\frac{c-1}{c}$	$\frac{1}{1-z}$
$(0\,1/c\,\infty)$	$\frac{1}{1-c}$	$\frac{1-c}{1-cz}$	$(0\,\infty\,1/c)$	$\frac{c-1}{c}$	$\frac{1-cz}{(1-c)z}$
$(1\,\infty\,1/c)$	$\frac{1}{1-c}$	$\frac{(c-1)z}{1-z}$	$(1\,1/c\,\infty)$	$\frac{c-1}{c}$	$\frac{cz}{cz-1}$

ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$	ρ	$\rho^\sharp(c)$	$\rho^\sharp(z)$
$(0\,1/c\,1\,\infty)$	$\frac{c}{c-1}$	$\frac{c-1}{c(1-z)}$	$(0\,1\,1/c\,\infty)$	$1-c$	$\frac{1}{1-cz}$	$(0\,1\,\infty\,1/c)$	$\frac{1}{c}$	$\frac{1-cz}{1-z}$
$(0\,\infty\,1\,1/c)$	$\frac{c}{c-1}$	$\frac{cz-1}{cz}$	$(0\,\infty\,1/c\,1)$	$1-c$	$\frac{1-z}{(c-1)z}$	$(0\,1/c\,\infty\,1)$	$\frac{1}{c}$	$\frac{c(1-z)}{1-cz}$

Table 1: The $\mathfrak{S}(\Sigma)$-action on $\mathbb{P}^1 \times S_0$.

Proposition 6.5 *Let $S' = \operatorname{Spec} A_0[\sqrt{c}, \sqrt{1-c}]$. Then there exists a $\mathfrak{S}_4$-action on $S' \to S_0$ such that the natural map $\mathfrak{S}_4 \to \operatorname{Aut}(S_0) = \mathfrak{S}(\{0, 1, \infty\})$ is surjective.*

Proof We have the following ring isomorphism:

$$(22) \qquad A_0\left[\sqrt{c}, \sqrt{1-c}\right] \xrightarrow{\sim} \mathbb{C}\left[\gamma, \tfrac{1}{\gamma(\gamma^4-1)}\right]; \sqrt{c}, \sqrt{1-c} \longmapsto \frac{\gamma+\frac{1}{\gamma}}{2}, \frac{\gamma-\frac{1}{\gamma}}{2\sqrt{-1}}.$$

Using the ring isomorphism (22), we can check directly that for each $\rho \in \operatorname{Aut}(S_0)$, there exists a lift of ρ in $\operatorname{Aut}(S')$. Moreover, the S_0-automorphism group of S' is isomorphic to $\mu_2 \times \mu_2$. Thus we have the following exact sequence of groups:

$$1 \longrightarrow \mu_2 \times \mu_2 \longrightarrow \operatorname{Aut}(S' \to S_0) \longrightarrow \operatorname{Aut}(S_0) \longrightarrow 1.$$

To prove the proposition, we should show that $\operatorname{Aut}(S' \to S_0)$ is isomorphic to $\mathfrak{S}_4$. We will construct an $\operatorname{Aut}(S' \to S_0)$-action on a set of cardinality 4. By the ring

isomorphism (22), we have $S' \simeq \mathbb{P}^1 - \{0, \infty, \pm 1, \pm\sqrt{-1}\}$. Thus, $\mathrm{Aut}(S' \to S_0)$ acts on the set $\{0, \infty, \pm 1, \pm\sqrt{-1}\}$. Let F be the set

$$F = \left\{ \begin{array}{ll} \{\{1, \sqrt{-1}, \infty\}, \{-1, -\sqrt{-1}, 0\}\}, & \{\{-1, \sqrt{-1}, \infty\}, \{1, -\sqrt{-1}, 0\}\} \\ \{\{1, -\sqrt{-1}, \infty\}, \{-1, \sqrt{-1}, 0\}\}, & \{\{-1, -\sqrt{-1}, \infty\}, \{1, \sqrt{-1}, 0\}\} \end{array} \right\}.$$

If we plot $\pm 1, \pm\sqrt{-1}, 0, \infty$ on the Riemann sphere, they are the vertexes of the regular octahedron. Then F is the set of pairs of opposite faces of the octahedron. We can check that $\mathrm{Aut}(S' \to S_0)$ acts on F and the group homomorphism $\mathrm{Aut}(S' \to S_0) \to \mathfrak{S}(F)$ is surjective. By the exact sequence above, the orders of $\mathrm{Aut}(S' \to S_0)$ and $\mathfrak{S}(F)$ coincide. Thus $\mathrm{Aut}(S' \to S_0) \simeq \mathfrak{S}(F)$. ∎

From Proposition 6.5, we have a morphism $(S', \mathfrak{S}_4) \to (S_0, \mathfrak{S}(\{0, 1, \infty\}))$. By taking its direct product, we have

$$(23) \qquad (S' \times S', \mathfrak{S}_4 \times \mathfrak{S}_4) \longrightarrow (S_0 \times S_0, G_{T_0}).$$

We denote the group $\mathfrak{S}_4 \times \mathfrak{S}_4$ by G_T. Since T is a fiber product of T_0 and $S' \times S'$ over $S_0 \times S_0$, this induces a morphism $(T, G_T) \to (T_0, G_{T_0})$.

We have already constructed the morphisms $(\mathcal{Y}_0, G_{\mathcal{Y}_0}) \to (T_0, G_{T_0})$ and $(T, G_T) \to (T_0, G_{T_0})$. Then the group

$$G_{\mathcal{Y}_0} \times_{G_{T_0}} G_T = (\mathfrak{S}(\Sigma)^2) \times_{\mathfrak{S}(\{0,1,\infty\})^2} (\mathfrak{S}_4^2) \simeq (\mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4)^2$$

acts on $\mathcal{Y} = \mathcal{Y}_0 \times_{T_0} T$ by (18). We denote this group by $G_{\mathcal{Y}}$. Thus, we have the morphism $(\mathcal{Y}, G_{\mathcal{Y}}) \to (T, G_T)$. We often denote an element of $G_{\mathcal{Y}}$ by a 3-tuple

$$(\rho_1, \rho_2, \tau) \in \mathfrak{S}(\Sigma) \times \mathfrak{S}(\Sigma) \times G_T$$

such that $\tau \in G_T$ and $(\rho_1, \rho_2) \in G_{\mathcal{Y}_0}$ have the same image in $G_{T_0} = \mathfrak{S}(\{0, 1, \infty\})^2$. We finish Step 2.

We will lift this group action to the double cover $\mathcal{X} \to \mathcal{Y}$. Let $\mathcal{L}$ be the line bundle on $\mathcal{Y}$ defined by

$$\mathcal{L} = pr_1^* \Omega^1_{\mathbb{P}^1 \times S_0 / S_0} \otimes pr_2^* \Omega^1_{\mathbb{P}^1 \times S_0 / S_0},$$

where pr_i is a morphism $\mathcal{Y} \to \mathcal{Y}_0 \hookrightarrow (\mathbb{P}^1 \times S_0) \times (\mathbb{P}^1 \times S_0) \xrightarrow{pr_i} \mathbb{P}^1 \times S_0$. Let s be the section of $\mathcal{L}^{\otimes(-2)}$ defined by

$$s = f(x)g(y)(dx \otimes dy)^{\otimes(-2)}.$$

We have the natural $G_{\mathcal{Y}}$-linearization $(\Phi_\rho)_{\rho \in G_{\mathcal{Y}}}$ of $\mathcal{L}$ since it is constructed from sheaves of differentials.

