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GENERALIZED RICCI FLOW ON ALIGNED HOMOGENEOUS SPACES

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Abstract The fixed points of the generalized Ricci flow are the Bismut Ricci flat (BRF) metrics, i.e., a generalized metric (g, H) on a manifold M, where g is a Riemannian metric and H a closed 3-form, such that H is g-harmonic and $\mathrm{Rc}(g) = \frac{1}{4}H^2_g$. Given two standard Einstein homogeneous spaces G_i/K, where each G_i is a compact simple Lie group and K is a closed subgroup of them holding some extra assumption, we consider $M = G_1 \times G_2/\Delta K$. Recently, Lauret and Will proved the existence of a BRF metric on any of these spaces. We proved that this metric is always asymptotically stable for the generalized Ricci flow on M among a subset of G-invariant metrics and, if $G_1 = G_2$, then it is globally stable.

Keywords: homogeneous spaces; Bismut Ricci flat metrics; generalized Ricci flow; stability

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1. Introduction

In the context of generalized Riemannian geometry has arisen an extension of the Ricci flow equation called *generalized Ricci flow*. Given a manifold M and a generalized metric encoded in the pair (g, H), where g is a Riemannian metric and H a closed 3-form on M, this flow studied in [5] is given by

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\,\mathrm{Rc}(g(t)) + \frac{1}{2}(H(t))^2_{g(t)}, \\ \\ \frac{\partial}{\partial t} H(t) = -dd^*_{g(t)} H(t), \end{cases} \tag{1}$$

where $H^2_g := g(\iota.H, \iota.H)$ and d^*_g is the adjoint of d with respect to g.

A pair (g, H) is naturally associated with *Bismut* connections, i.e., the unique metric connection on the Riemannian manifold (M, g) with torsion equal to H, in this sense the generalized Ricci flow is the natural evolution in the direction of the Ricci tensor of this connection.

In search of canonical generalized geometry, García-Fernandez and Streets in [5] proposed to look for the fixed points of this flow, the so-called *Bismut Ricci flat*

(BRF for short) generalized metrics or generalized Einstein metrics, i.e., (g, H) such that

$$\mathrm{Rc}(g) = \tfrac{1}{4}H_g^2 \quad \text{and} \quad H \text{ is } g\text{-harmonic}.$$

Let G_1, G_2 be compact, connected, simple Lie groups and $K \subset G_1, G_2$ a closed Lie subgroup. Suppose that for $i = 1, 2$, there exist constants $0 < a_1 \leq a_2 < 1$ such that $\mathrm{B}_{\mathfrak{k}} = a_i \mathrm{B}_{\mathfrak{g}_i}|_{\mathfrak{k}}$, where $\mathfrak{k}, \mathfrak{g}_i$ are the Lie algebras of K and G_i, respectively, and $\mathrm{B}_{\mathfrak{h}}$ is the Killing form of the Lie algebra $\mathfrak{h}$. The homogeneous space defined by $M = G_1 \times G_2/\Delta K$ is aligned with $(c_1, c_2) = (\frac{a_1+a_2}{a_2}, \frac{a_1+a_2}{a_1})$ and $\lambda_1 = \cdots = \lambda_t = \frac{a_1 a_2}{a_1+a_2}$, as defined in [8], and so its third Betti number is one. Recently in [7], Lauret and Will found a BRF G-invariant generalized metric on any aligned homogeneous space $M = G/K$, where G is a compact semisimple Lie group with two simple factors generalizing results obtained in [13, 14].

If $G := G_1 \times G_2$, then we consider its Killing metric given by

$$g_{\mathrm{B}} = (-\mathrm{B}_{\mathfrak{g}_1}) + (-\mathrm{B}_{\mathfrak{g}_2}), \tag{2}$$

and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the g_{B}-orthogonal reductive decomposition of $\mathfrak{g}$. We fix the standard metric of $M = G_1 \times G_2/\Delta K$ determined by $g_{\mathrm{B}}|_{\mathfrak{p}\times\mathfrak{p}}$ as a background metric and consider the g_{B}-orthogonal $\mathrm{Ad}(K)$-invariant decomposition

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3,$$

where each $\mathfrak{p}_i$ is equivalent to the isotropy representation of the homogeneous space $M_i = G_i/K$ for $i = 1, 2$ and $\mathfrak{p}_3$ is equivalent to the adjoint representation $\mathfrak{k}$ (see [8, Proposition 5.1]).

Following [8], the closed 3-form H_0 on M given by

$$H_0(X, Y, Z) := Q([X, Y], Z) + Q([X, Y]_{\mathfrak{k}}, Z) - Q([X, Z]_{\mathfrak{k}}, Y) + Q([Y, Z]_{\mathfrak{k}}, X) \quad \text{for all } X, Y, Z \in \mathfrak{p},$$

where $Q = \mathrm{B}_{\mathfrak{g}_1} - \frac{a_2}{a_1}\mathrm{B}_{\mathfrak{g}_2}$, is g-harmonic for any G-invariant metric $g = (x_1, x_2, x_3)_{g_{\mathrm{B}}}$ of the form:

$$g := x_1 g_{\mathrm{B}}|_{\mathfrak{p}_1\times\mathfrak{p}_1} + x_2 g_{\mathrm{B}}|_{\mathfrak{p}_2\times\mathfrak{p}_2} + x_3 g_{\mathrm{B}}|_{\mathfrak{p}_3\times\mathfrak{p}_3}, \qquad x_1, x_2, x_3 > 0, \tag{3}$$

these metrics will be called *diagonal*.

The G-invariant BRF generalized metric found in [7] is given, up to scaling, by (g_0, H_0) (see [10, Remark A.3]), where

$$g_0 := \left(1, \frac{a_2}{a_1}, \frac{a_1+a_2}{a_1}\right)_{g_{\mathrm{B}}}. \tag{4}$$

In this paper, we study the dynamical stability of this generalized metric as a fixed point of the generalized Ricci flow given in Equation 1 on homogeneous spaces of the form

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$M = G_1 \times G_2/\Delta K$ as above, such that the standard metric on $M_i = G_i/K$ is Einstein for $i = 1, 2$. Note that since each G_i is simple, the spaces M_i are given in [1] (see also [6] and [9]), there are 17 families and 50 isolated examples among irreducible symmetric, isotropy irreducible and non-isotropy irreducible homogeneous spaces. Note that we can consider $G_1 = G_2 = H$, in this case $a_1 = a_2$. Our main result is the following.

Theorem 1.1. *Let $M = G_1 \times G_2/\Delta K$ be a homogeneous space as above such that $\mathrm{B}_{\mathfrak{k}} = a_i \mathrm{B}_{\mathfrak{g}_i}|_{\mathfrak{k}}$ and $(M_i = G_i/K, g^i_{\mathrm{B}})$ is Einstein, where g^i_{B} is the standard metric on each M_i for $i = 1, 2$, then*

(i) *There exists a neighbourhood U of the metric $g_0 = (1, \frac{a_2}{a_1}, \frac{a_1+a_2}{a_1})_{g_{\mathrm{B}}}$ in the space of all diagonal metrics, such that the generalized Ricci flow converges to (g_0, H_0) starting at any generalized metric (g, H_0) with g in U (see Theorem 3.5 and Proposition 3.2).*
(ii) *Let $M = H \times H/\Delta K$ (i.e., $a_1 = a_2$) be a homogeneous space such that $(H/K, g_{\mathrm{B}})$ is Einstein and $\mathrm{B}_{\mathfrak{k}} = a\,\mathrm{B}_{\mathfrak{h}}|_{\mathfrak{k}}$, then any diagonal generalized Ricci flow solution converges to the BRF metric (g_0, H_0), where $g_0 = (1, 1, 2)_{g_{\mathrm{B}}}$ (see Theorem 4.3).*

Remark 1.2. If $M = G_1 \times G_2/\Delta K$ is multiplicity-free, i.e., M_1, M_2 are both isotropy irreducible, K is simple, the $\mathrm{Ad}(K)$-representations $\mathfrak{p}_1$, $\mathfrak{p}_2$ are inequivalent and neither of them is equivalent to the adjoint representation $\mathfrak{k}$, then the metrics of the form given in Equation 3 are all the G-invariant metrics on the homogeneous space M.

Finally, in § 5, we give an overview of the generalized Ricci flow and its fixed points on simple compact Lie groups and an analysis of the stability on $\mathrm{SO}(n)$, the only known simple Lie group admitting a nice basis. Our result is the following.

