STRUCTURE OF SUMMABLE TALL IDEALS UNDER KATĚTOV ORDER

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Abstract. We show that Katětov and Rudin–Blass orders on summable tall ideals coincide. We prove that Katětov order on summable tall ideals is Galois–Tukey equivalent to $(\omega^\omega, \leq^*)$. It follows that Katětov order on summable tall ideals is upwards directed which answers a question of Minami and Sakai. In addition, we prove that l_∞ is Borel bireducible to an equivalence relation induced by Katětov order on summable tall ideals.

§1. Introduction. A set $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an *ideal* on ω if it is closed under taking subsets and finite unions. In this paper we always assume that an ideal is *proper*, i.e., it contains all finite subsets of ω and it does not contain ω. Given an ideal $\mathcal{I}$ on ω, define $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$. Elements of $\mathcal{I}^+$ are called $\mathcal{I}$-*positive* sets. The *dual filter* of $\mathcal{I}$ is denoted by $\mathcal{I}^* = \{\omega \setminus A : A \in \mathcal{I}\}$. If Y is an $\mathcal{I}$-positive set, then $\mathcal{I}\big| Y = \{A \cap Y : A \in \mathcal{I}\}$ is an ideal on Y.

The set of all finite subsets of ω is denoted by **Fin** or $[\omega]^{<\omega}$. Note that **Fin** is an ideal on ω. The set of all infinite subsets of ω is denoted by $[\omega]^\omega$. We say that an ideal $\mathcal{I}$ on ω is *tall* if for any $A \in [\omega]^\omega$, there exists $B \in [A]^\omega$ such that $B \in \mathcal{I}$. Let X, Y be two countably infinite sets. Let $\mathcal{I}$ be an ideal on X and $\mathcal{J}$ be an ideal on Y. We write $\mathcal{I} \simeq \mathcal{J}$ if there exists a bijection $e : X \to Y$ such that $A \in \mathcal{I} \Leftrightarrow e[A] \in \mathcal{J}$ where $e[A]$ is the image of A under e. One may check that an ideal $\mathcal{I}$ is not tall if there exists an $\mathcal{I}$-positive set A such that $\mathcal{I}\big|A \simeq$ **Fin**.

All ideals are assumed to be tall throughout this paper.

The set of all non-negative rational numbers is denoted by $\mathbb{Q}_+$. The set of all non-negative real numbers is denoted by $\mathbb{R}_+$. An ideal $\mathcal{I}$ on ω is a *summable ideal* if there is a function $f : \omega \to \mathbb{R}_+$ with $\sum_{n<\omega} f(n) = \infty$ such that

$$\mathcal{I} = \mathcal{I}_f := \left\{ A \subseteq \omega : \sum_{n \in A} f(n) < \infty \right\}.$$

Every summable ideal is an F_σ subset of 2^ω via characteristic functions (see Theorem 2.1 in Section 2 or [3]). For each summable ideal $\mathcal{I}_f$, if we take a function $f' : \omega \to \mathbb{Q}_+$ such that

$$|f(n) - f'(n)| \leq \frac{1}{2^n} \text{ for each } n \in \omega,$$

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then we have $\mathcal{I}_f = \mathcal{I}_{f'}$, so we assume always that $f \in \mathbb{Q}_+^\omega$ whenever we say that $\mathcal{I}_f$ is a summable ideal. One may easily check that summable ideal $\mathcal{I}_f$ is tall if and only if $\lim_{n\to\infty} f(n) = 0$. Define

$$\textsf{summable ideals} = \{\mathcal{I}_f : f \in \mathbb{Q}_+^\omega,\ \sum_{n<\omega} f(n) = +\infty \text{ and } \lim_{n\to\infty} f(n) = 0\}.$$

The followings are important tools for studying ideals, we refer the readers to a survey written by Hrušák [3] for details:

(1) (***Katětov ordering***) $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $p : \omega \to \omega$ such that
$$\forall A \subseteq \omega (A \in \mathcal{I} \Rightarrow p^{-1}(A) \in \mathcal{J}).$$

(2) (***Katětov–Blass ordering***) $\mathcal{I} \leq_{KB} \mathcal{J}$ if there is a finite-to-one function $p : \omega \to \omega$ such that
$$\forall A \subseteq \omega (A \in \mathcal{I} \Rightarrow p^{-1}(A) \in \mathcal{J}).$$

(3) (***Rudin–Blass ordering***) $\mathcal{I} \leq_{RB} \mathcal{J}$ if there is a finite-to-one function $p : \omega \to \omega$ such that
$$\forall A \subseteq \omega (A \in \mathcal{I} \Leftrightarrow p^{-1}(A) \in \mathcal{J}).$$

Obviously, $\mathcal{I} \leq_{RB} \mathcal{J} \Rightarrow \mathcal{I} \leq_{KB} \mathcal{J} \Rightarrow \mathcal{I} \leq_K \mathcal{J}$. Denote $\mathcal{I} <_K \mathcal{J}$ if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \not\leq_K \mathcal{I}$. Denote $\mathcal{I} \simeq_K \mathcal{J}$ if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$. Notice that $\simeq_K$ is an equivalence relation. Similarly we define $\mathcal{I} <_{KB} \mathcal{J}$, $\mathcal{I} <_{RB} \mathcal{J}$, $\mathcal{I} \simeq_{KB} \mathcal{J}$, and $\mathcal{I} \simeq_{RB} \mathcal{J}$.

Farah [2] proved that the Rudin–Blass order on all summable ideals has neither maximal elements nor minimal elements. He also proved that it is a dense ordering which includes an isomorphic copy of $(\mathcal{P}(\omega)/\mathrm{Fin}, \subseteq^*)$. Let F_σ **ideals** be the family of all F_σ-ideals. Minami and Sakai [5] proved that $(F_\sigma\,\textsf{ideals}, \leq_K)$ and $(F_\sigma\,\textsf{ideals}, \leq_{KB})$ are both upward directed and asked that if this is true for summable ideals [5, Question 5.1]. We will give a positive answer to this question in Section 3.

Let us consider a variation of the definition of summable ideals. Let

$$\mathsf{F_{DST}} = \{f \in \mathbb{Q}_+^\omega : \sum_{n<\omega} f(n) = +\infty,\ \lim_{n\to\infty} f(n) = 0, \text{ and } \forall n(f(n) \geq f(n+1))\}$$

and

$$\mathsf{ST} = \{\mathcal{I}_f : f \in \mathsf{F_{DST}}\}.$$

Actually, **ST** and **summable ideals** are virtually the same class of ideals (up to isomorphism) and we will show it in Section 2 (see Propositions 2.2 and 2.3).

In Section 4, we give a characterization of $\leq_K$ on **ST** which is crucial for later sections. In particular, we prove that for every $\mathcal{I}_f, \mathcal{I}_g \in \mathsf{ST}$, $\mathcal{I}_f \leq_K \mathcal{I}_g$ if and only if $\mathcal{I}_f \leq_{RB} \mathcal{I}_g$ (see Theorem 4.1).

Section 5 and 6 deal with Galois–Tukey connections which is introduced by Vojtáš [7]. For definition of Galois–Tukey connections, we follow the terminology in [1]. Let $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ be triples such that $A \subseteq A_- \times A_+$ and $B \subseteq B_- \times B_+$. We say $\mathbf{A} \leq_{GT} \mathbf{B}$ if there is a pair $\rho = (\rho_-, \rho_+)$ of functions such that:

(1) $\rho_- : A_- \to B_-$,
(2) $\rho_+ : B_+ \to A_+$, and
(3) $\forall a \in A_- \forall b \in B_+ (\rho_-(a) B b \Rightarrow a A \rho_+(b))$.

We write $\mathbf{A} \simeq_{GT} \mathbf{B}$ if $\mathbf{A} \leq_{GT} \mathbf{B}$ and $\mathbf{B} \leq_{GT} \mathbf{A}$. We say that $\mathbf{A}$ is Galois–Tukey equivalent to $\mathbf{B}$ if $\mathbf{A} \simeq_{GT} \mathbf{B}$.

Minami and Sakai [5] proved that $(F_\sigma\,\textsf{\textbf{ideals}}, \leq_K)$ and $(F_\sigma\,\textsf{\textbf{ideals}}, \leq_{KB})$ are both Galois–Tukey equivalent to $(\omega^\omega, \leq^*)$. We prove that $(\textsf{\textbf{ST}}, \leq_K) \simeq_{GT} (\omega^\omega, \leq^*)$ in Section 5 and that $(\textsf{\textbf{ST}}, \geq_K) \simeq_{GT} (\omega^\omega, \leq^*)$ in Section 6.

The last section is devoted to the study of Borel reducibility. We say that a topological space X is a *Borel space* if X is a Borel subset of some Polish space. Let X, Y be Borel spaces. Let E and F be equivalence relations on X and Y, respectively. We say that E is *Borel reducible* to F (denote $E \leq_B F$) if there is a Borel map $\Phi : X \to Y$ such that $xEy \Leftrightarrow \Phi(x)F\Phi(y)$ for all $x, y \in X$. We say that E is *Borel bireducible* to F if $E \leq_B F$ and $F \leq_B E$. Let $l_\infty = \{f \in \mathbb{R}^\omega : \sup_{n<\omega} |f(n)| < \infty\}$. For each $x, y \in \mathbb{R}^\omega$, define

$$x l_\infty y \iff x - y \in l_\infty.$$

We will prove that l_∞ is Borel bireducible to $\simeq_K$ on $\mathsf{F_{DST}}$. Here, $f \simeq_K g$ means $\mathcal{I}_f \simeq_K \mathcal{I}_g$.

§2. Preliminary. We make two comments in this section. One is that there is a convenient tool for studying F_σ-ideals. The other is that **ST** and **summable ideals** are virtually the same class of ideals (up to isomorphism).

Every summable ideal is an F_σ-ideal. This can be inferred by Mazur's characterization of F_σ-ideals using submeasures. A *submeasure* on ω is a function $\mu : \mathcal{P}(\omega) \to [0, +\infty]$ with the following properties for $A, B \subseteq \omega$:

(1) $\mu(A) \leq \mu(B)$ if $A \subseteq B$,
(2) $\mu(A \cup B) \leq \mu(A) + \mu(B)$, and
(3) $\mu(\emptyset) = 0$.

A submeasure μ is *lower semicontinuous* (lsc) if for every $A \subseteq \omega$ we have that

$$\mu(A) = \lim_{n\to\infty} \mu(A \cap n).$$

We say that μ is *unbounded* if $\mu(\omega) = \infty$. Mazur proved the following theorem.

THEOREM 2.1. [4] *The following are equivalent for every ideal $\mathcal{I}$ on ω :*

(1) *$\mathcal{I}$ is an F_σ-ideal.*
(2) *$\mathcal{I} = Fin(\mu)$ for some unbounded lsc submeasure μ on ω, where*

$$Fin(\mu) = \{A \subseteq \omega : \mu(A) < \infty\}.$$

For each summable ideal $\mathcal{I}_f$, let $u_f(A) = \sum_{i\in A} f(i)$ for $A \subseteq \omega$. It is easy to see that u_f is an unbounded lsc submeasure on ω by

$$u_f(A) = \sum_{i\in A} f(i) = \lim_{n\to\infty} \sum_{i\in A\cap n} f(i) = \lim_{n\to\infty} u_f(A \cap n).$$

Now we turn to the second comment.

