ON A THEOREM IN THE GENERALISED FOURIER TRANSFORM

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1. Introduction. The function $\tilde{\omega}_{\mu,\nu}(x)$ was defined by G.N. Watson, [9, (i)] in 1931 by the integral relation *

$$\tilde{\omega}_{\mu,\nu}(x) = x^{\frac{1}{2}} \int_0^\infty J_\mu(t) J_\nu(x/t) t^{-1} dt$$

$$= \frac{x^{\nu+\frac{1}{2}} 2^{-2\nu-1} \Gamma(\frac{1}{2}\mu - \frac{1}{2}\nu)}{\Gamma(\nu+1)\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu)} {}_0F_3(\nu+1, 1-\tfrac{1}{2}\mu+\tfrac{1}{2}\nu, 1+\tfrac{1}{2}\mu+\tfrac{1}{2}\nu; x^2/16)$$

+ another term with μ and ν interchanged;

$$-R(\mu + \tfrac{3}{2}) < 0 < R(\nu + \tfrac{3}{2}).$$

He showed (without proof) that it is a symmetric Fourier kernel. Later K.P. Bhatnagar, [1, (i), (ii)] in 1953 and 1954 investigated in some details the properties of this kernel and extended it to n parameters and defined

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* The integral $\int_0^\infty J_\mu(t) J_\nu(\frac{x}{t}) t^\rho dt$ was originally evaluated by C.V.H. Rao. See Messenger of Maths, 47, (1918), 134-7. Also see Bessel Functions by Watson. (1922) 437.

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$$\tilde{\omega}_{\mu_1,\ldots,\mu_n}(x) = x^{\frac{1}{2}} \int_0^\infty \cdots \int_0^\infty J_{\mu_1}(t_1)\ldots J_{\mu_{n-1}}(t_{n-1}) J_{\mu_n}\left(\frac{x}{t_1\ldots t_{n-1}}\right) \times$$

$$\times\ (t_1\ldots t_{n-1})^{-1} dt_1\ldots dt_{n-1}$$

$$= \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_{n-1}}(x/t)\ J_{\mu_n}(t)^{-\frac{1}{2}} dt\ ,$$

where $R(\mu_k + \frac{1}{2}) \geq 0$, $k = 1, 2, \ldots, n$ and the μ's may be permuted among themselves.

V.P. Mainra, [3, (ii)] in 1958 defined the function

$$\tilde{\omega}^{\lambda}_{\mu,v}(x) = \int_0^\infty \tilde{\omega}_{\mu,v}(xt)\ J_\lambda(t)\sqrt{t}\, dt,$$

and further generalised this kernel as

$$\tilde{\omega}^{v_1,\ldots v_m}_{\mu_1,\ldots\mu_n}(x) = \int_0^\infty \cdots \int_0^\infty J_{v_1}(t_1)\ldots J_{v_m}(t_m) J_{\mu_1}(T_1)\ldots J_{\mu_{n-2}}(T_{n-2}) \times$$

$$\tilde{\omega}_{\mu_{n-1},\mu_n}\left(\frac{xt_1\ldots t_m}{T_1\ldots T_{n-2}}\right) \frac{(t_1\ldots t_m)^{\frac{1}{2}}}{T_1\ldots T_{n-2}} dT_1\ldots dT_{n-2} dt_1\ldots dt_m\ ,\quad m < n.$$

$$R(\mu_r, v_s) > -1, \qquad r = 1, 2, \ldots n, \quad s = 1, \ldots m\ .$$

In 1961, Charles Fox [2] showed that the G-Function as defined by C.S. Meijer, [4] is a symmetric Fourier kernel. For certain values of the parameters the kernel degenerates into the kernels of Bhatnagar and Mainra, but he did not investigate the properties of this kernel. B. Singh [5, (i)] in 1964, Y.P. Singh [7, (i)] and P. Singh [6] in 1966 showed that many of the results given in Watson's Bessel Functions follow as particular cases of the integrals involving the generalised kernels mentioned above.

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In the present paper, the author has proved certain interesting results involving the functions f(t), g(t) and their transforms F(x) and G(x) in these generalised transforms.

The following results are either known or can be proved easily.