The morphism $\mathcal{X} \to \mathcal{Y}$ is the double covering associated with $(\mathcal{L}, s)$. Since $\mathrm{div}(s)$ is equal to the branching locus of $\mathcal{X} \to \mathcal{Y}$, $\mathrm{div}(s)$ is stable under the $G_{\mathcal{Y}}$-action. Hence, we have the 1-cocycle $\eta : G_{\mathcal{Y}} \to \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^\times) = B^\times$ such that

$$\Phi_\rho^{\otimes(-2)}(\rho^*(s)) = \eta(\rho)^{-1} \cdot s$$

for any $\rho \in G_{\mathcal{Y}}$. We can compute η as follows.

Proposition 6.6 *For* $\rho = (\rho_1, \rho_2, \tau) \in G_{\mathcal{Y}}$, $\eta(\rho)$ *is determined by* $\underline{\rho}_1, \underline{\rho}_2 \in$ $\mathfrak{S}(\{0, 1, \infty\})$ *and calculated by the following formula:*

$$\eta(\rho) = \eta_1(\underline{\rho}_1) \cdot \eta_2(\underline{\rho}_2),$$

where $\eta_1(\underline{\rho}_1), \eta_2(\underline{\rho}_2) \in B^\times$ *are given by the following table.*

$\underline{\rho}_1$	$\eta_1(\underline{\rho}_1)$	$\underline{\rho}_1$	$\eta_1(\underline{\rho}_1)$		$\underline{\rho}_2$	$\eta_2(\underline{\rho}_2)$	$\underline{\rho}_2$	$\eta_2(\underline{\rho}_2)$
id	1	$(0\,1)$	-1		id	1	$(0\,1)$	-1
$(1\,\infty)$	$1-a$	$(0\,1\,\infty)$	$a-1$		$(1\,\infty)$	$1-b$	$(0\,1\,\infty)$	$b-1$
$(0\,\infty)$	a	$(0\,\infty\,1)$	$-a$		$(0\,\infty)$	b	$(0\,\infty\,1)$	$-b$

Proof By the definition of the $G_{\mathcal{Y}}$-linearization on $\mathscr{L}$, we have

$$\Phi_\rho(\rho^*(dx \otimes dy)) = \left(\frac{\partial}{\partial x}\rho_1^\sharp(x)\right)\left(\frac{\partial}{\partial y}\rho_2^\sharp(y)\right) \cdot (dx \otimes dy).$$

Using Table 1, we can compute η for each $\rho_1, \rho_2 \in \mathfrak{S}(\Sigma)$ and we get the result. ∎

To apply Proposition 6.3, we should construct a "square root" of the 1-cocycle η. We define coboundary 1-cocycles $\phi_1, \phi_2 : G_T \to B^\times$ by

$$\phi_1(\tau) = \tau^\sharp\left(\frac{\sqrt{a}\sqrt{1-a}}{a^2 - a + 1}\right) \cdot \left(\frac{\sqrt{a}\sqrt{1-a}}{a^2 - a + 1}\right)^{-1}, \quad \phi_2(\tau) = \tau^\sharp\left(\frac{\sqrt{b}\sqrt{1-b}}{b^2 - b + 1}\right) \cdot \left(\frac{\sqrt{b}\sqrt{1-b}}{b^2 - b + 1}\right)^{-1}$$

for $\tau \in G_T$. Then we can check that for any $\rho = (\rho_1, \rho_2, \tau) \in G_{\mathcal{Y}}$,

$$(24) \qquad\qquad\qquad \eta_i(\underline{\rho}_i) = \mathrm{sgn}(\underline{\rho}_i)\phi_i(\tau)^2$$

holds for $i = 1, 2$. Hence, $\phi_1 \cdot \phi_2$ coincides with a square root of η up to sign. To get a square root of sign, we enlarge the group $G_{\mathcal{Y}}$ as follows.

We define the group $G_{\mathcal{X}}$ as the fiber product $G_{\mathcal{Y}} \times_{\mu_2} \mu_4$, where the group homomorphisms $G_{\mathcal{Y}} \to \mu_2$ and $\mu_4 \to \mu_2$ is given by

$$G_{\mathcal{Y}} \to \mu_2; \quad (\rho_1, \rho_2, \tau) \mapsto \mathrm{sgn}(\underline{\rho}_1)\mathrm{sgn}(\underline{\rho}_2),$$

$$\mu_4 \to \mu_2; \quad \zeta \mapsto \zeta^2.$$

We denote an element of $G_{\mathcal{X}}$ by a 4-tuple

$$(\rho_1, \rho_2, \tau, \zeta) \in \mathfrak{S}(\Sigma) \times \mathfrak{S}(\Sigma) \times G_T \times \mu_4.$$

The group $G_{\mathcal{X}}$ acts on $\mathcal{Y}$ through the surjection $G_{\mathcal{X}} \twoheadrightarrow G_{\mathcal{Y}}$. For $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_{\mathcal{X}}$, we define

$$\chi(\rho) = \zeta \cdot \phi_1(\tau) \cdot \phi_2(\tau) \quad \in B^\times = \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^\times).$$

Then by (24), $\chi : G_{\mathcal{X}} \to \Gamma(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}^\times)$ is a 1-cocycle satisfying the relation

$$\chi(\rho)^2 = \eta(\rho).$$

Hence, we have the following result.

Proposition 6.7 *There exists a $G_{\mathcal{X}}$-action on $\mathcal{X}$ such that the morphism $\mathcal{X} \to \mathcal{Y}$ is $G_{\mathcal{X}}$-equivariant.*

Proof The $G_{\mathcal{Y}}$-linearization $(\Phi_\rho)_{\rho \in G_{\mathcal{Y}}}$ of $\mathscr{L}$ induces the natural $G_{\mathcal{X}}$-linearization of $\mathscr{L}$. This linearization satisfies the equation

$$\Phi_\rho^{\otimes(-2)}(\rho^*(s)) = \eta(\rho)^{-1} \cdot s = \chi(\rho)^{-2} \cdot s.$$

Hence, by applying Proposition 6.3, we have the result. ∎

Thus, we have the morphism $(\mathcal{X}, G_{\mathcal{X}}) \to (T, G_T)$. We finish Step 3.

Recall that the set Σ^2 on $\mathcal{Y}_0$ is stable under the $G_{\mathcal{Y}_0}$-action. Then its base change Σ^2 on $\mathcal{Y}$ is also stable under the $G_{\mathcal{Y}}$-action. Since $\Sigma^2 \subset \mathcal{Y}$ is contained in the branching locus of $\mathcal{X} \to \mathcal{Y}$ and $\mathcal{X} \to \mathcal{Y}$ is $G_{\mathcal{X}}$-equivariant, Σ^2 on $\mathcal{X}$ is also $G_{\mathcal{X}}$-stable. Then by the universality of the blowing-up, for any $\rho \in G_{\mathcal{X}}$, there exists the unique lift $\widetilde{\rho} \in \mathrm{Aut}(\widetilde{\mathcal{X}})$ of ρ. By the uniqueness, $G_{\mathcal{X}} \to \mathrm{Aut}(\widetilde{\mathcal{X}}); \rho \mapsto \widetilde{\rho}$ defines a $G_{\mathcal{X}}$-action on $\widetilde{\mathcal{X}}$. We finish Step 4.

Finally, we can prove the main result of this section.

