Canad. Math. Bull. Vol. **53** (1), 2010 pp. 163–170
doi:10.4153/CMB-2010-031-8
© Canadian Mathematical Society 2010

Variants of Arnold's Stability Results for 2D Euler Equations

Michael Taylor

Abstract. We establish variants of stability estimates in norms somewhat stronger than the H^1-norm under Arnold's stability hypotheses on steady solutions to the Euler equations for fluid flow on planar domains.

1 Introduction

Let Ω be a smoothly bounded planar region and let $u^\varepsilon(t,x)$ be solutions to Euler equations on $\mathbb{R}\times\Omega$,

$$\partial_t u^\varepsilon + \nabla_{u^\varepsilon} u^\varepsilon = \nabla q^\varepsilon, \quad \operatorname{div} u^\varepsilon = 0, \quad u^\varepsilon \parallel \partial\Omega, \tag{1.1}$$

with initial data $u^\varepsilon(0) = u_0^\varepsilon$. Assume $u^0(t,x) \equiv u_s(x)$ is a smooth, steady solution to (1.1). V. Arnold found conditions on u_s guaranteeing the stability estimate

$$\|u^\varepsilon(t) - u_s\|_{H^1(\Omega)} \le C\|u_0^\varepsilon - u_s\|_{H^1(\Omega)}, \quad \forall\, t\in\mathbb{R}, \tag{1.2}$$

at least as long as the right side of (1.2) is sufficiently small. The analysis was based on use of conserved quantities of the form

$$H(u) = \int_\Omega \Big[\frac{1}{2}|u|^2 + \varphi(\omega)\Big]\, dA + \sum a_j \int_{\Gamma_j} u\cdot dx.$$

Here $\omega = \operatorname{rot} u$ and Γ_j are the connected components of $\partial\Omega$. The function φ is obtained as follows. Set $\omega_s = \operatorname{rot} u_s$ and let ψ_s denote the stream function of u_s, satisfying $u_s = J\nabla\psi_s$, where J represents counterclockwise rotation by $90°$. Assume $\psi_s = \Phi(\omega_s)$, with Φ smooth and monotone, and take φ such that $\varphi'(\lambda) = \Phi(\lambda)$. We can assume φ' to be linear for λ large (positive or negative). It is then possible to specify $a_j \in \mathbb{R}$ such that u_s is a critical point of H. A calculation gives

$$D^2H(u_s)(v,v) = \int_\Omega \big[\,|v|^2 + (\operatorname{rot} v)^2\varphi''(\omega_s)\big]\, dA,$$

or equivalently $D^2H(u_s)(v,v) = Q(v,v)$, with

$$Q(v,v) = \|v\|_{L^2}^2 + (\operatorname{rot} v, \Phi'(\omega_s)\operatorname{rot} v)_{L^2}.$$

Received by the editors December 6, 2006.
Published electronically December 4, 2009.
Research partially supported by NSF grant DMS-0456861.
AMS subject classification: **35Q35**.

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

For more details, see [1, pp. 89–94] or [9, pp. 106–111].

The form $D^2H(u_s) = Q$ is positive definite on

$$V^1(\Omega) = \{v \in H^1(\Omega, \mathbb{R}^2) : \operatorname{div} v = 0, v\|\partial\Omega\}, \tag{1.3}$$

provided

$$\Phi'(\omega_s) \geq K > 0 \text{ on } \Omega. \tag{1.4}$$

On the other hand, $D^2H(u_s)$ is negative definite provided Ω is simply connected,

$$-\Phi'(\omega_s) \geq K > 0, \tag{1.5}$$

and, for some $\delta > 0$,

$$\|\nabla\psi\|_{L^2}^2 \leq (K-\delta)\|\Delta\psi\|_{L^2}^2, \quad \forall\, \psi \in H^2(\Omega) \cap H_0^1(\Omega). \tag{1.6}$$

In either such case, we have

$$|H(u^\varepsilon) - H(u_s)| \approx \|u^\varepsilon - u_s\|_{H^1}^2, \tag{1.7}$$

provided the right side of (1.7) is sufficiently small, and one has the stability result (1.2). (We mention that $J\nabla\psi_s = -\nabla^\perp\psi_s$, as defined in [9, (2.12)], which accounts for an apparent sign difference between (1.4)–(1.5) and the results stated there.)