(1) $\tilde{\omega}_{\mu}(x) = \sqrt{x}\, J_{\mu}(x)$, $\tilde{\omega}_{\mu,\mu+1}(x) = J_{2\mu+1}(2\sqrt{x})$, $R(\mu) > -1$.

(2) $$\tilde{\omega}_{\mu_1,\ldots\mu_n}(x) = \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_{n-1}}(xt)\, J_{\mu_n}(t^{-1})t^{-\frac{3}{2}}dt$$

$$= \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_r}(xt)\,\tilde{\omega}_{\mu_{n+1},\ldots\mu_n}(t^{-1})\frac{dt}{t},\ R(\mu_r) > -1.$$

$$r = 1, 2, \ldots n .$$

(3) $$\tilde{\omega}^{v_1,\ldots v_m}_{\mu_1,\ldots\mu_n}(x) = \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)\,\tilde{\omega}_{v_1,\ldots v_m}(t)dt,\ m < n,$$

$$R(\mu_r + V_s) > -2,$$

$$r = 1, \ldots n, \quad s = 1, \ldots m .$$

(4) $$\tilde{\omega}_{\mu_1,\ldots\mu_n}(x) = 0\,(x^{\mu_r+\frac{1}{2}}),\quad r = 1, 2, \ldots n \text{ for small } x$$

$$= x^{\frac{1-n}{2n}}\left[\cos\,(2nx^{\frac{1}{n}} + \alpha)(A + 0(x^{-\frac{2}{n}}))\right.$$

$$\left. + \sin(2nx^{\frac{1}{n}} + \alpha)\ 0(x^{\frac{1}{n}})\right]$$

(5) $$\tilde{\omega}^{v_1,\ldots v_m}_{\mu_1,\ldots\mu_n} = x^{\frac{1-n+m}{2(n-m)}}\left[\cos((2n-m)x^{\frac{1}{n-m}} + \alpha)(A_1 + 0(x^{-\frac{2}{n-m}})\right)$$

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$$+ \sin(2(n-m)\, x^{\frac{1}{n-m}} + \alpha_1)\, 0(x^{\frac{1}{n-m}})] + \sum_{n=1}^{m} x^{-2(\frac{3}{4} + \frac{a_n}{2})} \times$$

$$\{P_n + 0(x^{-2})\} ,$$

for large x and $\alpha, \alpha_1, A, A_1, P_n$ are constants, $= 0(x^{\mu/2 + \frac{1}{2}})$ for small x, $r = 1, 2, \ldots n$.

(6) The Mellin transforms of $\tilde{\omega}_{\mu_1, \ldots \mu_n}(x)$ and

$\tilde{\omega}^{v_1, \ldots v_m}_{\mu_1, \ldots \mu_n}(x)$ are $2^{n(s - \frac{1}{2})} \dfrac{\Gamma(\frac{\mu_1}{2} + \frac{s}{2} + \frac{1}{4}) \ldots \Gamma(\frac{\mu_n}{2} + \frac{s}{2} + \frac{1}{4})}{\Gamma(\frac{\mu_1}{2} - \frac{s}{2} + \frac{3}{4}) \ldots \Gamma(\frac{\mu_n}{2} - \frac{s}{2} + \frac{3}{4})}$

and

$$2^{(n-m)(s-\frac{1}{2})} \frac{\Gamma(\frac{\mu_1}{2} + \frac{s}{2} + \frac{1}{4}) \ldots \Gamma(\frac{\mu_n}{2} + \frac{s}{2} + \frac{1}{4}) \Gamma(\frac{v_1}{2} - \frac{s}{2} + \frac{3}{4}) \ldots \Gamma(\frac{v_m}{2} - \frac{s}{2} + \frac{3}{4})}{\Gamma(\frac{\mu_1}{2} - \frac{s}{2} + \frac{3}{4}) \ldots \Gamma(\frac{\mu_n}{2} - \frac{s}{2} + \frac{3}{4}) \Gamma(\frac{v_1}{2} + \frac{s}{2} + \frac{1}{4}) \ldots \Gamma(\frac{v_m}{2} + \frac{s}{2} + \frac{1}{4})}$$

and belong to $L(-\infty, \infty)$ if $\frac{1}{2} - \frac{1}{n} > \sigma$, $s = \sigma + it$.