Proof of Proposition 6.2 We have the morphisms

$$(\widetilde{\mathcal{X}}, G_{\mathcal{X}}) \longrightarrow (\mathcal{X}, G_{\mathcal{X}}) \longrightarrow (\mathcal{Y}, G_{\mathcal{Y}}) \longrightarrow (T, G_T).$$

By composing these morphisms, we have the $G_{\mathcal{X}}$-action on the family $\widetilde{\mathcal{X}} \to T$. ∎

At the end of this section, we see an explicit description of the $G_{\mathcal{X}}$-action on $\widetilde{\mathcal{X}}$ using the local coordinates x, y, v on V. Let $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_{\mathcal{X}}$. Since $\widetilde{\mathcal{X}} \to \mathcal{Y}$ is $G_{\mathcal{X}}$-equivariant, we see that

$$x \mapsto \rho_1^\sharp(x), \quad y \mapsto \rho_2^\sharp(y).$$

Locally on $U \subset \mathcal{Y}$, the morphism $\mathcal{X} \to \mathcal{Y}$ is described by the ring homomorphism

$$B[x, y] \longrightarrow B[x, y, u]/(u^2 - f(x)g(y)),$$

where u corresponds to the local section $(dx \otimes dy)$ of $\mathscr{L}$. By the construction of the $G_{\mathcal{X}}$-action, the local coordinate u transforms

$$u \mapsto \chi(\rho)^{-1} \frac{\partial}{\partial x}(\rho_1^\sharp(x)) \frac{\partial}{\partial y}(\rho_2^\sharp(y)) u.$$

Since $u = vf(x)$, by the formula $\rho_1^\sharp(f(x)) = \eta_1(\underline{\rho}_1)^{-1}\left(\frac{\partial}{\partial x}\rho_1^\sharp(x)\right)^2 f(x)$, v transforms as follows:

$$v \mapsto \frac{\eta_1(\underline{\rho}_1) \frac{\partial}{\partial y}(\rho_2^\sharp(y))}{\chi(\rho) \frac{\partial}{\partial x}(\rho_1^\sharp(x))} v.$$

7 The group action on the higher Chow cycles

In this section, we prove the main result in this paper by using some technical propositions which is proved Section 8.

7.1 The group action on the higher Chow cycles

First, we produce more families of higher Chow cycles from ξ_0, ξ_1 and ξ_∞ by using the group action. For $t \in T$, $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_{\mathcal{X}}$ induces the isomorphism $\rho_t : \widetilde{\mathcal{X}}_t \to \widetilde{\mathcal{X}}_{\tau(t)}$ such that

(25)
$$
\begin{array}{ccc}
\widetilde{\mathcal{X}}_t & \dashrightarrow^{\rho_t} & \widetilde{\mathcal{X}}_{\tau(t)} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{X}} & \xrightarrow{\ \rho\ } & \widetilde{\mathcal{X}}
\end{array}
$$

commutes. The morphism ρ_t induces a map $\rho_t^* : \mathrm{CH}^2(\widetilde{\mathcal{X}}_{\tau(t)}, 1) \to \mathrm{CH}^2(\widetilde{\mathcal{X}}_t, 1)$. For an algebraic family of higher Chow cycles $\xi = \{\xi_t\}_{t \in T}$ on $\widetilde{\mathcal{X}} \to T$, we define the algebraic family $\rho^* \xi$ of higher Chow cycles by $\rho^* \xi = \{\rho_t^*(\xi_{\tau(t)})\}_{t \in T}$.

In particular, we have defined families of higher Chow cycles $\rho^* \xi_0$, $\rho^* \xi_1$ and $\rho^* \xi_\infty$ for $\rho \in G_{\mathcal{X}}$. For each $t \in T$, we define the subgroup Ξ_t of $\mathrm{CH}^2(\widetilde{\mathcal{X}}_t, 1)_{\mathrm{ind}}$ by

$$
\Xi_t = \langle ((\rho^* \xi_\bullet)_t)_{\mathrm{ind}} : \bullet \in \{0, 1, \infty\}, \rho \in G_{\mathcal{X}} \rangle_{\mathbb{Z}}.
$$

Then we can state the main theorem.

Theorem 7.1 *For a very general $t \in T$, we have* $\mathrm{rank}\ \Xi_t \geq 18$. *In particular,* $\mathrm{rank}\ \mathrm{CH}^2(\mathcal{X}_t, 1)_{\mathrm{ind}} \geq 18$.

To be more specific, 18 linearly independent cycles in Ξ_t are given as follows. Let $\rho^1, \rho^2, \ldots, \rho^6 \in G_{\mathcal{X}}$ be lifts of $(\mathrm{id}, \mathrm{id})$, $(\mathrm{id}, (0\ 1))$, $(\mathrm{id}, (1\ \infty))$, $(\mathrm{id}, (0\ 1\ \infty))$, $(\mathrm{id}, (0\ \infty))$ and $(\mathrm{id}, (0\ \infty\ 1)) \in G_{\mathcal{Y}_0}$ by $G_{\mathcal{X}} \twoheadrightarrow G_{\mathcal{Y}_0} = \mathfrak{S}(\Sigma)^2$, respectively. Then we will show that $(((\rho^i)^* \xi_\bullet)_t)_{\mathrm{ind}}$ $(i = 1, 2, \ldots, 6, \bullet = 0, 1, \infty)$ are linearly independent in Ξ_t for very general $t \in T$. The key of the proof is to compute $\mathscr{D}(\nu_{\mathrm{tr}}((\rho^i)^* \xi_\bullet))$ for each $i = 1, 2, \ldots, 6$ and $\bullet = 0, 1, \infty$.

Before proceeding to the proof, we explain geometric constructions of cycles in Ξ_t. For $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_{\mathcal{X}}$, let $\mathcal{D}_\rho$ be the $(1,1)$-curve on $\mathbb{P}^1 \times \mathbb{P}^1$ which passes through $(\rho_1^{-1}(0), \rho_2^{-1}(0))$, $(\rho_1^{-1}(1), \rho_2^{-1}(1))$ and $(\rho_1^{-1}(\infty), \rho_2^{-1}(\infty)) \in \Sigma^2$. The local equation of $\mathcal{D}_\rho$ is given by $\rho_1^\sharp(x) = \rho_2^\sharp(y)$. Let $\mathcal{C}_\rho \subset \mathcal{X}_t$ be its pull-back by $\mathcal{X}_t \to \mathcal{Y}_t = \mathbb{P}^1 \times \mathbb{P}^1$ and $\widetilde{\mathcal{C}}_\rho \subset \widetilde{\mathcal{X}}_t$ be the strict transform of $\mathcal{C}_\rho$. By definition, $\mathcal{D}_\rho$ coincides with the inverse image of $\mathcal{D}$ by ρ, so we have $\widetilde{\mathcal{C}}_\rho = \rho_t^{-1}(\widetilde{\mathcal{C}})$. Moreover, by considering the $G_{\mathcal{X}}$-action on Σ^2, we have $Q_{(\rho_1^{-1}(\bullet), \rho_2^{-1}(\bullet))} = \rho_t^{-1}(Q_{(\bullet, \bullet)})$ for $\bullet = 0, 1, \infty$. Thus we see that $(\rho^* \xi_\bullet)_t = \rho_t^*((\xi_\bullet)_{\tau(t)})$ $(\bullet \in \{0, 1, \infty\})$ is represented by the formal sum

(26)
$$
(\rho^* \xi_\bullet) t = \left(\widetilde{\mathcal{C}}_\rho, \psi_\bullet \circ \rho_t \right) + \left(Q_{(\rho_1^{-1}(\bullet), \rho_2^{-1}(\bullet))}, \varphi_\bullet \circ \rho_t \right).
$$

For example, for $\rho^1, \rho^2, \ldots, \rho^6 \in G_{\mathcal{X}}$ defined above, $\mathcal{D}_{\rho^1}, \mathcal{D}_{\rho^2}, \ldots, \mathcal{D}_{\rho^6}$ are graphs of rational functions $z, 1-z, z/(1-z), 1/(1-z), 1/z$, and $(z-1)/z$ on $\mathbb{P}^1$, respectively. Hence, $((\rho^i)^* \xi_\bullet)_t$ are higher Chow cycles made from such graphs.

