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# ON SPECIAL GROUP-AUTOMORPHISMS AND THEIR COMPOSITION

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**0. Introduction.** Let $G$ be a group and $\varphi$ an automorphism of $G$. We say that $\varphi$ is a *pseudo-identity* (pi) if, for each $x \in G$, there exists a finitely generated (fg) subgroup $K = K_x(\varphi)$ of $G$ such that $x \in K$ and $\varphi|K$ is an automorphism of $K$. It has been shown [**1**, **3**] that such special automorphisms of abelian or nilpotent groups play an important role in homotopy theory; and it was indicated in [**2**] that their purely algebraic properties might well repay study.

The following facts about pi's are elementary.

PROPOSITION 0.1. *Let $\varphi$ be an automorphism of $G$ and let $n$ be a non-zero integer. Then $\varphi$ is* pi *if and only if $\varphi^n$ is* pi.

PROPOSITION 0.2. *Let $\varphi$ be a pseudo-identity of $G$ and $\alpha$ an automorphism of $G$. Then $\alpha\varphi\alpha^{-1}$ is a pseudo-identity.*

We note that Proposition 0.1 implies that periodic automorphisms are pi; and Propositions 0.1, 0.2 show that pi's, as elements of Aut $G$, behave very much like periodic elements. Thus one would not expect that, in general, the composite of two pi's would be pi, and the following counterexample confirms this expectation.

*Example* 0.1. Let $Q = \langle x, y; x^2 = y^2 = 1\rangle$ and let $G = \bigoplus_Q Z$. Then $Q$ acts on $G$ by permuting summands, so that $Q$ is embedded in Aut $G$. Since $x$, $y$ are periodic, they act as pi's of $G$. However, $xy$ is not only non-periodic; it is also obvious that it does not act as a pi since an element of one summand is moved by iterates of $xy$ into infinitely many summands.

However, guided by one's experience of periodic elements, it would be reasonable to hope that some sort of nilpotency condition (on an appropriate subgroup of Aut $G$) might ensure that the composite of two pi's be pi, and it is the main purpose of this note to verify this conjecture. However, to achieve this we need to impose a restriction on the groups $G$ in question. This restriction is very reasonable for a homotopy theorist as it allows us to discuss automorphisms of locally nilpotent groups and this

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is as far as we usually go. However, the restriction may appear very severe to the group-theorist.

We make one further remark before describing the restriction. For an automorphism $\varphi:G \to G$ define, for each $x \in G$, the subgroup $H_x = H_x(\varphi)$ to be

$$H_x = \langle \varphi^n x, n \geqq 0 \rangle.$$

We then say that $\varphi$ is a *strong* pi if, for all $x \in G$, $H_x$ is fg and $x \in \varphi H_x$. It is plain that a strong pi is a pi (one merely takes $K_x = H_x$). Now strong pi's have the following two properties, which are vital for the applications to homotopy theory (compare [**2**] ).

PROPOSITION 0.3. *Let $\varphi:G \to G$ be a strong* pi, *and let $G'$ be a subgroup of $G$ such that $\varphi G' \subseteq G'$. Then $\varphi|G'$ is a strong* pi *of $G'$.*

PROPOSITION 0.4. *Suppose given an endomorphism of a group extension*

$$\begin{tikzcd} G' \arrow[r, tail] \arrow[d, "\varphi'"] & G \arrow[r, two heads] \arrow[d, "\varphi"] & G'' \arrow[d, "\varphi''"] \\ G' \arrow[r, tail] & G \arrow[r, two heads] & G'' \end{tikzcd}$$

*Then $\varphi$ is a strong* pi *if and only if $\varphi'$, $\varphi''$ are strong* pi*'s.*

Since strong pi's have the nice properties and pi's generalize the notion of automorphisms of fg groups, it is reasonable to seek a class of groups $G$ such that, in Aut $G$, all pi's are strong. Such a class is the class of quasinoetherian groups; recall that a group $G$ is *quasinoetherian* (qn) if every subgroup of a fg subgroup of $G$ is fg. We give Stammbach's proof that if $G$ is qn then every pi is strong, in the form of Corollary 1.2.

