BOUNDARY VALUE PROBLEMS ASSOCIATED WITH THE TENSOR LAPLACE EQUATION

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Introduction. The boundary value problems considered in this paper relate to harmonic p-tensors on Riemannian manifolds with boundary. We study the equation of Beltrami-Laplace

$$\Delta\phi = 0$$

and formulate three boundary value problems which correspond to the Dirichlet, Neumann, and mixed boundary value problems of potential theory. Existence proofs are given by means of the theory of singular integral equations. Essential use is made of the kernal $g_p(x, y)$ for closed manifolds which was introduced by de Rham.

Harmonic fields, which satisfy

$$d\phi = 0, \quad \delta\phi = 0,$$

constitute a distinguished subclass of solutions of the Laplace equation. The harmonic fields are precisely the solutions of the homogeneous second boundary value problem, analogous to the constant solutions of scalar potential theory.

Properties of certain domain functionals are derived from the existence theorems. In order to give a reasonably short proof for the third boundary value problem, we assume the existence of a Green's form for a larger manifold. An eigenvalue problem is examined in the concluding section.

The formal analogy with potential theory is very close throughout. The results may be interpreted as generalizations of the classical existence theorems, as a characterization of certain systems of elliptic partial differential equations, and as an extension of the theory of harmonic integrals on a closed manifold.

1. Manifolds and tensors. Let M be an orientable Riemannian manifold of dimension n and of class C^∞. Let the boundary of M be a regular sub-manifold B of dimension $n - 1$ of M. We suppose that M is finite in the sense that M is covered by a finite number of fundamental coordinate neighbourhoods which are open cubes in suitable local coordinate systems. On M is carried a positive definite metric tensor g_{ij} of the class C^∞.

Associated with M is the double F of M, a closed Riemannian manifold consisting of M, and an oppositely oriented replica $\tilde{M}$ of M, with corresponding boundary points identified. The metric tensor g_{ij} can be extended to F so as to be C^∞ on F, though not necessarily the same at corresponding points of M and of $\tilde{M} = CM$ (complement of M in F) [**2b**].

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On M there exist skew symmetric covariant tensors

$$\phi_{i_1 \ldots i_p}$$

of rank p, $0 \leqslant p \leqslant n$, and their associated differential forms [**5**] of degree p:

$$\phi = \phi_{(i_1 \ldots i_p)} \, dx^{i_1} \wedge \ldots \wedge dx^{i_p}. \tag{1.1}$$

The differentials dx^i anti-commute.

We define the generalized metric tensor,

$$\Gamma_{i_1 \ldots i_p, \, j_1 \ldots j_p} = \begin{vmatrix} g_{i_1 j_1} & \ldots & g_{i_1 j_p} \\ \vdots & & \vdots \\ g_{i_p j_1} & \ldots & g_{i_p j_p} \end{vmatrix}, \tag{1.2}$$

the volume n-tensor density

$$e_{i_1 \ldots i_n} = \Gamma_{i_1 \ldots i_n}{}^{12 \ldots n} \sqrt{\Gamma_{12 \ldots n, 12 \ldots n}}, \tag{1.3}$$

the covariant derivative

$$D_i \, \phi_{i_1 \ldots i_p} = \frac{\partial}{\partial x^i} \phi_{i_1 \ldots i_p} - \sum_{n=1}^{p} \left\{ {h \atop i_n \, i} \right\} \phi_{i_1 \ldots i_{n-1} h i_{n+1} \ldots i_p}, \tag{1.4}$$

the differential

$$(d\phi)_{i_1 \ldots i_{p+1}} = \Gamma_{i_1 \ldots i_{p+1}}{}^{j(j_1 \ldots j_p)} D_j \, \phi_{(j_1 \ldots j_p)}, \tag{1.5}$$

the dual

$$(*\phi)_{j_1 \ldots j_{n-p}} = e_{(i_1 \ldots i_p) j_1 \ldots j_{n-p}} \, \phi^{(i_1 \ldots i_p)}, \tag{1.6}$$

the co-differential

$$(\delta\phi)_{i_1 \ldots i_{p-1}} = - \Gamma_{i i_1 \ldots i_{p-1}}{}^{(j_1 \ldots j_p)} g^{ij} D_j \phi_{(j_1 \ldots j_p)}, \tag{1.7}$$

and the Laplacian

$$(\Delta\phi)_{i_1 \ldots i_p} = ((d\delta + \delta d)\phi)_{i_1 \ldots i_p} = - D^i D_i \phi_{i_1 \ldots i_p} + \sum_{n=1}^{p} \Gamma_{i_1 \ldots i_p}{}^{i(j_1 \ldots j_p)} g^{lj} R^h{}_{j_n j i} \phi_{j_1 \ldots j_{n-1} h j_{n+1} \ldots j_p}, \tag{1.8}$$

where R_{ijkl} is the curvature tensor. The brackets enclosing a set of indices mean that summation is to be effected only over those values which are in increasing order.