Our goal in this paper is to estimate $u^\varepsilon(t) - u_s$ in stronger norms, under hypotheses on u_s that imply (1.2). In Section 2 we first establish a stability estimate for $\|u^\varepsilon(t) - u_s\|_{L^\infty}$, valid for all t, and then a slow growth estimate on $\|\operatorname{rot} u^\varepsilon(t) - \operatorname{rot} u_s\|_{L^\infty}$, *i.e.*, growth at most linear in $|t|$, with rate roughly proportional to $\|u_0^\varepsilon - u_s\|_{H^1}$ (cf. (2.5)). We then deduce such a slow growth estimate for $u^\varepsilon(t) - u_s$, in the norm of the Zygmund space $C_*^1(\overline{\Omega})$, and also in a bmo_1-norm. These are slightly weaker than the $C^1(\overline{\Omega})$-norm, but nevertheless have implications for the flow generated by $u^\varepsilon(t)$. Going from estimates in these slightly weaker norms to a $C^1(\overline{\Omega})$-estimate seems to involve a "phase shift" in the stability estimates, which shoot up to exponentially increasing in time, and further shoot up to doubly exponentially increasing for higher norm estimates. These matters are discussed in Section 3.

One ingredient in the analysis in Section 2 is an estimate similar in flavor to estimates of Brezis, Gallouet, and Wainger [3, 4]. We discuss such variants in Appendix A.

2 Stability/Slow Growth in Stronger Norms

As in Section 1, we assume that Ω is a smoothly bounded planar region and $u_s \in C^\infty(\overline{\Omega}, \mathbb{R}^2)$ is a stationary solution to (1.1), satisfying stability hypotheses that lead to (1.2). We assume u_0^ε has additional smoothness, and we desire to obtain long time estimates on $u^\varepsilon(t) - u_s$ in other norms. Let us set $\omega_s = \operatorname{rot} u_s$ and

$$v^\varepsilon(t) = u^\varepsilon(t) - u_s, \quad v_0^\varepsilon = u_0^\varepsilon - u_s, \quad \omega^\varepsilon(t) = \operatorname{rot} u^\varepsilon(t), \quad \Omega^\varepsilon(t) = \omega^\varepsilon(t) - \omega_s.$$

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

We assume $\|u_0^\varepsilon - u_s\|_{H^1} = \|v_0^\varepsilon\|_{H^1}$ is small enough that (1.2) holds. This implies

$$\|\Omega^\varepsilon(t)\|_{L^2} \le C_0\|v_0^\varepsilon\|_{H^1}, \quad \forall\, t \in \mathbb{R}. \tag{2.1}$$

We next want to estimate the L^∞-norm of $v^\varepsilon(t)$. We use the following inequality:

$$\|v^\varepsilon(t)\|_{L^\infty} \le C\Big(\log \frac{A\|\Omega^\varepsilon\|_{L^\infty}}{\|\Omega^\varepsilon\|_{L^2}}\Big)^{1/2}\|\Omega^\varepsilon(t)\|_{L^2} + C\|v^\varepsilon(t)\|_{L^2}. \tag{2.2}$$

This is similar to estimates arising in [3] and [4]. See Appendix A for a discussion of this estimate. Note that conservation of vorticity implies

$$\|\Omega^\varepsilon(t)\|_{L^\infty} \le \|\omega^\varepsilon(t)\|_{L^\infty} + \|\omega_s\|_{L^\infty} = \|\omega_0^\varepsilon\|_{L^\infty} + \|\omega_s\|_{L^\infty}.$$

Note also that

$$0 < \beta < \alpha,\ \beta \le 1 \Rightarrow \Big(\log \frac{\alpha}{\beta}\Big)^{1/2}\beta \le (\log\alpha)_+^{1/2}\beta + \Big(\log\frac{1}{\beta}\Big)^{1/2}\beta.$$

