ON CERTAIN INTERSECTION PROPERTIES OF CONVEX SETS

V. L. KLEE, JR.

1. Introduction. A collection of $n + 1$ convex subsets of a Euclidean space E will be called an *n-set in E* provided each n of the sets have a common interior point although the intersection of all $n + 1$ interiors is empty. It is well-known that if $\{C_0, C_1\}$ is a 1-set, then C_0 and C_1 can be separated by a hyperplane. In the present note this result is generalized (Theorem I) by showing that if $\{C_0, \ldots, C_n\}$ is an n-set in E, then there is a variety V of deficiency n in E such that V intersects no set[1] Int C_i although in each direction away from V, V has a translate which intersects some set Int C_i. This theorem is then used to prove the converse (Theorem II) of Horn's recent generalization [2] of Helly's theorem [1] on the intersection of convex sets. The method of proof is essentially an elaboration of that of [3] and [4]. All theorems are stated only for a finite-dimensional Euclidean space E, although most of the proofs apply in rather general linear spaces.

2. A preliminary result. The following result will be useful in the sequel.

(2.1) *Suppose that $C_0, \ldots, C_n$ are closed convex subsets of E, each n of which have a point in common, and that $\bigcup_0^n C_i$ is convex. Then there is a point in common to all the C_i's.*

Proof. We may assume without loss of generality that all the C_i's are compact. For $n = 0$ the theorem is trivial. Now suppose it holds for $n = k - 1$ and consider the case $n = k$. If $\bigcap_0^k C_i = \Lambda$ then C_0 and $P = \bigcap_1^k C_i$ are disjoint compact convex sets, so they can be separated by a hyperplane H disjoint from both of them. Let $C'_i = C_i \cap H (1 \leqslant i \leqslant k)$. For an arbitrary integer j between 1 and k let $X = \bigcap C_i \ (1 \leqslant i \leqslant k, i \neq j)$. Since each k of the C_i's have a point in common, X intersects C_0. And since furthermore $P \subset X$, X must intersect H and hence $\bigcap C'_i \neq \Lambda \ (1 \leqslant i \leqslant k, i \neq j)$. But $\bigcup_1^k C'_i$ is convex, so it follows from the inductive hypothesis that $\bigcap_1^k C'_i \neq \Lambda$. Since this contradicts the fact that $P \cap H = \Lambda$, the proof is complete.

This remark may also be of interest.

(2.2) *Suppose that Γ is a collection of closed convex subsets of E such that*
(i) *either Γ is finite or some set in Γ is compact;*
(ii) *every finite subcollection of Γ has either a convex union or a non-empty intersection.*

Then there is a point in common to all sets of Γ.

Received April 4, 1950.

[1] Int X means the interior of X.

Proof. We need merely show that Γ has the finite intersection property, and this follows easily by an inductive argument which uses (2.1).

3. The "separation theorem" for n-sets. Let $x_0, x_1, \ldots, x_n$ be points of E; then $[x_0, x_1, \ldots, x_n]$ will denote the convex hull of the set $\{x_0, x_1, \ldots, x_n\}$, and $[x_0, x_1, \ldots, x_n] \equiv [x_0, x_1, \ldots, x_n] - \{x_0\}$, etc. (The sign $-$ is used for both set and vector differences, since in each case the meaning is clear from the context; $+$ is used for vector sum; $\cup$ for set union.)

The proof of Theorem I is effected by means of two lemmas, the first of which is the following:

(3.1) *If $\{C_0, \ldots, C_n\}$ is an n-set in E, then there are convex sets $K_i \supset C_i$ such that $\{K_0, \ldots, K_n\}$ is an n-set which covers E.*

Proof. We will show that if x is an arbitrary point of E then there are convex sets $C'_i \supset C_i$ such that $\{C'_0, \ldots, C'_i\}$ is an n-set and, in addition, $x \in \cup_0^n C'_i$; (3.1) follows from this fact by a straightforward application of Zorn's lemma.