If $d\phi = 0$, ϕ is said to be closed; if $\phi = d\chi$, ϕ is derived; if $\delta\phi = 0$, ϕ is coclosed and if $\phi = \delta\chi$, coderived. If $\Delta\phi = 0$, ϕ is said to be a harmonic form and if $d\phi = 0$, $\delta\phi = 0$, ϕ is a harmonic field. Harmonic fields are harmonic forms. Harmonic forms [**8**] on M are of class C^∞.

For all forms ϕ of degree p, we have

$$d \cdot d\phi = 0, \quad \delta \cdot \delta\phi = 0, \quad **\phi = (-1)^{np+p}\phi, \quad \delta\phi = (-1)^{np+n+1} *d*\phi. \tag{1.9}$$

On the boundary B, ϕ defines a tangential p-form $t\phi$ whose components are precisely those components of ϕ with no index n, where x^n is a normal coordinate in some system. The residual part of ϕ is the normal component $n\phi$. The relations

$$*t = n*, \quad *n = t* \tag{1.10}$$

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hold on B, and the decomposition $\phi = t\phi + n\phi$ is invariant on B.

We introduce the scalar product of two p-tensors ϕ and ψ:

$$(1.11) \qquad (\phi, \psi) = \int_M \phi \wedge *\psi = \int_M \psi \wedge *\phi .$$

The scalar square $(\phi, \phi) = N(\phi)$ is positive definite. Let

$$(1.12) \qquad D(\phi, \psi) = (d\phi, d\psi) + (\delta\phi, \delta\psi),$$

then $D(\phi) = D(\phi, \phi)$ is the Dirichlet integral associated with the Laplace equation on M. Scalar products and functionals extended over domains other than M will be indicated by subscripts.

The Stokes formula for a $(p + 1)$-dimensional chain C with boundary bC is [**5**]

$$(1.13) \qquad \int_C d\phi = \int_{bC} \phi,$$

valid for every p-tensor ϕ of class C^1. From Stokes's formula and the formula for the differential of a product follows the formula of Green:

$$(1.14) \qquad (d\phi, \psi) - (\phi, \delta\psi) = \int_B \phi \wedge *\psi .$$

Here ϕ, ψ are of degree p, $p + 1$ respectively. Two other forms of Green's formula, which follow easily from (1.14), will be needed. If ϕ and ψ are of equal degree p, then

$$(1.15) \qquad (d\phi, d\psi) + (\delta\phi, \delta\psi) - (\phi, \Delta\psi) = \int_B (\phi \wedge *d\psi - \delta\psi \wedge *\phi)$$

and

$$(1.16) \qquad (\Delta\phi, \psi) - (\phi, \Delta\psi) = \int_B (\phi \wedge *d\psi - \psi \wedge *d\phi + \delta\phi \wedge *\psi - \delta\psi \wedge *\phi).$$

The theorem of Hodge [**5**] for harmonic fields in a closed manifold F states that there exists a unique harmonic field with given periods on the $R_p(F)$ independent (absolute) p-cycles of F. These harmonic fields ω_p^i $(i = 1, \ldots , R_p (F))$ have a reproducing kernel

$$(1.17) \qquad \alpha_p(x, y) = \sum_i \omega_p^i(x)\omega_p^i(y), \quad (\omega_p^i, \omega_p^j)_F = \delta_{ij},$$

in the metric (1.11).

For our existence proofs the de Rham kernel $g_p(x, y)$ of the double F will be needed. This kernel has the following properties, described in [**8**]:

(a) For every $\phi \in L^2(F)$ (i.e. such that $N_F(\phi) < \infty$), we have

$$(1.18) \qquad \Delta G\phi = G\Delta\phi = \phi - H\phi,$$

where

$$(1.19) \qquad G\phi = (g_p, \phi)_F,$$

and

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(1.20) $$H\phi = (\alpha_p, \phi)_F.$$

(b) The kernel satisfies

(1.21) $$\Delta_x g_p(x, y) = -\alpha_p(x, y) \qquad x \neq y,$$

(1.22) $$g_p(x, y) = g_p(y, x),$$

(1.23) $$g_p(x, y) \sim \gamma_p(x, y) \qquad x \in N(y),$$

where $\gamma_p(x, y)$ is a local fundamental singularity for the Laplace equation $\Delta\phi = 0$. The precise nature of this singularity need not concern us here; we note that

(1.24) $$g_p(x, y) \sim \gamma_p(x, y) = O(s^{2-n}),$$

where s is the geodesic distance from x to y. Also, for every form $\phi \in C^1$, we have [**1**]

(1.25) $$\lim_{\epsilon \to 0} \int_{s=\epsilon} (\phi \wedge *d\gamma - \delta\gamma \wedge *\phi) = \phi.$$

(c) The equation

$$\Delta\mu = \phi$$

is solvable in F if and only if $H\phi = 0$, a solution being

$$\mu = G\phi.$$

The orthogonality conditions $HG = GH = 0$ make $G\phi$ unique.