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PROPOSITION 2.2. *There is a Borel function* $F : \mathsf{F} \to \mathsf{F}_{\mathsf{DST}}$ *such that* $\mathcal{I}_f \simeq \mathcal{I}_{F(f)}$ *for every* $f \in \mathsf{F}$ *where*

$$\mathsf{F} = \left\{ f \in \mathbb{Q}_+^\omega : \sum_{n\in\omega} f(n) = \infty \mathit{and} \lim_{n\to\infty} f(n) = 0 \right\}.$$

PROOF. Let $f \in \mathsf{F}$. For each $n \in \omega$, let

$$X_n = \left\{ k \in \omega : \frac{1}{n+2} \leq f(k) < \frac{1}{n+1} \right\}$$

and $N_n = |X_n|$. Since $\mathcal{I}_f$ is tall, we have that $N_n < \infty$ for all $n \in \omega$. Denote $M_n = \sum_{i=0}^{n} N_i$. Let $Y_0 = [0, M_0)$ and $Y_n = [M_{n-1}, M_n)$ for each $n > 0$. It follows that $|X_n| = |Y_n|$ for all n. For each $n \in \omega$, define

$$m_1^n \text{ by } f(m_1^n) = \max f[X_n]$$

and

$$m_j^n \text{ by } f(m_j^n) = \max f[X_n \setminus \{m_1^n, \dots, m_{j-1}^n\}] \text{ for each } 1 < j \leq N_n.$$

For each $n \in \omega$, define a bijection $h_n : X_n \to Y_n$ by

$$h_n(m_j^n) = M_{n-1} + j - 1 \text{ for each } 1 \leq j \leq N_n.$$

For each $n \in \omega$, define $f_n' : Y_n \to \mathbb{Q}_+$ by

$$f_n'(h_n(m_j^n)) = f(m_j^n) \text{ for all } 1 \leq j \leq N_n.$$

Let $f' = \bigcup_{n\in\omega} f_n'$. Define $F(f) = f'$. It follows that $F(f)$ is nonincreasing by the definition of f'. Then $\mathcal{I}_f \simeq \mathcal{I}_{F(f)}$ is witnessed by $h = \bigcup_{n\in\omega} h_n$.

Next we show that F is Borel. For any $n \in \omega$ and $b > a \geq 0$, define

$$U = \{f \in \mathsf{F}_{\mathsf{DST}} : f(n) \in (a, b)\} \text{ and}$$
$$U_q = \{f \in \mathsf{F}_{\mathsf{DST}} : f(n) = q\} \text{ for all } q \in (a, b) \cap \mathbb{Q}_+.$$

It follows that U is a basic open set in $\mathsf{F}_{\mathsf{DST}}$ and $U = \bigcup_{q\in(a,b)\cap\mathbb{Q}_+} U_q$. Fix n, U, q, and U_q as above. By the definition of $\mathsf{F}_{\mathsf{DST}}$, for each $f \in U_q$ we have that

$$f(i) \geq q \text{ for all } 0 \leq i < n \text{ and } f(j) \leq q \text{ for all } j > n.$$

Let $A = \{q\}$, $B = [q, +\infty)$, and $C = [0, q]$. It is easy to see that the set $A \times B^n \times C^\omega$ is Borel in $\mathbb{Q}_+^\omega$. For each $K \in [\omega]^{n+1}$, let $P_K \subseteq \omega^{n+1}$ be the set of all permutations of K (i.e., all bijections from K to K). For each $a = (a_0, a_1, \dots, a_n) \in P_K$, define $S(a) = (S_0, S_1, \dots, S_n, \dots) \in \mathcal{P}(\omega)^\omega$ by

$$S_{a_0} = A, S_{a_1} = \cdots = S_{a_n} = B \text{ and } S_j = C \text{ for all } j \notin \{a_0, \dots, a_n\}.$$

Denote $\prod S(a) = \prod_{i\in\omega} S_i$. Then

$$F^{-1}(U) = \bigcup_{q\in(a,b)\cap\mathbb{Q}_+} F^{-1}(U_q) = \bigcup_{q\in(a,b)\cap\mathbb{Q}_+} \bigcup_{K\in[\omega]^{n+1}} \bigcup_{a\in P_K} \prod S(a).$$

It follows that $F^{-1}(U)$ is a Borel set and F is Borel. ⊣

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PROPOSITION 2.3. *Define a map* Λ : **summable ideals** $\to$ **ST** *by*

$$\Lambda(\mathcal{I}_f) = \mathcal{I}_{F(f)} \text{ for all } \mathcal{I}_f \in \textbf{summable ideals},$$

where F is the function taken from the ***Proposition 2.2****. Then for each pair* $\mathcal{I}_f, \mathcal{I}_g \in$ **summable ideals** *we have that*

$$\mathcal{I}_f \leq_K \mathcal{I}_g \Leftrightarrow \mathcal{I}_{F(f)} \leq_K \mathcal{I}_{F(g)}.$$

PROOF. ($\Rightarrow$): Let $\mathcal{I}_f, \mathcal{I}_g \in$ **summable ideals** such that $\mathcal{I}_f \leq_K \mathcal{I}_g$. Then we have that

$$\mathcal{I}_{F(f)} \simeq \mathcal{I}_f \leq_K \mathcal{I}_g \simeq \mathcal{I}_{F(g)}.$$

It follows that $\mathcal{I}_{F(f)} \leq_K \mathcal{I}_{F(g)}$.

($\Leftarrow$): Let $\mathcal{I}_f, \mathcal{I}_g \in$ **summable ideals** such that $\mathcal{I}_f \not\leq_K \mathcal{I}_g$ and $p : \omega \to \omega$ be a map. Let e_1 and e_2 be witnesses for $\mathcal{I}_f \simeq \mathcal{I}_{F(f)}$ and $\mathcal{I}_g \simeq \mathcal{I}_{F(g)}$, respectively. By $\mathcal{I}_f \not\leq_K \mathcal{I}_g$, there exists $A \in \mathcal{I}_f$ such that

$$(e_1^{-1} \circ p \circ e_2)^{-1}(A) \notin \mathcal{I}_g.$$

Then we have that $e_1[A] \in \mathcal{I}_{F(f)}$ and

$$e_2^{-1}(p^{-1}(e_1[A])) \notin \mathcal{I}_g \Leftrightarrow p^{-1}(e_1[A]) \notin \mathcal{I}_{F(g)}.$$

Thus $\mathcal{I}_{F(f)} \not\leq_K \mathcal{I}_{F(g)}$. ⊣

§3. An answer to Minami and Sakai's question. In this section, we prove the following theorem which give a positive answer to a question of Minami and Sakai's [5, Question 5.1].

THEOREM 3.1. (**Summable ideals**, $\leq_{KB}$) *is countably upward directed.*

PROOF. Suppose that $\mathcal{I}_f \in$ **summable ideals**. Then $\lim_{n\to\infty} f(n) = 0$ and $\sum_{n\in\omega} f(n) = \infty$. Inductively take $\{k_n : n \in \omega\}$ such that for each $n \in \omega$,

(1) $k_n < k_{n+1}$,
(2) $f(m) < 1/(n+1)$ for each $m \geq k_n$, and
(3) $u_f([k_n, k_{n+1})) \geq 1$.

Suppose we have already constructed $\{k_j : j \leq n\}$ such that (1)–(3) holds. Since $\lim_{n\to\infty} f(n) = 0$, we can find k_{n+1} large enough such that (1) and (2) hold. By $\Sigma_{n\in\omega} f(n) = \infty$ we have $u_f([k_n, \infty)) = \infty$, so we may find k_{n+1} such that (3) holds. Denote $N_n := u_f([k_n, k_{n+1})) \geq 1$ and $I_n := [k_n, k_{n+1})$ for all $n \in \omega$. Then

$$u_f(I_n) = N_n \geq 1.$$

Now let $\{\mathcal{I}_{f_m} : m \in \omega\} \subseteq$ **summable ideals**. For each $m \in \omega$, let k_n^m, I_n^m, N_n^m be as above. Then for all $n, m \in \omega$,

$$u_{f_m}(I_n^m) = N_n^m \geq 1.$$

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For each $n \in \omega$, let $X_n = \prod_{m<n} I_n^m \subseteq \omega^n$ and $X = \bigcup_{n\in\omega} X_n$. Fix $n \in \omega$. For any $(i_0, i_1, \dots, i_{n-1}) \in X_n$, define a function f by

$$f\big((i_0, i_1, \dots, i_{n-1})\big) = \frac{\prod_{m<n} f_m(i_m)}{\prod_{m<n} N_n^m}.$$

Then

$$\begin{aligned} u_f(X_n) &= \sum_{(i_0,\dots,i_{n-1})\in X_n} f\big((i_0,\dots,i_{n-1})\big) = \sum_{i_0\in I_n^0,\dots,i_m\in I_n^{n-1}} f\big((i_0,\dots,i_{n-1})\big) \\ &= \frac{1}{\prod_{m<n} N_n^m} \cdot \sum_{i_0\in I_n^0,\cdots,i_{n-1}\in I_n^{n-1}} \left(\prod_{m<n} f_m(i_m)\right) = \frac{1}{\prod_{m<n} N_n^m} \cdot \prod_{m<n} u_{f_m}(I_n^m) \\ &= 1, \end{aligned}$$

so $u_f(X) = \infty$.

By (2), for all $n \in \omega$ we have that

$$f\big((i_0, i_1, \dots, i_{n-1})\big) \leq \frac{1}{(n+1)^n} \text{ for every } (i_0, i_1, \dots, i_{n-1}) \in X_n,$$

so $\mathcal{I}_f$ is tall.

We will show that for each $m \in \omega$, $\mathcal{I}_{f_m} \leq_{KB} \mathcal{I}_f$. Fix $m \in \omega$. Define $\pi_m : X \to \omega$ by

$$\begin{gathered} \pi_m\big((i_0, \dots, i_{n-1})\big) = 0, n \leq m, \\ \pi_m\big((i_0, \dots, i_{n-1})\big) = i_m, n > m, \end{gathered}$$

Then $\left|\pi_m^{-1}(0)\right| < \infty$ and $u_f\left(\pi_m^{-1}(0)\right) < \infty$. Let $i \in \omega \setminus \{0\}$. If $i \in I_n^m$ for some $n \leq m$ then $\pi_m^{-1}(i) = \emptyset$. We assume that $i \in I_n^m$ for some $n > m$. It follows that $\left|\pi_m^{-1}(i)\right| \leq |X_n| < \infty$ and

$$\begin{aligned} u_f\left(\pi_m^{-1}(i)\right) &= u_f\Big(\big\{(i_0, \dots, i_m, \dots, i_{n-1}) \in X : i_m = i\big\}\Big) \\ &= \frac{f_m(i)}{N_n^m} \cdot \sum_{(i_0,\dots,i_{n-1})} \left(\frac{\prod_{j<n,\ j\neq m} f_j(i_j)}{\prod_{j<n, j\neq m} N_n^j}\right) = \frac{f_m(i)}{N_n^m} \leq f_m(i). \end{aligned}$$

Thus, for every $A \subseteq \omega \setminus \{0\}$ with $u_{f_m}(A) < \infty$ we have that $u_f\left(\pi_m^{-1}(A)\right) \leq u_{f_m}(A) < \infty$. ⊣

§4. Characterizations of Katětov order among summable ideals. In this section, we prove the following theorem which is crucial for later section.