Remark 7.2 The idea of the construction of higher Chow cycles from the curve $\mathcal{D}_\rho$ is of Terasoma. The referee pointed out that the explicit 18 cycles made from the graphs of six rational functions on $\mathbb{P}^1$ are enough for the rank estimate.

7.2 Proof of the main theorem

We define subgroups N^{can} and N of $\Gamma(T, \mathcal{Q})$ as follows:

$$N^{\mathrm{can}} = \langle \nu_{\mathrm{tr}}(\xi_0), \nu_{\mathrm{tr}}(\xi_1), \nu_{\mathrm{tr}}(\xi_\infty) \rangle_{\mathbb{Z}},$$
$$N = \langle \nu_{\mathrm{tr}}(\rho^* \xi_\bullet) : \rho \in G_{\mathcal{X}}, \bullet \in \{0, 1, \infty\} \rangle_{\mathbb{Z}}.$$

By Proposition 2.3, the transcendental regulator map induces a surjective map $\Xi_t \twoheadrightarrow \mathrm{ev}_t(N)$ for each $t \in T$. Hence, to prove Theorem 7.1, we have to examine the rank of N. To get a rank estimate for N, we use the following proposition, whose proof will be given in the next section.

Proposition 7.3 *For $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_{\mathcal{X}}$, let $\Theta_\rho : \tau^*(\mathcal{O}_T^{\mathrm{an}})^{\oplus 2} \to (\mathcal{O}_T^{\mathrm{an}})^{\oplus 2}$ be the morphism defined by*

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \longmapsto \begin{pmatrix} \chi(\rho)^{-1} \cdot \phi_1(\tau)^{-2} \cdot \tau^\sharp(\varphi_1) \\ \chi(\rho)^{-1} \cdot \phi_2(\tau)^{-2} \cdot \tau^\sharp(\varphi_2) \end{pmatrix},$$

where $\tau^\sharp : \tau^* \mathcal{O}_T^{\mathrm{an}} \to \mathcal{O}_T^{\mathrm{an}}; \varphi \mapsto \varphi \circ \tau$. Let ξ be an algebraic family of higher Chow cycles on $\widetilde{\mathcal{X}} \to T$. Then we have

$$\mathscr{D}(\nu_{\mathrm{tr}}(\rho^* \xi)) = \Theta_\rho(\mathscr{D}(\nu_{\mathrm{tr}}(\xi))),$$

where $\mathscr{D} : \mathcal{Q} \simeq \mathcal{Q}_\omega \to (\mathcal{O}_T^{\mathrm{an}})^{\oplus 2}$ is the Picard–Fuchs differential operator defined in Proposition 5.2.

Thanks to this proposition, we can compute $\mathscr{D}(N)$ explicitly and we get the desired rank estimate for N. First, we compute $\mathscr{D}(N^{\mathrm{can}})$.

Proposition 7.4 *The images of $\nu_{\mathrm{tr}}(\xi_0), \nu_{\mathrm{tr}}(\xi_1), \nu_{\mathrm{tr}}(\xi_\infty)$ under the Picard–Fuchs differential operator $\mathscr{D}$ are as follows:*

$$\mathscr{D}(\nu_{\mathrm{tr}}(\xi_0)) = \frac{2}{a-b} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \mathscr{D}(\nu_{\mathrm{tr}}(\xi_1)) = \frac{2}{a-b} \begin{pmatrix} \frac{\sqrt{1-b}}{\sqrt{1-a}} \\ -\frac{\sqrt{1-a}}{\sqrt{1-b}} \end{pmatrix}, \quad \mathscr{D}(\nu_{\mathrm{tr}}(\xi_\infty)) = \frac{2}{a-b} \begin{pmatrix} \frac{\sqrt{b}}{\sqrt{a}} \\ -\frac{\sqrt{a}}{\sqrt{b}} \end{pmatrix}.$$

In particular, rank $N^{\mathrm{can}} = 3$.

Proof The key for the proof is to find $\rho^a, \rho^b \in G_{\mathcal{X}}$ such that

$$(27) \qquad (\rho^a)^*(\xi_1 - \xi_0) = \xi_1 + \xi_0, \quad (\rho^b)^*(\xi_1 - \xi_0) = \xi_0 - \xi_\infty.$$

We consider the following elements.

(a) $\rho^a = (\mathrm{id}, \mathrm{id}, \tau^a, 1)$, where $\tau^a \in G_T$ satisfies $(\tau^a)^\sharp(\sqrt{a}) = \sqrt{a}$, $(\tau^a)^\sharp(\sqrt{1-a}) = -\sqrt{1-a}$, $(\tau^a)^\sharp(\sqrt{b}) = \sqrt{b}$, and $(\tau^a)^\sharp(\sqrt{1-b}) = \sqrt{1-b}$. Then we have $\phi_1(\tau^a) = -1$ and $\phi_2(\tau^b) = 1$.

(b) $\rho^b = ((1\ \infty), (1\ \infty), \tau^b, 1)$, where $\tau^b \in G_T$ satisfies $(\tau^b)^\sharp(\sqrt{a}) = \sqrt{1-a}$, $(\tau^b)^\sharp(\sqrt{1-a}) = \sqrt{a}$, $(\tau^b)^\sharp(\sqrt{b}) = \sqrt{1-b}$, and $(\tau^b)^\sharp(\sqrt{1-b}) = \sqrt{b}$. Then we have $\phi_1(\tau^b) = 1$ and $\phi_2(\tau^b) = 1$.

These elements stabilize the curve $\widetilde{\mathcal{C}}$. By the local description of the $G_{\mathcal{X}}$-action in the end of Section 6, we see that these elements satisfy (27).

By Proposition 7.3 and (27), we can compute $\mathscr{D}(v_{\mathrm{tr}}(\xi_0 + \xi_1))$ and $\mathscr{D}(v_{\mathrm{tr}}(\xi_0 - \xi_\infty))$ from Proposition 5.3 as follows:

$$\mathscr{D}(v_{\mathrm{tr}}(\xi_0 + \xi_1)) = \Theta_{\rho^a}\left(\mathscr{D}(v_{tr}(\xi_1 - \xi_0))\right) = \frac{2}{a-b}\begin{pmatrix} 1 + \frac{\sqrt{1-b}}{\sqrt{1-a}} \\ -1 - \frac{\sqrt{1-a}}{\sqrt{1-b}} \end{pmatrix},$$

$$\mathscr{D}(v_{\mathrm{tr}}(\xi_0 - \xi_\infty)) = \Theta_{\rho^b}\left(\mathscr{D}(v_{tr}(\xi_1 - \xi_0))\right) = \frac{2}{(1-a)-(1-b)}\begin{pmatrix} \frac{\sqrt{b}}{\sqrt{a}} - 1 \\ 1 - \frac{\sqrt{a}}{\sqrt{b}} \end{pmatrix}.$$

Hence, the images of $v_{\mathrm{tr}}(\xi_0)$, $v_{\mathrm{tr}}(\xi_1)$, and $v_{\mathrm{tr}}(\xi_\infty)$ under the Picard–Fuchs differential operator are as in the statement. Since they are linearly independent over $\mathbb{Q}$, the latter part follows from these expressions. ∎

Next, we give an estimate for the rank of N. The proof below was simplified by advice from Terasoma. Furthermore, Sreekantan pointed out an error in the Table 2 in the previous version of this paper.

Proposition 7.5 Rank $N \geq 18$.

Proof Let $\rho^1, \rho^2, \ldots, \rho^6 \in G_\chi$ be elements defined after Theorem 7.1. Using Propositions 7.3 and 7.4, we can compute $\mathscr{D}(v_{\mathrm{tr}}((\rho^i)^*\xi_\bullet))$ for $i = 1, \ldots, 6$ and $\bullet = 0, 1, \infty$ as Table 2. Note that this table contains the ambiguity of signs because we do not fix the G_T-components of ρ^i.