Hence we have

$$\begin{aligned}\|v^\varepsilon(t)\|_{L^\infty} \le\ & C\Big(\log A\big[\,\|\omega_0^\varepsilon\|_{L^\infty} + \|\omega_s\|_{L^\infty}\big]\Big)_+^{1/2}\|\Omega^\varepsilon(t)\|_{L^2} \\ & + C\Big(\log\frac{1}{\|\Omega^\varepsilon(t)\|_{L^2}}\Big)^{1/2}\|\Omega^\varepsilon(t)\|_{L^2} + C\|v^\varepsilon(t)\|_{L^2},\end{aligned} \tag{2.3}$$

provided $\|\Omega^\varepsilon(t)\|_{L^2} \le 1$. We now assume $C_0\|v_0^\varepsilon\|_{H^1} \le e^{-1/2}$, which then fits into (2.1). Noting that $\big(\log\frac{1}{y}\big)^{1/2} y \nearrow$, for $0 < y < e^{-1/2}$, we deduce that

$$\begin{aligned}\|v^\varepsilon(t)\|_{L^\infty} \le\ & C\Big(\log A\big[\,\|\omega_0^\varepsilon\|_{L^\infty} + \|\omega_s\|_{L^\infty}\big]\Big)_+^{1/2}\|v_0^\varepsilon\|_{H^1} \\ & + C\Big(\log\frac{1}{\|v_0^\varepsilon\|_{H^1}}\Big)^{1/2}\|v_0^\varepsilon\|_{H^1},\end{aligned} \tag{2.4}$$

for all $t \in \mathbb{R}$, noting that the term $C\|v^\varepsilon(t)\|_{L^2}$ in (2.3) can be absorbed.

The estimate (2.4) is complementary to but not stronger than (1.2). An advantage of (2.4) is that it gives us the ability to exploit the vorticity equation $\partial_t\omega^\varepsilon + \nabla_{u^\varepsilon}\omega^\varepsilon = 0$ as follows. We have

$$\partial_t\Omega^\varepsilon + \nabla_{u^\varepsilon}\Omega^\varepsilon = -\nabla_{v^\varepsilon}\omega_s, \quad \Omega^\varepsilon(0) = \omega_0^\varepsilon - \omega_s,$$

so $\Omega^\varepsilon(t,x)$ is obtained by integrating $-\nabla_{v^\varepsilon}\omega_s$ along integral curves of $\partial_t + \nabla_{u^\varepsilon}$. Hence

$$\|\Omega^\varepsilon(t)\|_{L^\infty} \le \|\Omega^\varepsilon(0)\|_{L^\infty} + C\sup_{0\le s\le t}\|v^\varepsilon(s)\|_{L^\infty}\cdot t,$$

for $t > 0$, with an analogous estimate for $t < 0$, so bringing in (2.4) gives the following conclusion.

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

Proposition 2.1 *Under hypotheses such as* (1.4) *or* (1.5)–(1.6)*, and assuming the right side of* (1.2) *is sufficiently small, one has*

$$\|\Omega^\varepsilon(t)\|_{L^\infty} \le \|\omega_0^\varepsilon - \omega_s\|_{L^\infty} + CK(u_0^\varepsilon, u_s)\|v_0^\varepsilon\|_{H^1} \cdot |t|, \quad t \in \mathbb{R}, \tag{2.5}$$

where

$$K(u_0^\varepsilon, u_s) = \Big(\log A\big[\,\|\omega_0^\varepsilon\|_{L^\infty} + \|\omega_s\|_{L^\infty}\big]\Big)_+^{1/2} + \Big(\log \frac{1}{\|v_0^\varepsilon\|_{H^1}}\Big)^{1/2}. \tag{2.6}$$

Remark 1. Of course, for large $|t|$ one has the bound

$$\|\Omega^\varepsilon(t)\|_{L^\infty} \le \|\omega^\varepsilon(t)\|_{L^\infty} + \|\omega_s\|_{L^\infty} = \|\omega_0^\varepsilon\|_{L^\infty} + \|\omega_s\|_{L^\infty}.$$

The content of (2.5) is that for given (small) $\delta > 0$, if $\|\omega_0^\varepsilon - \omega_s\|_{L^\infty} \le \delta$, then $\|\Omega^\varepsilon(t)\|_{L^\infty} \le 2\delta$ for a time interval of length

$$\approx C\frac{\delta}{\rho}, \quad \rho = \|u_0^\varepsilon - u_s\|_{H^1}\Big(\log \frac{1}{\|u_0^\varepsilon - u_s\|_{H^1}}\Big)^{1/2}.$$