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THEOREM 4.1. *Let $\mathcal{I}_f, \mathcal{I}_g \in \mathbf{ST}$. Then the following are equivalent:*

(1) $\mathcal{I}_f \leq_K \mathcal{I}_g$.
(2) *There exist $p : \omega \to \omega$ and $0 < C$ such that*

$$A[C] = \{n : u_g(p^{-1}(n)) \leq C \cdot f(n)\} \in \mathcal{I}_f^*$$

and $p^{-1}(A[C]) \in \mathcal{I}_g^*$.
(3) *There exists $0 < M \in \omega$ such that for all $l > M$ and $k_1 > k_0 \geq M$, if $u_g([k_0, k_1]) > M \cdot u_f([0,l])$, then $g(k_1) \leq M \cdot f(l)$.*
(4) *There exist an interval-to-one map $p : \omega \to \omega$ and $0 < c < C$ such that $c \cdot f(i) \leq u_g(p^{-1}(i)) \leq C \cdot f(i)$ for all i.*
(5) *There exist an interval-to-one map $p : \omega \to \omega$ and $0 < C$ such that $u_g(p^{-1}(i)) \leq C \cdot f(i)$ for all i.*
(6) *There exists $0 < M \in \omega$ such that for all $k, l \in \omega$, if $u_g([0,k]) > M \cdot u_f([0,l])$, then $g(k) \leq M \cdot f(l)$.*
(7) $\mathcal{I}_f \leq_{RB} \mathcal{I}_g$.

PROOF. (1)⇒(2): Let $p : \omega \to \omega$ be a witness for $\mathcal{I}_f \leq_K \mathcal{I}_g$. We show that there exists $C > 0$ such that $A[C] = \left\{n : u_g\left(p^{-1}(n)\right) \leq C \cdot f(n)\right\} \in \mathcal{I}_f^*$. Otherwise, $A[C] \notin \mathcal{I}_f^*$ for every $C > 0$. Thus, we can find pairwise disjoint finite sets $\{a_n : 1 \leq n < \omega\}$ such that for each $1 \leq n \in \omega$,

(i) $f(j) \leq \frac{1}{n^2}$ for any $j \in a_n$,
(ii) $a_n \subseteq \omega \setminus A[n]$, and
(iii) $\frac{1}{n^2} \leq u_f(a_n) \leq \frac{2}{n^2}$.

By (ii),

$$u_g\left(p^{-1}(a_n)\right) > n \cdot u_f(a_n) \geq \frac{1}{n}.$$

Let $B = \bigcup_{1 \leq n < \omega} a_n$. Then

$$u_f(B) \leq \sum_{n=1}^{\infty} \frac{2}{n^2} < \infty, \text{and } u_g\left(p^{-1}(B)\right) \geq \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

This contradicts the definition of p.

(2)⇒(3): Let p and C be such that $A[C] \in \mathcal{I}_f^*$ and $p^{-1}(A[C]) \in \mathcal{I}_g^*$. Then $p^{-1}(\omega \setminus A[C]) \in \mathcal{I}_g$. Take $M > C + 1$ such that $u_g\left(p^{-1}(\omega \setminus A[C]) \setminus M\right) < 1$ and $u_f([0,M]) > 2$. Assume $l > M$, $k_1 > k_0 \geq M$ with $u_g([k_0,k_1]) > Mu_f([0,l])$. Consider $t = [k_0,k_1] \setminus p^{-1}([0,l])$. The proof is divided into two cases.

Case 1: $p(t) \cap A[C] = \emptyset$.

Then $t \cap p^{-1}(A[C]) = \emptyset$. By $k_0 \geq M$, we have that $t \subset p^{-1}(\omega \setminus A[C]) \setminus M$. By the definition of M we have $u_g(t) < 1$. On the other hand we have

$$u_g\Big([k_0,k_1] \cap p^{-1}([0,l]) \cap p^{-1}(\omega \setminus A[C])\Big) \leq u_g\Big(p^{-1}(\omega \setminus A[C]) \setminus M\Big) < 1$$

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and

$$u_g\Big([k_0,k_1]\cap p^{-1}([0,l])\cap p^{-1}(A[C])\Big)\le C\cdot u_f\Big([0,l]\cap A[C]\Big)\le C\cdot u_f([0,l]).$$

Thus

$$\begin{aligned}u_g\Big([k_0,k_1]\cap p^{-1}([0,l])\Big)&\le 1+C\cdot u_f([0,l])\\&=2+C\cdot u_f([0,l])-1<(C+1)\cdot u_f([0,l])-1\end{aligned}$$

and

$$\begin{aligned}u_g(t)=u_g\Big([k_0,k_1]\setminus p^{-1}([0,l])\Big)&=u_g([k_0,k_1])-u_g\Big([k_0,k_1]\cap p^{-1}([0,l])\Big)\\&>M\cdot u_f([0,l])-((C+1)\cdot u_f([0,l])-1)>1.\end{aligned}$$

A contradiction.

Case 2: $p(t)\cap A[C]\ne\emptyset$.

Let $m\in t$ and $p(m)\in A[C]$. Then $m\le k_1$, $p(m)>l$, and $u_g\left(p^{-1}(p(m))\right)\le C\cdot f(p(m))$. By the monotonicity of f and g, we have

$$g(k_1)\le g(m)\le u_g\left(p^{-1}(p(m))\right)\le C\cdot f(p(m))\le C\cdot f(l)<M\cdot f(l).$$

(3)⇒(4): Choose k_0 such that

$$u_g([M,k_0))=u_g([M,k_0-1])>M\cdot u_f([0,M]).$$

We recursively choose a sequence $k_0<k_1<\cdots$ such that for each i, k_{i+1} is the minimal such that $u_g([k_i,k_{i+1}))\ge M\cdot f(M+1+i)$. Then we have

$$u_g([M,k_{i+1}))>M\cdot u_f([0,M])+M\cdot\sum_{j=0}^{i}f(M+1+j)=M\cdot u_f([0,M+1+i]).$$

Thus $g(k_{i+1}-1)\le M\cdot f(M+1+i)$ by the assumption of (3). The proof is divided into two cases.

Case 1. $k_{i+1}-1=k_i$. Clearly we have that $g(k_{i+1}-1)=u_g([k_i,k_{i+1}))=M\cdot f(M+1+i)$.

Case 2. $k_{i+1}-1>k_i$. By the definition of k_{i+1}, we have $u_g([k_i,k_{i+1}-1))<M\cdot f(M+1+i)$. This implies that

$$\begin{aligned}M\cdot f(M+1+i)\le u_g([k_i,k_{i+1}))&=u_g([k_i,k_{i+1}-1))+g(k_{i+1}-1)\\&<2M\cdot f(M+1+i).\end{aligned}$$

Let $p:\omega\to\omega$ be an interval-to-one map such that $p^{-1}([0,M])=[0,k_0)$ and $p^{-1}(M+1+i)=[k_i,k_{i+1})$ for every i. Define

$$C=\max\left\{2M,\max\left\{\frac{u_g(p^{-1}(n))}{u_f(n)}:n\le M\right\}\right\}$$

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and

$$c = \min\left\{M, \min\left\{\frac{u_g(p^{-1}(n))}{u_f(n)} : n \leq M\right\}\right\}.$$

Then, for each $n \leq M$ we have that

$$c \cdot f(n) \leq \frac{u_g\left(p^{-1}(n)\right)}{f(n)} \cdot f(n) = u_g\left(p^{-1}(n)\right) = \frac{u_g\left(p^{-1}(n)\right)}{f(n)} \cdot f(n) \leq C \cdot f(n).$$

(4)⇒(7): For every $A \subseteq \omega$, we have that

$$u_f(A) < \infty \Rightarrow u_g\left(p^{-1}(A)\right) \leq C \cdot u_f(A) < \infty$$

and

$$u_g\left(p^{-1}(A)\right) < \infty \Rightarrow u_f(A) \leq \frac{1}{c} \cdot u_g\left(p^{-1}(A)\right) < \infty.$$

(4)⇒(5), (5)⇒(1), (7)⇒(1) are clear.

(5)⇒(6): Let C be as in (5). Define $M = C$. We will show that M is as desired. Let $l, k \in \omega$ with $u_g([0,k]) > M \cdot u_f([0,l])$. By the assumption of (5), $u_g\left(p^{-1}([0,l])\right) \leq M \cdot u_f([0,l])$, so $[0,k] \setminus p^{-1}([0,l]) \neq \emptyset$. Take $m \in [0,k] \setminus p^{-1}([0,l])$. Then $m \leq k$ and $p(m) > l$. It follows that

$$g(k) \leq g(m) \leq u_g\left(p^{-1}(p(m))\right) \leq M \cdot f(p(m)) \leq M \cdot f(l).$$

(6)⇒(5): Choose k_0 such that

$$u_g([0,k_0)) = u_g([0,k_0-1]) > M \cdot f(0).$$

Recursively define a sequence $k_0 < k_1 < \cdots$ such that for each $i > 0$, k_i is the minimal such that $u_g([k_{i-1},k_i)) \geq M \cdot f(i)$. Then we have

$$u_g([0,k_i)) > M \cdot \sum_{j=0}^{i} f(j) = M \cdot u_f([0,i]).$$

Thus $g(k_i - 1) \leq M \cdot f(i)$ by the assumption of (5). The proof is divided into two cases.

Case 1. $k_i - 1 = k_{i-1}$. Clearly we have that $g(k_i - 1) = u_g([k_{i-1},k_i)) = M \cdot f(i)$.

Case 2. $k_i - 1 > k_{i-1}$. By the choice of k_i, we have $u_g([k_{i-1},k_i - 1)) < M \cdot f(i)$. This implies that

$$M \cdot f(i) \leq u_g([k_i,k_{i+1})) = u_g([k_i,k_{i+1}-1)) + g(k_{i+1}-1) < 2M \cdot f(i).$$

Let $p : \omega \to \omega$ be an interval-to-one map such that $p^{-1}(0) = [0,k_0)$ and $p^{-1}(i) = [k_{i-1},k_i)$ for every $i > 0$. It is easy to see that p and $C = 2M$. ⊣

Remark: It is worth to note that p in the proof of (3)⇒(4) and (6)⇒(5) is a surjection and $\max p^{-1}(n) < \min p^{-1}(n+1)$ for $n \in \omega$ (see Figure 1).

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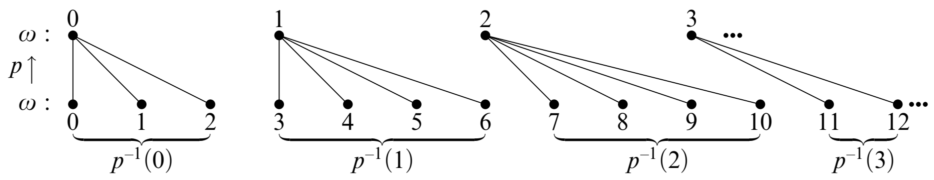

FIGURE 1. An example of interval-to-one map p in Remark.

§5. The structure of $(\mathsf{ST}, \leq_K)$ in the sense of Galois–Tukey connection. In this section we prove that $(\mathsf{ST}, \leq_K) \simeq_{GT} (\omega^\omega, \leq^*)$ (see Theorem 5.6). We first prove $(\mathsf{ST}, \leq_K) \leq_{GT} (\omega^\omega, \leq^*)$ (see Lemma 5.4).

We will define an order $(\mathbb{H}, \leq^\circ)$ such that $(\mathbb{H}, \leq^\circ)$ is upward directed and

$$(\mathsf{ST}, \leq_K) \leq_{GT} (\mathbb{H}, \leq^\circ) \leq_{GT} (\omega^\omega, \leq^*).$$

To define $(\mathbb{H}, \leq^\circ)$, we need the following.

First, we define a set $\Phi \subseteq \mathbb{Q}_+^{<\omega} \times \omega^{<\omega}$ by $(s, p) \in \Phi$ if and only if $(*)$ there exist $l_s, k_s \in \omega$ such that (s, p) satisfies the following (see Figure 2):

(i) $s : l_s \to \mathbb{Q}_+$ and $s(j) \geq s(j+1)$ for all $j < l_s - 1$,
(ii) $0 = p(1) < p(2) < \cdots < p(k_s) = l_s - 1$,
(iii) $u_s\big(\big[p(i), p(i+1)\big)\big) \geq 1$ for each $0 < i < k_s$, and
(iv) $s(j) \leq \frac{1}{i}$ for each $j \geq p(i)$ and $0 < i < k_s$.