Theorem 1.3. *There exists a neighbourhood U of the Killing metric g_{B} on the compact Lie group $\mathrm{SO}(n)$, such that any generalized Ricci flow solution starting at a diagonal metric in U converges to g_{B}.*

This implies that near the Killing metric of $\mathrm{SO}(n)$, the conjecture given in [5, Conjecture 4.14] holds, i.e., for any initial condition (g, H_0) close enough to (g_{B}, H_0), where g is a left-invariant metric on $\mathrm{SO}(n)$ diagonal with respect to g_{B} and H_0 is the Cartan 3-form, the generalized Ricci flow exists on $[0, \infty)$ and converges to the BRF structure (g_{B}, H_0).

2. Preliminaries

2.1. Aligned homogeneous spaces

The known results on compact homogeneous spaces G/K differs substantially between the cases of G simple and non-simple. One potential reason for this could be that the isotropy representation of G/K is rarely multiplicity-free when G is non-simple. The class of homogeneous spaces with the richest third cohomology was studied in [8], and they are called *aligned* due to their special properties concerning the decomposition in irreducibles of G and K and their Killing constants. We provide an overview of this definition and properties when G has only two simple factors ($s = 2$ in [7]).

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Let $M = G/K$ be a homogeneous space, where G is a compact, connected, semisimple with two simple factors Lie group and K is a connected closed subgroup. We fix the following decomposition for the Lie algebra of G and K:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \qquad \mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_t, \tag{5}$$

where $\mathfrak{g}_i$'s and $\mathfrak{k}_j$'s are simple ideals of $\mathfrak{g}$ and $\mathfrak{k}$, respectively, and $\mathfrak{k}_0$ is the center of $\mathfrak{k}$. We call $\pi_i : \mathfrak{g} \to \mathfrak{g}_i$ the usual projections and we set $Z_i := \pi_i(Z)$ for any $Z \in \mathfrak{g}$, $i = 1, 2$. The Killing form of any Lie algebra $\mathfrak{h}$ will always be denoted by $\mathrm{B}_{\mathfrak{h}}$.

Definition 2.1. *A homogeneous space G/K as above is said to be* aligned *if there exist $c_1, c_2 > 0$ such that:*

(i) *The Killing constants, defined by*

$$\mathrm{B}_{\pi_i(\mathfrak{k}_j)} = a_{ij}\, \mathrm{B}_{\mathfrak{g}_i}|_{\pi_i(\mathfrak{k}_j)\times\pi_i(\mathfrak{k}_j)},$$

satisfy the following alignment property:

$$(a_{1j}, a_{2j}) = \lambda_j(c_1, c_2) \quad \textit{for some} \quad \lambda_j > 0, \qquad \forall j = 1, \ldots, t.$$

(ii) *There exists an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{k}_0$ such that*

$$\mathrm{B}_{\mathfrak{g}_i}(Z_i, W_i) = -\tfrac{1}{c_i}\langle Z, W\rangle, \qquad \forall Z, W \in \mathfrak{k}_0, \quad i = 1, 2. \tag{6}$$

(iii) $\frac{1}{c_1} + \frac{1}{c_2} = 1$.

The ideals $\mathfrak{k}_j$'s are therefore uniformly embedded on each $\mathfrak{g}_i$ in some sense. From the definition, G/K is automatically aligned if $\mathfrak{k}$ is simple or one-dimensional and the following properties hold:

(i) $\pi_i(\mathfrak{k}) \simeq \mathfrak{k}$ for $i = 1, 2$.
(ii) The Killing form of $\mathfrak{k}_j$ is given by $\mathrm{B}_{\mathfrak{k}_j} = \lambda_j\, \mathrm{B}_{\mathfrak{g}}|_{\mathfrak{k}_j\times\mathfrak{k}_j}$, $\qquad \forall j = 1, \ldots, t$.

From now on, given homogeneous spaces $M_i = G_i/K$, $i = 1, 2$, such that each G_i is simple and their Killing constants satisfy $\mathrm{B}_{\mathfrak{k}} = a_i\, \mathrm{B}_{\mathfrak{g}_i}|_{\mathfrak{k}}$ for $i = 1, 2$, we consider the homogeneous space $M = G_1 \times G_2/\Delta K$, which is an aligned homogeneous space with

$$c_1 = \tfrac{a_1+a_2}{a_2}, \quad c_2 = \tfrac{a_1+a_2}{a_1} \quad \text{and} \quad \lambda := \lambda_1 = \cdots = \lambda_t = \tfrac{a_1 a_2}{a_1+a_2}. \tag{7}$$

If $a_1 \le a_2$, then $1 < c_1 \le 2 \le c_2$.

Let $G := G_1 \times G_2$ and consider the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, which is orthogonal with respect to g_{B}, the Killing metric of G. We fix the G-invariant metric on M called *standard* given by

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$$g_{\mathrm{B}} = (-\,\mathrm{B}_{\mathfrak{g}_1})|_{\mathfrak{p}\times\mathfrak{p}} + (-\,\mathrm{B}_{\mathfrak{g}_2})|_{\mathfrak{p}\times\mathfrak{p}},$$

as a background metric. Note that we denote by g_{B} both, the bi-invariant metric on the Lie group G and the G-invariant metric on M.

Consider the g_{B}-orthogonal $\mathrm{Ad}(K)$-invariant decomposition

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3,$$

where $\mathfrak{p}_i$ is equivalent to the isotropy representation of the homogeneous space $M_i = G_i/\pi_i(K)$ for $i = 1, 2$ and

$$\mathfrak{p}_3 := \left\{ \bar{Z} = \left(Z_1, -\tfrac{c_2}{c_1} Z_2 \right) : Z \in \mathfrak{k} \right\} \tag{8}$$

is equivalent to the adjoint representation $\mathfrak{k}$ (see [8, Proposition 5.1]).

In order to use some known results assume the following technical property:

Assumption 2.2. *None of the irreducible components of $\mathfrak{p}_1, \mathfrak{p}_2$ is equivalent to any of the simple factors of $\mathfrak{k}$ as $\mathrm{Ad}(K)$-representations and either $\mathfrak{z}(\mathfrak{k}) = 0$ or the trivial representation is not contained in any of $\mathfrak{p}_1, \mathfrak{p}_2$ (see [8, Section 6] for more details on this assumption).*

2.2. Bismut connection and generalized Ricci flow

For further information on the subject of this subsection, we refer to the recent book [5] and the articles [3, 4, 11, 13–17].

Given a compact Riemannian manifold (M, g) and a 3-form H on M, we call *Bismut* the unique metric connection on M with torsion T such that it satisfies the 3-covariant tensor

$$g(T_X Y, Z) := H(X, Y, Z), \qquad \forall X, Y, Z \in \chi(M),$$

is the 3-form H on M. When it holds, this connection ∇^B is given by

$$g(\nabla^B_X Y, Z) = g(\nabla^g_X Y, Z) + \tfrac{1}{2} H(X, Y, Z), \qquad \forall X, Y, Z \in \chi(M),$$

where ∇^g is the Levi Civita connection of (M, g).

If H is closed then the pair (g, H) is called a *generalized metric*. A way to make these structures evolve naturally is provided by the Ricci tensor of the Bismut connection giving rise to the evolution Equation 1 called *generalized Ricci flow* (see [5, 11] and references therein).

The fixed points of this flow are the BRF generalized metrics, also called *generalized Einstein metrics*, i.e., (g, H) such that

$$\mathrm{Rc}(g) = \tfrac{1}{4} H_g^2 \quad \text{and} \quad H \text{ is } g\text{-harmonic}. \tag{9}$$

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2.3. BRF metrics on aligned homogeneous spaces

We review in this section the homogeneous BRF generalized metrics recently found in [10], which is a new version of [7] including a Corrigendum.

According to [8], a bi-invariant symmetric bilinear form Q_0 on $\mathfrak{g}$ defined by

$$Q_0 = \mathrm{B}_{\mathfrak{g}_1} - (\tfrac{1}{c_1 - 1})\mathrm{B}_{\mathfrak{g}_2},$$

defines a G-invariant closed 3-form on M denoted H_0 and given by

$$H_0(X, Y, Z) := Q_0([X, Y], Z) + Q_0([X, Y]_{\mathfrak{k}}, Z) - Q_0([X, Z]_{\mathfrak{k}}, Y) + Q_0([Y, Z]_{\mathfrak{k}}, X) \quad \text{for all } X, Y, Z \in \mathfrak{p},$$

which is g-harmonic for any *diagonal* G-invariant metric $g = (x_1, x_2, x_3)_{g_{\mathrm{B}}}$ of the form:

$$g := x_1 g_{\mathrm{B}}|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + x_2 g_{\mathrm{B}}|_{\mathfrak{p}_2 \times \mathfrak{p}_2} + x_3 g_{\mathrm{B}}|_{\mathfrak{p}_3 \times \mathfrak{p}_3}.$$

Recently in [10, Theorem A.2], it was proved the existence of a G-invariant BRF generalized metric on any homogeneous space $M = G/K$ where G has two simple factors and the Assumption 2.2 holds. Expressed in terms of the standard metric as a background the result is as follows (see [10, Remark A.3]).