We will show that the vectors in Table 2 are linearly independent over $\mathbb{Q}$. It is enough to show that the first components of these vectors are linearly independent over $\mathbb{C}$. Note that the first components of these vectors are expressed by

$$2\zeta \cdot F_1 \cdot F_2,$$

where $\zeta \in \mu_4$, F_1 is either

$$(28) \quad \frac{1}{a-b}, \; \frac{1}{a+b-1}, \; \frac{1}{ab-a-b}, \; \frac{1}{ab-b+1}, \; \frac{1}{ab-1} \text{ or } \frac{1}{a-ab-1} \in \mathrm{Frac}(B_0)$$

and F_2 is either

$$(29) \quad 1, \; \frac{\sqrt{b}}{\sqrt{a}}, \; \frac{\sqrt{1-b}}{\sqrt{1-a}}, \; \frac{\sqrt{1-b}}{\sqrt{a}}, \; \frac{\sqrt{b}}{\sqrt{1-a}}, \; \frac{1}{\sqrt{1-a}}, \; \sqrt{1-b}, \; \frac{1}{\sqrt{a}} \text{ or } \sqrt{b} \in \mathrm{Frac}(B).$$

Since elements in (28) are linearly independent over $\mathbb{C}$ and elements in (29) are linearly independent over $\mathrm{Frac}(B_0)$, their products are linearly independent over $\mathbb{C}$. Hence, we have the result. ∎

Then we can prove the main theorem as follows.

Proof of Theorem 7.1 Since N is countable, by Lemma 2.4, $N \twoheadrightarrow \mathrm{ev}_t(N)$ is bijective for very general $t \in T$. By Proposition 7.5, rank $\mathrm{ev}_t(N) \geq 18$ for such t. Since $\mathrm{ev}_t(v_{\mathrm{tr}}(\rho^*\xi_\bullet)) = r((\rho^*\xi_\bullet)_t)$, $\mathrm{ev}_t(N)$ is generated by the images of $(\rho^*\xi_\bullet)_t$ ($\rho \in G_\chi, \bullet \in \{0, 1, \infty\}$) under the transcendental regulator map. By Proposition 2.3, the transcendental regulator map induces the surjective map $\Xi_t \twoheadrightarrow \mathrm{ev}_t(N)$. Hence, we have rank $\Xi_t \geq \mathrm{rank}\, \mathrm{ev}_t(N) \geq 18$ for very general $t \in T$. ∎

Its image in G_{y_0}	$\mathscr{D}(v_{\mathrm{tr}}((\rho^i)^*\xi_0)),\quad \mathscr{D}(v_{\mathrm{tr}}((\rho^i)^*\xi_1)),\quad \mathscr{D}(v_{\mathrm{tr}}((\rho^i)^*\xi_\infty))$
$(\mathrm{id},\mathrm{id})$	$\pm\dfrac{2}{a-b}\begin{pmatrix}1\\-1\end{pmatrix},\ \pm\dfrac{2}{a-b}\begin{pmatrix}\frac{\sqrt{1-b}}{\sqrt{1-a}}\\-\frac{\sqrt{1-a}}{\sqrt{1-b}}\end{pmatrix},\ \pm\dfrac{2}{a-b}\begin{pmatrix}\frac{\sqrt{b}}{\sqrt{a}}\\-\frac{\sqrt{a}}{\sqrt{b}}\end{pmatrix}$
$(\mathrm{id},(0\,1))$	$\pm\dfrac{2}{ab-a-b}\begin{pmatrix}\sqrt{1-b}\\\frac{1}{\sqrt{1-b}}\end{pmatrix},\ \pm\dfrac{2}{ab-a-b}\begin{pmatrix}\frac{1}{\sqrt{1-a}}\\\sqrt{1-a}\end{pmatrix},\ \pm\dfrac{2\sqrt{-1}}{ab-a-b}\begin{pmatrix}\frac{\sqrt{b}}{\sqrt{a}}\\-\frac{\sqrt{a}}{\sqrt{b}}\end{pmatrix}$
$(\mathrm{id},(1\,\infty))$	$\pm\dfrac{2\sqrt{-1}}{a+b-1}\begin{pmatrix}1\\-1\end{pmatrix},\ \pm\dfrac{2\sqrt{-1}}{a+b-1}\begin{pmatrix}\frac{\sqrt{b}}{\sqrt{1-a}}\\-\frac{\sqrt{1-a}}{\sqrt{b}}\end{pmatrix},\ \pm\dfrac{2\sqrt{-1}}{a+b-1}\begin{pmatrix}\frac{\sqrt{1-b}}{\sqrt{a}}\\-\frac{\sqrt{a}}{\sqrt{1-b}}\end{pmatrix}$
$(\mathrm{id},(0\,1\,\infty))$	$\pm\dfrac{2\sqrt{-1}}{a-ab-1}\begin{pmatrix}\sqrt{1-b}\\\frac{1}{\sqrt{1-b}}\end{pmatrix},\ \pm\dfrac{2}{a-ab-1}\begin{pmatrix}\frac{\sqrt{b}}{\sqrt{1-a}}\\-\frac{\sqrt{1-a}}{\sqrt{b}}\end{pmatrix},\ \pm\dfrac{2\sqrt{-1}}{a-ab-1}\begin{pmatrix}\frac{1}{\sqrt{a}}\\\sqrt{a}\end{pmatrix}$
$(\mathrm{id},(0\,\infty))$	$\pm\dfrac{2}{ab-1}\begin{pmatrix}\sqrt{b}\\\frac{1}{\sqrt{b}}\end{pmatrix},\ \pm\dfrac{2\sqrt{-1}}{ab-1}\begin{pmatrix}\frac{\sqrt{1-b}}{\sqrt{1-a}}\\-\frac{\sqrt{1-a}}{\sqrt{1-b}}\end{pmatrix},\ \pm\dfrac{2}{ab-1}\begin{pmatrix}\frac{1}{\sqrt{a}}\\\sqrt{a}\end{pmatrix}$
$(\mathrm{id},(0\,\infty\,1))$	$\pm\dfrac{2\sqrt{-1}}{ab-b+1}\begin{pmatrix}\sqrt{b}\\\frac{1}{\sqrt{b}}\end{pmatrix},\ \pm\dfrac{2\sqrt{-1}}{ab-b+1}\begin{pmatrix}\frac{1}{\sqrt{1-a}}\\\sqrt{1-a}\end{pmatrix},\ \pm\dfrac{2}{ab-b+1}\begin{pmatrix}\frac{\sqrt{1-b}}{\sqrt{a}}\\-\frac{\sqrt{a}}{\sqrt{1-b}}\end{pmatrix}$

Table 2: The images $v_{\mathrm{tr}}((\rho^i)^*\xi_\bullet)$ under the Picard–Fuchs differential operator $\mathscr{D}$.

7.3 Precise rank of N

Proposition 7.6 rank $N = 18$.

To prove this, we use the following proposition which we prove in Section 8.

Proposition 7.7

(1) *For $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_\mathcal{X}$, let $\Psi_\rho : \tau^* \mathcal{O}_T^{\mathrm{an}} \to \mathcal{O}_T^{\mathrm{an}}$ be the $\mathcal{O}_T^{\mathrm{an}}$-module isomorphism defined by*

$$\varphi \longmapsto \chi(\rho)^{-1} \cdot \tau^\sharp(\varphi).$$

Then $(\Psi_\rho)_{\rho \in G_\mathcal{X}}$ defines a $G_\mathcal{X}$-linearization on $\mathcal{O}_T^{\mathrm{an}}$. Furthermore, $(\Psi_\rho)_{\rho \in G_\mathcal{X}}$ induces a $G_\mathcal{X}$-linearization on $\mathcal{Q}_\omega = \mathcal{O}_T^{\mathrm{an}}/\mathcal{P}_\omega$. Under the isomorphism $\mathcal{Q}_\omega \simeq \mathcal{Q}$, $(\Psi_\rho)_{\rho \in G_\mathcal{X}}$ defines a $G_\mathcal{X}$-linearization on $\mathcal{Q}$.