For any $(s, p) \in \Phi$, define a subset of Φ by

$$\Phi(s, p) = \{(t, q) \in \Phi : s \sqsubseteq t, p \sqsubseteq q \text{ and } k_t = k_s + 1\}.$$

Define an order $\trianglelefteq_{(s,p)}$ on $\Phi(s, p)$ as follows: for each $(t_1, q_1), (t_2, q_2) \in \Phi(s, p)$, $(t_1, q_1) \trianglelefteq_{(s,p)} (t_2, q_2)$ if and only if there exists a map

$$\pi : \big[q_2(k_s), q_2(k_s+1)\big) \to \big[q_1(k_s), q_1(k_s+1)\big)$$

such that

$$u_{t_2}(\pi^{-1}(i)) \leq t_1(i) \text{ for each } i \in \big[q_1(k_s), q_1(k_s+1)\big).$$

It is easy to see that $\trianglelefteq_{(s,p)}$ is transitive.

LEMMA 5.1. *$(\Phi(s, p), \trianglelefteq_{(s,p)})$ is upward directed for all $(s, p) \in \Phi$.*

PROOF. Fix $(t_0, q_0), (t_1, q_1) \in \Phi(s, p)$. Define (t, q) as follows. Define $I_0 = \big[q_0(k_s), q_0(k_s+1)\big)$ and $I_1 = \big[q_1(k_s), q_1(k_s+1)\big)$ and $I = I_0 \times I_1$.

Fix $i \in \{0, 1\}$. Denote $N_i := u_{t_i}\big(I_i\big) \geq 1$. For every $(j_0, j_1) \in I$, define t by

$$t\big((j_0, j_1)\big) = \frac{t_0(j_0) \cdot t_1(j_1)}{N_0 \cdot N_1}.$$

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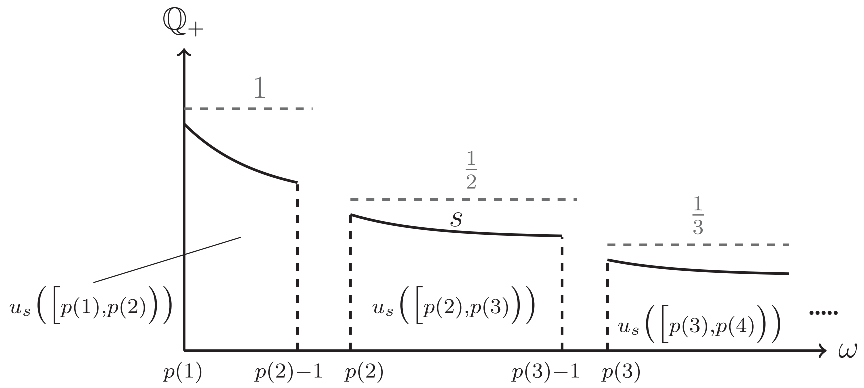

FIGURE 2. An element (s, p) of set Φ.

Let $M = |I_1| \cdot |I_2|$. We can find a bijection e from I to $[p(k_s)+1, p(k_s)+M]$ such that

$$t(e^{-1}(j)) \geq t(e^{-1}(j+1)) \text{ for all } j \in [p(k_s)+1, p(k_s)+M].$$

Let $q(k_s) = p(k_s)$ and $q(k_s+1) = p(k_s)+M+1$. Then

$$I = e^{-1}(\left[q(k_s), q(k_s+1)\right)).$$

Without loss of generality, we can regard I as $\left[q(k_s), q(k_s+1)\right)$.

We will show that $(t, q) \in \Phi(s, p)$ and $(t_i, q_i) \trianglelefteq_{(s,p)} (t, q)$ for $i \in \{0, 1\}$.

(1) $(t, q) \in \Phi(s, p)$:

Use

$$u_t(I) = \sum_{(j_0, j_1) \in I} \frac{t_0(j_0) \cdot t_1(j_1)}{N_0 \cdot N_1} = 1$$

and

$$t\big((j_0, j_1)\big) = \frac{t_0(j_0) \cdot t_1(j_1)}{N_0 \cdot N_1} \leq \frac{1}{N_0 \cdot N_1} \cdot \frac{1}{k_s^2} \leq \frac{1}{k_s} \text{ for all } (j_0, j_1) \in I.$$

(2) $(t_i, q_i) \trianglelefteq_{(s,p)} (t, q)$ for $i \in \{0, 1\}$:

Let π_0 and π_1 be the projection map onto the first coordinate and second coordinate, respectively. For any $j \in I_0$, we have that

$$\pi_0^{-1}(j) = \{(j, j_1) : j_1 \in I_1\}$$

and

$$u_t\left(\pi_0^{-1}(j)\right) = \frac{t_0(j)}{N_0} \cdot \sum_{j_1 \in I_1} \frac{t_1(j_1)}{N_1} = \frac{t_0(j)}{N_0} \leq t_0(j).$$

Similarly, we have $u_t\left(\pi_1^{-1}(j)\right) \leq t_1(j)$. ⊣

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For any $(s, p) \in \Phi$, define a cofinal subset $\widetilde{\Phi}(s, p)$ of $\Phi(s, p)$ such that $(\widetilde{\Phi}(s, p), \trianglelefteq_{(s,p)})$ is an increasing chain. We define $\widetilde{\Phi}(s, p)$ as follows. Enumerate $\Phi(s, p) = \{(s_n, p_n), n \in \omega\}$. Let $(t_0, q_0) = (s_0, p_0)$. Suppose we have already constructed $\{(t_i, q_i) : i < n\}$. Then we take (t_n, q_n) such that $(t_{n-1}, q_{n-1}) \trianglelefteq_{(s,p)} (t_n, q_n)$ and $(s_n, p_n) \trianglelefteq_{(s,p)} (t_n, q_n)$ by Lemma 5.1. Define

$$\widetilde{\Phi}(s, p) = \{(t_n, q_n) : n < \omega\}.$$

Define

$$\mathbb{H} = \left\{h \in \Phi^{\Phi} : h((s, p)) \in \widetilde{\Phi}(s, p) \text{ for all } (s, p) \in \Phi\right\}.$$

Define the order $\leq^{\circ}$ on $\mathbb{H}$ as follows: for each $h, h' \in \mathbb{H}$, $h \leq^{\circ} h'$ if and only if $h((s, p)) \trianglelefteq_{(s,p)} h'((s, p))$ for all but finitely many $(s, p) \in \Phi$. It is easy to see that $(\mathbb{H}, \leq^{\circ})$ is upward directed by the definition of $\mathbb{H}$.

Next, we prove the following:

Lemma 5.2. $(\mathbb{H}, \leq^{\circ}) \leq_{GT} (\omega^{\omega}, \leq^{*})$.

Proof. Enumerate

$$\Phi = \{(s_i, p_i), i \in \omega\}.$$

Enumerate

$$\widetilde{\Phi}(s_i, p_i) = \left\{\left(t_j^{(s_i,p_i)}, q_j^{(s_i,p_i)}\right) : j \in \omega\right\}$$

in such way that $(t_j^{(s_i,p_i)}, q_j^{(s_i,p_i)}) \trianglelefteq_{(s_i,p_i)} (t_k^{(s_i,p_i)}, q_k^{(s_i,p_i)})$ for all $j < k$.

Define $\rho_+ : \omega^{\omega} \to \mathbb{H}$ as follows. For every $g \in \omega^{\omega}$ and $i \in \omega$, let

$$\rho_+(g)((s_i, p_i)) = \left(t_{g(i)}^{(s_i,p_i)}, q_{g(i)}^{(s_i,p_i)}\right).$$

Define $\rho_- : \mathbb{H} \to \omega^{\omega}$ as follows. For every $h \in \mathbb{H}$ and $i \in \omega$, let

$$h((s_i, p_i)) = \left(t_{\rho_-(h)(i)}^{(s_i,p_i)}, q_{\rho_-(h)(i)}^{(s_i,p_i)}\right) \text{ for all } i \in \omega.$$

We claim that

$$\forall h \in \mathbb{H}\, \forall g \in \omega^{\omega}(\rho_-(h) \leq^{*} g \Rightarrow h \leq^{\circ} \rho_+(g)).$$

Suppose $h \in \mathbb{H}$, $g \in \omega^{\omega}$, and $\rho_-(h) \leq^{*} g$. There is $n \in \omega$ such that for each $i \geq n$ we have $\rho_-(h)(i) \leq g(i)$. Then $h((s_i, p_i)) \trianglelefteq_{(s_i,p_i)} \rho_+(g)((s_i, p_i))$ for all but finitely many $i \in \omega$, i.e., $h \leq^{\circ} \rho_+(g)$. ⊣

Now, we prove the following:

Lemma 5.3. $(\mathbf{ST}, \leq_K) \leq_{GT} (\mathbb{H}, \leq^{\circ})$.

Proof. Define $\rho_+ : \mathbb{H} \to \mathbf{ST}$ as follows. Define $q_{-1} \in \omega^1$ by $q_{-1}(1) = 0$ and $t_{-1}(0) = 1$. For each $h \in \mathbb{H}$, let $(t_0^h, q_0^h) = h((t_{-1}, q_{-1}))$ and $(t_{n+1}^h, q_{n+1}^h) = h((t_n^h, q_n^h))$

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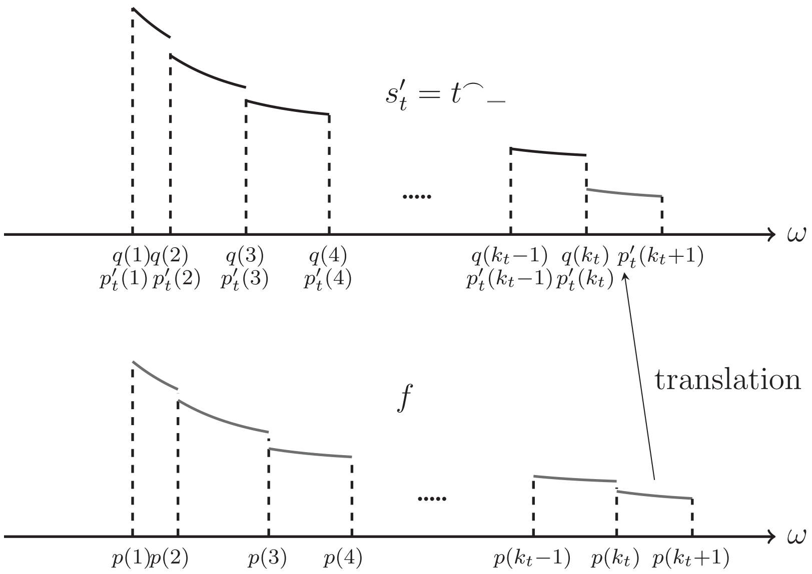

FIGURE 3. The definition of (s'_t, p'_t) in the proof of Lemma 5.3.

for all $n \in \omega$. Let $g = \bigcup_{n\in\omega} t_n^h$ and $\rho_+(h) = \mathcal{I}_g$. Since $q_{-1} = (q_{-1}(1))$, we have that $q_0^h = (q_0^h(1), q_0^h(2))$, and

$$q_n^h = (q_n^h(1), q_n^h(2), \dots, q_n^h(n+2)) \text{ for } n \in \omega, \text{ i.e., } k_{t_n^h} = n+2$$

(see $(*)$ at the beginning of Section 4 for the definition of $k_{t_n^h}$).