Theorem 2.3. *(Lauret and Will [10, Theorem A.2]) Let $M = G/K$ be an aligned homogeneous space with $s = 2$ such that Assumption 2.2 holds.*

(i) The G-invariant generalized metric (g_0, H_0) defined by

$$g_0 := \left(1, \tfrac{1}{c_1 - 1}, \tfrac{c_1}{c_1 - 1}\right)_{g_{\mathrm{B}}} \tag{10}$$

is BRF.

(ii) This is the only G-invariant BRF generalized metric on $M = G/K$ up to scaling of the form $(g = (x_1, x_2, x_3)_{g_{\mathrm{B}}}, H_0)$.

As these metrics are precisely the fixed points of the generalized Ricci flow, we will utilize the elements involved in the proof of their main theorem. We consider the homogeneous space $M_i = G_i/K$ for $i = 1, 2$, and its $\mathrm{B}_{\mathfrak{g}_i}$-orthogonal reductive decomposition $\mathfrak{g}_i = \mathfrak{k} \oplus \mathfrak{p}_i$. For each $i = 1, 2$, we endowed it with the standard metric, which we denote by g_{B}^i (i.e., $g_{\mathrm{B}}^i = -\mathrm{B}_{\mathfrak{g}_i}|_{\mathfrak{p}_i \times \mathfrak{p}_i}$). For what follows, we called $\mathrm{C}_{\chi_i} : \mathfrak{p}_i \to \mathfrak{p}_i$, the *Casimir operator* of the isotropy representation

$$\chi_i : \mathfrak{k} \to \mathrm{End}(\mathfrak{p}_i)$$

of M_i with respect to $-\mathrm{B}_{\mathfrak{g}_i}|_{\mathfrak{k} \times \mathfrak{k}}$, and we fix this notation:

$$A_3 := -\tfrac{c_2}{c_1}, \qquad B_3 := \tfrac{1}{c_1} + A_3^2 \tfrac{1}{c_2}, \qquad B_4 := \tfrac{1}{c_1} + \tfrac{1}{c_2} = 1.$$

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The following proposition is demonstrated in [7, Proposition 3.2] and gives the formula for the Ricci operator when condition Equation 7 holds ($\lambda := \lambda_1 = \cdots = \lambda_t$). Note that the added hypothesis only changes (*iii*) from the original proposition.

Proposition 2.4. *(Lauret and Will [7, Proposition 3.2]) If $s = 2$ and Equation 7 holds, then the Ricci operator of the metric $g = (x_1, x_2, x_3)_{g_B}$ is given as follows:*

(i) $\mathrm{Ric}(g)|_{\mathfrak{p}_1} = \frac{1}{4x_1} I_{\mathfrak{p}_1} + \frac{1}{2x_1}\left(1 - \frac{x_3}{x_1 c_1 B_3}\right) \mathrm{C}_{\chi_1}$.

(ii) $\mathrm{Ric}(g)|_{\mathfrak{p}_2} = \frac{1}{4x_2} I_{\mathfrak{p}_2} + \frac{1}{2x_2}\left(1 - \frac{x_3}{x_2 c_2 B_3} A_3^2\right) \mathrm{C}_{\chi_2}$.

(iii) $\mathrm{Ric}(g)|_{\mathfrak{p}_3} = r I_{\mathfrak{p}_3}$, *where*

$$
\begin{aligned}
r :=& \frac{\lambda}{4x_3 B_3}\left(\frac{2x_1^2 - x_3^2}{x_1^2} + \frac{(2x_2^2 - x_3^2)A_3^2}{x_2^2} - \frac{1 + A_3}{B_3}\left(\frac{1}{c_1} + \frac{1}{c_2}A_3^3\right)\right) \\
&+ \frac{1}{4x_3 B_3}\left(2\left(\frac{1}{c_1} + \frac{1}{c_2}A_3^2\right) - \frac{2x_1^2 - x_3^2}{x_1^2 c_1} - \frac{(2x_2^2 - x_3^2)A_3^2}{x_2^2 c_2}\right).
\end{aligned}
$$

(iv) $g(\mathrm{Ric}(g)\mathfrak{p}_i, \mathfrak{p}_j) = 0$ *for all $i \neq j$.*

The formula of the symmetric bilinear form $(H_0)_g^2$ used in the definition of the generalized Ricci flow is provided in the following results. In general, a bi-invariant symmetric bilinear form Q on $\mathfrak{g}$,

$$
Q = y_1 \mathrm{B}_{\mathfrak{g}_1} + y_2 \mathrm{B}_{\mathfrak{g}_2} \text{ such that } \frac{y_1}{c_1} + \frac{y_2}{c_2} = 0,
$$

defines a G-invariant closed 3-form on M given by

$$
\begin{aligned}
H_Q(X, Y, Z) :=& Q([X, Y], Z) + Q([X, Y]_{\mathfrak{k}}, Z) - Q([X, Z]_{\mathfrak{k}}, Y) + Q([Y, Z]_{\mathfrak{k}}, X), \\
&\text{for all } X, Y, Z \in \mathfrak{p}.
\end{aligned}
$$

Proposition 2.5. *(Lauret and Will [7, Proposition 4.2]) For any $X \in \mathfrak{p}_k$, $k = 1, 2$,*

$$
(H_Q)_g^2(X, X) = g_{\mathrm{B}}\left(\left(\left(\frac{2S_k}{x_k c_k} - \frac{2y_k^2}{x_k^2}\right)\mathrm{C}_{\chi_k} + \frac{y_k^2}{x_k^2} I_{\mathfrak{p}_k}\right) X, X\right),
$$

where

$$
C_3 = \frac{y_1}{c_1} + A_3 \frac{y_2}{c_2}, \qquad S_1 = \frac{1}{x_3 B_3}\left(y_1 + \frac{C_3}{B_4}\right)^2, \qquad S_2 = \frac{1}{x_3 B_3}\left(A_3 y_2 + \frac{C_3}{B_4}\right)^2.
$$

Proposition 2.6. *(Lauret and Will [10, Proposition A.1]) If $\bar{Z} \in \mathfrak{p}_3$ (see Equation 8), with $g_{\mathrm{B}}(\bar{Z}, \bar{Z}) = 1$, then*

$$
\begin{aligned}
(H_Q)_g^2(\bar{Z}, \bar{Z}) =& \frac{1}{x_1^2 B_3}\left(y_1 + \frac{C_3}{B_4}\right)^2 \frac{1 - c_1\lambda}{c_1} + \frac{1}{x_2^2 B_3}\left(y_2 A_3 + \frac{C_3}{B_4}\right)^2 \frac{1 - c_2\lambda}{c_2} \\
&+ \frac{\lambda}{x_3^2 B_3^3}\left(\frac{y_1}{c_1} + A_3^3 \frac{y_2}{c_2} + \frac{3C_3}{B_4}\left(\frac{1}{c_1} + A_3^2 \frac{1}{c_2}\right)\right)^2.
\end{aligned}
$$

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3. Generalized Ricci flow on aligned spaces

The aim of this paper is to study the generalized Ricci flow on homogeneous spaces of the form $M = G_1 \times G_2/K$, such that each G_i is a simple Lie group, K is a closed subgroup of them and their Killing constants satisfy $\mathrm{B}_{\mathfrak{k}} = a_i \,\mathrm{B}_{\mathfrak{g}_i}|_{\mathfrak{k}}$ for $i = 1, 2$. As in § 2.1, M is an aligned homogeneous space satisfying Equation 7, note that $a_i = \lambda c_i$, for $i = 1, 2$. We assume that the standard metric g_{B}^i on each homogeneous space $M_i := G_i/K$ is Einstein for $i = 1, 2$. In that sense, we look for invariant solutions $(g(t), H(t))$ to the equations given in Equation 1.

We set $x_1 := x_1(t)$, $x_2 := x_2(t)$, $x_3 := x_3(t)$ smooth positive functions and define

$$g(t) := x_1 g_{\mathrm{B}}|_{\mathfrak{p}_1\times\mathfrak{p}_1} + x_2 g_{\mathrm{B}}|_{\mathfrak{p}_2\times\mathfrak{p}_2} + x_3 g_{\mathrm{B}}|_{\mathfrak{p}_3\times\mathfrak{p}_3}.$$

As every scalar multiple of H_0 is $g(t)$-harmonic for every $t \geq 0$, the second equation of the flow in Equation 1 vanishes and $H(t) \equiv H_0$; therefore, we only need to focus on the first one, starting from some generalized metric of the form $(g(0), H_0)$, that is

$$\tfrac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)) + \tfrac{1}{2}(H_0)^2_{g(t)}. \tag{11}$$

Note that by definition, $g(t)$ is diagonal for all $t \geq 0$. In the next lemma we see that, under certain conditions, the set of diagonal metrics is invariant under the flow.