(d) The operator G commutes with d, $*$, and δ. Hence

(1.26) $$d_x g_p(x, y) = \delta_y g_{p+1}(x, y)$$

and

(1.27) $$g_{n-p}(x, y) = *_x *_y g_p(x, y).$$

2. Boundary value problems of the first and second kinds. The boundary value problem of the first kind consists of determining a harmonic form ϕ in M with given tangential and normal boundary components on B. It has been shown that a solution of the problem exists provided that the solution is unique [**2a**]. Uniqueness of the solution holds if the conditions

(2.1) $$\Delta\phi = 0, \quad t\phi = 0, \quad n\phi = 0$$

imply that ϕ is identically zero. Assume that (2.1) holds; it follows from (1.15) that ϕ is a harmonic field in M:

$$d\phi = 0, \quad \delta\phi = 0.$$

Let x^n be a normal coordinate in a neighbourhood of a given point P of B. From

$$0 = n(d\phi)_{i_1 \dots i_p n} = \Gamma_{i_1 \dots i_p n}{}^{j(j_1 \dots j_p)} D_j \phi_{(j_1 \dots j_p)} = (-1)^p D_n \phi_{i_1 \dots i_p},$$

and

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$$0 = t(\delta\phi)_{i_1\ldots i_{p-1}} = \Gamma_{ii_1\ldots i_{p-1}}{}^{(j_1\ldots j_p)} D^i \phi_{(j_1\ldots j_p)} = (-1)^{p-1} D^n \phi_{i_1\ldots i_{p-1}n}$$

it follows that the first derivatives in the normal direction of all components of ϕ vanish at P. Further differentiation shows that the higher normal derivatives of components of ϕ also vanish at P. It is, therefore, seen that all derivatives of ϕ vanish at P, and hence everywhere on B.

If M is an analytic manifold with analytic metric tensor, it can be shown [**6**] that harmonic forms are analytic. Therefore, the uniqueness holds in this case.

Under certain topological restrictions the uniqueness property holds for C^∞ manifolds. It was proved in [**2a, b**] that there exists a unique harmonic field ϕ with zero tangential boundary value such that either (a) ϕ has given periods on $R_{n-p}(M) = R_p(M, B)$ independent relative p-cycles or (b) $*\phi$ has given periods on $R_{n-p}(M)$ independent absolute p-cycles. Hence the number of independent harmonic fields with zero tangential and normal boundary components cannot exceed the number of p-cycles which are independent both as absolute and as relative cycles. If this number is zero, uniqueness for the first boundary value problem holds.

If uniqueness holds for M, it holds for any sub-manifold M_1 contained in M. For if $\phi \in C^\infty$ is harmonic in M_1 and vanishes on the boundary of M_1, all derivatives of ϕ vanish there and ϕ may be extended to a C^∞ harmonic form in M by defining it to be zero in $M - M_1$. The uniqueness holds if there is a maximum modulus or mean value theorem available; thus it holds for scalars in Euclidean space. Possibly uniqueness holds in general; this has not been proved. The existence proof which is to follow has the advantage that it is valid independently of the uniqueness.

The natural data of the second boundary value problem are $nd\phi$ and $t\delta\phi$, as may be seen from Green's formula. These data satisfy the condition of being self-dual, on account of (1.10). Together there are $\binom{n}{p}$ components, each containing one first normal derivative of a component of ϕ. If $1 \leqslant p \leqslant n - 1$, certain tangential derivatives also appear. From (1.15) it follows that

$$\Delta\phi = 0, \quad nd\phi = 0, \quad t\delta\phi = 0 \tag{2.2}$$

imply that, in M,

$$d\phi = 0, \quad \delta\phi = 0.$$

The harmonic fields are, therefore, precisely the solutions of the homogeneous boundary value problem of the second kind (zero data); we may refer to them as homogeneous solutions. Likewise, any solution of the non-homogeneous problem is undetermined to the extent of an added harmonic field.

Let ϕ be any harmonic form in M, τ any harmonic field in M. From (1.15) it is seen that

$$\int_B (\tau \wedge *d\phi - \delta\phi \wedge *\tau) = 0. \tag{2.3}$$

This orthogonality condition must be satisfied by the assigned data $nd\phi$ and $t\delta\phi$, for every harmonic field τ in M.

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These facts are well known in the scalar boundary value problem, in which the harmonic fields reduce to the constants. The orthogonality condition reduces to the assigned values of the normal derivative having a zero average. In the theory of elasticity [7] the equilibrium equations are the case $n = 3$, $p = 1$ of the slightly more general equation

$$(\delta d + \alpha d\delta)\phi = 0, \qquad \alpha > 0,$$

where α is a constant of the material. The uniqueness properties of this equation are the same as for $\alpha = 1$. The boundary value problem of the first kind in the theory of elasticity corresponds to the assignment of surface displacements, and has a unique solution. The second boundary value problem corresponds to the surface tractions being given; these must satisfy the conditions of rigid body equilibrium for the whole, and the solution is undetermined to the extent of a rigid body motion. Such motions, being irrotational and without divergence, are given precisely by harmonic fields.