Define $\rho_- : \mathbf{ST} \to \mathbb{H}$ as follows. For $\mathcal{I}_f \in \mathbf{ST}$, take $0 = m_1 < m_2 < \cdots < m_i < \cdots$ such that for each $i > 0$,

$$u_f([m_i, m_{i+1})) \geq 1 \text{ and } f(j) \leq \frac{1}{i} \text{ for every } j \geq m_i.$$

For each $n \geq 1$, let $p(n) = m_n$, $p_n = (p(1), \dots, p(n))$, and $s_n = f\big|_{[0,p(n))}$. For every $(t, q) \in \Phi$ with $q = (q(1), \dots, q(k_t))$, let

$$\begin{aligned} p'_t &= \big(q(1), \dots, q(k_t), q(k_t) + p(k_t+1) - p(k_t)\big) \\ &= \big(p'_t(1), \dots, p'_t(k_t), p'_t(k_t+1)\big). \end{aligned}$$

We have that:

- $p'_t(j) = q(j)$ for $1 \leq j \leq k_t$,
- $p'_t(k_t+1) = q(k_t) + p(k_t+1) - p(k_t)$, and
- $s'_t = t^\frown f\big|_{[p(k_t), p(k_t+1))}$.

Then $(s'_t, p'_t) \in \Phi(t, q)$ (see Figure 3). Take $(s^*_t, p^*_t) \in \widetilde{\Phi}(t, q)$ such that $(s'_t, p'_t) \trianglelefteq_{(t,q)} (s^*_t, p^*_t)$. Define $\rho_-(\mathcal{I}_f)((t, q)) = (s^*_t, p^*_t)$.

We claim that

$$\forall \mathcal{I}_f \in \mathbf{ST}\ \forall h \in \mathbb{H}(\rho_-(\mathcal{I}_f) \leq^\circ h \Rightarrow \mathcal{I}_f \leq_K \rho_+(h)).$$

Suppose $\mathcal{I}_f \in \mathbf{ST}$, $h \in \mathbb{H}$, and $\rho_-(\mathcal{I}_f) \leq^\circ h$. Let $\{s_n : n \in \omega\}$ and p be like in the definition of $\rho_-(\mathcal{I}_f)$. Then we have that

$$\rho_-(\mathcal{I}_f)((t,q)) \trianglelefteq_{(t,q)} h((t,q))$$

for all but finitely many $(t,q) \in \Phi$. By the definition of ρ_+, there exists $\{(t_n^h, q_n^h) : n \in \omega\}$. Then there is N such that for $n > N$, we have that

$$\rho_-(\mathcal{I}_f)((t_n^h, q_n^h)) \trianglelefteq_{(t_n^h, q_n^h)} h((t_n^h, q_n^h)) = (t_{n+1}^h, q_{n+1}^h).$$

Since $\rho_-(\mathcal{I}_f)((t_n^h, q_n^h)) = (s^*_{t_n^h}, p^*_{t_n^h})$, we have that

$$(s'_{t_n^h}, p'_{t_n^h}) \trianglelefteq_{(t_n^h, q_n^h)} (s^*_{t_n^h}, p^*_{t_n^h}) \trianglelefteq_{(t_n^h, q_n^h)} (t_{n+1}^h, q_{n+1}^h).$$

Then for each $n > N$, there exists a map

$$\pi'_n : [q_{n+1}^h(n+2), q_{n+1}^h(n+3)) \to [p'_{t_n^h}(n+2), p'_{t_n^h}(n+3))$$

such that

$$u_{t_{n+1}^h}(\pi'^{-1}_n(i)) \leq s'_{t_n^h}(i) \text{ for all } i \in [p'_{t_n^h}(n+2), p'_{t_n^h}(n+3)).$$

Define $\sigma_n : [p'_{t_n^h}(n+2), p'_{t_n^h}(n+3)) \to [p(n+2), p(n+3))$ by

$$\sigma_n(j) = j - p'_{t_n^h}(n+2) + p(n+2) \text{ for all } j \in [p'_{t_n^h}(n+2), p'_{t_n^h}(n+3)).$$

Define $\pi_n = \sigma_n \circ \pi'_n$ for each $n > N$. We have that

$$u_{t_{n+1}^h}(\pi_n^{-1}(i)) \leq s_{n+3}(i) \text{ for each } i \in [p(n+2), p(n+3)).$$

There exist $\pi_{-1} : [0, q_0^h(2)) \to [0, p(2))$ and $c_{-1} > 0$ such that

$$u_{t_0^h}(\pi_{-1}^{-1}(i)) \leq c_{-1} \cdot s_2(i) \text{ for each } i \in [0, p(2)).$$

For every $m \leq N$, there exist $\pi_m : [q_m^h(m+2), q_m^h(m+3)) \to [p(m+2), p(m+3))$ and $c_m > 0$ such that

$$u_{t_{m+1}^h}(\pi_m^{-1}(i)) \leq c_m \cdot s_{m+3}(i) \text{ for each } i \in [p(m+2), p(m+3)).$$

Let $\pi = \bigcup_{n \in \omega \cup \{-1\}} \pi_n$ and $C = \max\{1, c_{-1}, c_0, \dots, c_N\}$. Then we have that

$$u_g(\pi^{-1}(i)) \leq C \cdot f(i) \text{ for } i \in \omega \text{ (see Figure 4).}$$

Then π witnesses $\mathcal{I}_f \leq_K \mathcal{I}_g = \rho_+(h)$ by Theorem 4.1(5). ⊣

Combining Lemmas 5.2 and 5.3 we have:

LEMMA 5.4. $(\mathbf{ST}, \leq_K) \leq_{GT} (\omega^\omega, \leq^*)$.

The proof of the other side is short.

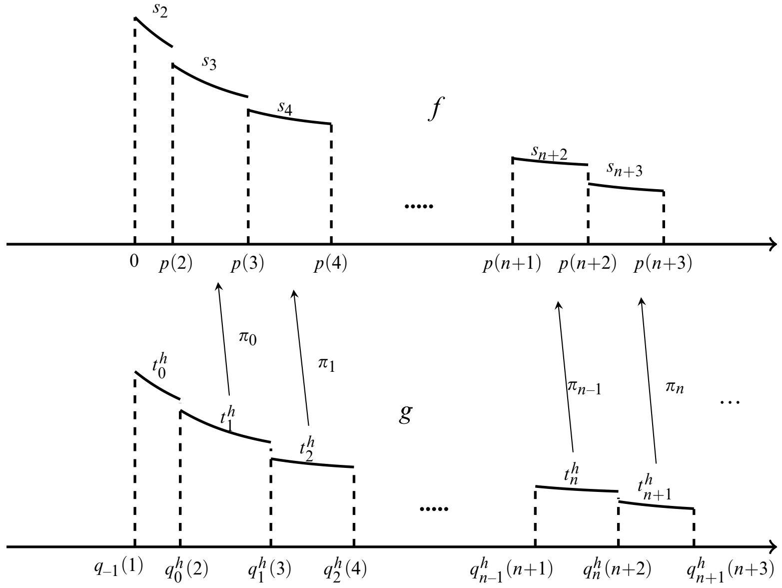

FIGURE 4. π witnesses $\mathcal{I}_f \leq_K \mathcal{I}_g$ in the proof of Lemma 5.3.

LEMMA 5.5. $(\mathbf{ST}, \leq_K) \geq_{GT} (\omega^\omega, \leq^*)$.

PROOF. Define $\rho_+ : \mathbf{ST} \to \omega^\omega$ as follows. For each $\mathcal{I}_g \in \mathbf{ST}$ and $n \geq 1$, there exists $\rho_+(\mathcal{I}_g)$ such that

$$u_g\big([\rho_+(\mathcal{I}_g)(n-1), \rho_+(\mathcal{I}_g)(n))\big) \geq n^2.$$

Define $\rho_- : \omega^\omega \to \mathbf{ST}$ as follows. For each $r \in \omega^\omega$, take a partition $(A_n^r : n \in \omega)$ of ω into successive finite intervals such that $|A_0^r| \geq 1$,

$$\min(A_n^r) \geq r(n), \text{ and } |A_n^r| \geq \max\{n, |A_{n-1}^r|\} \text{ for each } n \in \omega \setminus \{0\}.$$

Then define $f_r : \omega \to \mathbb{Q}_+$ by

$$f_r(k) = 1/|A_n^r| \text{ where } n \text{ such that } k \in A_n^r.$$

Let $\rho_-(r) = \mathcal{I}_{f_r}$.

We claim that

$$\forall r \in \omega^\omega\ \forall \mathcal{I}_g \in \mathbf{ST}(\rho_-(r) \leq_K \mathcal{I}_g \Rightarrow r \leq^* \rho_+(\mathcal{I}_g)).$$

Take arbitrary $r \in \omega^\omega$ and $\mathcal{I}_g \in \mathbf{ST}$ such that $\mathcal{I}_{f_r} = \rho_-(r) \leq_K \mathcal{I}_g$. By Theorem 4.1(5), there exist a map $p : \omega \to \omega$ and $C > 0$ such that

$$u_g(p^{-1}(i)) \leq C \cdot f_r(i) \text{ for all } i \in \omega.$$

By the **Remark** (above Figure 1), we may assume that p is a nondecreasing surjection and $p(i) \leq i$. Thus $\max p^{-1}(i) \geq i$ for all $i \in \omega$. Then for large enough n, there is j_n such that

$$u_g([0, j_n]) = u_g\Big(p^{-1}\Big(\bigcup_{0 \leq m \leq n} A^r_m\Big)\Big) \leq C \cdot u_{f_r}\Big(\bigcup_{0 \leq m \leq n} A^r_m\Big) = n \cdot C \leq n^2$$
$$\leq u_g\left(\left[\rho_+(\mathcal{I}_g)(n-1), \rho_+(\mathcal{I}_g)(n)\right)\right) \leq u_g\left(\left[0, \rho_+(\mathcal{I}_g)(n)\right)\right).$$

Thus $j_n \leq \rho_+(\mathcal{I}_g)(n)$ for large enough n. Then for large enough n we have that

$$r(n) \leq \max\Big(\bigcup_{0 \leq m \leq n} A^r_m\Big) \leq \max p^{-1}\Big(\bigcup_{0 \leq m \leq n} A^r_m\Big) = j_n.$$

Therefore we have that $r(n) \leq j_n \leq \rho_+(\mathcal{I}_g)(n)$ for large enough n. ⊣

THEOREM 5.6. $(\mathsf{ST}, \leq_K) \simeq_{GT} (\omega^\omega, \leq^*)$.

PROOF. Combine Lemma 5.4 with Lemma 5.5. ⊣

§6. The structure of $(\mathsf{ST}, \geq_K)$ in the sense of Galois–Tukey connection. In this section we prove that $(\mathsf{ST}, \geq_K) \simeq_{GT} (\omega^\omega, \leq^*)$. First, we prove the following:

LEMMA 6.1. $(\mathsf{ST}, \geq_K) \leq_{GT} (\omega^\omega, \leq^*)$.

PROOF. Let $\omega^{\uparrow\omega}$ be all strictly increasing functions from ω to $\omega \setminus \{0\}$. It suffices to show that $(\mathsf{ST}, \geq_K) \leq_{GT} (\omega^{\uparrow\omega}, \leq^*)$ because $(\omega^{\uparrow\omega}, \leq^*) \leq_{GT} (\omega^\omega, \leq^*)$.

Define $\rho_- : \mathsf{ST} \to \omega^{\uparrow\omega}$ as follows. For each $\mathcal{I}_f \in \mathsf{ST}$, define $\rho_-(\mathcal{I}_f) \in \omega^{\uparrow\omega}$ by $\rho_-(\mathcal{I}_f)(0) = 1$ and

$$\rho_-(\mathcal{I}_f)(k) = \min\left\{n > \rho_-(\mathcal{I}_f)(k-1) : \forall m \geq n\left(f(m) \leq \frac{1}{k}\right)\right\}$$

for all $k \geq 1$.