Results (2) and (3) can be proved by an application of Parseval's theorem [8, p. 54].

<u>Notations employed</u>

$$f_{\mu_1, \ldots \mu_n}(x) = \int_0^\infty f(t)\, \tilde{\omega}_{\mu_1, \ldots \mu_n}(xt)\, dt$$

$$f^{v_1, \ldots v_m}_{\mu_1, \ldots \mu_n}(x) = \int_0^\infty f(t)\, \tilde{\omega}^{v_1, \ldots v_m}_{\mu_1, \ldots \mu_n}(xt)\, dt$$

If $g(x) = \int_0^\infty f(t)\tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)dt$, $g(x)$ is called the $\tilde{\omega}_{\mu_1,\ldots\mu_n}(x)$ transform of $f(x)$.

If $g(x) = f(x)$, $f(x)$ is said to be $R_{\mu_1,\ldots\mu_n}$.

THEOREM 1. *Let* $f(t)$ *and* $G(t)$ *be continuous and belong to* $L(0,\infty)$ *and* $F(x)$, $G(x)$ *be the* $\tilde{\omega}_{\mu_1,\ldots\mu_n}(x)$ *transforms of* $f(t)$ *and* $g(t)$ *respectively. Then*

$$\int_0^\infty F(x)\,G(x)dx = \int_0^\infty f(t)g(t)dt\ .$$

Proof.

$$\int_0^\infty F(x)G(x)dx = \int_0^\infty G(x)dx \int_0^\infty f(t)\,\tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)dt$$

$$= \int_0^\infty f(t)dt \int_0^\infty G(x)\tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)dt$$

$$= \int_0^\infty f(t)g(t)dt\ .$$

This is the Parseval theorem for the transform $\tilde{\omega}_{\mu_1,\ldots\mu_n}(x)$ introduced by Bhatnagar [1, (i)].

THEOREM 2. *Let* $f(t)$, $G(t)$ *and* $g(t)$ *satisfy the conditions of Theorem 1. Then*

$$\int_0^\infty F(x)G(x)dx = \int_0^\infty f_{\mu_1}(t)g_{\mu_1}(t)dt$$

$$= \ldots = \int_0^\infty f_{\mu_n}(t)g_{\mu_n}(t)dt\ .$$

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Proof.

$$\int_0^\infty f(t)\,\tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)\,dt$$

$$= \int_0^\infty \sqrt{t}\, f(t)dt \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\mu_{r+1}\cdots\mu_n}(xy) J_{\mu_r}(t/y) y^{-\frac{3}{2}} dy,$$

$R(\mu_s) > -1, \quad s = 1, 2, \ldots n ,$

$$= \int_0^\infty f_{\mu_r}(y^{-1})\,\tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\ldots\mu_n}(xy)\, y^{-1}\, dy \quad .$$

Therefore

$$\int_0^\infty G(x)dx \int_0^\infty f(t)\,\tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)dt$$

$$= \int_0^\infty G(x)dx \int_0^\infty f_{\mu_r}(y^{-1})\,\tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\mu_{r+1},\ldots\mu_n}(xy)dy/y$$

$$= \int_0^\infty f_{\mu_r}(y^{-1})y^{-1}dy \int_0^\infty G(x)\,\tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\mu_{r+1},\ldots\mu_n}(xy)dx \;.$$

Now

$$\int_0^\infty G(x)\,\tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\mu_{r+1},\ldots\mu_n}(x)dx$$

$$= \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\mu_{r+1},\ldots\mu_n}(xy)dx \int_0^\infty g(t)\,\tilde{\omega}_{\mu_1,\ldots\mu_n}(xt)dt$$

$$= \int_0^\infty g(t)dt \int_0^\infty \tilde{\omega}_{\mu_1,\ldots\mu_{r-1},\mu_{r+1},\ldots\mu_n}(x)\tilde{\omega}_{\mu_1,\ldots\mu_n}\left(\frac{xt}{y}\right)y^{-1}dx$$

