A LOCAL MEAN VALUE THEOREM FOR THE COMPLEX PLANE

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1. Introduction

The equation

$$f(z_1)-f(z_0) = f'(z)(z_1-z_0) \tag{1}$$

need not have a solution z in the complex plane, even when f is entire. For example, let $f(z) = e^z$, $z_1 = z_0+2k\pi i$. Thus the classical mean value theorem does not extend to the complex plane. McLeod has shown (**2**) that if f is analytic on the segment joining z_1 and z_0, then there are points w_1 and w_2 on the segment such that $f(z_1)-f(z_0) = (z_1-z_0)(\lambda_1 f'(w_1)+\lambda_2 f'(w_2))$ where $\lambda_i \geqq 0$ and $\lambda_1+\lambda_2 = 1$.

The purpose of this article is to give a local mean value theorem in the complex plane. We show that there is at least one point z satisfying (1), which we will call a *mean value point*, near z_1 and z_0 but not necessarily on the segment joining them, provided z_1 and z_0 are sufficiently close. The proof uses Rouché's Theorem (**1**).

2. The local mean value theorem

Theorem. *If $f(z)$ is analytic in a domain containing z_0, then there is a neighbourhood N of z_0 such that if z_1 is in N then* (1) *has at least one solution z in $M = \{z: |z-z_0| < |z_1-z_0|\}$.*

Proof. Write $f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$. Then assuming $z_1 \neq z_0$, we may write by direct computation

$$f'(z)-\frac{f(z_1)-f(z_0)}{z_1-z_0} = \sum_{k=2}^{\infty} \frac{f^{(k)}(z_0)}{(k-1)!}\left\{(z-z_0)^{k-1}-\frac{1}{k}(z_1-z_0)^{k-1}\right\}. \tag{2}$$

We now look for zeros of this expression and note that the zeros are independent of the constant and linear term of the power series. So, there is nothing to prove if f is linear, since any z is a zero of (2) in this case. Actually one can easily show that z' is a mean value point for g and the pair z_1, z_0 if and only if z' is a mean value point for $g(z)+a+bz$ and the pair z_1, z_0. The proof is given in two cases since more can be said about the position of the mean value point if $f''(z_0) \neq 0$.

Case 1: $f''(z_0) \neq 0$.

Set $g_1(z) = f''(z_0)\{(z-z_0)-\frac{1}{2}(z_1-z_0)\} = f''(z_0)\{z-\frac{1}{2}(z_1+z_0)\}$, the first term of the series in (2). Set

$$g_2(z) = \sum_{k=3}^{\infty} \frac{f^{(k)}(z_0)}{(k-1)!}\left\{(z-z_0)^{k-1} - \frac{1}{k}(z_1-z_0)^{k-1}\right\}.$$

Let $C = \{z\colon |z-\frac{1}{2}(z_1+z_0)| = \frac{1}{2}|z_1-z_0|\}$. Then, with $r = \frac{1}{2}|z_1-z_0|$, the following statements are valid for all z on C:

$$|g_1(z)| = |f''(z_0)|\,.\,r,$$

$$|g_2(z)| \leqq \sum_{k=3}^{\infty} \frac{|f^k(z_0)|}{(k-1)!}\left\{(2r)^{k-1} + \frac{1}{k}(2r)^{k-1}\right\} = \sum_{k=3}^{\infty} \frac{|f^k(z_0)|}{(k-1)!}\,.\,\frac{k+1}{k}\,.(2r)^{k-1},$$

$$\left|\frac{g_2(z)}{g_1(z)}\right| \leqq \frac{\sum_{k=3}^{\infty} \frac{|f^k(z_0)|}{(k-1)!}\frac{k+1}{k}\,.\,(2r)^{k-1}}{|f''(z_0)|\,.\,r}$$

$$= \frac{1}{|f''(z_0)|}\sum_{k=3}^{\infty} \frac{|f^{(k)}(z_0)|}{(k-1)!}\frac{k+1}{k}\,.\,2^{k-1}r^{k-2}. \quad (3)$$

