SUBGROUPS OF INFINITE INDEX IN THE MODULAR GROUP III

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As in [**4**], a *specification* is a list $(r, s, t_1, h_0, h_\infty, c(1), \ldots, c(h_0))$ such that

(i) each of r, s, t_1, h_0, h_∞ is a non-negative integer or ∞,

(ii) for each i, $c(i)$ is a positive integer,

(iii) if $h_\infty = 0$ then $h_0 = \infty$,

(iv) if $h_\infty = 1$ and $t_1 + h_0$ is finite then t_1 is even,

(v) $r + s + t_1 + h_0 + h_\infty = \infty$.

Each subgroup of infinite index in the modular group Γ has associated with it a specification. The sublist $(r, s, t_1, h_0, h_\infty)$ is the *short specification* (of the subgroup). The sequence $(c(i))$ is the *cusp-split* (of the subgroup). For each $k \geq 1$, $h(k)$ is the number of i such that $c(i) \leq k$.

In [**6**] it was shown that a specification with $h_\infty = 0$ is the specification of a subgroup if and only if

$$\sum (c(j) - 6) \geq 3r + 4s + 6t_1 - 6.$$

Here we consider cases with $h_\infty > 0$. We shall prove the following theorems.

THEOREM 1. *A specification with $h_\infty \geq 2$ is the specification of a subgroup of Γ.*

THEOREM 2. *A specification with $h_\infty = 1$ is the specification of a subgroup of Γ if either of the following is satisfied*:

(i) $h_0 - h(5) = \infty$,

(ii) t_1 *is even or infinite and* $r + s + t_1 + h(1) \geq 1$.

The latter gives a sufficient condition.

THEOREM 3. *If a subgroup in Γ has $h_\infty = 1$ and $t_1 + h_0 - h(5)$ finite then t_1 is even.*

This generalises Theorem 4.4 of [**4**] and shows that the first condition in Theorem 2(ii) is necessary. Example 3.1 shows that the other condition is not necessary. In a sense, the case $h_\infty = 1$ is similar to that for subgroups of finite index, see [**1**], [**2**], [**3**].

These theorems will be proved by the diagrams discussed in [**4**], [**5**] and [**6**].

1. Basic diagrams. Suppose that $(A(1), A(2), \ldots)$ is a finite or infinite sequence of D-diagrams such that, for each j, $A(j)$ has a green polygon with red loops at vertices $V(j, 1)$ and $V(j, 2)$. For $j = 1, 2, \ldots$, we can join the pair $(V(j, 2), V(j+1, 1))$. The result is a D-diagram with a green polygon including all of the $V(j, k)$ and having a red loop at $V(1, 1)$; see Lemma 3.2(i) of [**4**]. We refer to this process as the joining of the sequence, and write $D(A(j))$ for the resultant diagram.

To construct the diagrams we shall require later, we use some of the diagrams introduced in [**4**]. We need some new diagrams to allow us to introduce green polygons with less than six sides.

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DEFINITION 1.1. The diagrams $P(i)$ $(i=2,3,4,5,\ldots)$.

Let $i\geq 2$. We construct an L-diagram $Q(i)$ with vertex set U, V. We add i non-intersecting edges from U to V, ordered anticlockwise at each vertex. We add i free edges at V, one between each adjacent pair of edges from U. These free edges are included in the ordering at V in the obvious way. Then we have an L-diagram (each triangle consists of a pair of edges adjacent at U and the free edge between them at V). The corresponding D-diagram $P(i)$ has $3i$ vertices, one green polygon with i sides (corresponding to U) and one with $2i$ sides and i red loops (corresponding to V with its free edges). Since $Q(i)$ has no degenerate triangles, $P(i)$ has no blue loops.

Suppose that $(i_1, i_2, \ldots)$ is a sequence of integers, each greater than one. Since $P(i_j)$ has i_j (≥ 2) red loops on the larger green polygon, we can join the sequence $(P(i_j))$. By Lemma 3.2 of [**4**], the green i_j-gon from $P(i_j)$ is unaffected, so that $D(P(i_j))$ has green polygons of size $i_1, i_2, \ldots$. Further, there is one other green polygon of size $\sum i_j$ with at least one red loop.

DEFINITION 1.2. The diagrams B_j $(j=2,3,4,5)$.

There are three regular solids with triangular faces. In each case we can order the edges at each vertex anticlockwise with respect to the orientation. We obtain an L-diagram with no degenerate triangles and no free edges.

From the tetrahedron, which has vertices of degree three, we get the D-diagram A_3 with four green polygons of size three and no loops. Each red edge connects vertices in distinct green polygons since the L-diagram has no loops. We cut one edge of A_3 to obtain B_3. By Lemma 3.5 of [**4**], B_3 has two green 3-gons and one green 6-gon, the last having the two red loops produced by cutting. B_3 has no other loops of any colour.

From the octahedron, we obtain B_4 with four green 4-gons and one green 8-gon with two red loops.

From the icosahedron, we obtain B_5 with ten green 5-gons and one green 10-gon with two red loops.

We can regard a triangle as a two faced "solid". From this we obtain B_2 with one green 2-gon and one green 4-gon with two red loops. This is identical to the diagram $P(2)$ of Definition 1.1.