For $0 \leqslant j \leqslant n$, $D_j \equiv \cap_{i \neq j} \text{Int } C_i$. If for some j we cannot have $d_j \in (c_j, x)$, with $d_j \in D_j$ and $c_j \in C_j$, then we merely let C'_j be the convex hull of $C_j \cup \{x\}$, $C'_i = C_i$ for $i \neq j$, and the sets C'_i will have the desired properties. Suppose this is not the case; that is, that there are points $d_0, \ldots, d_n, c_0, \ldots, c_n$ such that for $0 \leqslant i \leqslant n$, $c_i \in C_1$, and $d_i \in D_i \cap (c_i, x)$. For each j let $X_j = (c_j, d_0, \ldots, d_{j-1}, d_{j+1}, \ldots, d_n)$. Then $X_j \subset \text{Int } C_j$. But by use of Cramer's rule it can be shown that all the X_i's have a point in common and hence that $\cap_0^n \text{Int } C_i \neq \Lambda$, which is a contradiction, completing the proof of (3.1).

A linear subset of E is called a *subspace*, and each translate of a subspace is a *variety*. The *deficiency* in E of a subspace (and of its translates) is the dimension of a subspace complementary to it.

(3.2) *Suppose that $\{K_0, \ldots, K_n\}$ is an n-set which covers E and that $V \equiv \cap_0^n \overline{K}_i$. Then $V = E - \cup_0^n \text{Int } K_i$, and is a variety of deficiency n in E.*

Proof. Let $W = E - \cup_0^n \text{ Int } K_i$ and (for each j) $\pi_j = \cap_{i \neq j} \text{ Int } K_i$. From (2.1) we see that V is non-empty. We show first that $V \subset W$. For if not, there is a point p and an integer j such that $p \in V \cap \text{Int } K_j$. Let $q \in \pi_j$. Then, since for each $i \neq j$ we have $p \in \overline{K}_i$ and $q \in \text{Int } K_i$, $(p, q) \subset \pi_j$. But also $p \in \text{Int } K_j$, so (p, q) intersects Int K_j and $\cap_0^n \text{ Int } K_i \neq \Lambda$, which is a contradiction.

To see that $W \subset V$, let $y \in W$ and $z \in \pi_j$ for some j. Consider an arbitrary point x such that $y \in (x, z)$. If, for any $i \neq j$, $x \in K_i$, then we have $y \in$ Int K_i, which contradicts the fact that $y \in W$. Hence $x \in K_j$. Thus we have shown that $y \in \overline{K}_j$ for each j, and consequently $y \in V$. Since $W \subset V$ and $V \subset W$, $V = W$.

Obviously V is convex. To prove that it is actually a variety we must show that if $y \in V$, $z \in V$, and $y \in (x, z)$, then $x \in V$. But if x non $\in V$,

https://doi.org/10.4153/CJM-1951-031-2 Published online by Cambridge University Press

then (since $V = W$) $x \in \text{Int } K_j$ for some j and hence $y \in \text{Int } K_j$, which contradicts the fact that $y \in W$. Hence V is a variety and it remains only to show that the deficiency of V in E is n.

We assume without loss of generality that V contains the origin. Let S be a subspace of E complementary to V. Each point x of E has a unique expression in the form $v_x + x^*$ where $v_x \in V$ and $x^* \in S$. From the fact that $V = W$ it follows that no translate of V other than V itself can intersect all the sets K_i. This in turn implies that $\{K^*_0, K^*_1, \ldots, K^*_n\}$ is an n-set in S. Now Helly's theorem applies to an arbitrary *finite* collection of convex sets even though they may not be compact, so we can conclude that S is at least n-dimensional.

If $p_i \in \pi_i$ for each i then the variety U determined by $\{p_0, \ldots, p_n\}$ is n-dimensional. Let x be an arbitrary point of E. For a sufficiently small positive t we have $p_i + tx \in \pi_i$ for each i. Now for each i let $K_i{}^t = K_i \cap (U + tx)$. $\{K_0{}^t, K_1{}^t, \ldots, K_n{}^t\}$ is an n-set in $U + tx$, so V must intersect $U + tx$. From this it follows that V must intersect every translate of U, and hence that the deficiency of V in E is no greater than n. This completes the proof of (3.2).

THEOREM I. *If* $\{C_0, \ldots, C_n\}$ *is an n-set in E, then there is a variety V of deficiency n in E such that*

(a) *V intersects no set* Int C_i;

(b) *if V′ is any variety of deficiency* $n-1$ *which contains V, and H is either of the half-spaces into which V separates V′, then H intersects some set* Int C_i.