Define $\rho_+ : \omega^{\uparrow\omega} \to \mathsf{ST}$ as follows. For each $x \in \omega^{\uparrow\omega}$, define $F : \omega^{\uparrow\omega} \to \mathsf{F_{DST}}$ by $F(x)(k) = 1$ for all $0 \leq k < x(1)$, and

$$F(x)(k) = \frac{1}{n} \text{ where } n \text{ is such that } k \in [x(n), x(n+1)).$$

Then $\mathcal{I}_{F(x)}$ is tall for each $x \in \omega^{\uparrow\omega}$. Let $\rho_+(x) = \mathcal{I}_{F(x)}$.

We claim that

$$\forall \mathcal{I}_f \in \mathsf{ST}\, \forall x \in \omega^{\uparrow\omega}\left(\rho_-(\mathcal{I}_f) \leq^* x \Rightarrow \mathcal{I}_f \geq_K \rho_+(x)\right).$$

Let $\mathcal{I}_f \in \mathsf{ST}$ and $x \in \omega^{\uparrow\omega}$ such that $\rho_-(\mathcal{I}_f) \leq^* x$. We will show that $f \leq^* F(x)$ and then $\mathbf{id} : \omega \to \omega$ will be a witness for $\mathcal{I}_f \geq_K \mathcal{I}_{F(x)}$. To see that $f \leq^* F(x)$, take $N > 0$ such that

$$\rho_-(\mathcal{I}_f)(n) \leq x(n) \text{ for each } n \geq N.$$

Then for each $n \geq N$ and $k \in [x(n), x(n+1))$ we have $k \geq \rho_-(\mathcal{I}_f)(n)$. By the definition of $\rho_-(\mathcal{I}_f)$, we have $f(k) \leq \frac{1}{n} = F(x)(k)$. It follows that $f \leq^* F(x)$. ⊣

LEMMA 6.2. $(\mathsf{ST}, \geq_K) \geq_{GT} (\omega^\omega, \leq^*)$.

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PROOF. Define $\rho_- : \omega^\omega \to \mathbf{ST}$ as follows. For each $x \in \omega^\omega$, take a partition $\{A_n^x : n \in \omega \setminus \{0\}\}$ of ω into successive finite intervals such that for all $n > 0$:

(1) $\min A_n^x \geq \min\{x(n), n\}$ and
(2) $|A_n^x| \geq n^2(x(n)+1)$.

Then define $g_x \in \mathbf{F_{DST}}$ by

$$g_x(k) = \frac{1}{n} \text{ where } n \text{ is such that } k \in A_n^x.$$

Let $\rho_-(x) = \mathcal{I}_{g_x}$. It follows that $\rho_-(x)$ is tall for all $x \in \omega^\omega$.

Define $\rho_+ : \mathbf{ST} \to \omega^\omega$ as follows. Suppose $\mathcal{I}_f \in \mathbf{ST}$. For each $n > 0$, define $\rho_+(\mathcal{I}_f)$ by $\rho_+(\mathcal{I}_f)(0) = 0$ and $\rho_+(\mathcal{I}_f)(n) =$

$$\min\left\{m \geq n+1 : m \geq u_f\big([0, \rho_+(\mathcal{I}_f)(n-1)]\big) \;\&\; \forall k \geq m\left(f(k) \leq \frac{1}{(n+1)^2}\right)\right\}.$$

We claim that

$$\forall \mathcal{I}_f \in \mathbf{ST}\; \forall x \in \omega^\omega (\rho_-(x) \geq_K \mathcal{I}_f \Rightarrow x \leq^* \rho_+(\mathcal{I}_f)).$$

Let $\mathcal{I}_f \in \mathbf{ST}$ and $x \in \omega^\omega$ such that $x \not\leq^* \rho_+(\mathcal{I}_f)$. Let $\rho_-(x) = \mathcal{I}_{g_x}$. We will show for each $M > 0$, there are $l > M$, $k_1 > k_0 \geq M$ such that:

(3) $u_{g_x}([k_0, k_1]) > M \cdot u_f([0, l])$ and
(4) $g_x(k_1) > M \cdot f(l)$.

Then, $\rho_-(x) \not\geq_K \mathcal{I}_f$ follows from Theorem 4.1. To prove (3) and (4), fix $M > 0$ and let $n_0 > M$ be such that $x(n_0) > \rho_+(\mathcal{I}_f)(n_0)$. Define $l = \rho_+(\mathcal{I}_f)(n_0 - 1)$ and $k_0 < k_1$ such that $[k_0, k_1] = A_{n_0}^x$. It follows that $l > M$, $k_1 > k_0 \geq M$ and

$$u_{g_x}([k_0, k_1]) \geq n_0 \cdot (x(n_0) + 1) > n_0 \cdot x(n_0) > M \cdot u_f([0, l]).$$

Thus (3) holds. By the definition of l, we have that $f(l) \leq \frac{1}{n_0^2}$ and

$$g_x(k_1) = \frac{1}{n_0} = n_0 \cdot \frac{1}{n_0^2} > M \cdot f(l).$$

Thus (4) holds. ⊣

THEOREM 6.3. $(\mathbf{ST}, \geq_K) \simeq_{GT} (\omega^\omega, \leq^*)$.

PROOF. Use Lemmas 6.1 and 6.2. ⊣

§7. $\simeq_K$ on $\mathbf{F_{DST}}$ is Borel bireducible to l_∞. In this section we will prove that l_∞ is Borel bireducible to $\simeq_K$ on $\mathbf{F_{DST}}$.

DEFINITION 7.1. (1) Let $C = \{(A_n) \in \mathcal{P}(\omega)^\omega : \forall n (A_n \subseteq A_{n+1})\}$ and for each $(A_n), (B_n) \in C$,

$$(A_n) H (B_n) \Longleftrightarrow \exists n \forall m (A_m \subseteq B_{n+m} \wedge B_m \subseteq A_{n+m}).$$

(2) Let $X_0 = \prod_{n<\omega} n$, where $n = \{0, 1, \ldots, n-1\}$. For each $\alpha, \beta \in X_0$, define

$$\alpha E_{K_\sigma} \beta \Longleftrightarrow \exists n \forall m (|\alpha(m) - \beta(m)| \leq n).$$

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It is proved in [6, Proposition 19] that $H \simeq_B l_\infty \simeq_B E_{K_\sigma}$, so it suffices to prove that $\simeq_K \leq_B H$ and $E_{K_\sigma} \leq_B \simeq_K$.

7.1. The proof of $\simeq_K \leq_B H$. This will be proved in Corollary 7.10. The proof consists of two steps. We first show that the so-called decomposable equivalence relations are all Borel reducible to H (Theorem 7.5). Then we prove that $\simeq_K$ is decomposable (Theorem 7.9). Before this, we need some preparations.

LEMMA 7.2. *$\simeq_K$ is an F_σ subset of $\mathsf{F}^2_{\mathsf{DST}}$.*

PROOF. Denote $\preceq_K = \{(f,g) \in \mathsf{F}^2_{\mathsf{DST}} : \mathcal{I}_f \leq_K \mathcal{I}_g\}$. By Theorem 4.1(6), $\preceq_K =$

$$\{(f,g) : \exists n \in \omega \setminus \{0\}, \forall k \in \omega, \forall l \in \omega \big[u_g([0,k]) \leq n \cdot u_f([0,l]) \text{ or } g(k) \leq n \cdot f(l)\big]\}$$

$$= \bigcup_{n \in \omega \setminus \{0\}} \bigcap_{k \in \omega} \bigcap_{l \in \omega} \big[\{(f,g) : u_g([0,k]) \leq n \cdot u_f([0,l])\} \cup \{(f,g) : g(k) \leq n \cdot f(l)\}\big].$$

For each $n > 1$, let

$$F_n = \bigcap_{k \in \omega} \bigcap_{l \in \omega} \big[\{(f,g) : u_g([0,k]) \leq n \cdot u_f([0,l])\} \cup \{(f,g) : g(k) \leq n \cdot f(l)\}\big].$$

Then F_n is closed. Thus $\preceq_K$ is F_σ.

Denote $\succeq_K = \{(f,g) : \mathcal{I}_f \geq_K \mathcal{I}_g\}$. Similarly we can prove that $\succeq_K$ is F_σ. It follows that $\simeq_K = \preceq_K \cap \succeq_K$ is F_σ. ⊣

We need the following characterization of $\simeq_K$.

LEMMA 7.3. *Let $f, g \in \mathsf{F}_{\mathsf{DST}}$. Then $f \simeq_K g$ if and only if there exists $n > 0$ such that for each $k \in \omega$ we have that*

$$\frac{f(l_k)}{n} \leq g(k) \leq n \cdot f(l'_k),$$

where l_k, l'_k are such that

$$\frac{u_f([0, l_k - 1]])}{n} \leq u_g([0,k]) < \frac{u_f([0, l_k])}{n}$$

and

$$n \cdot u_f([0, l'_k]) < u_g([0,k]) \leq n \cdot u_f([0, l'_k + 1]).$$

PROOF. $(\Rightarrow)$: Recall that $f \simeq_K g$ means $\mathcal{I}_f \leq_K \mathcal{I}_g$ and $\mathcal{I}_g \leq_K \mathcal{I}_f$. By Theorem 4.1(6), there exists M_1 such that for all k and l' we have that

$$u_g([0,k]) > M_1 \cdot u_f([0,l']) \Rightarrow g(k) \leq M_1 \cdot f(l').$$

For the same reason, there exists M_2 such that for all k and l we have that

$$u_f([0,l]) > M_2 \cdot u_g([0,k]) \Rightarrow f(l) \leq M_2 \cdot g(k).$$

Let $n = \max\{M_1, M_2\}$. Then we have that

$$u_g([0,k]) > n \cdot u_f([0,l']) \geq M_1 \cdot u_f([0,l']) \Rightarrow g(k) \leq M_1 \cdot f(l') \leq n \cdot f(l')$$

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and

$$u_f([0,l]) > n \cdot u_g([0,k]) \geq M_2 \cdot u_g([0,k]) \Rightarrow f(l) \leq M_2 \cdot g(k) \leq n \cdot g(k).$$

Define

$$l_k = \min\left\{l \in \omega : u_g([0,k]) < \frac{u_f([0,l])}{n}\right\}$$

and

$$l'_k = \max\left\{l' \in \omega : u_g([0,k]) > n \cdot u_f([0,l'])\right\}.$$

We have that

$$n \cdot u_f([0,l'_k]) < u_g([0,k]) \leq n \cdot u_f([0,l'_k+1])$$

and

$$\frac{u_f([0,l_k-1]])}{n} \leq u_g([0,k]) < \frac{u_f([0,l_k])}{n}.$$

It follows that

$$\frac{f(l_k)}{n} \leq g(k) \leq n \cdot f(l'_k).$$

($\Leftarrow$): For each $k \in \omega$, we have $l_k = \min\{l \in \omega : u_g([0,k]) < \frac{u_f([0,l])}{n}\}$. For each $l \in \omega$ we have that

$$u_g([0,k]) < \frac{u_f([0,l])}{n} \Rightarrow l \geq l_k.$$

For each $l \geq l_k$ we have that

$$\frac{f(l)}{n} \leq \frac{f(l_k)}{n} \leq g(k).$$

It follows that for each $k \in \omega$

$$u_f([0,l]) > n \cdot u_g([0,k]) \Rightarrow f(l) \leq n \cdot g(k).$$

By Theorem 4.1(6), we have $\mathcal{I}_g \leq_K \mathcal{I}_f$.

$\mathcal{I}_f \leq_K \mathcal{I}_g$ can be proved in a similar way. ⊣

Now we define decomposable equivalence relations.