DEFINITION 1.3. The diagrams $B(j,n)$ $(j=2,3,4,5;\ n=2,3,4,\ldots)$.

For each j, n, $B(j,n)$ is a D-diagram with a number of green j-gons, one green n-gon and one further green polygon. This last polygon will have two red loops, and these will be the only loops of the diagram. Once again, we begin with L-diagrams.

For $j=2,3$, we use the diagrams shown in Figure 1. We give the case $B(2,2n+1)$ in detail. We take one copy of 1(a) and n copies of 1(d). We replace the broken edge of 1(a) with the copies of 1(d), identifying all the vertices U, to give a single vertex U, and all the vertices V. This is done in such a way that the edges of the new diagram do not intersect. The result is an L-diagram with n vertices of degree two, one of degree $2n+1$, viz. U, and one of degree $2n+5$, viz. V. The diagram has two free edges, both at V. The corresponding D-diagram is $B(2,2n+1)$.

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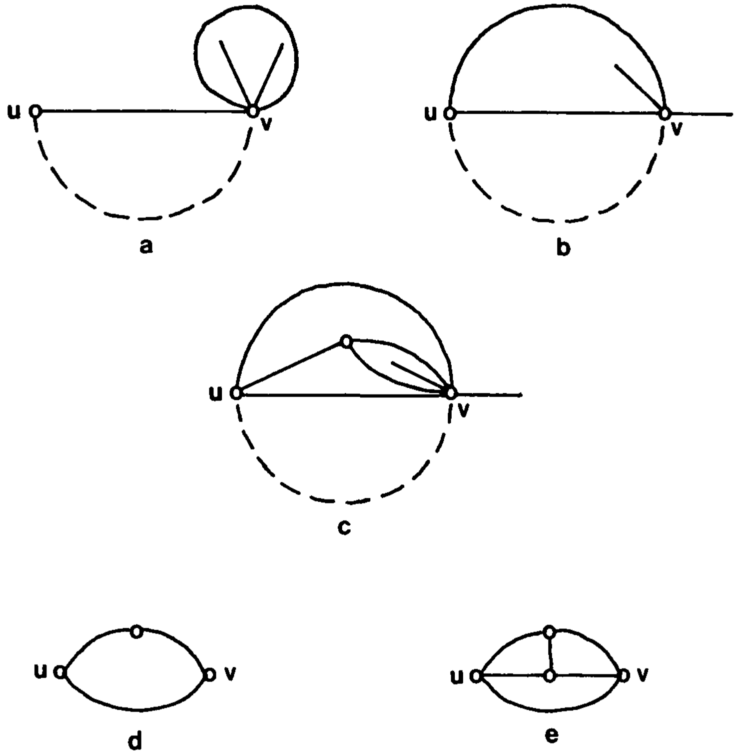

Figure 1.

To obtain $B(2, 2n+2)$, we use 1(b) in place of 1(a) in the above construction.

To obtain $B(3, 3n+1)$ (resp. $B(3, 3n+2)$, $B(3, 3n+3)$), we use 1(a) (resp. 1(b), 1(c)) together with n copies of 1(e).

For $j=4, n \geq 2$, we define the L-diagram $A(4, n)$. This has vertex set $\{U, V, X_1, \ldots, X_n\}$. The X_i are the successive vertices of a convex n-gon P whose sides are edges of $A(4, n)$. The vertex U is inside P and V outside. Each is joined all of the X_i, the ordering of the edges round U and V being determined by the suffices on the X_i. Round X_i, the edges are, in order, $X_iU, X_iX_{i-1}, X_iV, X_iX_{i+1}$, the suffices being taken modulo n. Clearly, $A(4, n)$ has no free edges and no degenerate triangles. The corresponding D-diagram has n green 4-gons (corresponding to the X_i) and two green n-gons (U and V), and has no loops of any colour. The red edge corresponding to UX_i joins vertices in a 4-gon and an n-gon. By Lemma 3.5 of [**4**], the cut of this edge gives the D-diagram $B(4, n)$ with the following properties. $B(4, n)$ has a green n-gon (from V), $n-1$ green 4-gons (from $X_2, \ldots, X_n$) and a green $(n+4)$-gon (from U and X_1). The last has two red loops, these being produced by the cut and being the only loops on the diagram.

For $j=5, n \geq 2$, we define the L-diagram $A(5, n)$. This has vertex set $\{U, V, X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$. Let P_1, P_2 be convex polygons with P_1 inside P_2. We label the vertices of P_1 (resp. P_2) anticlockwise by the X_i (resp. Y_i). The edges of each polygon are edges of $A(5, n)$. Also, for each i, X_i is joined to Y_i and to Y_{i+1}. The vertex U is inside P_1 and joined to each X_i, and V outside P_2 and joined to each Y_i. With the edges at each vertex ordered in an obvious way, $A(5, n)$ is an L-diagram without free edges or degenerate triangles. The vertices U and V have degree n, while each X_i and each Y_i has degree 5. As in the previous case, we cut the edge corresponding to UX_1 in the D-diagram corresponding to $A(5, n)$. This yields the D-diagram $B(5, n)$ with a green n-gon (from V), $2n-1$ green 5-gons and one green $(n+5)$-gon, the last having two red loops. These are the only loops.