To proceed with further estimates on $v^\varepsilon(t) = u^\varepsilon(t) - u_s$, we use the fact that since $v^\varepsilon(t) \in V^1(\Omega)$ and $\operatorname{rot} v^\varepsilon(t) = \Omega^\varepsilon(t)$, we have

$$v^\varepsilon(t) = J\nabla\Delta^{-1}\Omega^\varepsilon(t) + Pv^\varepsilon(t), \tag{2.7}$$

where Δ^{-1} solves the Dirichlet problem and P is the orthogonal projection of $L^2(\Omega, \mathbb{R}^2)$ onto a finite dimensional space of harmonic vector fields in $C^\infty(\overline{\Omega}, \mathbb{R}^2)$, (cf. [10, Chapter 17, Lemma 3.5]). ($P = 0$ if Ω is simply connected.) Given the estimate (1.2) on $\|v^\varepsilon\|_{H^1}$, we have global control on $Pv^\varepsilon(t)$ in quite strong norms. To estimate $J\nabla\Delta^{-1}\Omega^\varepsilon(t)$ via (2.5), we note the following mapping property of Δ^{-1}:

$$\Delta^{-1} : L^\infty(\Omega) \longrightarrow C_*^2(\overline{\Omega}), \tag{2.8}$$

where $C_*^2(\overline{\Omega})$ is a Zygmund space; (cf. [10, Chapter 13, Section 9]). Combining (2.5)–(2.8), we have the following.

Proposition 2.2 *In the setting of Proposition* 2.1,

$$\|v^\varepsilon(t)\|_{C_*^1(\overline{\Omega})} \le C\|\omega_0^\varepsilon - \omega_s\|_{L^\infty} + C\|v_0^\varepsilon\|_{H^1} + CK(u_0^\varepsilon, u_s)\|v_0^\varepsilon\|_{H^1} \cdot |t|. \tag{2.9}$$

One significant aspect of such an estimate as (2.9) is the log-Lipschitz modulus of continuity possessed by elements of $C_*^1(\overline{\Omega})$:

$$|v(x) - v(y)| \le C \log \frac{1}{|x-y|}\,|x-y| \cdot \|v\|_{C_*^1}, \quad |x-y| \le \frac{1}{2}. \tag{2.10}$$

Because of this modulus of continuity, Osgood's theorem applies to show that the t-dependent vector field $u^\varepsilon(t)$ generates a uniquely defined flow, though estimates on

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

such a flow are not as good as they would be if the $C^1_*(\overline{\Omega})$ estimate could be replaced by an equally strong $C^1(\overline{\Omega})$ estimate. In Section 3 we will obtain $C^1(\overline{\Omega})$ estimates, but the upper bounds will be larger than they are in (2.9).

The work of Chang, Dafni and Stein [5] produces a slightly sharper result than (2.9). By Theorem 5.8 of that paper,

$$|\alpha| \le 2 \Longrightarrow ED^\alpha \Delta^{-1} \colon L^\infty(\Omega) \to \mathrm{bmo}(\mathbb{R}^2),$$

where for a function f on Ω, one sets $Ef(x) = f(x)$ for $x \in \Omega$, 0 for $x \in \mathbb{R}^2 \setminus \Omega$. Consequently, (2.9) is sharpened to

$$\|E\nabla v^\varepsilon(t)\|_{\mathrm{bmo}(\mathbb{R}^2)} \le C\|\omega_0^\varepsilon - \omega_s\|_{L^\infty} + C\|v_0^\varepsilon\|_{H^1} + CK(u_0^\varepsilon, u_s)\|v_0^\varepsilon\|_{H^1} \cdot |t|. \tag{2.11}$$

While (2.11) is stronger than (2.9), it does not yield a modulus of continuity estimate stronger than (2.10).