Lemma 3.1. *Let $M = G_1 \times G_2/K$ be an aligned homogeneous space satisfying Equation 7 and assume that $(M_i = G_i/K, g_{\mathrm{B}}^i)$ is Einstein, where g_{B}^i is the standard metric on M_i, then the set of diagonal metrics with respect to the decomposition $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3$ is invariant under the generalized Ricci flow.*

Proof. As (M_i, g_{B}^i) is Einstein, there exist constants $\kappa_i \in (0, \frac{1}{2}]$ such that $\mathrm{C}_{\chi_i} = \kappa_i \,\mathrm{Id}_{\mathfrak{p}_i}$ for $i = 1, 2$. Therefore, from Propositions 2.4, 2.5 and 2.6, we see that $\frac{d}{dt} g(t)$ is tangent to the space of diagonal metrics, because all the operators involved are multiples of the identity in each $\mathfrak{p}_j$ for $j = 1, 2, 3$. □

Proposition 3.2. *Let $M = G_1 \times G_2/K$ be as in Lemma 3.1 such that $\mathrm{C}_{\chi_i} = \kappa_i \,\mathrm{Id}_{\mathfrak{p}_i}$ for $i = 1, 2$. Fix the standard metric of M, g_{B}, as a background metric, then the generalized Ricci flow for metrics of the form $g = (x_1, x_2, x_3)_{g_{\mathrm{B}}}$ is given by the following system of ordinary differential equations:*

$$\begin{cases} x_1'(t) = \dfrac{2\kappa_1 x_1 x_3^2 + c_1 x_3\left(-1 + x_1^2 + 2\kappa_1(1 + x_1^2 - 2x_1x_3)\right) + c_1^2\left(x_3 - x_1^2 x_3 - 2\kappa_1\left(x_3 + x_1^2 x_3 - x_1(1 + x_3^2)\right)\right)}{2(c_1-1)c_1 x_1^2 x_3}, \\ x_2'(t) = -\dfrac{c_1^3(1+2\kappa_2)x_2^2x_3 - 2\kappa_2 x_2 x_3^2 + c_1 x_3\left(-1 + x_2^2 + 2\kappa_2(1 + x_2^2 + 2x_2x_3)\right) - 2c_1^2 x_2\left(x_2x_3 + \kappa_2(1 + 2x_2x_3 + x_3^2)\right)}{2(c_1-1)^2 c_1 x_2^2 x_3}, \\ x_3'(t) = \dfrac{\left(x_3^2 - 2c_1x_3^2 + c_1^2(x_3^2-1)\right)\left(-c_1^2(1+2\lambda)x_2^2x_3^2 + (x_1^2 - x_2^2)x_3^2 + c_1\left((\lambda-1)x_1^2 + (2+\lambda)x_2^2\right)x_3^2 + c_1^3\lambda x_2^2(x_3^2 - x_1^2)\right)}{2(c_1-1)^3 c_1 x_1^2 x_2^2 x_3^2}. \end{cases} \tag{12}$$

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Proof. From Definition 2.1 (*iii*), $c_2 = \frac{c_1}{c_1-1}$. Hence, as in [7, Section 5], we have that $A_3 = -\frac{1}{c_1-1}$and $B_3 = \frac{1}{c_1-1}$. Note that for $j = 1, 2, 3$,

$$g(\mathrm{Ric}(g)\mathfrak{p}_j, \mathfrak{p}_j) = x_j g_{\mathrm{B}}(\mathrm{Ric}(g)\mathfrak{p}_j, \mathfrak{p}_j),$$

and $\mathrm{Ric}(g)|_{\mathfrak{p}_j}$ is a multiple of the identity in $\mathfrak{p}_j$ given by Proposition 2.4.

Given Q_0 with $y_1 = 1$ and $y_2 = -\frac{1}{c_1-1}$, the formula of $(H_0)^2_g$ follows from Propositions 2.5 and 2.6 with

$$C_3 = \tfrac{1}{c_1-1}, \quad S_1 = \tfrac{c_1^2}{(c_1-1)x_3}, \text{and} \quad S_2 = \tfrac{c_1^2}{(c_1-1)^3 x_3}.$$

Therefore, replacing these on Equation 11, the proposition holds. □

Note that we can write these equations using the usual notation of nonlinear systems of differential equations

$$x'(t) = f(x), \tag{13}$$

where $x := (x_1, x_2, x_3)$ and $f : \mathbb{R}^3_{>0} \to \mathbb{R}^3$ is given by the equations in Equation 12.

According to Theorem 2.3, (g_0, H_0) defined by Equation 10 is a BRF generalized metric, which means a fix or an equilibrium point of the flow. If we set $x_0 := (1, \frac{1}{c_1-1}, \frac{c_1}{c_1-1})$ then $f(x_0) = 0$ and the local behaviour of Equation 12 is qualitatively determined by the behaviour of the linear system $x' = Ax$ near the origin, where $A = Df(x_0)$, the derivative of f at x_0.

To initiate our study of the generalized Ricci flow on this class of aligned homogeneous spaces, we define some invariant subspaces and show some plots. Regardless of whether K is abelian ($\lambda = 0$) or not, we see that:

- The plane given by $x_3 = \frac{c_1}{c_1-1}$ is invariant by the flow.
- Within the plane, the lines defined by $x_1 = 1$ and $x_2 = \frac{1}{c_1-1}$ are invariant, and
- when $\kappa_1 = \kappa_2$, the plane $x_1 = (c_1 - 1)x_2$ and the line in it given by $x_3 = \frac{c_1}{c_1-1}$ are also invariant.

Just to illustrate the flow in those invariant planes we consider $M = \mathrm{SU}(7) \times \mathrm{SO}(8)/\mathrm{SO}(7)$, a homogeneous space of dimension 55 such that $c_1 = \frac{10}{7}$, $\kappa_1 = \kappa_2 = \frac{1}{2}$ and $\lambda = \frac{1}{4}$. In this case, the BRF metric is $x_0 = (1, \frac{7}{3}, \frac{10}{3})$ and the plots of the invariant planes of the flow are shown in Figure 1 and Figure 2.

For the remainder of this section, we recall some definitions of the local theory of nonlinear systems. Consider the system given in Equation 13,

Definition 3.3. *An equilibrium point x_0 (i.e., $f(x_0) = 0$) is called hyperbolic if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.*

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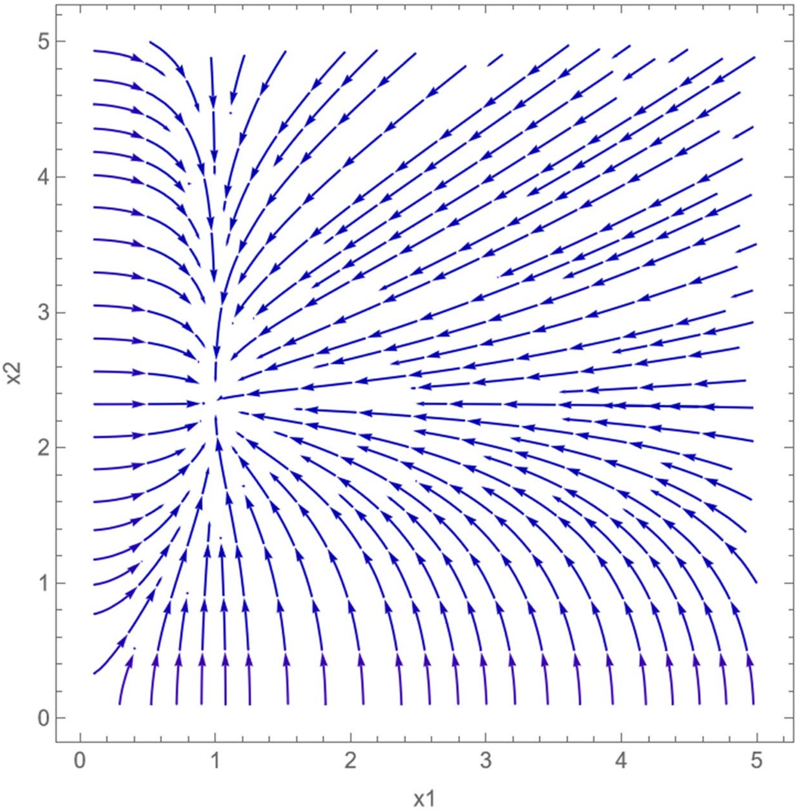

Figure 1. Flow in the invariant plane $x_3 = \frac{10}{3}$.