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$$= y^{-1} \int_0^{\infty} g(t)\, \tilde{\omega}_{\mu_1, \ldots \mu_n}^{\mu_1, \ldots \mu_{r-1}, \mu_{r+1}, \ldots \mu_n} (t/y)dt$$

$$= y^{-1} \int_0^{\infty} g(t)\, \tilde{\omega}_{\mu_r} (t/y)dt = y^{-1} \int_0^{\infty} g(t)\sqrt{\frac{t}{y}}\; J_{\mu_r} (t/y)\, dt$$

$$= y^{-1}\, g_{\mu_r} (y^{-1})dy \, .$$

Therefore

$$\int_0^{\infty} F(x)\, G(x)dx = \int_0^{\infty} f_{\mu_r} (y^{-1})g_{\mu_r} (y^{-1})y^{-2}dy$$

$$= \int_0^{\infty} f_{\mu_r} (y)g_{\mu_r} (y)dy \, , \quad r = 1, 2, \ldots n.$$

All the conditions of De la Vallée Poussin's theorem (see Carslaw, H. S. Introduction to the Theory of Fourier's Series and Integrals. Art. 89, p.209) are satisfied and a change in the order of integrations can be effected.

COROLLARY 1. Let

$$F(x) = \int_0^{\infty} f(t)\, \tilde{\omega}_{\mu_1, \ldots \mu_n} (xt)dt, \quad G(x) = \int_0^{\infty} g(t)\, \tilde{\omega}_{\mu_1, \ldots \mu_n} (xt)dt \, .$$

Then

$$\int_0^{\infty} F(x)G(x)dx = \int_0^{\infty} f_{\mu_1, \ldots \mu_r} (y)\, g_{\mu_1, \ldots \mu_r} (y)dy, \quad r = 1, 2, \ldots n,$$

$$R(\mu_r) > -1, \quad r = 1, 2, \ldots r,$$

under the conditions of the theorem.

COROLLARY 2. Let

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$$F(x) = \int_0^\infty f(t)\,\tilde{\omega}_{\mu_1, \ldots \mu_m}(xt)dt$$

$$G(x) = \int_0^\infty g(t)\,\tilde{\omega}_{\mu_1, \ldots \mu_n}(xt)dt, \quad m < n ,$$

then

$$\int_0^\infty F(x)G(x)dx = \int_0^\infty f_{\mu_1, \mu_2 \ldots \mu_r}(y) g_{\mu_1, \ldots \mu_r, \mu_{m+1}, \ldots \mu_n}(y)dy,$$

$$r < m < n, \qquad R(\mu_s) > -1 , \qquad s = 1, 2, \ldots n,$$

under the conditions of the theorem.

Proof.

$$\int_0^\infty F(x)G(x)dx = \int_0^\infty G(x)dx \int_0^\infty f_{\mu_1, \ldots \mu_r}(y)\tilde{\omega}_{\mu_{r+1}, \ldots \mu_m}(x/y)y^{-1}dy \quad *$$

$$= \int_0^\infty f_{\mu_1, \ldots \mu_r}(y)y^{-1}dy \int_0^\infty g(t)\,\tilde{\omega}_{\mu_1, \ldots \mu_n}^{\mu_{r+1}, \ldots \mu_m}(yt)\, ydt$$

$$= \int_0^\infty f_{\mu_1, \ldots \mu_r}(y)\, g_{\mu_1, \ldots \mu_r, \mu_{m+1}, \ldots \mu_n}(y)dy .$$

Examples

1. Let $f(t) = g(t)$, so that $F(x) = G(x)$.

$$\int_0^\infty x^{-\frac{1}{4}}(1+ax)^{-\frac{3}{2}}\,\tilde{\omega}_{2b-\frac{1}{2}, 0}(p^{\frac{1}{2}}x^{\frac{1}{2}})\, dx$$

* The second integral on the right can be deduced from equation (2).