3. Potentials. Let $g = g_p(x, y)$ be the de Rham kernel of the double F. We introduce the potentials

$$\mu = \int_B (\rho \wedge *dg - \delta g \wedge *\rho), \tag{3.1}$$

$$\nu = \int_B (g \wedge *d\sigma - \delta\sigma \wedge *g), \tag{3.2}$$

where $t\rho$, $n\rho$, $nd\sigma$, $t\delta\sigma$ are continuous on B. We also suppose that ν satisfies the orthogonality condition (2.3). Both μ and ν are defined and are of class C^∞ in M and in CM. It follows that

$$\Delta\mu = \int_B (\rho \wedge *d\alpha - \delta\alpha \wedge *\rho) = 0,$$

since $d\alpha = 0$, $\delta\alpha = 0$. From (1.16) and (2.3) we have

$$N(\Delta\nu) = -\int_B (\Delta\nu \wedge *d\nu - \delta\nu \wedge *\Delta\nu) = 0,$$

since $d\Delta\nu = 0$, $\delta\Delta\nu = 0$; so that μ and ν are harmonic forms in M and in CM. Thus we are able to use $g_p(x, y)$ as kernel for the potentials in spite of the fact that $g_p(x, y)$ is not a harmonic form.

The potentials (3.1), (3.2) and their derivatives have discontinuities across B, which we now calculate [2b]. We note that

$$\Delta(\phi(x), g_p(x, y)) = -(\phi(x), \alpha_p(x, y)) + \begin{cases} \phi(y), & y \in M \\ 0, & y \in CM \end{cases} \tag{3.3}$$

and observe that $\alpha_p\ (x, y)$ is continuous in F. Also

$$\begin{aligned} \Delta(\phi, g_p) &= \delta d(\phi, g_p) + d\delta(\phi, g_p) \\ &= \delta(\phi, \delta g_{p+1}) + d(\phi, dg_{p-1}) \\ &= d\int_B g_{p-1} \wedge *\phi - \delta\int_B \phi \wedge *g_{p+1} + d(\delta\phi, g_{p-1}) + \delta(d\phi, g_{p+1}). \end{aligned} \tag{3.4}$$

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The two last terms on the right-hand side of (3.4) are continuous on B. Since $t\phi$ and $t_*\phi$ may be chosen independently, it follows that the discontinuities of the remaining terms as y crosses B into CM are as follows:

$$\begin{aligned}(3.5)\quad & t{*}d\int_B g_{p-1}\wedge *\phi \text{ decreases by } t{*}\phi,\quad t\delta\int_B \phi\wedge *g_{p+1} \text{ increases by } t\phi,\\ & td\int_B g_{p-1}\wedge *\phi \text{ and } t{*}\delta\int_B \phi\wedge *g_{p+1} \text{ continuous.}\end{aligned}$$

For $x \in B$, y in a neighbourhood of x, we have [**2b**]

$$td\, g_p = O(s^{2-n}),\ \ s = s(x, y)$$

hence on B we have

$$(3.6)\qquad t\delta\int_B g_{p-1}\wedge *\phi \text{ and } t{*}d\int_B \phi\wedge *g_{p+1} \text{ continuous.}$$

Noting that

$$\mu = \int_B (\rho\wedge *dg + *\rho\wedge *d{*}g),$$

$$\nu = \int_B (g\wedge *\, d\sigma + *g\wedge *d{*}\sigma),$$

and applying these results, we find that $t\mu$, $t_*\mu$, $t_*d\nu$, and $t_*d_*\nu$ have the discontinuities $t\rho$, $t_*\rho$, $-\ t_*d\sigma$, and $-\ t_*d_*\sigma$, respectively, as the argument point passes from M into CM. We conclude that on B,

$$\begin{aligned}(3.7)\qquad t\mu &= \tfrac{1}{2}t\rho + t\int_B(\rho\wedge *dg + *\rho\wedge *d{*}g),\\ t{*}\mu &= \tfrac{1}{2}t{*}\rho + t{*}\int_B(\rho\wedge *dg + *\rho\wedge *d{*}g),\\ t{*}d\nu &= -\ \tfrac{1}{2}t{*}d\sigma + t{*}d\int_B(g\wedge *d\sigma + *g\wedge *d{*}\sigma),\\ t{*}d{*}\nu &= -\ \tfrac{1}{2}t{*}d{*}\sigma + t{*}d{*}\int_B(g\wedge *d\sigma + *g\wedge *d{*}\sigma).\end{aligned}$$

The integrals on the right are to be interpreted as principal values.