Remark 2. In addition to applicability to results on flows generated by the velocity field u^ε, another advantage of the estimates in Proposition 2.1 over the H^1-estimate (1.2) arises from the following consideration (pointed out by the referee). One does not have a uniqueness result for weak solutions to the Euler equation (1.1) with initial data u_0^ε in $V^1(\Omega)$, defined by (1.3). However, under the additional condition that rot u_0^ε belong to $L^\infty(\Omega)$, one does have global existence and uniqueness; (cf. [7, 8]).

3 C^1 and H^k Estimates

We desire to complement the estimates in Section 2 on $v^\varepsilon(t) = u^\varepsilon(t) - u_s$ with estimates in the C^1 norm and in H^k norm. A major ingredient will be estimates in these norms of $u^\varepsilon(t)$, given as in (1.1). A crucial connection between these estimates is given by the estimate from [2] of the following type:

$$\|\nabla u^\varepsilon\|_{L^\infty} \le C\Big(1 + \log \frac{A\|u^\varepsilon\|_{H^3}}{\|\omega^\varepsilon\|_{L^\infty}}\Big)\|\omega^\varepsilon\|_{L^\infty} + C\|\nabla u^\varepsilon\|_{L^2}, \tag{3.1}$$

established in the context of bounded regions in [10, Section 3, Chapter 17]. As we have seen, conservation of vorticity gives

$$\|\omega^\varepsilon(t)\|_{L^\infty} \le C. \tag{3.2}$$

A standard attack on estimating $\|u^\varepsilon(t)\|_{H^k}$ starts with

$$\frac{d}{dt}\|u^\varepsilon\|_{H^k}^2 = -2(P\nabla_{u^\varepsilon} u^\varepsilon, u^\varepsilon)_{H^k}, \tag{3.3}$$

where P is the Helmholtz projection. Then an integration by parts combined with Gagliardo–Nirenberg–Moser estimates give

$$|(P\nabla_{u^\varepsilon} u^\varepsilon, u^\varepsilon)_{H^k}| \le C\|u^\varepsilon\|_{C^1}\|u^\varepsilon\|_{H^k}^2; \tag{3.4}$$

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

(cf. (3.24) in [10, Chapter 17]. It follows that

$$\frac{d}{dt}\|u^\varepsilon\|_{H^k}^2 \le C\|u^\varepsilon\|_{C^1}\|u^\varepsilon\|_{H^k}^2. \tag{3.5}$$

Let us set $G_k^\varepsilon(t) = \|u^\varepsilon(t)\|_{H^k}^2$. Using (3.1) and (3.2) we see that if $k \ge 3$,

$$\frac{d}{dt}G_k^\varepsilon(t) \le C\big(1 + \log^+ G_k^\varepsilon(t)\big)\, G_k^\varepsilon(t). \tag{3.6}$$

Gronwall's inequality then yields an estimate

$$\|u^\varepsilon(t)\|_{H^k} \le e^{Ce^{Ct}}, \quad 0 \le t. \tag{3.7}$$

Taking $k = 3$ and using (3.1) and (3.2) again, we have

$$\|\nabla u^\varepsilon(t)\|_{L^\infty} \le Ce^{Ct}, \quad 0 \le t. \tag{3.8}$$

Remark 3. The estimates (3.1)–(3.8) are valid for any smooth initial data $u^\varepsilon(0,x) = u_0(x)$, not necessarily producing a stationary solution at $\varepsilon = 0$.

Now assume $u^0(t,x) \equiv u_s(x)$ is a stationary solution satisfying either the hypotheses (1.4) or (1.5) and (1.6), so we have estimates on $v^\varepsilon = u^\varepsilon - u_s$ and on $\Omega^\varepsilon = \operatorname{rot} v^\varepsilon$ given in (1.2), (2.4), and (2.5). Parallel to (3.1), we have

$$\|\nabla v^\varepsilon\|_{L^\infty} \le C\Big(1 + \log\frac{A\|v^\varepsilon\|_{H^3}}{\|\Omega^\varepsilon\|_{L^\infty}}\Big)\|\Omega^\varepsilon\|_{L^\infty} + C\|\nabla v^\varepsilon\|_{L^2}.$$

