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EMBEDDINGS IN COSET MONOIDS

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Abstract

A submonoid S of a monoid M is said to be *cofull* if it contains the group of units of M. We extract from the work of Easdown, East and FitzGerald (2002) a sufficient condition for a monoid to embed as a cofull submonoid of the coset monoid of its group of units, and show further that this condition is necessary. This yields a simple description of the class of finite monoids which embed in the coset monoids of their group of units. We apply our results to give a simple proof of the result of McAlister [D. B. McAlister, 'Embedding inverse semigroups in coset semigroups', *Semigroup Forum* **20** (1980), 255–267] which states that the symmetric inverse semigroup on a finite set X does not embed in the coset monoid of the symmetric group on X. We also explore examples, which are necessarily infinite, of embeddings whose images are not cofull.

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1. Cofull embeddings

In [5], Easdown *et al.* used groups of units and coset monoids to develop a theory of factorizable inverse monoids which is dual to the classical construction of fundamental inverse semigroups from semilattices. In the classical theory, an inverse subsemigroup S of an inverse semigroup M is *full* if it contains the semilattice E_M of idempotents of M. If M is a monoid, then one also has the alternative group-theoretic point of view, and may focus instead on the group G_M of units of M, and say that a submonoid S of M is *cofull* if S contains G_M. In [5] it was shown that cofull inverse submonoids of coset monoids (defined below) play a role which is dual to that played by full inverse subsemigroups of the Munn inverse semigroup [13].

An inverse monoid M is *factorizable* if $M = E_M G_M (= G_M E_M)$. The study of factorizable inverse monoids was initiated in [2]; for related studies see [4–6, 8–10, 12, 16, 17] and references therein. Key examples of factorizable inverse monoids include finite symmetric inverse semigroups, as well as the so-called coset monoids which we now describe. Let G be a group and denote by $\mathcal{S}(G)$ the join semilattice of all subgroups of G. (The join $H \vee K$ of two subgroups $H, K \in \mathcal{S}(G)$ is defined to be $\langle H \cup K \rangle$, the smallest subgroup of G containing HK.) Now let

$$\mathcal{C}(G) = \{Hg \mid H \in \mathcal{S}(G),\ g \in G\},$$

be the set of all cosets of all subgroups of G. An associative product $*$ is defined on $\mathcal{C}(G)$, for $H, K \in \mathcal{S}(G)$ and $g, l \in G$, by

$$(Hg) * (Kl) = (H \vee gKg^{-1})gl,$$

the smallest coset of G containing $HgKl$. This operation turns $\mathcal{C}(G)$ into a factorizable inverse monoid, with identity $\{1\}$, known as the *coset monoid* of G. Coset monoids were introduced in [14, 15]; see also [8]. We now collect a number of elementary properties of coset monoids which, along with other properties, were stated in [11].

LEMMA 1. *Let G be a group. Then:*
(i) $E_{\mathcal{C}(G)} = \mathcal{S}(G)$;
(ii) $G_{\mathcal{C}(G)} = \{\{g\} \mid g \in G\} \cong G$; *and*
(iii) *the subgroups of $\mathcal{C}(G)$ are precisely the sections of G. (A section of G is a quotient of a subgroup of G.)* □

For $g \in G$ write $\overline{g} = \{g\}$ and for $S \subseteq G$ write $\overline{S} = \{\overline{g} \mid g \in S\}$. In particular, we have $G_{\mathcal{C}(G)} = \overline{G}$ by Lemma 1(ii).