By reasoning similar to that used in the scalar potential theory we see that the solution of the first boundary value problem is equivalent to the solution of the equations

$$(3.8)\qquad t\mu = t\phi,\quad t{*}\mu = t{*}\phi$$

where $t\phi$, $t_*\phi$ are the given continuous data of the problem. Similarly, the second boundary value problem is solved by means of the equations

$$(3.9)\qquad t{*}d\nu = t{*}d\phi,\quad t{*}d{*}\nu = t{*}d{*}\phi,$$

for given continuous data $t_*d\phi$, $t_*d_*\phi$ on B.

The kernel of the equations (3.8) is

$$\begin{pmatrix} t_x\, t_y *_x d_x\, g_p(x, y), & t_x\, t_y *_x d_x *_x g_p(x, y) \\ t_x\, t_y *_y *_x d_x\, g_p(x, y), & t_x\, t_y *_y *_x d_x *_x g_p(x, y) \end{pmatrix}$$

and the transposed kernel

$$\begin{pmatrix} t_x\, t_y *_y d_y\, g_p(x, y), & t_x\, t_y *_x *_y d_y\, g_p(x, y) \\ t_x\, t_y *_y d_y *_y g_p(x, y), & t_x\, t_y *_x *_y d_y *_y g_p(x, y) \end{pmatrix}$$

is the kernel of the equations (3.9). When $x = y$, the kernels are singular of order $(n - 1)$. Thus (3.8) and (3.9) are systems of singular integral equations.

4. Solution of the integral equations. The condition for the compatibility of (3.8) and (3.9) is that the non-homogeneous terms be orthogonal (in the boundary metric) to every solution of the homogeneous transposed equation [**4**]. In each case the transposed equation arises when we try to solve the boundary value problem of the complementary type for CM.

For the boundary value problem of the first kind we must show that

$$\int_B (\phi \wedge *d\sigma + *\phi \wedge *d*\sigma) = 0, \tag{4.1}$$

where σ is any solution of the equations

$$\begin{aligned} 0 &= \tfrac{1}{2}t*d\sigma + t*d\int_B (g \wedge *d\sigma + *g \wedge *d*\sigma), \\ 0 &= \tfrac{1}{2}t*d*\sigma + t*d*\int_B (g \wedge *d\sigma + *g \wedge *\delta*\sigma). \end{aligned} \tag{4.2}$$

The notations t_-, t_+ will be used to indicate tangential boundary components with limiting values from the interiors of M and CM respectively. The equations (4.2) imply

$$t_+\delta\nu = 0, \quad t_+*\delta\nu = 0. \tag{4.3}$$

We show that ν is a harmonic field in CM. From (1.21) and (3.2) we have

$$\delta d\delta\nu = 0, \quad d\delta d\nu = 0 \qquad \text{in } CM.$$

Now

$$(d\delta\nu, d\delta\nu)_{CM} = (\delta\nu, \delta d\delta\nu)_{CM} + \int_{-B} \delta\nu \wedge *d\delta\nu = 0,$$

so

$$d\delta\nu = 0 \qquad \text{in } CM.$$

Hence

$$(\delta\nu, \delta\nu)_{CM} = (\nu, d\delta\nu)_{CM} - \int_{-B} \delta\nu \wedge *\nu = 0,$$

so

$$\delta\nu = 0 \qquad \text{in } CM.$$

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Similarly it is easily shown that

$$dv = 0 \qquad \text{in } CM.$$

As the argument point y crosses B, the normal derivatives of components of ν have discontinuities given by (3.5) and (3.6). We find

(4.4) $$t\_d\nu = 0, \quad t\_{*}d\nu = t{*}d\sigma, \quad t\_\delta\nu = t\delta\sigma, \quad t\_{*}\delta\nu = 0.$$

In M, therefore,

(4.5) $$\nu = \int_B (g \wedge *d\nu - \delta\nu \wedge *g).$$

It follows that

(4.6) $$\Delta\nu = -\int_B (\alpha \wedge *d\nu - \delta\nu \wedge *\alpha) = (\alpha, \Delta\nu),$$

since $\alpha = \alpha_p(x, y)$ is a harmonic field. We may assume that the harmonic fields ω_p^i of (1.17) are orthogonal, though not normalized, over M. If any of these fields vanish on B they can be omitted from the kernel $\alpha_p(x, y)$ without effecting the validity of (4.6). We therefore have

$$(\omega_p^i, \omega_p^j) = r_i \delta_{ij}, \qquad 0 < r_i < 1,$$

for the remaining ω_p^i. From (4.6),

$$\Delta\nu = \sum x_i \omega_p^i, \quad x_i = (\Delta\nu, \omega_p^i)_F,$$

and also

$$\Delta\nu = (\alpha, \sum x_i \omega_p^i) = \sum x_i r_i \omega_p^i.$$

The ω_p^i are linearly independent, so

$$x_i = r_i x_i,$$

implying $x_i = 0$ and also $\Delta\nu = 0$ in M.