Proof. Let the K_i's be as in (3.1) and V as in (3.2). For each j let $z_j \in D_j = \bigcap_{i \neq j} C_i$ and let S be the variety determined by $\{z_0, \ldots, z_n\}$. S is a variety which is intersected by V in a single point P, and $\sigma = [z_0, \ldots, z_n]$ is an n-simplex whose boundary (relative to S) is contained in the union of the C_i's. In fact, if F_j is the face determined by $\{z_i | i \neq j\}$, then $F_j \subset \text{Int } C_j$. Now if V' is a variety of deficiency $n - 1$ (in E) which contains V, then V' intersects S in a line through P. Hence to prove Theorem I we need merely show that $P \in \sigma$. But if P non $\in \sigma$ then for some j there is a point $q \in F_j$ such that either $q \in (P, z_j)$ or $z_j \in (q, P)$. In the first case this implies that $P \in \pi_j$, in the second that $z_j \in \text{Int } C_j$, so in either case that $\bigcap_0^n \text{Int } C_i \neq \Lambda$, which is a contradiction, completing the proof.

4. Horn's generalization of Helly's theorem.

THEOREM II. *Suppose that* Γ *is a collection of compact convex subsets of E. Then the following statements are equivalent*:

(i) *every n members of* Γ *have a point in common*;

(ii) *each variety of deficiency n in E is contained in a variety of deficiency* $n-1$ *which intersects every member of* Γ;

(iii) *each variety of deficiency* $n-1$ *has a translate which intersects every member of* Γ.

Proof. That (i) implies (ii) is Horn's Theorem 4 [2; p. 928]. This (and that (i) implies (iii)) can also be proved by an argument which, like the proof of (2.1) above, leans heavily on the separation theorem and closely resembles Helly's proof [1] of his theorem.

To see that (ii) implies (iii), consider an arbitrary variety V of deficiency $n-1$ in E. We assume without loss of generality that V contains the origin. Now let S be a subspace of V, of deficiency 1 in V, and let $p \in V - S$. It follows from (ii) that for each integer k there is a variety V_k of deficiency $n-1$ in E such that $S + kp \subset V_k$ and V_k intersects all the members of Γ Now the sequence of sets $V_1, V_2, \ldots,$ must have a convergent subsequence [5, pp. 10-12]; say $\lim_{i \to \infty} V_{n_i} = W$. Since the members of Γ are compact, W intersects each member of Γ, and it follows by a simple argument that W is a translate of V. Thus (ii) implies (iii).

To complete the proof of Theorem II we show that if (i) is false, then so is (iii). For if (i) is false then for some $m < n$ there are sets $B_0, \ldots, B_m$ in Γ and open convex sets $C_i \supset B_i$ such that $\{C_0, \ldots, C_m\}$ is an m-set. Let V be the variety of Theorem I. Then V is of deficiency $m \leqslant n$ and intersects no set Int C_i. We assume without loss of generality that V contains the origin. Now consider an arbitrary translate $V + x$ of V (for x non $\in V$). For some $t > 0$, $V - tx$ intersects some set Int C_j. But then if $V + x$ intersects B_j, V must intersect Int C_j, which is a contradiction. Hence $V + x$ does not intersect all the sets B_i, and the proof is complete.

REFERENCES

[1] E. Helly, *Über Mengen konvexer Körper mit gemeinschaftlichen Punkten*, Jber. Deutschen Math. Verein., vol. 32 (1923), 175-176.

[2] Alfred Horn, *Some generalizations of Helly's theorem on convex sets*, Bull. Amer. Math. Soc., vol. 55 (1949), 923-929.

[3] V. L. Klee, Jr., *Convex sets in linear spaces*, Dissertation (University of Virginia, 1949).

[4] Shizuo Kakutani, *Ein Beweis des Satzes von M. Eidelheit über konvexe Mengen*, Proc. Imp. Acad., Tokyo, vol. 13 (1937), 93-94.

[5] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloquium Publications, vol. 28, (New York, 1942).

University of Virginia

https://doi.org/10.4153/CJM-1951-031-2 Published online by Cambridge University Press