If the series $\sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$ converges for $|z-z_0|<R$, then the last series in (3) converges for $r<\frac{R}{2}$, and represents a continuous function of r in that range with the value zero at $r = 0$. Hence there is an $r_0>0$ such that if $r<r_0$ the series in (3) converges to a number less than one. So if $|z_1-z_0|<2r_0$ we may say $|g_2(z)| < |g_1(z)|$ for all z on C. Thus by Rouché's Theorem $g_1(z)$ and $g_1(z)+g_2(z)$ have the same number of zeros inside C, $g_1(z)$ has one zero at $\frac{z_1+z_0}{2}$. Hence there is one mean value point inside C.

Case 2: $0 = f''(z_0) = \ldots = f^{(n-1)}(z_0)$, and $f^{(n)}(z_0) \neq 0$.

Set $g_1(z) = \frac{f^{(n)}(z_0)}{(n-1)!}\left((z-z_0)^{n-1} - \frac{1}{n}(z_1-z_0)^{n-1}\right)$, the first non-vanishing term in (2). Set

$$g_2(z) = \sum_{k=n+1}^{\infty} \frac{f^{(k)}(z_0)}{(k-1)!}\left((z-z_0)^{k-1} - \frac{1}{k}(z_1-z_0)^{k-1}\right).$$

Let $C = \{z\colon |z-z_0| = |z_1-z_0|\}$ and $r = |z_1-z_0|$. Then for z on C we have

$$\left|\frac{g_2(z)}{g_1(z)}\right| \leqq \frac{\sum_{k=n+1}^{\infty} \frac{|f^{(k)}(z_0)|}{(k-1)!}\left(r^{k-1} + \frac{1}{k}r^{k-1}\right)}{\frac{|f^n(z_0)|}{(n-1)!}\left(r^{n-1} - \frac{1}{n}r^{n-1}\right)}$$

$$= \frac{n!}{(n-1)|f^n(z_0)|}\sum_{k=n+1}^{\infty} \frac{|f^{(k)}(z_0)|}{(k-1)!}\frac{k+1}{k}r^{k-n}.$$

As in Case 1, we can argue this last series is convergent for $r<R$ and the series converges to zero at $r=0$. So there is an r_0 such that if $|z_1-z_0|<r_0$ then $|g_2(z)|<|g_1(z)|$ on C. Hence $g_1(z)$ and $g_1(z)+g_2(z)$ have the same number of zeros inside C. $g_1(z)=0$ has $n-1$ roots at $z_0+(z_1-z_0).n^{-1/(n-1)}$ inside C. Hence there are $n-1$ mean value points (not necessarily distinct) inside C. This completes the proof.

3. Example

Let $f(z)=e^z$ so that Case 1 holds. In this case on C we have

$$|g_1(z)|=|e^{z_0}|.r,$$

and

$$|g_2(z)|\leqq|e^{z_0}|\sum_{k=3}^{\infty}\frac{k+1}{k}\frac{(2r)^{k-1}}{(k-1)!}<\tfrac{4}{3}|e^{z_0}|\sum_{k=3}^{\infty}\frac{(2r)^{k-1}}{(k-1)!}=\tfrac{4}{3}|e^{z_0}|(e^{2r}-1-2r).$$

Hence $|g_1(z)|>|g_2(z)|$ on C provided $r>\frac{4}{3}(e^{2r}-1-2r)$ or $4e^{2r}-4-11r<0$. This function has a minimum at $\frac{1}{2}\log 11/8$ which is greater than 0.15. We know then that if $|z_1-z_0|<0.3$ then there is a mean value point for e^z and the pair z_0, z_1 inside the circle with the segment z_0z_1 as a diameter.

REFERENCES

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(2) R. M. McLeod, Mean value theorems for vector valued functions, *Proc. Edinburgh Math. Soc.* (2) **14** (1965), 197-209.

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