Since ν is a harmonic form in M, (2.3) holds for ν:

(4.7) $$\int_B (\tau \wedge *d\nu - \delta\nu \wedge *\tau) = 0 \qquad (d\tau = 0, \delta\tau = 0 \text{ in } M).$$

From Green's formula (1.15) we have

$$D(\nu) = (d\nu, d\nu) + (\delta\nu, \delta\nu) = \int_B (\nu \wedge *d\nu - \delta\nu \wedge *\nu).$$

Supplying for ν the expression (4.5), we have

(4.8)
$$D(\nu) = \int_B \nu{*}d\left(\int_B g \wedge *d\nu - \delta\nu \wedge *g\right) - \int_B \delta\left(\int_B g \wedge *d\nu - \delta\nu \wedge *g\right) \wedge *\nu.$$

Taking the first term of the four, we denote by B_ϵ the boundary B with a sphere of radius ϵ around the point $y = x$ removed. Then

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$$\int_B \nu * d\left(\int_B g_p \wedge *d\nu\right) = \lim_{\epsilon\to 0}\int_B \nu \wedge *d\left(\int_{B_\epsilon} g_p \wedge *d\nu\right)$$
$$= \lim_{\epsilon\to 0}\int_B \nu \wedge *\left(\int_{B_\epsilon} \delta g_{p+1} \wedge *d\nu\right)$$
$$= \lim_{\epsilon\to 0}\int_B \delta\left(\int_{B_\epsilon} \nu \wedge *g_{p+1}\right) \wedge *d\nu$$
$$= \int_B \delta\left(\int_B \nu \wedge *g_{p+1}\right) \wedge *d\nu.$$

The remaining terms may be inverted in a similar way, using (1.26) also. From (1.14),

$$\int_B \nu \wedge *g_{p+1} = -(d\nu, g_{p+1})_{CM} + (\nu, \delta g_{p+1})_{CM} = (\nu, \delta g_{p+1})_{CM},$$

since $d\nu = 0$ in CM. The reversal of sign is due to the orientation of B. The other terms of (4.8) may be transformed in an analogous way, and the result is

$$D(\nu) = \int_B [\delta(\nu, \delta g_{p+1})_{CM} + d(\nu, dg_{p-1})_{CM}] \wedge *d\nu$$
$$- \int_B \delta\nu \wedge *[d(\nu, dg_{p-1})_{CM} + \delta(\nu, \delta g_{p+1})_{CM}]$$
$$= -\int_B [(\nu, \alpha)_{CM} \wedge *d\nu - \delta\nu \wedge *(\nu, \alpha)_{CM}] = 0,$$

in view of (4.7), since $(\nu, \alpha)_{CM}$ is a harmonic field. Hence, finally,

$$d\nu = 0, \quad \delta\nu = 0, \qquad \text{in } M,$$

so that from (4.4) we see that (4.1) is satisfied.

THEOREM I. *Let $t\chi$, $t_*\chi$ be continuous forms on B. There exists a harmonic form ϕ such that on B, $t\phi = t\chi$, $t_*\phi = t_*\chi$.*

The second boundary value problem may be treated in similar fashion. Let $nd\phi$, $t\delta\phi$ be given continuous boundary values satisfying the condition (2.3). Then the integral equations (3.9) are compatible if and only if

$$\int_B (\rho \wedge * d\phi - \delta\phi \wedge *\rho) = 0 \tag{4.9}$$

for every solution ρ of the equations

$$t_+\mu = 0, \quad t_+*\mu = 0. \tag{4.10}$$

Let μ, defined by (3.1), satisfy (4.10). In CM, μ is a harmonic form with zero tangential and normal boundary components, hence

$$d\mu = 0, \quad \delta\mu = 0, \qquad \text{in } CM.$$

The discontinuity conditions (3.5) and (3.6) show that

$$t_-\mu = -t\rho, \quad t_-*\mu = -t_*\rho. \tag{4.11}$$

Hence in M,

$$\mu = -\int_B (\mu \wedge *dg_p - \delta g_p \wedge *\mu)$$
$$= -\delta \int_B \mu \wedge *g_{p+1} + d\int_B g_{p-1} \wedge *\mu.$$

In CM, $\delta\mu$ is zero, that is

$$\delta d \int_B g_{p-1} \wedge *\mu = 0.$$

However [**2b**],

$$\delta d \int_B g_{p-1} \wedge *\mu = -d\delta \int_B g_{p-1} \wedge *\mu - \int_B \alpha_{p-1} \wedge *\mu$$
$$(4.12) \qquad = -d\int_B dg_{p-2} \wedge *\mu - \int_B \alpha_{p-1} \wedge *\mu$$
$$= -d\int_B g_{p-2} \wedge *\delta\mu - \int_B \alpha_{p-1} \wedge *\mu,$$

Stokes's theorem being applied in the last step. Since

$$tdg = O(s^{-n+2}),$$

the expression (4.12) is continuous across B. Therefore,

$$t\_\delta\mu = 0$$

and, dually,

$$n\_d\mu = 0.$$

In M, now, μ is a harmonic form with these boundary values; from (1.16) we see that μ is a harmonic field in M.