DEFINITION 1.4. Diagrams from [**4**].

The first few of these have a single red loop, so that any one of them can be joined to the join of a sequence as defined in the first paragraph of this section.

S: this increases "s" by one.

T: this increases "t_1" by two.

P: this increases "h_0" by one, the new "$c(i)$" being 1.

X: this gives a diagram with $h_\infty = \infty$, whatever the original value.

To the list, we add

R^*: the null diagram, used conventionally to leave a red loop not to be used for a subsequent join.

We also require R, a blue triangle with three red loops, and C, an unbranched chain of blue triangles joined by red edges extending infinitely in one direction. C has infinitely many red loops and so can be used to combine an infinite collection of S, T, etc. into a single diagram.

DEFINITION 1.5. Finite trees.

On several occasions, we shall have to combine a finite collection $D(1), \ldots, D(n)$ of D-diagrams, each of which has a red loop at a vertex on an infinite green component.

If $n > 2$, we take $n-2$ copies of R and form a connected diagram by $n-3$ joining operations. Since we have used a minimal number of joins (i.e. red edges) to achieve connectedness, the resultant diagram has no cycles other than the blue triangles. Since the diagram is finite, $h_\infty = 0$, and it is easy to see that the diagram has short specification $(n, 0, 0, 1, 0)$. We make a *tree* of $\{D(i)\}$ by joining each red loop on this finite diagram to one of the $D(i)$.

If $n = 2$, we define the *tree* as the join of $D(1)$ and $D(2)$. The next proposition follows by a straight-forward application of 3.1, 3.2, 3.3 of [**4**].

PROPOSITION 1.6. *Suppose that $D(1), \ldots, D(n)$ is a collection of D-diagrams such that, for each i, $D(i)$ has short specification $(1 + r(i), s(i), t_1(i), \infty, h_\infty(i))$, and has a red loop at $A(i)$ on an infinite green component. Then the tree of $\{D(i)\}$ is a D-diagram with short specification $(\sum r(i), \sum s(i), \sum t_1(i), \infty, \sum h_\infty(i))$. Further, the cusp-split of the tree is obtained by combining those of the $D(i)$.*

2. Proof of Theorem 1.

LEMMA 2.1. *A specification with* $h_\infty \geq 2$ *and* $r+s+t_1+h(1)+h_\infty=\infty$ *is the specification of a subgroup of* Γ.

Proof. Suppose first that we have a specification of the given type which also has t_1 even or infinite.

For each i with $c(i)\geq 2$, we take one copy of $P_{c(i)}$. If $h_0-h(2)$ is finite, we add to this collection an infinite number of copies of R. Let $\mathcal{S}$ denote the collection obtained. Observe that P_j $(j\geq 3)$ and R have a green polygon with at least three red loops. By the construction of $\mathcal{S}$, we have infinitely many parts with more than two loops. Then $D(\mathcal{S})$ will have infinitely many red loops on its (only) infinite green polygon.

If h_∞ is finite then we join $h_\infty-1$ copies of C to $D(\mathcal{S})$ to form the diagram D^*. Applying 3.2 of [**4**], we find that this has h_∞ infinite green components. As in Theorem 4.3 of [**4**], we can join, to the (infinitely many) red loops of D^*, r copies of R^*, s of S, $\frac{1}{2}t_1$ of T, with the obvious interpretation if t_1 is infinite, and $h(1)$ of P. As h_∞ is finite, the second hypothesis implies that $r+s+t_1+h(1)$ is infinite, so that this process can be carried out in such a way that each red loop of D^* is used. With this proviso, it is easy to see that the resultant diagram has the required values of the parameters.

If h_∞ is infinite, we add infinitely many copies of X to the list above, and join all of them to $D(\mathcal{S})$ itself.

Note that, in the case where t_1 is even or infinite, we can construct a diagram whenever $h_\infty\geq 1$.

Now suppose that t_1 is odd. To $D(\mathcal{S})$ we join one copy of R. We then compose this with a copy of C (see [**4**]). This gives a diagram $D(\mathcal{S})^*$ with two infinite green components, each having infinitely many red loops. We use $D(\mathcal{S})^*$ in place of $D(\mathcal{S})$, except that we use $\frac{1}{2}(t_1-1)$ copies of T and, where h_∞ is finite, $h_\infty-2$ copies of C. Here we do need the hypothesis that $h_\infty\geq 2$.

COROLLARY 2.2. *A specification with* $h_\infty\geq 1$ *and* $r+s+t_1+h(1)=\infty$, *and with* t_1 *even or infinite is the specification of a subgroup of* Γ.

The proof of the next result is a refinement of that of Theorem 2.2 of [**5**]. We construct a basic L-diagram using the technique of the earlier result, making some alterations to get vertices of low degree. We then amend the corresponding D-diagram to get a diagram with the desired specification.

LEMMA 2.3. *A specification with* $h_\infty\geq 1$ *and* $h_0-h(5)$ *infinite is the specification of a subgroup of* Γ.