(2) *Let ξ be an algebraic family of higher Chow cycles on $\widetilde{\mathcal{X}} \to T$. For any $\rho \in G_\mathcal{X}$, we have*

$$\nu_{\mathrm{tr}}(\rho^* \xi) = \Psi_\rho(\nu_{\mathrm{tr}}(\xi)).$$

The following proof of Proposition 7.6 was simplified by the advice from T. Saito.

Proof of Proposition 7.6   It is enough to prove rank $N \leq 18$. By the $G_\mathcal{X}$-linearization $(\Psi_\rho)_{\rho \in G_\mathcal{X}}$ of $\mathcal{Q}$, N becomes a right $G_\mathcal{X}$-module.

We consider the following two subgroups H and I of $G_\mathcal{X}$. We define the normal subgroup H by $\mathrm{Ker}(G_\mathcal{X} \to G_T)$. By the definition of Ψ_ρ, H acts on N as ± 1. Next, we define the subgroup I of $G_\mathcal{X}$ by

$$I = \{ (\rho_1, \rho_2, \tau, \zeta) \in G_\mathcal{X} : \rho_1 = \rho_2 \in \mathfrak{S}(\{0, 1, \infty\}) \text{ and } \zeta = 1 \},$$

where we regard $\mathfrak{S}(\{0, 1, \infty\})$ as a subgroup of $\mathfrak{S}(\Sigma)$ by $\{0, 1, \infty\} \subset \Sigma$. By definition, I stabilizes the subset $\{(0,0), (1,1), (\infty, \infty)\}$ of Σ^2 on $\mathcal{Y}$. Thus $\mathcal{C}$ is stable under the I-action. Hence, for $\bullet = 0, 1, \infty$ and $\rho \in I$, $(\rho^* \xi_\bullet)_t$ coincides with either $\pm(\xi_0)_t, \pm(\xi_1)_t$, or $\pm(\xi_\infty)_t$ up to decomposable cycles (see (26)). By Proposition 2.3, $\nu_{\mathrm{tr}}(\xi)$ is determined by ξ_{ind}, therefore $\nu_{\mathrm{tr}}(\rho^* \xi_\bullet)$ coincides with either $\pm \nu_{\mathrm{tr}}(\xi_0), \pm \nu_{\mathrm{tr}}(\xi_1)$ or $\pm \nu_{\mathrm{tr}}(\xi_\infty)$. This implies that N^{can} is an I-submodule of N.

Therefore N^{can} is a HI-submodule of N. Then $N^{\mathrm{can}} \hookrightarrow N$ induces the following $G_\mathcal{X}$-module homomorphism:

$$(30) \qquad\qquad \mathrm{Ind}_{HI}^{G_\mathcal{X}} N^{\mathrm{can}} \longrightarrow N,$$

where $\mathrm{Ind}_{HI}^{G_\mathcal{X}} N^{\mathrm{can}}$ is the induced representation. Since the $G_\mathcal{X}$-module N is generated by N^{can}, the map (30) is surjective. Then we have

$$\mathrm{rank}\, N \leq \mathrm{rank}\, \mathrm{Ind}_{HI}^{G_\mathcal{X}} N^{\mathrm{can}} = |G_\mathcal{X}/HI| \cdot \mathrm{rank}\, N^{\mathrm{can}} = 3 \cdot |G_\mathcal{X}/HI|.$$

Hence it is enough to show $|G_\mathcal{X}/HI| = 6$. First, we show $H \cap I = \{\mathrm{id}\}$. Let $\rho = (\rho_1, \rho_2, \tau, \zeta) \in H \cap I$. Then we have $\rho_1 = \rho_2 \in \mathfrak{S}(\{0, 1, \infty\})$, $\tau = \mathrm{id}$ and $\zeta = 1$. Furthermore, the image of $(\rho_1, \rho_2) \in G_{\mathcal{Y}_0}$ under $G_{\mathcal{Y}_0} \to G_{T_0}$ is $(\mathrm{id}, \mathrm{id})$ by the condition of the fiber product. Since $\mathfrak{S}(\{0, 1, \infty\}) \hookrightarrow \mathfrak{S}(\Sigma) \twoheadrightarrow \mathrm{Aut}(S_0)$ is the isomorphism, we see that $(\rho_1, \rho_2) = (\mathrm{id}, \mathrm{id})$. Hence $\rho = \mathrm{id}$. Second, we calculate $|G_\mathcal{X}|$ and $|H|$. We have that $|G_\mathcal{X}| = 2 \cdot |G_\mathcal{Y}| = 2 \cdot |\mathfrak{S}_4 \times_{\mathfrak{S}_3} \mathfrak{S}_4|^2 = 2^{11} \cdot 3^2$. By $|G_T| = |\mathfrak{S}_4^2| = 2^6 \cdot 3^2$ and the fact that $G_\mathcal{X} \to G_T$ is surjective, $|H| = |G_\mathcal{X}|/|G_T| = 2^5$. Third, we calculate $|I|$. For each $\rho \in \mathfrak{S}(\{0, 1, \infty\})$, the number of elements in $G_\mathcal{X}$ of the form $(\rho, \rho, *, 1)$ equals to the

cardinality of $\mathrm{Ker}(G_T \to G_{T_0})$, i.e., 2^4. Hence $|I| = 2^4 \cdot |\mathfrak{S}(\{0, 1, \infty\})| = 2^5 \cdot 3$. To sum up these results, we have

$$|G_{\mathcal{X}}/HI| = \frac{|G_{\mathcal{X}}|}{|HI|} = \frac{|G_{\mathcal{X}}|}{|H| \cdot |I|} = 6. \qquad \blacksquare$$

8 The group action on the Picard–Fuchs differential operator

In this section, we prove Propositions 7.3 and 7.7 and complete the proof of the main theorem. They are proved by examining the $G_{\mathcal{X}}$-actions on the periods and the Picard–Fuchs differential operator.

8.1 The group action on periods

First, we prove Proposition 7.7. By the property of 1-cocycles, we see that $(\Psi_\rho)_{\rho \in G_{\mathcal{X}}}$ defines a $G_{\mathcal{X}}$-linearization on $\mathcal{O}_T^{\mathrm{an}}$. To prove that this induces the $G_{\mathcal{X}}$-linearization on $\mathcal{Q}_\omega$, we should prove that Ψ_ρ preserves the subsheaf $\mathcal{P}_\omega$ which is the local system consisting of period function with respect to ω.

Let $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_{\mathcal{X}}$. By the local description of the $G_{\mathcal{X}}$-action on $\widetilde{\mathcal{X}}$, we can check that $\rho^* \omega = \chi(\rho) \cdot \omega$. Pulling-back this relation at $\tau(t) \in T$, we have

(31) $\qquad (\rho_t^{-1})^*(\omega_t) = \chi(\rho^{-1})(\tau(t)) \cdot \omega_{\tau(t)} \quad \in \Gamma(\widetilde{\mathcal{X}}_{\tau(t)}, \Omega^2_{\widetilde{\mathcal{X}}_{\tau(t)}/\mathbb{C}})$.