This then is the restriction we will impose; notice that, in the situations encountered in Propositions 0.3, 0.4 it suffices to assume $G$ qn to infer that $G'$, $G''$ are qn. Our main theorem then asserts that if $G$ is qn and if $N$ is a locally nilpotent subgroup of Aut $G$ generated by pi's, then $N$ consists of pi's. The proof is achieved by applying Philip Hall's commutator collecting process.

In Section 2 we discuss a generalization, natural to homotopy theorists, to the study of $P$-automorphisms. Here $P$ is a family of primes and, by a $P$-automorphism, we understand an endomorphism which is simultaneously $P$-injective and $P$-surjective. Such notions are only useful, it would seem, where, for any family of primes $Q$, there is a $Q$-torsion subgroup, $T_Q(G)$, of $G$. Thus we now confine attention to locally nilpotent groups $G$. We say that a homomorphism $\varphi:G_1 \to G_2$ of locally nilpotent groups is *$P$-injective* if ker $\varphi$ is a $P'$-torsion group, where $P'$ is the complement of $P$; and *P-surjective* if, for all $y \in G_2$, there exists a $P'$-number $n$ with $y^n \in \operatorname{im} \varphi$.

Whereas such a generalization is of evident value in homotopy theory, the more natural formulation in group theory of this kind of extension of the discourse would be to the study of *P-local* groups. Here a group $G$ is $P$-local if it admits unique $q^{\text{th}}$ roots for all $P'$-numbers $q$. We may then proceed with the study of automorphisms and pseudo-identities in the category of $P$-local groups exactly as for groups; indeed, we may consider the case of groups simply as the special case $P = \Pi$, the family of all primes. In particular we have no difficulty in generalizing the main theorem of Section 1. We then relate the 'homotopy-oriented' generalization with this generalization via the $P$-localizing functor, and this works very well provided we confine attention to locally nilpotent groups. The details are set out in Section 2; we remark, in particular, that this approach leads to a notion of $P$-pseudo-identity of nilpotent groups which is more general than that given in [**2**].

**1. The main theorem.** We first prove two results which will be needed in the proof of our main theorem. Given any automorphism $\varphi$ of a group $G$, we define, for each $x \in G$,

$$H_x = \langle \varphi^n x, n \geqq 0 \rangle, \; H_x^- = \langle \varphi^n x, n \leqq 0 \rangle, \; \bar{H}_x = \langle H_x, H_x^- \rangle.$$

THEOREM 1.1. (a) $x \in \varphi H_x \Leftrightarrow H_x^-$ *is* fg.
(b) $\bar{H}_x$ *is* fg $\Leftrightarrow H_x, H_x^-$ *are* fg.

*Proof.*

$$\begin{aligned}\text{(a) } H_x^- \text{ is fg} &\Leftrightarrow \exists\, n \geqq 0, \varphi^{-n}(x) \in \langle x, \varphi^{-1}x, \dots, \varphi^{-(n-1)}x \rangle \\ &\Leftrightarrow \exists\, n \geqq 0, x \in \langle \varphi^n x, \varphi^{n-1}x, \dots, \varphi x \rangle \\ &\Leftrightarrow x \in \varphi H_x.\end{aligned}$$

(b) Plainly $\bar{H}_x$ is fg if $H_x$, $\bar{H}_x$ are fg. Conversely, if $\bar{H}_x$ is fg, then $\exists\, m \geqq 0$ such that

$$\varphi^n(x) \in \langle \varphi^i(x), -m \leqq i \leqq m \rangle,$$

for all $n$. But then

$$\varphi^n(x) \in \langle \varphi^i(x), 0 \leqq i \leqq 2m \rangle \text{ for all } n,$$

so that $H_x$ is fg. Similarly, $H_x^-$ is fg.