DEFINITION 7.4. Let F be a F_σ equivalence relation on Borel space X. We call F is *decomposable* on X if there is a sequence $\{F_n : n \in \omega\}$ of closed subsets of X^2 such that:

(1) For each $n < \omega$, $F_n \subseteq F_{n+1}$ and $F_n \circ F_n \subseteq F_{n+1}$ (i.e., $xF_ny \wedge yF_nz \Rightarrow xF_{n+1}z$).
(2) $F = \bigcup_{n\in\omega} F_n$.
(3) $[U]_n = \{x \in X : \exists z \in U(zF_nx)\}$ is Borel for each open subset U of X and $n \in \omega$.

THEOREM 7.5. *Let F be an F_σ equivalence relation such that F is decomposable on Borel space X. Then $F \leq_B H$.*

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PROOF. Let $\{F_n : n < \omega\}$ be a sequence which witnesses that F is decomposable. Fix a basis $\{U_n : n \in \omega\}$ of X. For each $n \in \omega$, define a function $f_n : X \to \mathcal{P}(\omega)$ by

$$f_n(x) = \{k \in \omega : \exists z \in U_k(zF_nx)\}.$$

By (3) of Definition 7.4, for each $m \in \omega$ we have that

$$f_n^{-1}(\{A \subseteq \omega : m \in A\}) = [U_m]_n$$

is Borel. It follows that f_n is a Borel function for each $n \in \omega$. By (1) of Definition 7.4, we have that

$$f_n(x) \subseteq f_{n+1}(x) \text{ for each } n \in \omega \text{ and } x \in X.$$

We prove that $\Phi : x \mapsto (f_n(x))$ is a Borel reduction from F to H. Φ is a Borel map by the following: For any open subset $\prod_{n\in\omega} \mathcal{U}_n$ of C, we have

$$\Phi^{-1}\left(\prod_{n\in\omega} \mathcal{U}_n\right) = \bigcap_{n\in\omega} f_n^{-1}(\mathcal{U}_n).$$

It follows that Φ is Borel by f_n being Borel for all $n \in \omega$.

Then we show that Φ is a reduction from F to H. Let $x, y \in X$ such that xFy. Then there exists $n \in \omega$ such that xF_ny. Therefore, for any $z \in X$ such that zF_mx for some $m \in \omega$, we have that

$$zF_{\max\{n,m\}+1}y.$$

It follows that $f_m(x) \subseteq f_{n+1+m}(y)$ for all $m \in \omega$. Similarly, there exists n' such that $f_m(y) \subseteq f_{n'+1+m}(x)$ for all $m \in \omega$. Let $N = \max\{n + 1, n' + 1\}$. We have that

$$\forall m \in \omega(f_m(x) \subseteq f_{N+m}(y) \ \wedge \ f_m(y) \subseteq f_{N+m}(x)).$$

Conversely, let $x, y \in X$ such that $(f_n(x))H(f_n(y))$. Then there exists $n \in \omega$ such that

$$f_m(x) \subseteq f_{n+m}(y) \text{ for all } m \in \omega.$$

Fix n as above. For each $m \in \omega$, define $F_m^x = \{z : zF_mx\}$. Then for each $k \in \omega$ we have that

$$U_k \cap F_m^x \neq \emptyset \implies k \in f_m(x) \implies k \in f_{n+m}(y) \implies U_k \cap F_{n+m}^y \neq \emptyset.$$

Since F_{n+m}^y is closed, we have $F_m^x \subseteq F_{n+m}^y$. Take m large enough such that xF_mx, then we have that

$$xF_mx \Rightarrow x \in F_m^x \Rightarrow x \in F_{n+m}^y \Rightarrow xF_{n+m}y.$$

It follows that xFy. ⊣

Next, we will show that $\simeq_K$ is decomposable. We need some observations.

DEFINITION 7.6. For each $n \in \omega$, define R_n, S_n, E_n, and F_n on $\mathbf{F_{DST}}$ as follows:

(1) fR_ng if and only if there exists an interval-to-one map $p : \omega \to \omega$ such that $u_g\left(p^{-1}(i)\right) \leq n \cdot f(i)$ for all $i \in \omega$.

(2) fS_ng if and only if for all $k, l \in \omega$, $u_g([0, k]) > n \cdot u_f([0, l])$ implies $g(k) \leq n \cdot f(l)$.

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(3) fE_ng if and only if fR_ng and gR_nf.
(4) fF_ng if and only if fS_ng and gS_nf.

LEMMA 7.7. *Let $f,g,h \in \mathsf{F}_{\mathsf{DST}}$. For each pair $n \leq m$ we have follows*:

(1) $fR_ng \Rightarrow fR_mg$; $fS_ng \Rightarrow fS_mg$; $fE_ng \Rightarrow fE_mg$; $fF_ng \Rightarrow fF_mg$.
(2) $fR_ng \Rightarrow fS_ng$; $fS_ng \Rightarrow fR_{2n}g$.
(3) $fR_ng \wedge gR_nh \Rightarrow fR_{n^2}h$; $fE_ng \wedge gE_nh \Rightarrow fE_{n^2}h$.
(4) $fS_ng \wedge gS_nh \Rightarrow fS_{4n^2}h$; $fF_ng \wedge gF_nh \Rightarrow fF_{4n^2}h$.

PROOF. (1): The proof is obvious.

(2): Use the proof of Theorem 4.1 (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (5).

(3): Let $f,g,h \in \mathsf{F}_{\mathsf{DST}}$ such that fR_ng and gR_nh. Then there exist p_1 and p_2 such that

$$u_g(p_1^{-1}(i)) \leq n \cdot f(i) \text{ and } u_h(p_2^{-1}(i)) \leq n \cdot g(i) \text{ for all } i \in \omega.$$

Then

$$u_h((p_1 \circ p_2)^{-1}(i)) = u_h(p_2^{-1}(p_1^{-1}(i))) \leq n \cdot u_g(p_1^{-1}(i)) \leq n^2 \cdot f(i)$$

for all $i \in \omega$. It follows that $fR_{n^2}h$.

Similarly, we can prove that $fE_ng, gE_nh \Rightarrow fE_{n^2}h$.

(4): By (2) we have that $fS_ng \Rightarrow fR_{2n}g$ and $gS_nh \Rightarrow gR_{2n}h$. Then by (3) we have that $fR_{2n}g \wedge gR_{2n}h \Rightarrow fR_{4n^2}h \Rightarrow fS_{4n^2}h$. ⊣

LEMMA 7.8. *$[U]_n = \{f \in \mathsf{F}_{\mathsf{DST}} : \exists g \in U(fF_ng)\}$ is Borel for every open subset U of $\mathsf{F}_{\mathsf{DST}}$ and $n \geq 1$.*

PROOF. Fix $n \geq 1$. Without loss of generality, assume U is the form of $(\prod_{i<m}(p_i,q_i) \times \prod_{i \geq m} \mathbb{Q}_+) \cap \mathsf{F}_{\mathsf{DST}}$ for some $m \in \omega$, where $0 \leq p_i < q_i \in \mathbb{Q}_+$ for each $i < m$. Fix m as above. Denote

$$S = \{s \in \mathbb{Q}_+^m : \forall i < m-1\big(s(i) \geq s(i+1)\big) \ \wedge \ \forall i < m\big(p_i < s(i) < q_i\big)\}.$$

For each $s \in S$, let T_s be the set of all $t \in \mathbb{Q}_+^{<\omega}$ such that:

(1) For each $l < |t|-1$, $t(l) \geq t(l+1)$.
(2) $\frac{u_t([0,|t|-2])}{n} \leq u_s([0,m-1]) < \frac{u_t([0,|t|-1])}{n}$.
(3) For each $l' < |t|$ and $k < m$,

$$u_s([0,k]) > n \cdot u_t([0,l']) \Rightarrow s(k) \leq n \cdot t(l').$$

(4) For each $l < |t|$ and $k < m$,

$$u_s([0,k]) < \frac{u_t([0,l])}{n} \Rightarrow s(k) \geq \frac{t(l)}{n}.$$

Claim. $[U]_n = \bigcup_{s \in S} \bigcup_{t \in T_s} \Big\{f \in \mathsf{F}_{\mathsf{DST}} : f\big|_{|t|} = t\Big\}$.

PROOF. ($\subseteq$): For each $f \in [U]_n$ there exists $g \in U$ such that fF_ng. Then $g\big|_m = s \in S$. Define l_s by

$$l_s = \min\{l > 1 : u_f([0,l-1]) > n \cdot u_s([0,m-1])\}.$$

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Let $t = f\big|_{l_s}$. It is easy to see that t satisfies (1) by $f \in \mathsf{F_{DST}}$. (2) follows from the definition of l_s. By $f F_n g$ we have (3) and (4). It follows that $t \in T_s$.

($\supseteq$): Let $s \in S$ and $t \in T_s$ and $f \in \mathsf{F_{DST}}$ such that $f\big|_{|t|} = t$. Let $l = |t|$. Then we have that

$$\frac{u_f([0, l-2])}{n} \leq u_s([0, m-1]) < \frac{u_f([0, l-1])}{n}.$$

We will find g extending s such that $g \in U$ and $f F_n g$. It suffices to construct a sequence $\{g(i) \in \mathbb{Q}_+ : i < \omega\}$ such that:

(5) $g(i) = s(i)$ for each $i < m$ and $g(i-1) \geq g(i)$ for each $i \geq m$.

(6) For each $i \geq m$,

$$\frac{u_f([0, l-2+i-m])}{n} \leq u_g([0, i-1]) < \frac{u_f([0, l-1+i-m])}{n}$$

and $g(i-1) \geq \frac{f(l-1+i-m)}{n}$.

(7) For each $i \geq m$, if

$$n \cdot u_f([0, l']) < u_g([0, i-1]) \leq n \cdot u_f([0, l'+1]),$$

then $l' < l-1+i-m$ and $g(i-1) \leq n \cdot f(l')$.

(8) $\lim\limits_{n\to\infty} g(n) = 0$.

Then (5) $\Rightarrow g \in U$, (3)(5)(7) $\Rightarrow f S_n g$, and (4)–(6) $\Rightarrow g S_n f$.

Suppose we have already constructed $\{g(i) : i < j\}$ such that (5)–(7) hold for each $i < j$. Let

$$\varepsilon_j = \frac{u_f([0, l-1+j-m])}{n} - u_g([0, j-1]).$$

Define

$$g(j) = \max\{\varepsilon_j, \frac{f(l+j-m)}{n}\}.$$

By (6) for $j-1$, we have

$$\varepsilon_j \leq \frac{f(l-1+j-m)}{n} \leq g(j-1).$$

It follows that $g(j) \leq g(j-1)$ and $g(j)$ satisfies (5).

By $g(j) = \max\{\varepsilon_j, \frac{f(l+j-m)}{n}\}$, i.e.,

$$\frac{f(l+j-m)}{n} \leq g(j) \text{ and } \varepsilon_j \leq g(j) < \varepsilon_j + \frac{f(l+j-m)}{n},$$

we have that

$$\begin{aligned}\frac{u_f([0, l-1+j-m])}{n} \leq u_g([0, j]) &< \frac{u_f([0, l-1+j-m])}{n} + \frac{f(l+j-m)}{n} \\ &= \frac{u_f([0, l+j-m])}{n}.\end{aligned}$$

It is follows that $g(j)$ satisfies (6).

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Assume that

$$n \cdot u_f([0, l']) < u_g([0, j]) \leq n \cdot u_f([0, l' + 1]).$$

By $n \geq 1$ and

$$u_g([0, j]) < \frac{u_f([0, l + j - m])}{n} \leq n \cdot u_f([0, l + j - m]),$$

we have that $l' < l + j - m$ and

$$g(j) \leq \frac{f(l - 1 + j - m)}{n} \leq n \cdot f(l - 1 + j - m) \leq n \cdot f(l').$$

It is follows that $g(j)$ satisfies (7).