Here, $\rho_t : \widetilde{\mathcal{X}}_t \to \widetilde{\mathcal{X}}_{\tau(t)}$ is the isomorphism in the diagram (25) and $\chi(\rho^{-1})(\tau(t)) \in \mathbb{C}$ is the value of $\chi(\rho^{-1}) \in B$ at $\tau(t)$. From the equation (31), for any 2-chain $\Gamma_{\tau(t)}$ on $\widetilde{\mathcal{X}}_{\tau(t)}$, we have

(32)
$$\int_{\rho_t^{-1}(\Gamma_{\tau(t)})} \omega_t = \int_{\Gamma_{\tau(t)}} (\rho_t^{-1})^*(\omega_t) = \chi(\rho^{-1})(\tau(t)) \cdot \int_{\Gamma_{\tau(t)}} \omega_{\tau(t)}$$
$$= \tau^\sharp\left(\chi(\rho^{-1})\right)(t) \cdot \int_{\Gamma_{\tau(t)}} \omega_{\tau(t)} = \chi(\rho)(t)^{-1} \cdot \int_{\Gamma_{\tau(t)}} \omega_{\tau(t)},$$

where the last equality follows from the property of 1-cocycles. This is the key for the proof of Proposition 7.7.

Proof of Proposition 7.7 We prove (1). We prove that $\Psi_\rho(\tau^* \mathcal{P}_\omega) = \mathcal{P}_\omega$. Since $\rho^*(\Psi_{\rho^{-1}}) = \Psi_\rho^{-1}$, it is enough to show only ($\subset$). Let U' be an open subset of T in the classical topology and $\varphi \in \tau^* \mathcal{P}_\omega(U') = \mathcal{P}_\omega(\tau(U'))$. Then there exists a C^∞-family of 2-cycles $\{\Gamma_t\}_{t \in \tau(U)}$ such that $\varphi(t) = \int_{\Gamma_t} \omega_t$. For any $t \in U'$, we have

$$(\Psi_\rho(\varphi))(t) = \chi(\rho)(t)^{-1} \cdot \varphi(\tau(t)) = \chi(\rho)(t)^{-1} \cdot \int_{\Gamma_{\tau(t)}} \omega_{\tau(t)} \underset{(32)}{=} \int_{\rho_t^{-1}(\Gamma_{\tau(t)})} \omega_t \in \mathcal{P}(\omega_t).$$

Hence, we have $\Psi_\rho(\varphi) \in \mathcal{P}_\omega(U')$. Thus, $(\Psi_\rho)_{\rho \in G_{\mathcal{X}}}$ induces the $G_{\mathcal{X}}$-linearization on $\mathcal{Q}_\omega = \mathcal{O}_T^{\mathrm{an}}/\mathcal{P}_\omega$.

Next, we prove (2). By Corollary 2.5, it is enough to show that

(33) $\qquad \langle \mathrm{ev}_t\left(\nu_{\mathrm{tr}}(\rho^* \xi)\right), [\omega_t] \rangle = \langle \mathrm{ev}_t\left(\Psi_\rho(\nu_{\mathrm{tr}}(\xi))\right), [\omega_t] \rangle$

for any $t \in T$. We take a neighborhood U' of $\tau(t)$ and a C^∞-family of 2-chains $\{\Gamma_{t'}\}_{t' \in U'}$ such that $\Gamma_{t'}$ is a 2-chain associated with $\xi_{t'}$. We will compute the right-hand side of (33). Under the isomorphism $\Omega \xrightarrow{\sim} \Omega_\omega$, $\Psi_\rho(\nu_{\mathrm{tr}}(\xi))$ is represented by the local holomorphic function

$$\tau^{-1}(U') \ni t' \longmapsto \chi(\rho)(t')^{-1} \int_{\Gamma_{\tau(t')}} \omega_{\tau(t')} \underset{(32)}{=} \int_{\rho_{t'}^{-1}(\Gamma_{\tau(t')})} \omega_{t'} \in \mathbb{C}.$$

The right-hand side of (33) is represented by the value of this function at t. That is,

$$(34) \qquad \langle \mathrm{ev}_t\left(\Psi_\rho(\nu_{\mathrm{tr}}(\xi))\right), [\omega_t] \rangle = \int_{\rho_t^{-1}(\Gamma_{\tau(t)})} \omega_t \quad \mathrm{mod}\ \mathcal{P}(\omega_t).$$

On the other hand, the left-hand side of (33) is $\langle r((\rho^*\xi)_t), [\omega_t] \rangle = \langle r(\rho_t^*(\xi_{\tau(t)})), [\omega_t] \rangle$. Since $\rho_t^{-1}(\Gamma_{\tau(t)})$ is a 2-chain associated with $\rho_t^*(\xi_{\tau(t)})$, the left-hand side of (33) also coincides with (34). Hence we have the result. ∎

8.2 The group action on the Picard–Fuchs operator

Next, we will investigate the $G_\mathcal{X}$-action on the differential operator $\mathscr{D}$. For $\tau \in G_T$, we define differential operators $\mathscr{D}_1^\tau, \mathscr{D}_2^\tau : \mathcal{O}_T^{\mathrm{an}} \to \mathcal{O}_T^{\mathrm{an}}$ as follows:

$$\mathscr{D}_1^\tau = a'(1-a')\frac{\partial^2}{(\partial a')^2} + (1-2a')\frac{\partial}{\partial a'} - \frac{1}{4},$$

$$\mathscr{D}_2^\tau = b'(1-b')\frac{\partial^2}{(\partial b')^2} + (1-2b')\frac{\partial}{\partial b'} - \frac{1}{4},$$

where $a' = \tau^\sharp(a)$ and $b' = \tau^\sharp(b)$. By definition,

$$(35) \qquad \mathscr{D}_i^\tau(\tau^\sharp(\varphi)) = \tau^\sharp(\mathscr{D}_i(\varphi)) \quad (i = 1, 2)$$

holds for any local section φ of $\mathcal{O}_T^{\mathrm{an}}$. We have the following transformation formulas.

Proposition 8.1 *For $\rho = (\rho_1, \rho_2, \tau, \zeta) \in G_\mathcal{X}$, we have the following relations in the ring of differential operators on $\mathcal{O}_T^{\mathrm{an}}$:*

$$\mathscr{D}_1 \cdot \chi(\rho)^{-1} = \chi(\rho)^{-1} \cdot \phi_1(\tau)^{-2} \cdot \mathscr{D}_1^\tau,$$

$$\mathscr{D}_2 \cdot \chi(\rho)^{-1} = \chi(\rho)^{-1} \cdot \phi_2(\tau)^{-2} \cdot \mathscr{D}_2^\tau,$$

where we regard $\chi(\rho), \phi_1(\tau)$ and $\phi_2(\tau)$ as differential operators by multiplication.

Proof We prove the first equation. The second equation is proved similarly. Since $\chi(\rho) = \zeta\phi_1(\tau)\phi_2(\tau)$ and $\zeta\phi_2(\tau)$ commutes with $\mathscr{D}_1$, it is enough to show that

$$(36) \qquad \phi_1(\tau)^3 \cdot \mathscr{D}_1 \cdot \phi_1(\tau)^{-1} = \mathscr{D}_1^\tau.$$

Let $\tau = (\tau_1, \tau_2) \in G_T = \mathfrak{S}_4^2$. We denote the image of τ_1 under $\mathfrak{S}_4 \to \mathrm{Aut}(S_0) = \mathfrak{S}(\{0, 1, \infty\})$ by $\underline{\tau}_1$. By (24) and the table in Proposition 6.6, $\phi_1(\tau)$ is determined by $\underline{\tau}_1$ up to sign and coincides with either $\pm 1, \pm\sqrt{-1}\sqrt{1-a}$ or $\pm\sqrt{-1}\sqrt{a}$. Using $\frac{\partial}{\partial a} \cdot a^\lambda = \lambda a^{\lambda-1} + a^\lambda \frac{\partial}{\partial a}$ and $\frac{\partial}{\partial a} \cdot (1-a)^\lambda = -\lambda(1-a)^{\lambda-1} + (1-a)^\lambda \frac{\partial}{\partial a}$, we can compute the left-hand side of (36) as follows:

$$a(1-a)\frac{\partial^2}{\partial a^2} + (1-2a)\frac{\partial}{\partial a} - \frac{1}{4} \qquad (\underline{\tau}_1 = \mathrm{id}, (0\ 1)),$$