With the help of (4.11), the orthogonality condition becomes

$$\int_B (\mu \wedge *d\phi - \delta\phi \wedge *\mu) = 0,$$

which is satisfied in view of (2.3), since μ is a harmonic field in M. This proves that the second boundary value problem is solvable.

THEOREM II. *Let $nd\chi$, $t\delta\chi$ be given continuous forms on B, such that*

$$(4.13) \qquad \int_B (\tau \wedge *d\chi - \delta\chi \wedge *\tau) = 0$$

for every harmonic field τ in M. Then there exists in M a harmonic form ϕ such that $nd\phi = nd\chi$, $t\delta\phi = t\delta\chi$.

5. Domain functionals. In this section we assume that the uniqueness condition for the first boundary value problem relative to M is satisfied. It has been shown [**3**] that there exists a fundamental singularity in the large $\gamma_p(x, y)$ for M, in this case. The singularity of $\gamma_p(x, y)$ is of the type (1.24).

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Subtraction of a suitable harmonic p-form from γ_p gives the Green's form $G_p(x, y)$ of M, with the properties

$$G_p(x, y) \sim \gamma_p(x, y), \qquad x \in N(y),$$

(5.1)
$$\Delta_x G_p(x, y) = 0, \qquad x \neq y,$$

$$t_x G_p(x, y) = 0, \quad n_x G_p(x, y) = 0,$$

and

$$G_p(x, y) = G_p(y, x),$$

the symmetry being a consequence of Green's formula. The Green's form of M is unique. In terms of $G_p(x, y)$ the solution of the first boundary value problem is given by

(5.2)
$$\phi(y) = \int_B (\phi(x) \wedge *dG_p(x, y) - \delta G_p(x, y) \wedge *\phi(x)).$$

Let ϕ be a p-form of class C^2 in M, and vanishing on B. Then the equation

$$\Delta\phi = \rho$$

defines a p-form ρ which is continuous. If ρ is zero, clearly ϕ is zero, so that the correspondence of ϕ to ρ is one-one. With the aid of (1.16) we can express ϕ in terms of ρ by the equation

(5.3)
$$\phi = (G_p, \rho).$$

Since Δ commutes with d and $*$, we have the relations

(5.4)
$$*_x *_y G_p(x, y) = G_{n-p}(x, y),$$

and

(5.5)
$$\delta_y G_{p+1}(x, y) - d_x G_p(x, y) = \int_B \delta_z G_{p+1}(x, z) \wedge *d_z G_p(y, z),$$

similar to the formulae (1.26) and (1.27).

Let ϕ be a p-form with finite Dirichlet integral over M; we have from (1.15) the formula

(5.6)
$$D(\phi, G) = \chi - \phi,$$

where

(5.7)
$$\chi = \int_B (\phi \wedge *dG - \delta G \wedge *\phi)$$

is a harmonic form with the same boundary values as ϕ.

6. The third boundary value problem. Let

$$A = A_{i_1 \ldots i_p,\ j_1 \ldots j_p}$$

be a double alternating p-tensor, symmetric in the two sets of indices $i_1 \ldots i_p$ and $j_1 \ldots j_p$. We define the p-form $A\phi$ as follows:

(6.1)
$$(A\phi)_{i_1 \ldots i_p} = A_{i_1 \ldots i_p,\ (j_1 \ldots j_p)} \phi^{(j_1 \ldots j_p)}.$$

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If the invariant

$$A_{(i_1 \ldots i_p), (j_1 \ldots j_p)} \phi^{(i_1 \ldots i_p)} \phi^{(j_1 \ldots j_p)} > 0 \tag{6.2}$$

for every non-zero ϕ, A is positive definite. If $A = A(x)$ is positive definite in M, then

$$(\phi, A\phi) > 0 \tag{6.3}$$

for every $\phi = \phi(x)$ not identically zero in M.

The boundary conditions of mixed type which we discuss may be formulated as follows: Let A_p and A_{n-p} be two double p-tensors, symmetric and positive definite on the $(n - 1)$-dimensional boundary B of M. We require as boundary conditions

$$\begin{aligned} t{*}d\phi + {*}_B(A_p\, t\phi) &= \chi_{n-p-1}, \\ t{*}d{*}\phi + {*}_B(A_{n-p} t{*}\phi) &= \chi_{p-1}, \end{aligned} \tag{6.4}$$

where χ_{p-1}, χ_{n-p-1} are continuous forms, of the degrees indicated, on B.

The uniqueness condition (2.1) will be assumed to hold for M. A harmonic form ϕ which satisfies the homogenous conditions (6.4) is then identically zero. For, from (1.15) we find

$$\begin{aligned} D(\phi) &= \int_B (\phi \wedge {*}d\phi + {*}\phi \wedge {*}d{*}\phi) \\ &= -\int_B (t\phi \wedge {*}_B(A_p t\phi) + t{*}\phi \wedge {*}_B(A_{n-p} t{*}\phi)) \\ &= -(t\phi, A_p t\phi)_B - (t{*}\phi, A_{n-p} t{*}\phi)_B \leqslant 0. \end{aligned} \tag{6.5}$$

Hence both sides of (6.5) must vanish, so that $t\phi = 0$, $n\phi = 0$ on B. The uniqueness condition now implies that ϕ is zero in M. If, therefore, a harmonic form satisfies the non-homogeneous conditions (6.4), it is unique.