Since $\|v^\varepsilon\|_{H^k} \le \|u^\varepsilon\|_{H^k} + \|u_s\|_{H^k}$, we can use (3.7) to deduce that

$$\|\nabla v^\varepsilon(t)\|_{L^\infty} \le C\Big(Ce^{C|t|} + \log\frac{1}{\|\Omega^\varepsilon(t)\|_{L^\infty}}\Big)\|\Omega^\varepsilon(t)\|_{L^\infty} + C\|\nabla v^\varepsilon(t)\|_{L^2}.$$

We can insert (2.5) and (1.2) into this estimate, to obtain

$$\|\nabla v^\varepsilon(t)\|_{L^\infty} \le Ce^{C|t|}\Big(\|\omega_0^\varepsilon - \omega_s\|_{L^\infty} + CK(u_0^\varepsilon, u_s)\|v_0^\varepsilon\|_{H^1}\cdot|t|\Big).$$

It would be interesting to know whether one could replace the exponential factor $e^{C|t|}$ by something smaller. Such estimates are obtained in [6] in a related setting, but with dissipation (and small forcing) added to (1.1) (and with Ω replaced by a torus). For estimates there, dissipation plays a crucial role.

A Discussion of the BGW-type Estimate (2.2)

We discuss the estimate (2.2), *i.e.*,

$$\|u\|_{L^\infty} \le C\Big(1 + \log\frac{A\|\omega\|_{L^\infty}}{\|\omega\|_{L^2}}\Big)^{1/2}\|\omega\|_{L^2} + C\|u\|_{L^2}, \tag{A.1}$$

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

and variants, which are similar to estimates arising in [3] and [4]. (The slight difference in appearance between (2.2) and (A.1) can be accounted for by adjusting A.) Here $\omega \in L^\infty(\Omega)$, where Ω is a smoothly bounded planar domain, and $\omega = \operatorname{rot} u$, with $u \in V^1(\Omega)$, defined by (1.3). One has, as in (2.7), $u = J\nabla\Delta^{-1}\omega + Pu$, where Δ^{-1} solves the Dirichlet problem and P is an orthogonal projection of $L^2(\Omega, \mathbb{R}^2)$ onto a finite dimensional space of harmonic vector fields in $C^\infty(\overline{\Omega}, \mathbb{R}^2)$. In particular, $\|u\|_{H^1} \approx \|\omega\|_{L^2} + \|u\|_{L^2}$, and for any given $r \in (0,1)$, $\|u\|_{C^r} \le C\|\omega\|_{L^\infty} + C\|u\|_{L^2}$. Hence (A.1) follows from

$$\|u\|_{L^\infty} \le C\Big(1 + \log \frac{A\|u\|_{C^r}}{\|u\|_{H^1}}\Big)^{1/2} \|u\|_{H^1},$$

given $u \in H^1(\Omega) \cap C^r(\overline{\Omega})$, where Ω is a smoothly bounded planar domain. Standard extension maps allow us to work instead on $\mathbb{T}^2$.

More generally, working on $\mathbb{T}^n$, we claim that

$$\|u\|_{L^\infty} \le C\Big(1 + \log \frac{A\|u\|_{C^r}}{\|u\|_{H^{n/p,p}}}\Big)^{1-1/p} \|u\|_{H^{n/p,p}}, \tag{A.2}$$

given $1 < p < \infty$.

To get this, take $\Psi \in C_0^\infty(\mathbb{R})$, with $\Psi(s) = 1$ for $|s| \le 1$, 0 for $|s| \ge 2$, and write

$$u = \Psi(\varepsilon D)u + (I - \Psi(\varepsilon D))u.$$

There is the elementary estimate $\|(I - \Psi(\varepsilon D))u\|_{L^\infty} \le C\varepsilon^r \|u\|_{C^r}$. We claim that

$$\|\Psi(\varepsilon D)u\|_{L^\infty} \le C\Big(\log \frac{1}{\varepsilon}\Big)^{1-1/p} \|u\|_{H^{n/p,p}}. \tag{A.3}$$

Given this, picking ε such that $\varepsilon^r = \frac{\|u\|_{H^{n/p,p}}}{\|u\|_{C^r}}$ then gives (A.2).