Proof. The second condition implies that $h_0=\infty$ and that there are infinitely many values of i for which $c(i)\geq 6$. Let $\mathcal{S}$ denote the subsequence consisting of these "large" $c(i)$.

Case 1: $h_\infty=1$, t_1 *even or infinite.* We split $\mathcal{S}$ into two parts:

$$\mathcal{S}_1=(d(n):n=1,2,\ldots,a=r+s+[\tfrac{1}{2}t_1]+h(2)),$$

$$\mathcal{S}_2=(e(j):j\in\mathbb{Z}).$$

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We note that $\mathcal{S}_2$ is infinite, $\mathcal{S}_1$ may be finite.

From $\mathcal{S}_1$ we define the sequence $\mathcal{S}_1^+=(f(n):n=1,\ldots,a)$, where $f(n)$ is equal to $d(n)-1$ for $r+s+[\frac{1}{2}t_1]+h(1)$ values of n and to $d(n)-2$ for the remaining $h(2)-h(1)$ values. Note that, for all n, $f(n)\geq 4$. We also define

$$\mathcal{S}_1^-=(f(-m):m=1,2,\ldots,b=h(5)-h(2)),$$

a renumbering of the subsequence of the cusp-split consisting of the $c(i)$ equal to 3, 4 or 5. $\mathcal{S}_1^-$ may be finite. Note that, for $n<0$, and so for all n, $f(n)\geq 3$.

Let Q be the origin of the usual coordinate plane and, for $k\in\mathbb{N}$, let $C(k)$ be the line $y=k$.

We begin the construction of E_1 by drawing (straight) edges from Q to each of the integer points on $C(1)$. For $j\in\mathbb{Z}$, $(j,1)$ will be a vertex of E_1. For $n\in\{-b,\ldots,-1,1,\ldots,a\}$, we add a new vertex X_n at $(2n+\frac{1}{2},\frac{1}{2})$, i.e. in the triangle with vertices Q, $(2n,1),(2n+1,1)$. We add $f(n)-3$ new vertices on $C(1)$ between $(2n,1)$ and $(2n+1,1)$; this is possible as $f(n)\geq 3$. We add edges from X_n to Q, to $(2n,1)$, to the new vertices on $C(1)$, and to $(1,2n+1)$. Then X_n has degree $f(n)$. Finally, we add edges consisting of all segments of $C(1)$ lying between vertices on that line. These edges give the boundary of E_1 (in the sense of [**6**]).

Figure 2 shows the start of this construction in a case where $f(-1)=3$, $f(1)=4$.

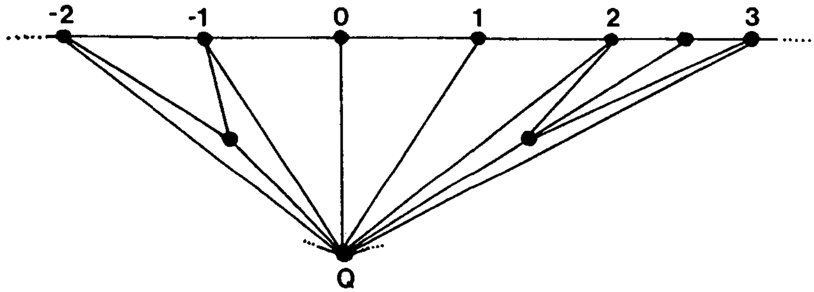

Figure 2.

As in 2.2 of [**5**], we define inductively the sequence (E_k), with E_k having as its boundary a set of vertices on $C(k)$ with the segments between these as edges. For $k\geq 2$, we construct E_k so that the vertices on $C(k-1)$ have degrees equal to the $e(j)$ with $j=2^{k-1}(2m+1)$, $m\in\mathbb{Z}$. This is possible since, for all j, $e(j)\geq 6$. The vertices below $C(k)$ are unaltered after E_{k+1} is constructed, so we get as a limit an L-diagram L with cusp-split given by $\mathcal{S}_2\cup\mathcal{S}_1^+\cup\mathcal{S}_1^-$ and $h_\infty=1$. L has no free edges or degenerate triangles. Further, the vertices X_n include those with degrees given in $\mathcal{S}_1^+$ and all are joined to Q, the vertex of infinite degree.

We now amend L to increase the degree of the X_n for the n with $n<0$, if any.

First, consider the $r+s+[\frac{1}{2}t_1]+h(1)$ cases with $f(n)=d(n)-1$. For these, we add an additional edge from Q to X_n, following the original edge QX_n at each vertex. We add a free edge at Q between the two edges to X_n. This increases the degree of X_n to $d(n)$.

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Now consider the cases with $f(n)=d(n)-2$. Now we add an additional edge QX_n as above, and a new vertex Y_n in the region between these edges. We join Y_n to Q and to X_n, in each case the edge inserted between the edges QX_n at the vertex. The result is an L-diagram with X_n of degree $d(n)$ and Y_n of degree 2.

When both types of alteration have been made, we have L^*, an L-diagram with short specification $(r+s+[\frac{1}{2}t_1]+h(1), 0, 0, \infty, 1)$. (We postpone the proof of the value of t_1 in this case and in case 2 until we have disposed of the other cases.)