(8) follows from (7) for $j \geq m$. ⊣

By the **Claim** above, we have that $[U]_n$ is Borel.

THEOREM 7.9. *$\simeq_K$ is decomposable on a Borel space* $\mathbf{F_{DST}}$.

PROOF. $\simeq_K$ is decomposable which is witnessed by $\mathbf{F_{DST}}$ being Borel and $\{F_{4n^2} : n \in \omega\}$ from Definition 7.6. We show that $\mathbf{F_{DST}}$ is a Borel subset of $\mathbb{Q}_+^\omega$. Recall that

$$f \in \mathbf{F_{DST}} \Leftrightarrow (\sum_{n=0}^{\infty} f(n) = +\infty) \wedge (\lim_{n \to \infty} f(n) = 0) \wedge (\forall n \in \omega (f(n) \geq f(n+1)).$$

Define

$$A = \{f \in \mathbb{Q}_+^\omega : \sum_{n=0}^{\infty} f(n) = +\infty\},$$
$$B = \{f \in \mathbb{Q}_+^\omega : \lim_{n \to \infty} f(n) = 0\}, \text{ and}$$
$$C_n = \{f \in \mathbb{Q}_+^\omega : f(n) \geq f(n+1)\} \text{ for each } n \in \omega.$$

We have

$$f \in A \Leftrightarrow \forall M \in \omega \, \exists N \in \omega \left(\sum_{n=0}^{N} f(n) \geq M \right)$$

and

$$f \in B \Leftrightarrow \forall m \in \omega \, \exists N \in \omega \, \forall n \geq N \left(f(n) < \frac{1}{m} \right).$$

Thus A and B are Borel.

Obviously, C_n is Borel for each $n \in \omega$. It follows that $\mathbf{F_{DST}}$ is Borel by $\mathbf{F_{DST}} = A \cap B \cap (\bigcap_{n \in \omega} C_n)$. ⊣

COROLLARY 7.10. $\simeq_K \leq_B H$.

PROOF. Use Theorems 7.5 and 7.9. ⊣

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7.2. The proof of $E_{K_\sigma} \leq_B \simeq_K$. Now we turn to the proof of $E_{K_\sigma} \leq_B \simeq_K$.

THEOREM 7.11. $(X_0, E_{K_\sigma}) \leq_B (\mathsf{F_{DST}}, \simeq_K)$.

PROOF. First, we define a map $\Phi : X_0 \to \mathsf{F_{DST}}$ as follows. Take $a_0 = 1$ and $a_{n+1} = 2^n \sum_{i=0}^{n} a_i$ for each $n < \omega$. For $\alpha \in X_0$, define a sequence $\{c_n^\alpha : n \geq 1\}$ and $f_\alpha \in \mathsf{F_{DST}}$ as follows. For each $n \geq 1$:

(1) $c_1^\alpha = 1$;

(2) $\left|[c_n^\alpha, c_{n+1}^\alpha)\right| = a_n \cdot 2^{\frac{n(n-1)}{2} + \alpha(n)}$;

(3) $f_\alpha(j) = 2^{-\frac{n(n-1)}{2} - \alpha(n)}$ for each $j \in [c_n^\alpha, c_{n+1}^\alpha)$.

Then f_α is constant on every $[c_n^\alpha, c_{n+1}^\alpha)$ and $u_{f_\alpha}([c_n^\alpha, c_{n+1}^\alpha)) = a_n$ for each $n \geq 1$. Let $\Phi(\alpha) = f_\alpha$. We will show that Φ is Borel. Take a basic open subset V of $\Phi[X_0]$, i.e., there exists $\mathcal{A} \in [\mathbb{Q}_+^{<\omega}]^\omega$ such that $V = \bigcup_{s \in \mathcal{A}} V_s$ and $V_s = [s]$[1] for all $s \in \mathcal{A}$. Fix $s \in \mathcal{A}$. Then there exist $\alpha \in X_0$ and $m \in \omega$ such that $s = f_\alpha|_{[0,m]}$. Let $n \geq 1$ be such that $m \in [c_n^\alpha, c_{n+1}^\alpha)$. Then we have that

$$\Phi^{-1}(V_s) = \{\beta \in X_0 : \beta(j) = \alpha(j), j \leq n\} \text{ is open.}$$

It follows that $\Phi^{-1}(V) = \bigcup_{s \in \mathcal{A}} \Phi^{-1}(V_s)$ is open. Therefore Φ is continuous, hence Borel.

We claim that

$$\forall \alpha, \beta \in X_0 (\alpha E_{K_\sigma} \beta \Leftrightarrow \Phi(\alpha) \simeq_K \Phi(\beta)).$$

($\Rightarrow$): We will find $n \in \omega$ such that for each $k \in \omega$,

$$\frac{\Phi(\beta)(l_k)}{n} \leq \Phi(\alpha)(k) \leq n \cdot \Phi(\beta)(l_k'),$$

where l_k is such that

$$\frac{u_{\Phi(\beta)}([0, l_k - 1]])}{n} \leq u_{\Phi(\alpha)}([0, k]) < \frac{u_{\Phi(\beta)}([0, l_k])}{n},$$

and l_k' is such that

$$n \cdot u_{\Phi(\beta)}([0, l_k']) < u_{\Phi(\alpha)}([0, k]) \leq n \cdot u_{\Phi(\beta)}([0, l_k' + 1]).$$

Then $\Phi(\alpha) \simeq_K \Phi(\beta)$ by Lemma 7.3.

By $\alpha E_{K_\sigma} \beta$, there exists N such that $|\alpha(m) - \beta(m)| \leq N$ for $m \geq 1$. Let $n = 2^N$. For each $k \in \omega$, take l_k such that

$$\frac{u_{\Phi(\beta)}([0, l_k - 1]])}{n} \leq u_{\Phi(\alpha)}([0, k]) < \frac{u_{\Phi(\beta)}([0, l_k])}{n}.$$

Take n_k such that $k \in [c_{n_k}^\alpha, c_{n_k+1}^\alpha)$. We have that

$$u_{\Phi(\alpha)}([0, c_{n_k}^\alpha)) = u_{\Phi(\beta)}([0, c_{n_k}^\beta)) = \sum_{i=1}^{n_k-1} a_i.$$

[1]For $s \in \mathbb{Q}_+^{<\omega}$, $[s] = \{f \in \mathbb{Q}_+^\omega : s \sqsubseteq f\}$.

Then we have that

$$\frac{u_{\Phi(\beta)}([0, c^{\beta}_{n_k}))}{n} = \frac{u_{\Phi(\alpha)}([0, c^{\alpha}_{n_k}))}{n} < u_{\Phi(\alpha)}([0, k)) < \frac{u_{\Phi(\beta)}([0, l_k])}{n}.$$

It follows that

$$l_k > c^{\beta}_{n_k} \text{ and } \Phi(\beta)(l_k) \leq 2^{-\frac{n_k(n_k-1)}{2} - \beta(n_k)}.$$

By

$$\Phi(\alpha)(k) = 2^{-\frac{n_k(n_k-1)}{2} - \alpha(n_k)}$$

and $\alpha(n_k) \leq \beta(n_k) + N$, we have that

$$\Phi(\alpha)(k) \geq 2^{-\frac{n_k(n_k-1)}{2} - \beta(n_k) - N} \geq \frac{\Phi(\beta)(l_k)}{2^N}.$$

Take l'_k such that

$$n \cdot u_{\Phi(\beta)}([0, l'_k]) < u_{\Phi(\alpha)}([0, k]) \leq n \cdot u_{\Phi(\beta)}([0, l'_k + 1]).$$

Then we have that

$$n \cdot u_{\Phi(\beta)}([0, l'_k]) < u_{\Phi(\alpha)}([0, k]) \leq n \cdot u_{\Phi(\alpha)}([0, c^{\alpha}_{n_k+1})) = n \cdot u_{\Phi(\beta)}([0, c^{\beta}_{n_k+1})).$$

It follows that

$$l'_k < c^{\beta}_{n_k+1} \text{ and } \Phi(\beta)(l'_k) \geq 2^{-\frac{n_k(n_k-1)}{2} - \beta(n_k)}.$$

By

$$\Phi(\alpha)(k) = 2^{-\frac{n_k(n_k-1)}{2} - \alpha(n_k)}$$

and $-\alpha(n_k) \leq -\beta(n_k) + N$, we have that

$$\Phi(\alpha)(k) \leq 2^{-\frac{n_k(n_k-1)}{2} - \beta(n_k) + N} \leq 2^N \cdot \Phi(\beta)(l'_k).$$

Then by $n = 2^N$ we have that

$$\frac{\Phi(\beta)(l_k)}{n} \leq \Phi(\alpha)(k) \leq n \cdot \Phi(\beta)(l'_k).$$

($\Leftarrow$): Let $\alpha, \beta \in X_0$ such that $(\alpha, \beta) \notin E_{K_\sigma}$. We will show $\Phi(\alpha) \not\approx_K \Phi(\beta)$. By $(\alpha, \beta) \notin E_{K_\sigma}$, for each $N > 0$ there exists $m_N > N$ such that

$$|\alpha(m_N) - \beta(m_N)| > N.$$

Fix N. Take $k_N = c^{\alpha}_{m_N+1} - 1$ and $l_N = c^{\beta}_{m_N}$. Then

$$\Phi(\alpha)(k_N) = 2^{-\frac{m_N(m_N-1)}{2} - \alpha(m_N)}$$

and

$$\Phi(\beta)(l_N) = 2^{-\frac{m_N(m_N-1)}{2} - \beta(m_N)}.$$

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Assume $\beta(m_N) > \alpha(m_N) + N$. Thus

$$\Phi(\alpha)(k_N) > 2^N \cdot \Phi(\beta)(l_N).$$

By

$$u_{\Phi(\alpha)}([0, k_N]) = \sum_{i=1}^{m_N} a_i \geq 2^{m_N} \cdot \sum_{i=1}^{m_N-1} a_i + \sum_{i=1}^{m_N-1} a_i$$

and

$$u_{\Phi(\beta)}([0, l_N]) = \sum_{i=1}^{m_N-1} a_i + \Phi(\beta)(l_N) \leq \sum_{i=1}^{m_N-1} a_i + 2^{-N},$$

we have that

$$u_{\Phi(\alpha)}([0, k_N]) > 2^N \cdot u_{\Phi(\beta)}([0, l_N]).$$

Without loss of generality, we can assume that there exists an infinite set $\{N_i : i \in \omega\}$ such that for each i, $\beta(m_{N_i}) > \alpha(m_{N_i}) + N_i$. Then for all $0 < M < \omega$, there exists $i \in \omega$ such that $M \leq 2^{N_i}$. It follows that there exist k_{N_i} and l_{N_i} such that

$$u_{\Phi(\alpha)}([0, k_{N_i}]) > 2^{N_i} \cdot u_{\Phi(\beta)}([0, l_{N_i}]) \text{ and } \Phi(\alpha)(k_{N_i}) > 2^{N_i} \cdot \Phi(\beta)(l_{N_i}).$$

By $M \leq 2^{N_i}$,

$$u_{\Phi(\alpha)}([0, k_{N_i}]) > M \cdot u_{\Phi(\beta)}([0, l_{N_i}]) \text{ and } \Phi(\alpha)(k_{N_i}) > M\Phi(\beta)(l_{N_i}).$$

It follows that $\Phi(\alpha) \not\simeq_K \Phi(\beta)$ by Theorem 4.1(6). ⊣

THEOREM 7.12. *$\simeq_K$ on $\mathsf{F}_{\mathsf{DST}}$ is Borel bireducible to l_∞.*

PROOF. Use Corollary 7.10, Theorem 7.11, and $H \leq_B l_\infty \leq_B E_{K_\sigma}$. ⊣

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