The estimate (A.3) is equivalent to

$$\|\Psi(\varepsilon D)\Lambda^{-n/p} v\|_{L^\infty} \le C\Big(\log \frac{1}{\varepsilon}\Big)^{1-1/p} \|v\|_{L^p}, \tag{A.4}$$

where $\Lambda = (1-\Delta)^{1/2}$. We have $\Lambda^{-s} v = J_s * v$ where, for $0 < s < n$, $J_s \in C^\infty(\mathbb{T}^n \setminus 0)$ and $J_s(x) \sim C|x|^{s-n}$, $\quad |x| \le 1$. It follows that $\Psi(\varepsilon D)\Lambda^{-s} v = K_{s,\varepsilon} v$, where

$$|K_{s,\varepsilon}(x)| \le \begin{cases} C\varepsilon^{s-n} & \text{for } |x| \le \varepsilon, \\ C|x|^{s-n} & \text{for } |x| \ge \varepsilon. \end{cases}$$

It then follows that if $qs - qn + n = 0$,

$$\|K_{s,\varepsilon}\|_{L^q}^q \le C\varepsilon^{q(s-n)}\varepsilon^n + \int_\varepsilon^1 r^{q(s-n)} r^{n-1}\, dr = C + C\log\frac{1}{\varepsilon}.$$

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press

Note that

$$s = \frac{n}{p}, \ \frac{1}{p} + \frac{1}{q} = 1 \Longrightarrow qs - qn + n = nq\Big(\frac{1}{p} - 1 + \frac{1}{q}\Big) = 0.$$

Thus, with $q = p'$,

$$\|\Psi(\varepsilon D)\Lambda^{-n/p}v\|_{L^\infty} \le \|K_{n/p,\varepsilon}\|_{L^q}\|v\|_{L^p} \le C\Big(\log\frac{1}{\varepsilon}\Big)^{1/q}\|v\|_{L^p},$$

which yields the asserted estimate (A.4). The proof of (A.2) is complete.

Acknowledgment Thanks to an anonymous referee for insightful comments.

References

[1] V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics.* Applied Mathematical Sciences, Springer-Verlag, New York, 1998.

[2] T. Beale, T. Kato, and A. Majda, *Remarks on the breakdown of smooth solutions of the* 3*-D Euler equations.* Comm. Math. Phys. **94**(1984), no. 1, 61–66. doi:10.1007/BF01212349

[3] H. Brézis and T. Gallouet, *Nonlinear Schrödinger evolutions.* Nonlinear Anal. **4**(1980), no. 4, 677–681. doi:10.1016/0362-546X(80)90068-1

[4] H. Brézis and S. Wainger, *A note on limiting cases of Sobolev embeddings and convolution inequalities.* Comm. Partial Differential Equations **5**(1980), no. 7, 773–789. doi:10.1080/03605308008820154

[5] D.-C. Chang, G. Dafni, and E. Stein, *Hardy spaces, BMO, and boundary value problems for the Laplacian on a smooth domain in* $\mathbb{R}^n$. Trans. Amer. Math. Soc. **351**(1999), no. 4, 1605–1661. doi:10.1090/S0002-9947-99-02111-X

[6] E. Grenier, C. Jones, F. Rousset, and B. Sandstede, *Viscous perturbations of marginally stable Euler flow and finite-time Melnikov theory.* Nonlinearity **18**(2005), no. 2, 465–483. doi:10.1088/0951-7715/18/2/001

[7] T. Kato, *On classical solutions of the two dimensional nonstationary Euler equation.* Arch. Rational Mech. Anal. **25**(1967), 188–200.

[8] V. I. Judovič, *Non-stationary flows of an ideal incompressible fluid.* Ž. Vyčisl. Mat. i Mat. Fiz **3**(1963), 1032–1066.

[9] C. Marchioro and M. Pulvirenti, *Mathematical theory of incompressible nonviscous fluids.* Applied Mathematical Sciences 96, Springer-Verlag, New York, 1994.

[10] M. Taylor, *Partial differential equations*, Vols. 1–3. Applied Mathematical Sciences 115–116. Springer-Verlag, New York, 1996.

Mathematics Department, University of North Carolina, Chapel Hill, NC 27599, USA
e-mail: met@email.unc.edu

https://doi.org/10.4153/CMB-2010-031-8 Published online by Cambridge University Press