Definition 3.4. *Let ϕ_t denote the flow of the differential equation in Equation 13 defined for all $t \in \mathbb{R}$. An equilibrium point x_0 is stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in N_\delta(x_0)$ and $t \geq 0$ we have*

$$\phi_t(x) \in N_\varepsilon(x_0),$$

where $N_\alpha(x_0)$ is the open ball of positive radius α centred at x_0.

x_0 is asymptotically stable if it is stable and there exists $\delta > 0$ such that for all $x \in N_\delta(x_0)$ we have

$$\lim_{t\to\infty} \phi_t(x) = x_0.$$

Since the stability of an equilibrium point is a local property it is reasonable to expect that it would be the same as the stability at the origin of the linear system $x'(t) = Df(x_0)x$. This expectation is not always met, but it holds for hyperbolic equilibrium points.

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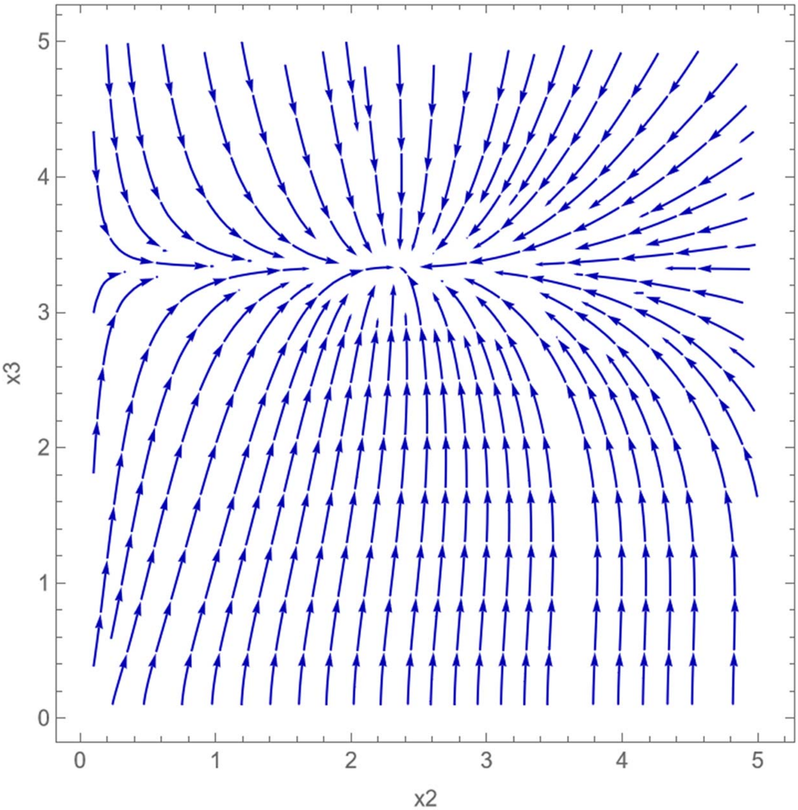

Figure 2. Flow in the invariant plane $x_1 = \frac{3}{7}x_2$.

The next result gives us a better understanding of the local behaviour of the flow near the BRF generalized metric.

Theorem 3.5. *The metric* $g_0 = (1, \frac{1}{c_1-1}, \frac{c_1}{c_1-1})g_{\mathrm{B}}$ *is asymptotically stable for the dynamical system Equation 12.*

Proof. From Theorem 2.3, (g_0, H_0) is a BRF generalized metric, which means an equilibrium point of the generalized Ricci flow.

If $f := (f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_2, x_3))$ satisfies

$$\begin{cases} x_1'(t) = f_1(x_1, x_2, x_3), \\ x_2'(t) = f_2(x_1, x_2, x_3), \\ x_3'(t) = f_3(x_1, x_2, x_3), \end{cases}$$

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as in Equation 12, then $f(g_0) = 0$ and its differential at g_0 is given by:

$$Df(g_0) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 - c_1 & 0 \\ 0 & 0 & c_1(-1 + \lambda) \end{pmatrix}.$$

As all the eigenvalues of the matrix are real and negative, the equilibrium point $g_0 = (1, \frac{1}{c_1-1}, \frac{c_1}{c_1-1})_{g_B}$ is asymptotically stable by [12, Section 2.9]. □

4. Global stability

In the previous section, we study the local behaviour of the nonlinear dynamical system given in Equation 12, the study of its global behaviour is substantially harder due to the possibility of chaos.

Definition 4.1. *An equilibrium point x_0 of a nonlinear system of differential equations $x'(t) = f(x)$ is globally stable if it is stable and globally attractive, which means*

$$\lim_{t\to\infty} \phi_t(x) = x_0 \text{ for all } x \in D,$$

where D is the domain of f.

This definition says that an equilibrium point is globally stable if the set of points in the space that are asymptotic to it is the whole space. The Lyapunov stability theorems provide sufficient conditions for this type of stability as well as asymptotic stability, this approach is based on finding a scalar function of a state that satisfies certain properties. Namely, this function has to be continuously differentiable and positive definite. Besides, if the first derivative of this function with respect to time is negative semidefinite along the state trajectories, then we can conclude that the equilibrium point is stable. Further, if the first derivative along every state trajectories is negative definite then we can conclude that the equilibrium point is globally stable.

As there is no universal method for creating Lyapunov functions for ordinary differential equations, this problem is far from trivial.

Theorem 4.2. *[12, Section 2.9] Let E be an open subset of $\mathbb{R}^n$ containing x_0, the equilibrium point. Suppose that $f \in C^1(E)$ and that $f(x_0) = 0$. Suppose further that there exists a real-valued function $V \in C^1(E)$ (called Lyapunov function) satisfying $V(x_0) = 0$ and $V(x) > 0$ if $x \neq x_0$. Then,*

(a) if $V'(x) \leq 0$ for all $x \in E$, x_0 is stable.
(b) If $V'(x) < 0$ for all $x \in E \setminus \{x_0\}$, x_0 is asymptotically stable.
(c) If $V'(x) > 0$ for all $x \in E \setminus \{x_0\}$, x_0 is unstable.

Given the space H/K, where H is a simple Lie group and $K \subseteq H$, we consider the homogeneous space $M = H \times H/\Delta K$, such that $G_1 = G_2 = H$ and $c_1 = 2$. We aim to establish global stability for this specific case, when hypothesis used before hold.

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Theorem 4.3. *Let $M = H \times H/\Delta K$, ($c_1 = 2$) a homogeneous space such that $(H/K, g_{\mathrm{B}})$ is Einstein (i.e., $\mathrm{C}_\chi = \kappa \,\mathrm{Id}_{\mathfrak{p}}$ for a constant $\kappa \in (0, \frac{1}{2}]$) and $\mathrm{B}_{\mathfrak{k}} = 2\lambda \,\mathrm{B}_{\mathfrak{h}}|_{\mathfrak{k}}$. Then the BRF metric $g_0 := (1, 1, 2)_{g_{\mathrm{B}}}$ is globally stable.*

Proof. Consider the function,

$$V(x_1, x_2, x_3) := \frac{\frac{\lambda}{10}(x_1 - 1)^2 + \frac{\lambda}{10}(x_2 - 1)^2 + (x_3 - 2)^2}{2},$$

we will prove that this is a Lyapunov function for the dynamical system given in Equation 12 taking $c_1 = 2$ and $\kappa := \kappa_1 = \kappa_2$, i.e.,

$$\begin{cases} x_1'(t) = \frac{x_3 - x_1^2 x_3 + \kappa\left(-2x_3 - 2x_1^2 x_3 + x_1(4 + x_3^2)\right)}{2x_1^2 x_3}, \\ x_2'(t) = \frac{x_3 - x_2^2 x_3 + \kappa\left(-2x_3 - 2x_2^2 x_3 + x_2(4 + x_3^2)\right)}{2x_2^2 x_3}, \\ x_3'(t) = -\frac{(x_3^2 - 4)\left((x_1^2 + x_2^2)x_3^2 - 2\lambda\left(x_2^2 x_3^2 + x_1^2(-4x_2^2 + x_3^2)\right)\right)}{4x1^2 x2^2 x3^2}. \end{cases} \tag{14}$$