COROLLARY 1.2. (U. Stammbach). *Let $G$ be quasinoetherian* (qn) *and let $\varphi$ be an automorphism of $G$. Then $\varphi$ is* pi *if, and only if, for all $x \in G$, $\bar{H}_x$ is* fg. *In other words, all* pi*'s of* qn *groups are strong.*

*Proof.* If $\bar{H}_x$ is fg, then by Theorem 1.1, $H_x$ is fg and $x \in \varphi H_x$. Thus $\varphi$ is (strongly) pi (with $K_x = H_x$). Conversely, suppose that $\varphi$ is pi. For each $x \in G$, we have

$$K_x \subseteq G,\ K_x \text{ fg},\ x \in K_x, \text{ and } \varphi|K_x{:}K_x \cong K_x.$$

It follows immediately that $\bar{H}_x \subseteq K_x$ and, since $G$ is qn, $\bar{H}_x$ is fg.

We now consider a finite set $\varphi_1, \ldots, \varphi_k$ of automorphisms of the qn group $G$; we write $\bar{H}_x^i$ for $\bar{H}_x(\varphi_i)$.

THEOREM 1.3. *Let $\varphi_i$ be a* pi *of the* qn *group $G$, $i = 1, 2, \ldots, k$. Then, for each $x \in G$, the group*

$$\langle \varphi_k^{i_k}\varphi_{k-1}^{i_{k-1}} \ldots \varphi_1^{i_1}x,\ i_j \in \mathbf{Z}, j = 1, 2, \ldots, k\rangle$$

*is* fg.

*Proof.* We prove this by induction on $k$. For $k = 1$ this is just Corollary 1.2. We assume the assertion true for $(k - 1)$ so that

$$\langle \varphi_{k-1}^{i_{k-1}} \ldots \varphi_1^{i_1}x\rangle \text{ is fg.}$$

Let $y_1, \ldots, y_m$ be a set of generators. For each $y_l$, $1 \leqq l \leqq m$, the group

$$\bar{H}_{y_l}^k = \langle \varphi_k^n y_l,\ n \in \mathbf{Z}\rangle$$

is fg; and plainly

$$\langle \varphi_k^{i_k}\varphi_{k-1}^{i_{k-1}} \ldots \varphi_1^{i_1}x\rangle = \langle \bar{H}_{y_1}^k, \bar{H}_{y_2}^k, \ldots, \bar{H}_{y_m}^k\rangle.$$

Thus $\langle \varphi_k^{i_k}\varphi_{k-1}^{i_{k-1}} \ldots \varphi_1^{i_1}x\rangle$ is fg, and the induction is complete.

We are now ready to prove the main theorem.

THEOREM. *Let $G$ be a* qn *group and let $N$ be a locally nilpotent subgroup of* Aut $G$. *Then if $N$ is generated by* pi*'s, $N$ consists of* pi*'s.*

*Proof.* It plainly suffices to prove that if $\varphi, \psi \in N$ and $\varphi, \psi$ are pi, then $\varphi\psi$ and $\varphi^{-1}$ are pi. Since the second conclusion is trivial we may concentrate on the first. Thus we may assume $N = \langle \varphi, \psi\rangle$ so that $N$ is nilpotent. We argue by induction on nil $N$ that $\varphi\psi$ is pi if $\varphi, \psi$ are pi.

If nil $N = 1$ then $\varphi, \psi$ commute. Now by Theorem 1.3 $\langle \varphi^i\psi^j x;\ i, j \in \mathbf{Z}\rangle$ is fg for all $x \in G$. Since $G$ is qn it follows that

$$\langle \varphi^i\psi^i x,\ i \in \mathbf{Z}\rangle \text{ is fg for all } x \in G,$$

so that $\varphi\psi$ is pi by Corollary 1.2.