Let D^* be the D-diagram corresponding to L^*. Then D^* has $r+s+[\frac{1}{2}t_1]+h(1)$ red loops. We can use these to join to D^*, r copies of R^*, s of S, $[\frac{1}{2}t_1]$ of T and $h(1)$ of Q. The result is a D-diagram with the required specification.

Case 2: $h_\infty=1$, t_1 *odd.* We begin by splitting $\mathscr{S}$ into two infinite parts $\mathscr{S}^0$ and $\mathscr{S}^1$. We treat $\mathscr{S}^0$ like $\mathscr{S}$ in case 1. We proceed as before except that we use six vertices of degree three on the boundary of E_2 to add a new boundary component as in (h) of §3 in [**6**]. We "fill" this with vertices whose degrees form the sequence $\mathscr{S}^1$. The only other change is that we add $\frac{1}{2}(t_1-1)$ copies of T at the final stage. We get a diagram of the required specification, the value of t_1 being verified below.

Case 3: $h_\infty=\infty$. We proceed as in cases 1, 2, but create $r+s+[\frac{1}{2}t_1]+h(1)+1$ red loops in D^*. This additional red loop is used to join a copy of X. This gives $h_\infty=\infty$ without affecting the other parameters, since the red loop of D^* lies on the infinite green component, and 3.2, 3.3 of [**4**] apply.

Case 4: $1<h_\infty<\infty$. We begin by dividing $(c(i))$ into h_∞ infinite subsequences $\mathscr{S}(1), \ldots, \mathscr{S}(h_\infty)$, each with infinitely many terms greater than five (recall that $h_0-h(5)$ is infinite).

As above, we can construct a diagram $D(1)$ with short specification $(r+1, s, t_1, \infty, 1)$ and $\mathscr{S}(1)$ as cusp-split. For $i>1$, we construct $D(i)$ with short specification $(1, 0, 0, \infty, 1)$ and $\mathscr{S}(i)$ as cusp-split. By the above method, we obtain diagrams with a red loop on the infinite green component. By 1.6, the tree of the $D(i)$ has the required specification.

To verify the values of t_1 in the first two cases, we use the ideas of §2 of [**4**]. We translate the tree T_2 of a D-diagram and the associated graph G into the language of L-diagrams.

First observe that the spanning tree T_2 of a D-diagram D consists of green edges and red edges joining vertices in distinct green components. In L, the corresponding L-diagram, the green components of D correspond to vertices of L and the red edges of D to edges of L other than free edges. Hence T_2 corresponds to a spanning tree of L. We use the term T_2 for this tree as well.

Now suppose that L can be drawn without intersections on the plane in such a way that the (L-diagram) ordering of the edges at each vertex is anticlockwise ordering. Then each (L-diagram) triangle defines a bounded region of the plane, and these regions do not overlap. The diagrams produced in cases 1 and 2 have this property, and also are free from loops. In the absence of loops, each "triangle" has two or three sides, the former when a free edge is involved. The blue triangles of D correspond to these regions and two triangles have vertices joined by a red edge if and only if the regions have a common side. The pseudograph showing blue triangles and red edge connections is then the graphtheoretic dual of $L-\{\text{free edges}\}$. We refer to this as G^*. If we delete from G^* those edges which cross T_2 edges of L then we get the graph G of [**4**].

In Lemma 2.2 of [**4**], we directed some of the edges of G so that each vertex has one edge directed towards it. By Lemma 2.5 of [**4**], t_1 is equal to the number of undirected edges of G. We shall show that in the L-diagram for case 1 every edge of G must be directed, while in case 2 there must be exactly one undirected edge. Thus the relevant L-diagram has t_1 equal to zero (resp. one). The addition of the stated number of copies of T then gives the final value required.

Case 1. We begin by constructing a suitable T_2. For $k \geq 1$, we take all edges which are segments of $C(k)$ and one edge e_k from a vertex on $C(k)$ to a vertex on $C(k+1)$. We add the edge $Q(0, 1)$. Finally, for relevant n, we add one of the edges QX_n (when there is no Y_n) or QY_n and Y_nX_n (when there is a Y_n). This gives a spanning tree.

Now consider the G for the diagram with this particular T_2. In the region between $C(k)$ and $C(k+1)$, each triangle has one side on $C(k)$ or one on $C(k+1)$. These edges belong to T_2. The remaining edges contribute a single unbranched chain to G^*. The removal of the edge corresponding to e_k (which also belongs to T_2) gives two unbranched chains, each extending infinitely in one direction, to G. Now consider the region between $y=0$ and $C(1)$.