Set $E := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \;/\; x_1, x_2, x_3 > 0\}$, it is evident that $V(1, 1, 2) = 0$ and V is clearly positive for all other $(x_1, x_2, x_3) \in E$, and thus V is a Lyapunov function for the dynamical system Equation 14 if its derivative F is negative definite on $E \setminus \{(1, 1, 2)\}$, where

$$\begin{aligned} F(x_1, x_2, x_3) :=& \frac{\lambda}{10}(x_1 - 1)x_1' + \frac{\lambda}{10}(x_2 - 1)x_2' + (x_3 - 2)x_3' = \\ =& -\frac{1}{4x_1^2 x_2^2 x_3^2}\left(64\lambda x_1^2 x_2^2\right. \\ &+ \tfrac{8}{10}\lambda\kappa x_1^2 x_2 x_3 + \tfrac{8}{10}\lambda\kappa x_1 x_2^2 x_3 - \tfrac{8}{10}\lambda\kappa x_1^2 x_2^2 x_3 - \tfrac{8}{10}\lambda\kappa x_1^2 x_2^2 x_3 - 32\lambda x_1^2 x_2^2 x_3 \\ &+ \tfrac{2}{10}\lambda x_1^2 x_3^2 + 8x_1^2 x_3^2 - \tfrac{4}{10}\lambda\kappa x_1^2 x_3^2 - 16\lambda x_1^2 x_3^2 - \tfrac{2}{10}\lambda x_1^2 x_2 x_3^2 + \tfrac{4}{10}\lambda\kappa x_1^2 x_2 x_3^2 \\ &+ \tfrac{2}{10}\lambda x_2^2 x_3^2 + 8x_2^2 x_3^2 - \tfrac{4}{10}\lambda\kappa x_2^2 x_3^2 - 16\lambda x_2^2 x_3^2 - \tfrac{2}{10}\lambda x_1 x_2^2 x_3^2 + \tfrac{4}{10}\lambda\kappa x_1 x_2^2 x_3^2 \\ &- \tfrac{2}{10}\lambda x_1^2 x_2^2 x_3^2 - \tfrac{2}{10}\lambda x_1^2 x_2^2 x_3^2 - \tfrac{4}{10}\lambda\kappa x_1^2 x_2^2 x_3^2 - \tfrac{4}{10}\lambda\kappa x_1^2 x_2^2 x_3^2 - 16\lambda x_1^2 x_2^2 x_3^2 \\ &+ \tfrac{2}{10}\lambda x_1^3 x_2^2 x_3^2 + \tfrac{4}{10}\lambda\kappa x_1^3 x_2^2 x_3^2 + \tfrac{2}{10}\lambda x_1^2 x_2^3 x_3^2 + \tfrac{4}{10}\lambda\kappa x_1^2 x_2^3 x_3^2 \\ &- 4x_1^2 x_3^3 + 8\lambda x_1^2 x_3^3 + \tfrac{2}{10}\lambda\kappa x_1^2 x_2 x_3^3 - 4x_2^2 x_3^3 + 8\lambda x_2^2 x_3^3 + \tfrac{2}{10}\lambda\kappa x_1 x_2^2 x_3^3 \\ &- \tfrac{2}{10}\lambda\kappa x_1^2 x_2^2 x_3^3 - \tfrac{2}{10}\lambda\kappa x_1^2 x_2^2 x_3^3 + 8\lambda x_1^2 x_2^2 x_3^3 \\ &\left.- 2x_1^2 x_3^4 + 4\lambda x_1^2 x_3^4 - 2x_2^2 x_3^4 + 4\lambda x_2^2 x_3^4 + x_1^2 x_3^5 - 2\lambda x_1^2 x_3^5 + x_2^2 x_3^5 - 2\lambda x_2^2 x_3^5\right). \end{aligned}$$

Given the symmetries of x_1 and x_2, to establish the negative condition of F, we define a function g as follows,

$$F(x_1, x_2, x_3) = -\frac{1}{4x_1^2 x_2^2 x_3^2}\left(x_1^2 g(x_2, x_3) + x_2^2 g(x_1, x_3)\right),$$

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where

$$g(x,y) := 32\lambda x^2 + \tfrac{8}{10}\lambda\kappa xy - \tfrac{8}{10}\lambda\kappa x^2 y - 16\lambda x^2 y + \tfrac{2}{10}\lambda y^2 + 8y^2 - \tfrac{4}{10}\lambda\kappa y^2 - 16\lambda y^2$$
$$- \tfrac{2}{10}\lambda xy^2 + \tfrac{4}{10}\lambda\kappa xy^2 - \tfrac{2}{10}\lambda x^2y^2 - \tfrac{4}{10}\lambda\kappa x^2y^2 - 8\lambda x^2y^2 + \tfrac{2}{10}\lambda x^3y^2 + \tfrac{4}{10}\lambda\kappa x^3y^2$$
$$- 4y^3 + 8\lambda y^3 + \tfrac{2}{10}\lambda\kappa xy^3 - \tfrac{2}{10}\lambda\kappa x^2y^3 + 4\lambda x^2y^3 - 2y^4 + 4\lambda y^4 + y^5 - 2\lambda y^5.$$

Hence, it is enough to prove that $g(x,y) \geq 0$ on $\tilde{E}$, where $\tilde{E} := \{(x,y) \in \mathbb{R}^2 \ / \ x,y > 0\}$, and equality is only attained at $(x,y) = (1,2)$. We split the function g considering the terms where λ or κ is involved,

$$g(x,y) := \frac{1}{10}\lambda h_1(x,y) + h_2(x,y) + \frac{2}{10}\lambda\kappa h_3(x,y) + 2\lambda h_4(x,y),$$

where

$$h_1(x,y) := 2y^2 - 2xy^2 - 2x^2y^2 + 2x^3y^2,$$
$$h_2(x,y) := 8y^2 - 4y^3 - 2y^4 + y^5,$$
$$h_3(x,y) := 4xy - 4x^2y - 2y^2 + 2xy^2 - 2x^2y^2 + 2x^3y^2 + xy^3 - x^2y^3,$$
$$h_4(x,y) := 16x^2 - 8x^2y - 8y^2 - 4x^2y^2 + 4y^3 + 2x^2y^3 + 2y^4 - y^5.$$

Now, from its factorization, it is clear that h_1 and h_2 are greater than zero on $\tilde{E}\setminus\{(1,2)\}$,

$$h_1(x,y) := 2(x-1)^2(1+x)y^2,$$
$$h_2(x,y) := (y-2)^2y^2(2+y).$$

We now define the functions g_1, g_2 as follows:

$$g_1(x,y) := h_4(x,y) + \tfrac{1}{20}h_1(x,y),$$
$$g_2(x,y) := \tfrac{1}{20}h_3(x,y),$$

such that

$$g(x,y) := h_2(x,y) + 2\lambda g_1(x,y) + 4\kappa\lambda g_2(x,y).$$

The proof that $g(x,y) > 0$ on $\tilde{E} \setminus \{(1,2)\}$ is going to be by cases, as h_2 is already positive, we have to differenciate if g_1 and g_2 are positive or not on $\tilde{E} \setminus \{(1,2)\}$.

- **Case 1:** $g_1 < 0$ and $g_2 < 0$, The $(x,y) \in \tilde{E}$ satisfying these conditions are represented in Figure 3. Note that $g_1 < 0$ if and only if $h_4 < 0$.

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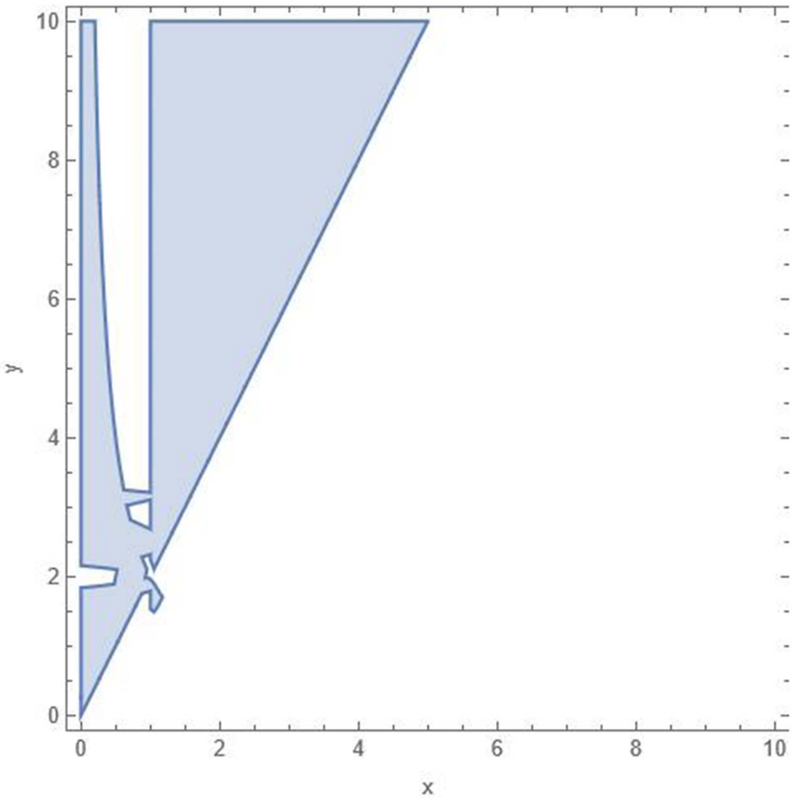

Figure 3. Region of Case 1: $g_1 < 0$ and $g_2 < 0$.