We now suppose the theorem proved if nil $N \leqq c - 1$ and assume nil $N = c \geqq 2$. We will show, arguing by induction on weight, that all basic commutators in the generators $\varphi, \psi$ of $N$ are pi. This is certainly true for the basic commutators of weight 1, since these are the generators $\varphi, \psi$ themselves. Thus we suppose our assertion proved for basic commutators of weight $< k$ and consider a basic commutator $\alpha$ of weight $k$ where $k \geqq 2$. Then $\alpha = [\beta, \gamma]$, where $\beta$ is a basic commutator of weight $l$, $\gamma$ is a basic commutator of weight $m$, and $l + m = k$. Our inductive hypothesis implies that $\beta, \gamma$ are pi. Now

$$\langle \beta\gamma\beta^{-1}, \gamma^{-1} \rangle = \langle \beta\gamma\beta^{-1}\gamma^{-1}, \gamma \rangle \subseteq \langle N', \gamma \rangle,$$

where $N'$ is the commutator subgroup of $N$. Thus

$$\text{nil}\langle \beta\gamma\beta^{-1}, \gamma^{-1} \rangle \leqq c - 1.$$

Moreover $\beta\gamma\beta^{-1}$, $\gamma^{-1}$ are pi, so that, by our main inductive hypothesis, $[\beta, \gamma] = \beta\gamma\beta^{-1}\gamma^{-1}$ is pi. We have therefore established that all basic commutators in the generators $\varphi$, $\psi$ of $N$ are pi.

Finally, we remark that, since nil $N = c$, every element of $N$ may be expressed in the form

$$\gamma_1^{i_1}\gamma_2^{i_2} \dots \gamma_q^{i_q},$$

where $\gamma_1, \gamma_2, \dots, \gamma_q$ is the ordered list of basic commutators of $\varphi$, $\psi$ of weight $\leqq c$. Since each $\gamma_i$ is pi, it follows from Theorem 1.3 that, for each $x \in G$, the group $\langle \gamma_1^{i_1}\gamma_2^{i_2} \dots \gamma_q^{i_q}x \rangle$ is fg. But the group $\langle (\varphi\psi)^n x \rangle$ is a subgroup of this group and hence, $G$ being qn, also fg. Thus, by Corollary 1.2, $\varphi\psi$ is pi.

**2. A generalization.** We consider the category of $P$-local groups, where $P$ is a family of primes. A $P$-local group $G$ is said to be *finitely generated* (fg) if there exists a finite set of elements $S$ in $G$ such that no proper $P$-local subgroup of $G$ contains $S$; such a set $S$ may be called a generating set and we write $G = \langle S \rangle_P$ to indicate that $S$ generates $G$ as a $P$-local group. We say that an automorphism $\varphi$ of the $P$-local group $G$ is a *pseudo-identity* if, for each $x \in G$, there exists a fg $P$-local subgroup $K_x$ of $G$ such that $x \in K_x$ and $\varphi|K_x$ is an automorphism of $K_x$. We also have the obvious notion of a quasinoetherian $P$-local group. Notice that, of course, all these notions reduce to those discussed in the previous section if $P$ is the family of all primes, $\Pi$.

If we define, for the automorphism $\varphi$ of the $P$-local group $G$, the $P$-local subgroup $\bar{H}_x$, $x \in G$, by the rule

$$\bar{H}_x = \langle \varphi^n x, n \in \mathbf{Z} \rangle_P,$$

then we may prove, just as in the special case $P = \Pi$,

THEOREM 2.1. *Let $G$ be a* qn *$P$-local group, and let $\varphi$ be an automorphism of $G$. Then $\varphi$ is* pi *if and only if $\bar{H}_x$ is* fg *for each $x \in G$. In other words, all* pi*'s of* qn *$P$-local groups are strong.*

We then proceed to the generalization of our main theorem.