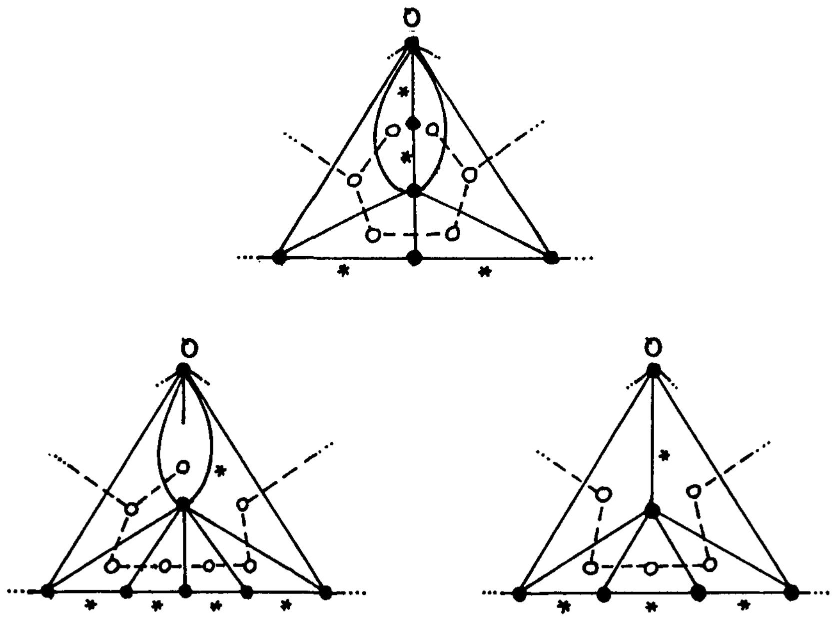

Figure 3.

Figure 3 shows the situation round each type of X_n. The starred edges are in T_2, the broken edges are those of G. Taking into account the removal of the edge $Q(0, 1)$, we see that this region contributes to G two chains, each extending infinitely in one direction,

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one is unbranched, the other has $h(1)+2(h(2)-h(1))$ branches of length one (from the corresponding X_n and Y_n).

It is clear that a tree extending infinitely from a point A and which has only finite branches from the "trunk" can be directed in one way to meet the criterion above. Every edge must be directed. It follows that each edge of each component of G is directed. Hence the diagram has t_1 equal to zero.

Case 2. We proceed as above, taking care to choose e_1, e_2 to the right of the vertices used to create the new boundary component introduced between $C(2)$ and $C(3)$. We now show how we can extend T_2 and G into this region.

For (relative) ease of description, we deform the new boundary into the unit circle round the origin. Each boundary vertex has degree 3 or 4 and we continue the construction inwards, the kth stage consists of vertices on and arcs of $C^*(k)=\{(x, y): x^2+y^2=k^{-2}\}$. For T_2, we use one edge f_k from $C^*(k)$ to $C^*(k+1)$ and all but one of the arcs of $C^*(k)$. It is convenient to choose the f_k so that they form a connected chain, and, for each k, to omit an arc which meets f_k. To connect this with the rest of the spanning tree we add an edge from $C(2)$ which lies on the new boundary.

The corresponding G consists of components like those of case 1 except in the region between $C(1)$ and $C(2)$ to the right of e_1 and that within the new boundary. The region within the new boundary contributes an infinite chain (from the f_k and the adjacent edges not in T_2) with finite branches (from the regions between the $C^*(k)$). This structure is connected (via the edge deleted from $C(2)$) to the infinite component (possibly with branches of length 1) from the region between $C(1)$ and $C(2)$. As before, the other components consist entirely of directed edges. This new type of component has a vertex with two infinite branches attached. As was seen in the proof of 2.2 of **[4]**, we must leave one undirected edge at this point. Thus G has one undirected edge, so the diagram has t_1 equal to one.

Proof of Theorem 1. Suppose that we have a specification with $h_\infty \geq 2$. After 2.1 and 2.3, we may assume that $r+s+t_1+h(1)+h_\infty$ and $h_0-h(5)$ are finite. Then $n=r+s+[\frac{1}{2}t_1]+h(1)$ is finite. Also, by Lemma 4.2 of **[4]**, h_0 and hence $h(5)-h(1)$ are infinite.

If $n>0$, then we take a D-diagram consisting of n blue triangles, $n-1$ red edges, and $n+2$ red loops. To n of these loops, we join r copies of R^*, s of S, $[\frac{1}{2}t_1]$ of T and $h(1)$ of P. We refer to the result as the diagram N for the specification. It has $h(1)$ green loops and one other green polygon, the last having two red loops available for joining.

As $h(5)-h(1)$ is infinite, there is a $j\in\{2, 3, 4, 5\}$ such that $c(i)=j$ for infinitely many i. For such a j, and for each i with $c(i)\neq 1, j$, we take a copy of $B(j, c(i))$. To this collection, we add an infinite number of copies of B_j, and, if $n>0$, a copy of the diagram N defined above. We divide the collection into h infinite sequences. We form the join of each sequence. Each join has $h_\infty=1$, and has a red loop on the infinite green component.

If t_1 is even, we make a tree of these diagrams. By 1.6, this has the required specification.

If t_1 is odd, we construct N^* by joining a copy of R to the relevant N. We make h_∞

sequences as for t_1 even, and form the join of each. We make a tree of $h_\infty - 1$ of these, including that involving N^*. To the remaining join of a sequence, we join a copy of R. The tree has a "spare" red loop (on N^*); we join a copy of R to this. Now we have two diagrams, each with a triangle having two red loops. We compose these. Using 3.10(ii) of [**4**], we obtain a diagram with the correct specification.

If $n = 0$, the diagram N may be taken to be the null diagram.

3. Proofs of Theorems 2 and 3.

Proof of Theorem 2. The first condition is sufficient by Lemma 2.3.