Since $g_1 < 0$ and $2\lambda < 1$, we have that $g_1 < 2\lambda g_1$. In the same way, $g_2 < 4\lambda\kappa g_2$ and the following inequality hold for all $(x, y) \in \tilde{E}$,

$$h_2 + g_1 + g_2 \; < \; h_2 + 2\lambda g_1 + 4\lambda\kappa g_2.$$

Our aim now is to demonstrate that the function $h_2 + g_1 + g_2$ is greater than zero on $\tilde{E}$. It is a cubic function on x such that $p(x)$ defined below is quadratic,

$$\begin{aligned}(h_2 + g_1 + g_2)(x, y) &= x\left(\tfrac{1}{20}y(4 + y^2) + x\tfrac{1}{20}(320 - 164y - 84y^2 + 39y^3) + x^2\tfrac{y^2}{5}\right)\\ &:= xp(x).\end{aligned}$$

To understand $p(x) \in \mathbb{R}[y][x]$, we look for its critical point, which is always a minimum given that the quadratic coefficient $\frac{y^2}{5}$ is always positive.

$$p'(x) = 0 \text{ if and only if } x = \bar{x} := \tfrac{1}{y^2}\left(\tfrac{21}{2}y^2 - 40 + \tfrac{41}{2}y - \tfrac{39}{8}y^3\right).$$

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Since $p(0) = \frac{1}{20}y(4+y^2)$ is positive for $y > 0$, and due to the continuity of $p(x)$, we conclude that if $\bar{x} < 0$ then $p(x) \geq 0$ for all $x, y > 0$ and hence $h_1 + g_1 + g_2 \geq 0$ for all $x, y > 0$, with equality only occurring a $(x, y) = (1, 2)$.

Thus, the values of y we have to study are those that lead to $\bar{x} > 0$, signifying $y \in I := \left(\frac{1}{39}\left(81 - \sqrt{321}\right), \frac{1}{39}\left(81 + \sqrt{321}\right)\right)$. For this interval, we want that $p(\bar{x}) > 0$, as the positivity of the minimum implies the positivity of the entire function.

$$p(\bar{x}) := -\frac{(y-2)^2(25600 - 640y - 13756y^2 - 484y^3 + 1521y^4)}{320y^2},$$

and this is positive in the interval we are interested in if and only if $q(y) := (25600 - 640y - 13756y^2 - 484y^3 + 1521y^4)$ is negative for all $y \in I$. Note that as $q(\frac{7}{5}) > 0$, $q(\frac{3}{2}) < 0$ and $q(\frac{13}{5}) < 0$, $q(3) > 0$ we can localize the positive roots of q and conclude that there are no roots of q in I (see Figure 6), besides $q(2) = -10240 < 0$.

This implies that $p(\bar{x}) > 0$ for all $y \in I$ and therefore, $h_2 + g_1 + g_2 > 0$ on $\tilde{E} \setminus \{(1, 2)\}$ as desired.

- **Case 2:** $g_1 < 0$ and $g_2 > 0$,

  The $x, y > 0$ satisfying these conditions are plotted in Figure 4. As in Case 1, $g_1 < 0$ implies that $h_4 < 0$ and therefore $h_4 < 2\lambda h_4$. It is easy to see that

$$(h_2 + h_4)(x, y) = 2x^2(y-2)^2(2+y),$$

  and hence,

$$0 < h_2 + h_4 < h_2 + 2\lambda h_4 < h_2 + 2\lambda h_4 + 2\lambda \tfrac{1}{20} h_1,$$

  proving that $g(x, y) > 0$ for all $(x, y) \in \tilde{E} \setminus \{(1, 2)\}$ in this case.
- **Case 3:** $g_1 > 0$ and $g_2 < 0$,

  This represents the last case for our analysis, it includes the cases when x tends to infinity and y tends to 0, as illustrated in Figure 5.

  Consider g factorized as follows:

$$g(x, y) = h_2 + 2\lambda(g_1 + 2\kappa g_2).$$

  Since $g_2 < 0$ in this case, $g_2 < 2\kappa g_2$, therefore if $g_1 + g_2$ is positive this case is proved.

  On the other hand, if $g_1 + g_2 < 0$, since $g_1 > 0$, we have the following inequalities

$$g_1 + g_2 < 2\kappa(g_1 + g_2) < 2\kappa g_1 + 2\kappa g_2 < g_1 + 2\kappa g_2.$$

  If the last expression is still negative for some $x, y > 0$ satisfying the hypothesis of this case, one obtains that:

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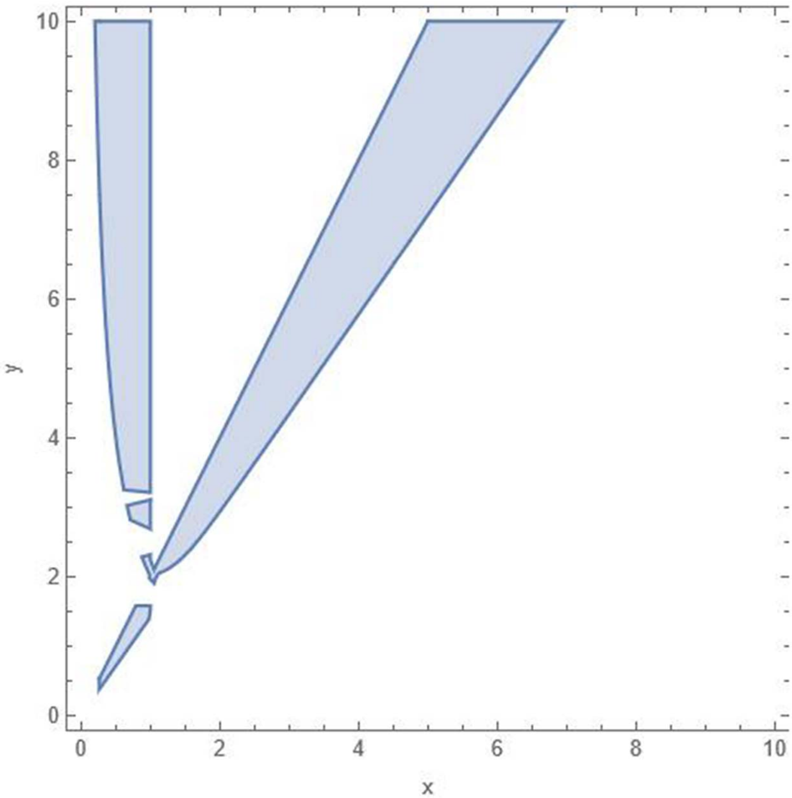

Figure 4. Region of Case 2: $g_1 < 0$ and $g_2 > 0$.

$$h_2 + g_1 + g_2 < h_2 + 2\kappa(g_1 + g_2) < h_2 + (g_1 + 2\kappa g_2) < h_2 + 2\lambda((g_1 + 2\kappa g_2) \\ = g(x, y),$$

and the proof is complete due to the proof of Case 1. □

5. Generalized Ricci flow on the Lie group SO(n)

5.1. Setup for any compact Lie group

Before presenting the main problem in this section, we review some known facts for compact Lie groups, see [7, Section 6].