THEOREM 2.2. *Let $G$ be a* qn *$P$-local group and let $N$ be a locally nilpotent subgroup of* Aut *$G$. Then if $N$ is generated by* pi*'s, $N$ consists of* pi*'s.*

In order to relate these results to the study of endomorphisms of arbitrary groups, we apply the technique of $P$-localization. To do so effectively, however, we should restrict attention to locally nilpotent groups. Such groups $G$ have $Q$-torsion subgroups, $T_Q(G)$, for any family of primes $Q$, and a very satisfactory $P$-localization theory [**4**, **5**, **6**]. Thus we have a $P$-localizing map $e:G \to G_P$ which is characterized by the condition that $G_P$ is $P$-local and $e$ is $P$-bijective. Here a homomorphism $\varphi:M \to N$ of locally nilpotent groups is *$P$-injective* if ker $\varphi$ is a $P'$-torsion group, where $P'$ is the complement of $P$; and *$P$-surjective* if, $\forall\, y \in N$, $\exists$ a $P'$-number $n$ such that $y^n \in \operatorname{im} \varphi$. Moreover, $\varphi$ is $P$-injective ($P$-surjective) if and only if $\varphi_P:M_P \to N_P$ is injective (surjective).

To apply the theorems above, we need the following.

THEOREM 2.3. *A locally nilpotent $P$-local group is* qn.

*Proof.* Let us first show that a fg locally nilpotent $P$-local group $K$ is nilpotent. Let $K = \langle S\rangle_P$ with $S$ finite, and let $L = \langle S\rangle$. Then $L$ is nilpotent and $K$ is the $P$-localization of $L$; it thus follows that $K$ is nilpotent. (Notice that this means that, if $G$ is a $P$-local group, then it is locally nilpotent as $P$-local group if and only if it is locally nilpotent as group. A similar observation even applies to the concept 'nilpotent' itself.)

Now let $H \subseteq K \subseteq G$ be inclusions of $P$-local groups with $G$ locally nilpotent and $K$ fg. Form $L$ as above, and let $M = H \cap L$. Then $M$ as a subgroup of $L$ is fg and $M_P = H$. It follows that $H$ is fg as a $P$-local group.

Let $\varphi:G \to G$ be an endomorphism of the locally nilpotent group $G$ with $P$-localization $\varphi_P:G_P \to G_P$. Then $\varphi$ is a $P$-automorphism if and only if $\varphi_P$ is an automorphism. It thus seems natural to look among the $P$-automorphisms of $G$ for those deserving to be called $P$-pseudo-identities. A definition was given in [**2**, Remark, p. 16], but that was too restrictive for our purposes here. Obviously, it is desirable that $\varphi$ is $P$-pi if and only if $\varphi_P$ is pi. We achieve this effect by adopting the appropriate definition of a $P$-fg locally nilpotent group.

*Definition* 2.1. A locally nilpotent group $K$ is $P$-fg if $K_P$ is fg. A $P$-automorphism $\varphi:G \to G$ of the locally nilpotent group $G$ is a *$P$-pseudo-identity* if, for all $x \in G$, $\exists$ $P$-fg subgroup $K$ of $G$ such that $x \in K$ and $\varphi|K$ is a $P$-automorphism of $K$. The $P$-automorphism $\varphi$ is a *strong* $P$-pi if, for all $x \in G$, the subgroup

$$H_x = \langle \varphi^n x,\ n \geqq 0\rangle$$

is $P$-fg and $x^m \in \varphi H_x$ for some $P'$-number $m$.

*Remark.* We defined the concept of $P$-fg in [**2**] by asking that $K/T_{P'}(K)$ be fg. Thus our present definition of $P$-fg, as of $P$-pi and strong $P$-pi, is more general than that in [**2**]. The following example confirms that our new notions are strictly more general.

*Example* 2.1. Let $\varphi:\mathbf{Q} \to \mathbf{Q}$ be given by $\varphi(x) = \frac{1}{3}x$. Then $\varphi$ is 2-pi according to our present definition. For let

$$x = \frac{a}{2^n b} \quad \text{with } a, b \text{ odd}, n \in \mathbf{Z}.$$

Then if $K = \left\langle \frac{1}{2^n} \right\rangle_2$, $x \in K$, and $K = K_2$ is a cyclic $\mathbf{Z}_2$-module. Moreover $\varphi|K$ is a 2-automorphism of $K$. On the other hand, $K$ is certainly not fg as a group, and indeed it is easy to see that it is not possible to find any fg group $L$ containing $x$ such that $\varphi|L$ is a 2-automorphism of $L$. We need not discuss the increased generality of the concept of strong $P$-pi. For it was pointed out in [**2**] that all $P$-pi's in the sense of [**2**] are strong; and we have Corollary 2.5 below.