With this and 2.2, we may assume that the second condition holds and that $h_0 - h(5)$ and $r + s + t_1 + h(1)$ are finite. We define n as in Theorem 1. From (ii), $n > 0$. A connected diagram made by joining $n - 1$ copies of R, r of R^*, s of S, $\frac{1}{2}t_1$ of T and $h(1)$ of P has one red loop available for further joining.

Much as for Theorem 1, we construct the join of $B(j, c(i))$'s and B_j's for a suitable j. Joining this to the finite diagram produced above, we obtain the result.

The parity condition on t_1 in Theorem 2(ii) is necessary when h_0 is finite, indeed we included it in the definition of a specification. In fact, it is necessary whenever $h_0 - h(5)$ is finite.

Proof of Theorem 3. Assume that H is a subgroup with $h_\infty = 1$ and $t_1 + h_0 - h(5)$ finite. We consider the D_2-diagram for H (see [**5**]). In this, we choose the red edges for T_2 so that each finite green component is connected to the infinite one by a minimal sequence of red edges in T_2. We observe that there are only finitely many green polygons with more than five sides.

Let L be the L-diagram of H, and D the D-diagram. As $h_\infty = 1$, L has one vertex, V say, of infinite degree. For each vertex A of L, we define the sequence $(E_i(A) : i = 1, 2, \ldots)$, where $E_{i+1}(A)$ is obtained from $E_i(A)$ by adding all vertices which are adjacent to vertices of $E_i(A)$; see [**6**]. We show that there is an integer n such that V is a vertex of $E_n(A)$ for every vertex A. We define the distance $d(-, -)$ between vertices of L in the obvious way. Then we have to show that $d(V, A) \leq n$ for each A.

Since r and s are finite, the number of vertices of L associated with free edges or degenerate triangles is finite. Since L is connected, there is an integer n_1 such that, if A is associated with one of these features, then $d(V, A) \leq n_1$.

By the analysis preceding 2.3 of [**6**], a vertex B of degree one is adjacent to a vertex of degree five or more. Since there are finitely many vertices of degree greater than five, there is an integer n_2 such that each is at most n_2 distant from V. It follows that, if B is adjacent to one of these then $d(V, B) \leq n_2 + 1$. If B is adjacent to a vertex of degree five then the analysis in [**6**] shows that the latter is of a special kind (obtained by the process of composition) and is joined by two edges to a vertex of degree six or more. Then it is clear that $d(V, B) \leq n_2 + 2$.

Again from [**6**], we know that a loop in L which is not associated with a free edge (red loop of D), a degenerate triangle (blue loop of D) or vertex of degree one (green loop of D) must occur at a vertex of degree at least six. Thus, if a vertex B has a loop, then $d(V, B) = \max\{n_1, n_2\}$.

Suppose that $B \in \mathscr{V}(L)$ is such that $d(V, B) > 8 + \max\{n_1, n_2 + 2\}$. Then $E_8(B)$ contains no vertex with a loop or a free edge and involves no degenerate triangle. Now the argument of 2.1, 2.2 of [**6**] applies to show that, as $E_7 \subseteq C(E_8)$,

$$\sum \{\deg(V) - 6\} \geq -6, \tag{1}$$

where the sum is over the vertices of E_7. (We can ignore vertices of $C(E_8) - E_7$ since each has degree five or less and would contribute a negative amount on the left.) Now $V \notin \mathscr{V}(E_7)$, so $E_7 \neq L$ and, as L is connected, each E_i involves at least one new vertex. Hence $\mathscr{V}(E_7) \geq 7$. This contradicts (1). Hence, for all B, $d(V, B) \leq 8 + \max\{n_1, n_2 + 2\}$ and we have established a bound of the required kind.

By our choice of T_2, each vertex of L is connected to V by a path of at most n edges corresponding to red edges in T_2. In D, the T_2 paths consist of red and green edges. Since the size of the finite green polygons is bounded, the length of the T_2 path from a vertex of D to the nearest vertex on the infinite green component is uniformly bounded.

Suppose that we cut all the red T_2 edges of the D-diagram D. As a T_2 edge connects vertices in distinct green components, each cut combines the components, see 3.4, 3.5 of [**4**]. In general, this procedure will result in a collection of D-diagrams, but here we show that we get one diagram with just one (necessarily infinite) green component.

Let U be a vertex of D. We may as well assume that the distinguished vertex Q of D is on the infinite green component, since otherwise we could consider a conjugate of H with this property. In D, there is a T_2 path from U to a vertex W on the infinite component. By earlier remarks, the subtrees of T_2 rooted at vertices on the infinite green component are finite (indeed bounded) in length. Consider all of these which are rooted between Q and W. The number of red T_2 edges involved is finite. Cutting these, we obtain a green path from Q via W to U (in the resultant diagram). Cutting the other red T_2 edges does not affect this path, so that, in the result, Q and U lie on the same green component. Since U was arbitrary, we must have a single diagram with $h_\infty = 1$, $h_0 = 0$.