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$$= \frac{4}{a} p^{\frac{1}{4}} K_{2b-\frac{1}{2}}\left(2^{\frac{1}{2}} a^{-\frac{1}{4}} p^{\frac{1}{4}}\right) J_{2b-\frac{1}{2}}\left(2^{\frac{1}{2}} a^{-\frac{1}{4}} p^{\frac{1}{4}}\right), \quad *$$

or

$$\int_0^\infty x^{\frac{1}{2}} (1+a^2x^2)^{-\frac{3}{2}} \tilde{\omega}_{2b-\frac{1}{2},0}(px)dx$$

$$= 2a^{-2} p^{\frac{1}{2}} K_{2b-\frac{1}{2}}\left(\sqrt{\frac{2p}{a}}\right) J_{2b-\frac{1}{2}}\left(\sqrt{\frac{2p}{a}}\right).$$

$$f(x) = x^{\frac{1}{2}}/(1+a^2x^2)^{\frac{3}{2}}, \quad F(x) = 2a^{-2}x^{\frac{1}{2}} K_{2b-\frac{1}{2}}\left(\sqrt{\frac{2x}{a}}\right) J_{2b-\frac{1}{2}}\left(\sqrt{\frac{2x}{a}}\right)$$

and

$$\int_0^\infty x\left\{K_{2b-\frac{1}{2}}\left(\sqrt{\frac{2x}{a}}\right) J_{2b-\frac{1}{2}}\left(\sqrt{\frac{2x}{a}}\right)\right\}^2 dx = \frac{a^4}{4}\int_0^\infty \frac{xdx}{(1+a^2x^2)^3} = \frac{a^2}{16},$$

$$R(b) > -\frac{1}{4}.$$

2. Let $f(x) = F(x) = x^{\frac{1}{2}} K_v(x)$ which is $R_{-v,v}$, $-1 < R(v) < 1$.

$$f_v(y) = \int_0^\infty x^{\frac{1}{2}} K_v(x)\sqrt{xy}\, J_v(xy)dx = y^{v+\frac{1}{2}}/(1+y^2).$$

* See [1], References. This can be proved by an application of Parseval's theorem on Mellin Transform.

Therefore

$$\int_0^\infty \{f_v(y)\}^2 dy = \int_0^\infty \frac{y^{2v+1}}{(1+y^2)^2} dy = \frac{v\pi}{2 \sin v\pi} .$$

Also

$$\int_0^\infty \{F(x)\}^2 dx = \int_0^\infty \{x^{\frac{1}{2}} K_v(x)\}^2 dx = \frac{v\pi}{2 \sin v\pi} .$$

Hence

$$\int_0^\infty \{F(x)\}^2 dx = \int_0^\infty \{f(y)\}^2 dy .$$

REFERENCES

1. K.P. Bhatnagar, (i) On certain theorems on self-reciprocal functions, Acad. Roy. Belgique. Bull. cl. Sci. (5) 39, (1953), 42-69. (ii) Two theorems on self-reciprocal functions and a new transform. Bull. Calcutta Math. Soc. 45, (1953), 109-112.

2. Charles Fox, (i) The Gand H Functions as symmetrical Fourier Kernels, Trans. American Math. Soc. (1961) 98, 395-429.

3. V.P. Mainra, (i) On Poisson's formulae, Bull. Cal. Math. Soc. 49 (1957), 163-176. (ii) A new transform, Bull. Cal. Math. Soc. 1958 Supplement 76-93.

4. C.S. Meijer, (i) On the G. Function. Proc. Neder. Akad. Wetensch. (1946), 49, 227-237, 344-356, 457-469, 632-641, 765-772, 936-943, 1062-1072, 1165-1175.

5. B. Singh, (i) On certain expansions and integrals involving $\tilde{\omega}_{\mu,v}(x)$. Proc. Rajasthan Acad. Sci. 9 (1962) 9-22.

6. P. Singh, (i) Thesis for the Ph.D. degree. B.I.T.S. (Pilani) To appear in the Proc. Nat. Acad. of Science. Allahalad (1966).

https://doi.org/10.4153/CMB-1967-072-4 Published online by Cambridge University Press

7. Y.P. Singh, (i) Thesis for the Ph.D. degree B.I.T.S. (Pilani) India (1960). See (Scientific Journal, Benaras Univ. 1966).

8. E.C. Titchmarsh, (i) Theory of Fourier Integrals, 1962 (Oxford).

9. G.N. Watson, (i) Some self-reciprocal functions. Quart. J. Math., (Oxford) 2, (1931) 298-309.
(ii) A treatise on the Theory of Bessel Functions, Camb. Univ. Press (1966).

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