Let $M^d = G$ be a compact, semisimple, connected Lie group and $\mathfrak{g}$ its Lie algebra. By [2, Chapter V], we know that every class in the set of all closed 3-forms of G, $H^3(G)$ has a unique bi-invariant representative called *Cartan 3-forms* of the form:

$$\overline{Q}(X, Y, Z) := Q([X, Y], Z), \quad \forall X, Y, Z \in \mathfrak{g}.$$

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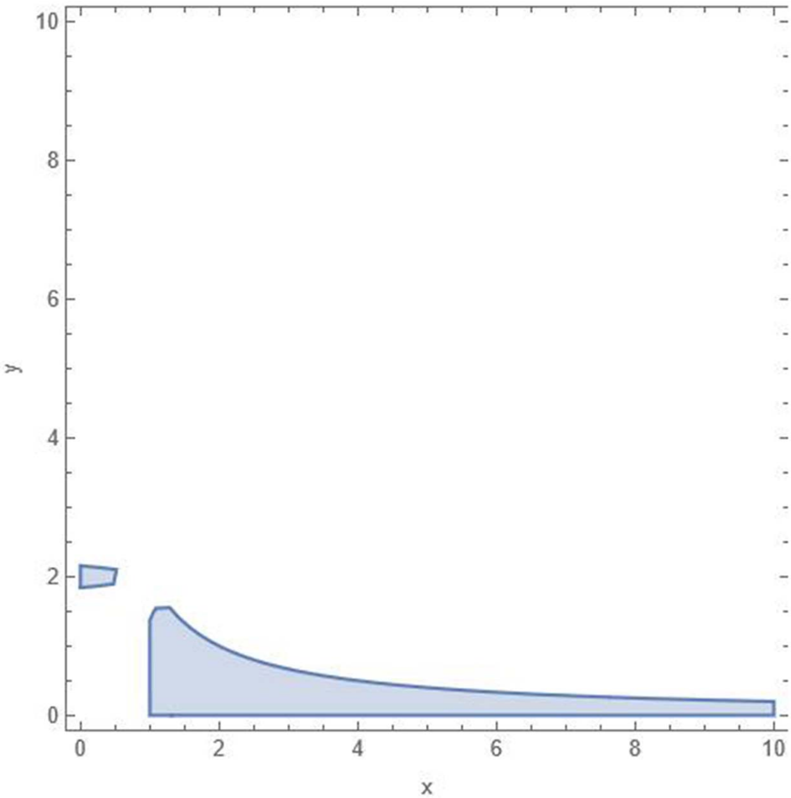

Figure 5. Region of Case 3: $g_1 > 0$ and $g_2 < 0$.

We note that if G is simple, then $H^3(G) = \mathbb{R}\,[H_{\mathrm{B}}]$ where $H_{\mathrm{B}} := \overline{\mathrm{B}_{\mathfrak{g}}}$, and $\mathrm{B}_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. It is well-known that Cartan 3-forms are harmonic with respect to any bi-invariant metric on G.

Following [8, Section 3], we consider a bi-invariant metric g_b on G, a g_b-orthonormal basis $\{e_1, \dots e_d\}$ and a left-invariant metric $g = (x_1, \dots, x_d)$ on G, such that $g(e_i, e_j) = x_i \delta_{ij}$. The ordered basis $\{e_1, \dots e_d\}$ determines structural constants given by

$$c_{ij}^k := g_b([e_i, e_j], e_k),$$

and by [8, Corollary 3.2 (ii)]:

$$H_{\mathrm{B}} \text{ is } g\text{-harmonic if and only if, } \sum_{1 \le i,j \le n} \frac{c_{ij}^k c_{ij}^l}{x_i x_j} = 0 \quad \forall k, l \text{ such that } x_k \neq x_l. \tag{15}$$

Therefore, within the context of the generalized Ricci flow, if Equation 15 holds, then H will remain constant along any solution and the flow will be governed by the equation

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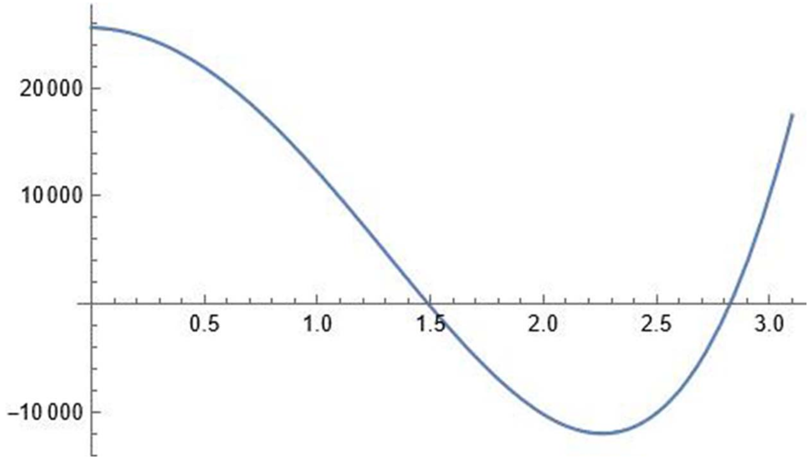

Figure 6. q(y).

$$\frac{\partial}{\partial t} g(t) = -2\,\mathrm{Ric}(g(t)) + \tfrac{1}{2}(H(t))^2_{g(t)}. \tag{16}$$

We will use the formulas derived in [7, Section 6], which are as follows for $H_b := g_b([\cdot,\cdot],\cdot)$:

$$(H_b)^2_g(e_k, e_l) = \sum_{i,j} \tfrac{1}{x_i x_j} g_b([e_k, e_i], e_j) g_b([e_l, e_i], e_j) = \sum_{i,j} \frac{c_{ij}^k c_{ij}^l}{x_i x_j}, \qquad \forall k, l. \tag{17}$$

Concerning the Ricci curvature, it is well-known that

$$\mathrm{Rc}(g)(e_k, e_l) = \tfrac{1}{2} \sum_{i,j} c_{ij}^k c_{ij}^l - \tfrac{1}{4} \sum_{i,j} c_{ij}^k c_{ij}^l \frac{x_i^2 + x_j^2 - x_k x_l}{x_i x_j}, \qquad \forall k, l. \tag{18}$$

5.2. Case SO(n)

For this section, we consider $G = \mathrm{SO}(n)$, the usual basis of its Lie algebra is $\beta = \{e_{rs} := E_{rs} - E_{sr}\}$, which is *nice* as it satisfies $c_{ij}^k c_{ij}^l = 0$ for all $k \neq l$.

For computational purposes, we label the elements of the basis as $\beta := \{e_1, e_2, \ldots, e_{\frac{n(n-1)}{2}}\}$. This basis is orthogonal with respect to g_{B}, the Killing metric of $\mathrm{SO}(n)$, and any diagonal left-invariant metric can be expressed as $g = (x_1, \ldots, x_{\frac{n(n-1)}{2}})_{g_{\mathrm{B}}}$.

Proposition 5.1. *Consider $G = \mathrm{SO}(n)$ and fix the Killing metric g_{B} as a background metric. For $\left(g(t) = (x_1(t), \ldots, x_{\frac{n(n-1)}{2}}(t))_{g_{\mathrm{B}}}, H_{\mathrm{B}}\right)$, the generalized Ricci flow is given by*

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$$x_k'(t) = -\sum_{i,j}(c_{ij}^k)^2 + \frac{1}{2}\sum_{i,j}(c_{ij}^k)^2\frac{x_i^2+x_j^2-x_k^2}{x_ix_j} + \frac{1}{2}\sum_{i,j}\frac{(c_{ij}^k)^2}{x_ix_j} \quad \forall k = 1,\dots,\tfrac{n(n-1)}{2}. \tag{19}$$

Proof. Replacing Equation 17 and (Equation 18) in Equation 16, the proposition follows as the basis β is nice. □

It is clear that $g_{\mathrm{B}} = (1, 1, \dots, 1)_{g_{\mathrm{B}}}$ is a BRF generalized metric, which means a fix point of the above system. To study the dynamical stability of this point, we examine the linearization of the provided nonlinear system.

Theorem 5.2. *The Killing metric g_{B} on the compact Lie group $\mathrm{SO}(n)$ is asymptotically stable for the dynamical system given in Equation 19.*

Proof. We define functions

$$f_k(x_1,\dots,x_{\frac{n(n-1)}{2}}) := -\sum_{i,j}(c_{ij}^k)^2 + \frac{1}{2}\sum_{i,j}(c_{ij}^k)^2\frac{x_i^2+x_j^2-x_k^2+1}{x_ix_j}, \quad \text{for } k = 1,\dots,\tfrac{n(n-1)}{2},$$

such that $x'(t) = f(x)$. Therefore, the differential of f is given by:

$$\begin{cases} \dfrac{\partial f_k}{\partial x_k} = \dfrac{1}{2}\displaystyle\sum_{i,j}\frac{(c_{ij}^k)^2}{x_ix_j}(-2x_k) & \forall k = 1,\dots,\frac{n(n-1)}{2}, \\ \dfrac{\partial f_k}{\partial x_i} = \dfrac{1}{2}\displaystyle\sum_{i,j}(c_{ij}^k)^2\frac{x_i^2-x_j^2+x_k^2-1}{x_i^2x_j} & \forall k \neq i. \end{cases}$$

Hence, on the Killing metric g_{B},

$$Df(1,1,\dots,1) = \begin{pmatrix} -\sum_{i,j}(c_{ij}^k)^2 & 0 & \dots & \dots & 0 \\ 0 & -\sum_{i,j}(c_{ij}^k)^2 & \dots & \dots & 0 \\ 0 & 0 & \ddots & \dots & 0 \\ 0 & \dots & \dots & 0 & -\sum_{i,j}(c_{ij}^k)^2 \end{pmatrix}.$$

As all its eigenvalues are reals and negative the proof is complete. □

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