(37)
$$-a(1-a)^2\frac{\partial^2}{\partial a^2} - (1-a)^2\frac{\partial}{\partial a} - \frac{1}{4} \qquad (\underline{\tau}_1 = (1\ \infty), (0\ 1\ \infty)),$$

$$-a^2(1-a)\frac{\partial^2}{\partial a^2} + a^2\frac{\partial}{\partial a} - \frac{1}{4} \qquad (\underline{\tau}_1 = (0\ \infty), (0\ \infty\ 1)).$$

On the other hand, $\mathscr{D}_1^\tau$ is also determined by $\underline{\tau}_1 \in \mathrm{Aut}(S_0) = \mathfrak{S}(\{0, 1, \infty\})$ by definition. We can check (36) holds for each $\underline{\tau}_1$. For example, when $\underline{\tau}_1 = (1\ \infty)$, the differential operator $\mathscr{D}_1^\tau$ is computed as follows: since $a' = \frac{a}{a-1}$, we have

$$\frac{\partial}{\partial a'} = \frac{\partial a}{\partial a'} \cdot \frac{\partial}{\partial a} = -\frac{1}{(a'-1)^2} \cdot \frac{\partial}{\partial a} = -(a-1)^2 \cdot \frac{\partial}{\partial a}$$

$$\frac{\partial^2}{(\partial a')^2} = \left(-(a-1)^2 \cdot \frac{\partial}{\partial a}\right)^2 = (a-1)^4 \frac{\partial^2}{\partial a^2} + 2(a-1)^3\frac{\partial}{\partial a}.$$

By substituting $a', \frac{\partial}{\partial a'}, \frac{\partial^2}{(\partial a')^2}$ in $\mathscr{D}_1^\tau$ by them, we can confirm that $\mathscr{D}_1^\tau$ coincides with the second differential operator in (37). ∎

Finally, we prove Proposition 7.3.

Proof of Proposition 7.3 By Proposition 8.1, for a local section φ of $\mathcal{O}_T^{\mathrm{an}}$, we have

$$\mathscr{D}_1\left(\Psi_\rho(\varphi)\right) = \mathscr{D}_1\left(\chi(\rho)^{-1} \cdot \tau^\sharp(\varphi)\right) = \chi(\rho)^{-1} \cdot \phi_1(\tau)^{-2} \cdot \mathscr{D}_1^\tau(\tau^\sharp(\varphi))$$

$$\underset{(35)}{=} \chi(\rho)^{-1} \cdot \phi_1(\tau)^{-2} \cdot \tau^\sharp(\mathscr{D}_1(\varphi)),$$

$$\mathscr{D}_2\left(\Psi_\rho(\varphi)\right) = \chi(\rho)^{-1} \cdot \phi_2(\tau)^{-2} \cdot \tau^\sharp(\mathscr{D}_2(\varphi)).$$

In particular, we have $\mathscr{D}(\Psi_\rho(\varphi)) = \Theta_\rho(\mathscr{D}(\varphi))$. Then by Proposition 7.7, we have

$$\mathscr{D}(\nu_{\mathrm{tr}}(\rho^*\xi)) = \mathscr{D}(\Psi_\rho(\nu_{\mathrm{tr}}(\xi))) = \Theta_\rho(\mathscr{D}(\nu_{\mathrm{tr}}(\xi))). \qquad ■$$

References

[AO21] M. Asakura and N. Otsubo, *Regulators of K_1 of hypergeometric fibrations*. In: P. Charollois, G. Freixas i Montple and V. Maillot (eds.), Arithmetic L-functions and differential geometric methods, Progress in Mathematics, 338, Birkhäuser/Springer, Cham, 2021, pp. 1–30.

[BHPV04] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, *Compact complex surfaces*. 2nd ed., Springer, Berlin, 2004.

[Blo86] S. Bloch, *Algebraic cycles and higher K-theory*. Adv. Math. **61**(1986), no. 3, 267–304.

[CDKL16] X. Chen, C. Doran, M. Kerr, and J. D. Lewis, *Normal functions, Picard–Fuchs equations, and elliptic fibrations on K3 surfaces*. J. Reine Angew. Math. **721**(2016), 43–79.

[CMP02] J. S Carlson, S. Müller-Stach, and C. Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, 85, Cambridge University Press, Cambridge, 2003, xvi+430 pp.

[CL05] X. Chen and J. D. Lewis, *The Hodge-D-conjecture for K3 and abelian surfaces*. J. Algebraic Geom. **14**(2005), no. 2, 213–240.

[Col99] A. Collino, *Indecomposable motivic cohomology classes on quartic surfaces and on cubic fourfolds*. In: H. Bass, A. O. Kuku and C. Pedrini (eds.), Algebraic K-theory and its applications (Trieste, 1997), World Scientific, River Edge, NJ, 1999, pp. 370–402.

[dAMS02] P. L. del Angel and S. Müller-Stach, *The transcendental part of the regulator map for K_1 on a mirror family of K3-surfaces*. Duke Math. J. **112**(2002), no. 3, 581–598.

[dAMS08] P. L. del Angel and S. Müller-Stach, *Differential equations associated to families of algebraic cycles*. Ann. Inst. Fourier **58**(2008), 2075–2085.

[Fla92] M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*. Invent. Math. **109**(1992), no 2, pp 307–327.

[Ker13] M. Kerr, *K_1 of elliptically fibered K3 surfaces: a tale of two cycles*. In: R. Laza, M. Schütt and N. Yui (eds.), Arithmetic and geometry of K3 surfaces and Calabi–Yau threefolds, Springer, New York, 2013, pp. 387–409.

[Lev88] M. Levine, *Localization on singular varieties*. Invent. Math. **91**(1988), no. 3, 423–464.

[Mil92] S. J. M. Mildenhall, *Cycles in a product of elliptic curves, and a group analogous to the class group*. Duke Math. J. **67**(1992), no. 2, 387–406.

[MS97] S. J. Müller-Stach, *Constructing indecomposable motivic cohomology classes on algebraic surfaces*. J. Algebraic Geom. **6**(1997), no. 3, 513–543.

[MS98] S. J. Müller-Stach, *Algebraic cycle complexes: Basic properties*. In: B. B. Gordon, J. D. Lewis, S. Müller-Stach, S. Saito and N. Yui (eds.), The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), NATO Science Series C: Mathematical and Physical Sciences, 548, Springer, Dordrecht, 2000, pp. 285–305.

[Sas21] T. Sasaki, *Limits and singularities of normal functions*. Eur. J. Math. 7(2021), no. 4, 1401–1437.

[Spi99] M. Spieß, *On indecomposable elements of K1 of a product of elliptic curve*. K-Theory **17**(1999), 363–383.

[Sre14] R. Sreekantan, *Higher Chow cycles on abelian surfaces and a non-archimedean analogue of the Hodge-D-conjecture*. Compos. Math. **150**(2014), no. 4, 691–711.

[Voi02] C. Voisin, *Hodge theory and complex algebraic geometry. I*. Translated from the French by Leila Schneps. Reprint of the 2002 English edition, Cambridge Studies in Advanced Mathematics, 76, Cambridge University Press, Cambridge, 2007, x+322 pp.

[WW62] E. T. Whittaker and G. N. Watson, *A course of modern analysis. An introduction to the general theory of infinite processes and of analytic functions: with an account of the principal transcendental functions*. 4th ed., reprinted Cambridge University Press, New York, 1962, vii+608 pp.

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