We now establish the expected link between pi's and $P$-pi's.

THEOREM 2.4. *Let $\varphi:G \to G$ be an endomorphism of the locally nilpotent group $G$. Then*

(a) $\varphi$ *is* $P$-pi $\Leftrightarrow \varphi_P$ *is* pi;

(b) $\varphi$ *is strongly* $P$-pi $\Leftrightarrow \varphi_P$ *is strongly* pi.

*Proof.* (a) Assume $\varphi$ $P$-pi and let $y \in G_P$. Then $y^m = ex$ for some $x \in G$, and $P'$-number $m$. Choose $K$ $P$-fg in $G$ with $x \in K$ and $\varphi|K$ a $P$-automorphism of $K$. Then $ex \in K_P$, $K_P$ is fg $P$-local and $\varphi_P|K_P$ is an automorphism of $K_P$. Since $y^m = ex$ and $K_P$ and $G_P$ are $P$-local, it follows that $y \in K_P$, so that $\varphi_P$ is pi.

Conversely, assume $\varphi_P$ pi and let $x \in G$. Then $\exists$ $K$ fg $P$-local in $G_P$ with $ex \in K$ and $\varphi_P|K$ an automorphism of $K$. Let $L = e^{-1}K$. Then $e|L:L \to K$ is the $P$-localizing map, $K = L_P$, so that $x \in L$ and $L$ is $P$-fg. Moreover it is clear that $\varphi$ maps $L$ to itself and, since $\varphi|L$ localizes to $\varphi_P|K$, it follows that $\varphi|L$ is a $P$-automorphism of $L$.

(b) In the light of what we proved under (a), it is plain that all we have to prove is that

(2.1) $\quad H_x(\varphi)_P = H_{ex}(\varphi_P)$;

(2.2) $\quad H_y(\varphi_P) = H_z(\varphi_P)$, where $y \in G_P$, $z = y^m$, with $m$ a $P'$-number.

As to (2.1), since $H_x(\varphi) = \langle \varphi^n x, n \geqq 0 \rangle$, it follows that

$$H_x(\varphi)_P = \langle e\varphi^n x \rangle_P = \langle \varphi_P^n ex \rangle_P = H_{ex}(\varphi_P).$$

As to (2.2), it is plain that

$$H_z(\varphi_P) \subseteq H_y(\varphi_P);$$

but conversely, since $(\varphi_p^n y)^m \in H_z(\varphi_P)$ and $H_z(\varphi_P)$ is $P$-local,

$$\varphi_p^n y \in H_z(\varphi_P), \quad n \geqq 0,$$

so that $H_y(\varphi_P) \subseteq H_z(\varphi_P)$.

COROLLARY 2.5. *All $P$-pi's of a locally nilpotent group are strong.*

We can now translate Theorem 2.2, restricted to locally nilpotent groups, if we wish into a statement about $P$-pi's. We could formulate the result as follows.

THEOREM 2.6. *Let $\varphi$, $\psi$ be $P$-pi's of the locally nilpotent group $G$. If the subgroup $\langle \varphi_P, \psi_P \rangle$ of* Aut $G_P$ *is nilpotent, then $\varphi\psi$ is $P$-pi.*

Finally we remark that, in the case that $G$ is abelian, there is a very natural polynomial criterion for a $P$-pi. We first prove a proposition.