Let D' be the new diagram. The obvious T_2 consists of the green component. The "remaining" red edges are precisely the red edges of D not in the T_2 of that diagram. Thus the two diagrams have the same associated graph G. It follows that the hyperbolic generators in the standard presentations are in one-one correspondence. Now, D' has $h_\infty = 1$, $h_0 = 0$, $t_1 (= t_1(D)) < \infty$, so that $h_0 + t_1$ is finite. By Theorem 4.4 of [**4**], t_1 is even.

The following example shows that the above conditions are not necessary.

EXAMPLE 3.1. There is a subgroup of Γ with short specification $(0, 0, 0, \infty, 1)$ and cusp-split $(3, 2, 2, 2, \ldots)$.

It is easy to see that there is a diagram A with three blue triangles, one red loop and cusp-split $(2, 3, 4)$. With one copy of A and an infinite number of copies of B_2, we can form an infinite chain starting from A. It is clear that this has the correct cusp-split and has $r = s = 0$, $h_0 = \infty$. The value of t_1 is easily verified by constructing a T_2 and a G.

On the other hand, we have a negative result for a very similar specification.

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PROPOSITION 3.2. *There is no subgroup of* Γ *with short specification* $(0, 0, 0, \infty, 1)$ *and cusp-split* $(2, 2, 2, \ldots)$.

Proof. We begin by considering the situation of a green 2-gon AB in a D-diagram. Suppose first that there is a blue loop at A. As a green edge is equivalent to a red edge followed by a blue edge, we must have a red edge AB and hence a loop at B. Since this accounts for the red and blue edges at A and at B, we have a two-point diagram P'. Now suppose that there is a red loop at A. Then there is a blue edge AB. Let C be the third vertex of the blue triangle including A and B. Since we have a green edge from B to A, we must have a red edge BC. It follows that there is a green loop at C. This gives the three-vertex diagram P of 1.4. A similar conclusion follows if we have a red loop at B. There remains the case where there are no loops at A or at B. Then A (resp. B) is a vertex of a blue triangle Δ_1 (resp. Δ_2). We cannot have $\Delta_1 = \Delta_2$ or we should have a red edge between two vertices and hence a red loop at either A or B. Let A' (resp. B') be the vertex preceding A (resp. B) on Δ_1 (resp. Δ_2). The green edges AB, BA require that there are red edges AB', $A'B$. If the third vertex of either triangle is in a green 2-gon then our analysis shows that the associated pair of triangles is Δ_1, Δ_2, for Δ_1 is one and the other is linked to it by two red edges and Δ_1 has only one vertex left. Then we have a third red edge between these triangles. This gives a six-vertex diagram P''. If we do not have P'' then the third vertex of Δ_1 has a red loop or a red edge not involved with a green 2-gon.

From this analysis, we require three conclusions. First, a diagram without blue or green loops cannot involve green 2-gons of the first two types. Secondly, a diagram with more than six vertices and no red loops has a red edge not associated with a green 2-gon. Thirdly, in a diagram other than P, P', P'', each green 2-gon has associated with it a distinct pair of blue triangles.

Suppose that H is a subgroup with the specification of the statement. Let D be the D-diagram of H, and D_1 the D_1-diagram (i.e. the result of deleting the green edges from D). As D is infinite, the green 2-gons cannot be either of the first two types. Using the specification and Lemma 2.4 of [**4**], the only cycles of D_1 are the blue triangles and the red/blue quadrilaterals corresponding to the green 2-gons. Since D is infinite, it is not P, P' or P''. Hence D has a red edge e which is not associated with a green 2-gon. The edge e cannot belong to any cycle of D_1, so cutting e must split D_1 and hence D into two parts. Also, as neither end of e is in a 2-gon, both must be on the infinite green component. By 3.1, 3.2 of [**4**], one part, D' say, is finite. All but one of the green polygons of D' come from D, and so are 2-gons. The other green polygon has a red loop, this being produced by cutting. Since D has no loops of any colour, this is the only loop of D'. As D' has a red loop but no blue or green loop, $D' \neq P, P', P''$. Let $(D')_1$ be the D_1-diagram for D'. Suppose that $(D')_1$ has m triangles and k red/blue quadrilaterals. By earlier remarks, each quadrilateral is associated with a distinct pair of triangles, so that $m \geq 2k$. Cutting a red edge cannot give a new cycle, so that this accounts for the cycles of $(D')_1$ apart from the blue triangles. The connectedness of $(D')_1$ requires $m-1$ red edges to join the triangles. The k quadrilaterals require a further k. Since there are no more cycles, there are precisely $m-1+k$ red edges. These involve $2(m-1+k)$ of the $3m$ vertices of $(D')_1$. Each other vertex must have a red loop. But $(D')_1$ has one red loop, so that

$$1 = 3m - 2(m - 1 + k) = m - 2k + 2 \geq 2,$$

the last since $m \geq 2k$. This contradiction shows that there is no such subgroup.

Example 3.1 and Proposition 3.2 indicate that no straight-forward extension of Theorem 2 is possible. No complete result is known, though it is easy to prove some generalisations of 3.1, e.g. there are subgroups with short specification $(0, 0, 0, \infty, 1)$, with $h(1) = 0$ and with $c(i) = 2$ infinitely often and some $c(i)$ odd, or with $c(i) = 3$ infinitely often and $c(i) \not\equiv 0 \pmod 3$ for some i.

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