For the existence proof we shall assume that M is contained in the interior of a manifold M' having a Green's form (5.1). The difference $M' - M$ may be chosen to be a product of B with an interval, so that the uniqueness property holds for $M' - M$.

The potential

$$\nu = \int_B (G_p \wedge {*}d\sigma + {*}G_p \wedge {*}d{*}\sigma) \tag{6.6}$$

is a harmonic form in M and in $M' - M$. The analysis of the discontinuities on B carries over unchanged from §3. We suppose that ν satisfies (6.4) and find the integral equations (with principal values understood, as before)

$$\begin{aligned} \chi_{n-p-1} = -\tfrac{1}{2} t{*}d\sigma &+ \int_B (t{*}d_y\, G + {*}_B\, A_p\, t_y\, G) \wedge {*}d\sigma \\ &+ \int_B (t{*}d_y\, {*}_x\, G + {*}_B\, A_p\, t_y\, {*}_x\, G) \wedge {*}d{*}\sigma, \end{aligned} \tag{6.7}$$

$$\chi_{p-1} = -\tfrac{1}{2}t{*}d{*}\sigma + \int_B t{*}d{*}_y G + {*}_B A_{n-p} t{*}_y G) \wedge {*}d\sigma$$
$$+ \int_B (t{*}d{*}_y {*}_x G + {*}_B A_{n-p}\, t{*}_y {*}_x G) \wedge {*}d{*}\sigma.$$

Define the harmonic form

$$\chi = \int_B \rho \wedge {*}({*}dG + {*}_B A_p\, tG) + {*}\rho \wedge ({*}d{*}G + {*}_B A_{n-p}\, t{*}G), \tag{6.8}$$

then the homogeneous transposed equation associated with (6.7) arises from solving

$$t_+\chi = 0, \quad t_+{*}\chi = 0, \tag{6.9}$$

where the $+$ sign denotes values from $M' - M$. Let ρ be any solution of (6.9); since χ vanishes on the boundary of M' and on B, it follows that χ is zero in $M' - M$.

From the interior of M, χ has the boundary values

$$t_-\chi = -t\rho, \quad t_-{*}\chi = -t{*}\rho\ ; \tag{6.10}$$

as follows from (3.5) and (3.6). Noting (4.12) and (5.5) as well, we find that

$$t_-{*}d\chi = {*}_B A_p\, t\rho\ ; \quad t_-{*}d{*}\chi = {*}_B A_{n-p}\, t{*}\rho. \tag{6.11}$$

It follows that χ satisfies the homogeneous boundary conditions (6.4) from the interior of M, hence χ is zero in M. This implies that $t\rho = 0$, $t{*}\rho = 0$, and proves that the equations (6.7) have a unique solution.

THEOREM III. *Let M be interior to a manifold M' for which the uniqueness condition* (2.1) *holds; then the boundary value problem* (6.4) *is uniquely solvable for given continuous χ_{p-1}, χ_{n-p-1} on B.*

7. Eigenvalues. Let M be a finite manifold, with non-zero boundary B, which satisfies the conditions of §1, and for which the uniqueness condition (2.1) holds. Consider the equation

$$\Delta\phi = \lambda\phi \tag{7.1}$$

with the boundary condition

$$t\phi = 0, \quad n\phi = 0. \tag{7.2}$$

From (1.15) it follows easily that (7.1) has no negative eigenvalues, and the uniqueness condition shows that zero is not an eigenvalue. The integral equation which corresponds to (7.1) and (7.2) is

$$\phi = \lambda(G, \phi). \tag{7.3}$$

The iterated kernels of sufficiently high order of this equation are continuous. In view of (5.3), it follows that there exists a set of eigenvalues $\lambda_n > 0$, and eigenforms ϕ_n, complete in the L^2 space of p-forms which satisfy (7.2). The ϕ_n may be chosen to satisfy

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$$(\phi_n, \phi_m) = \delta_{mn}, \quad D(\phi_n, \phi_m) = \lambda_n \delta_{mn}. \tag{7.4}$$

Suppose now that M is a closed manifold ($B = 0$). In view of Hodge's theorem, zero is an eigenvalue of multiplicity $R_p(M)$; and all other eigenvalues are positive. Clearly the harmonic component of any solution of (7.1) with $\lambda \neq 0$ is zero. The integral equation is now

$$\phi = \lambda(g, \phi). \tag{7.5}$$

There exists a set of eigenforms complete in the L^2 space of p-forms on M with zero harmonic component, in view of (1.18). These, together with the Hodge forms, are complete in $L^2(M)$.

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