PROPOSITION 2.7. *Let $\varphi$ be a $P$-automorphism of the $P$-fg abelian group $A$. Then there exists a polynomial $F(t)$, over $\mathbf{Z}$,*

$$(2.3) \quad F(t) = c_0 t^n + c_1 t^{n-1} + \ldots + c_{n-1} t + c_n$$

*with $|c_0|$, $|c_n|$ P′-numbers and $F(\varphi)A \subseteq T_{P'}A$. (We call such a polynomial $P$-adapted.)*

*Proof.* Now $\varphi_P$ is an automorphism of the fg $\mathbf{Z}_P$-module $A_P$. We may write

$$A_P = M_P \oplus \bigoplus_{p \in P} T_p(A),$$

where $M_P$ is a free fg $\mathbf{Z}_P$-module and $T_p(A)$, the $p$-torsion subgroup of $A$, is finite. Just as in [**2**] we may show that there is a polynomial $F'(t)$, with leading coefficient 1 and constant term $\pm 1$, such that

$$F'(\varphi')T(A_P) = 0,$$

where $\varphi' : T(A_P) \to T(A_P)$ is the restriction of $\varphi_P$. Now let $\varphi_P$ induce $\varphi'' : M_P \to M_P$ by passing to quotients. Then $\varphi''$ is an automorphism, so there is a characteristic polynomial $f(t)$ over $\mathbf{Z}_P$ such that

$$f(\varphi'')M_P = 0;$$

moreover, the constant term of $f$ is $\pm\det \varphi''$ where $\det \varphi''$ is invertible in $\mathbf{Z}_P$. It follows that, by multiplying $f(t)$ by a suitable $P'$-number, we create a $P$-adapted polynomial $F''(t)$, such that

$$F''(\varphi'')M_P = 0.$$

It is then plain that if $F(t) = F'(t)F''(t)$, then $F(t)$ is $P$-adapted and

$$F(\varphi_P)A_P = 0.$$

Finally, the commutative diagram

$$\begin{tikzcd} A \arrow[r, "F(\phi)"] \arrow[d, "e"] & A \arrow[d, "e"] \\ A_P \arrow[r, "F(\phi_P)"] & A_P \end{tikzcd}$$

shows that $F(\phi)A \subseteq \ker e = T_{P'}A$.

*Remark.* It is easy to see that, conversely, the existence of such a polynomial shows that $\phi$ is a $P$-automorphism.

We now produce the criterion, generalizing the fundamental tool used in [**1**].

THEOREM 2.8. *Let $\phi$ be a $P$-automorphism of the abelian group $A$. Then $\phi$ is a $P$-*pi *if and only if, for each $a \in A$, there exists a $P$-adapted polynomial $F_a(t)$ such that $F_a(\phi)(a) = 0$.*

*Proof.* Let $\phi$ be $P$-pi. Then, for each $a \in A$, $\exists$ $P$-fg $K$ such that $a \in K$ and $\phi|K$ is a $P$-automorphism. By Proposition 2.7, $\exists$ a $P$-adapted polynomial $F$ such that

$$F(\phi)K \subseteq T_{P'}K.$$

Thus $F(\phi)a$ is a $P'$-torsion element. We may multiply $F$ by a $P'$-number to create a polynomial $F_a$ which will also be $P$-adapted and which satisfies

$$F_a(\phi)(a) = 0.$$

Conversely, suppose that such $P$-adapted polynomials $F_a(t)$ exist. Consider $H_a(\phi)$. Then we know that

$$c_0\phi^n(a) \in \langle a, \phi a, \dots, \phi^{n-1}a\rangle,$$

where $F_a$ is given by (2.3). Thus

$$c_0\phi_P^n e(a) \in \langle ea, \phi_P ea, \dots, \phi_P^{n-1} ea\rangle_P$$

so that, $|c_0|$ being a $P'$-number,

$$\phi_P^n e(a) \in \langle ea, \phi_P ea, \dots, \phi_P^{n-1} ea\rangle_P.$$

We conclude that $H_{ea}(\phi_P)$ is fg, so that, by (2.1), $H_a(\phi)$ is $P$-fg. Second, we know that

$$c_n a \in \phi H_a(\phi).$$

Since $|c_n|$ is a $P'$-number we conclude that $\phi$ is (strongly) $P$-pi according to Definition 2.1.

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