THEOREM 2 (McAlister [11]). *Let X be a set. Then:*
(i) *the symmetric inverse semigroup $\mathcal{I}_X$ embeds in the coset monoid $\mathcal{C}(\mathcal{G}_Y)$ of the symmetric group $\mathcal{G}_Y$ where Y is any set with $|Y| = |X| + 1$; and*
(ii) *$\mathcal{I}_X$ does not embed in $\mathcal{C}(\mathcal{G}_X)$ if X is finite and nonempty.* □

It follows by Theorem 2(i) and the Wagner–Preston theorem that any inverse monoid (indeed any inverse semigroup) embeds in the coset monoid of some group. An interesting question that arises is '*given an inverse monoid M, what minimal features (such as cardinality) are required of a group G to ensure that an embedding*[1] *$M \to \mathcal{C}(G)$ exists*'? By Lemma 1(iii), G would at least have to contain a section isomorphic to G_M, so in particular the cardinality of G would be bounded below by that of G_M. Thus, another natural question arises: '*which inverse monoids M embed in $\mathcal{C}(G_M)$*'? The goal of this article is to give necessary and sufficient conditions for an inverse monoid M to embed as a *cofull* submonoid of $\mathcal{C}(G_M)$. This allows us to give an answer to the second question, above, in the case that M is finite.

Let M be an inverse monoid and write $E = E_M$ and $G = G_M$. For $e \in E$ let

$$G_e = \{g \in G \mid eg = e\} = \{g \in G \mid ge = e\}.$$

[1] Unless stated otherwise, all embeddings are assumed to be *semigroup* embeddings.

It is easy to check that G_e is a subgroup of G, and that $G_e \vee G_f \subseteq G_{ef}$ for each $e, f \in E$. Define a map

$$\psi_M : E \to \mathcal{S}(G) : e \mapsto G_e \quad \text{for each } e \in E.$$

Let $\mathscr{C}$ denote the class of factorizable inverse monoids M for which ψ_M is a *semilattice embedding*; that is, ψ_M is injective, and $G_{ef} = G_e \vee G_f$ for all $e, f \in E$. Motivating examples of members of $\mathscr{C}$ include the *merge and part braid monoid* introduced in [5] both geometrically (in terms of braids for which certain strings associated with an equivalence class are allowed to interact) and algebraically (as a quotient of a semidirect product of a semilattice of equivalence relations by the braid group), and the largest factorizable inverse submonoid of the dual symmetric inverse monoid on a finite set [9]. (See [5, Section 4] for proofs and details.) The recently discovered class of *dual reflection monoids* [7] also belongs to $\mathscr{C}$. We aim to show that membership of $\mathscr{C}$ is a necessary and sufficient condition for a monoid M to embed as a cofull submonoid of $\mathcal{C}(G_M)$. The following fact is well known and is implicit in [2].

LEMMA 3. *Suppose that M is a factorizable inverse monoid. Then, for all $e, f \in E_M$ and $g, h \in G_M$,*

$$eg = fh \iff e = f \quad \text{and} \quad gh^{-1} \in G_e. \qquad \square$$

LEMMA 4. *Any cofull submonoid of a factorizable inverse monoid is itself a factorizable inverse monoid.*

PROOF. Suppose that N is a cofull submonoid of a factorizable inverse monoid M, and choose $m \in N$. Then $m = eg$ for some $e \in E_M$ and $g \in G_M$. Since N is cofull, we have $g^{-1} \in N$ and so $m^{-1} = g^{-1}e = g^{-1}(eg)g^{-1} = g^{-1}mg^{-1} \in N$, showing that N is inverse. We also have $e = mg^{-1} \in N$ so that N is factorizable. $\square$

THEOREM 5. *A monoid M embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if and only if $M \in \mathscr{C}$.*

PROOF. Write $E = E_M$ and $G = G_M$, and suppose first that $M \in \mathscr{C}$. Define $\theta : M \to \mathcal{C}(G)$ by $eg \mapsto G_e g$ for $e \in E$ and $g \in G$. Then θ is well defined by Lemma 3, and injective since ψ_M is. That θ is a homomorphism follows from the rule for $*$ in the coset monoid, and the facts that $G_{ef} = G_e \vee G_f$ and $g^{-1}G_e g = G_{g^{-1}eg}$ for all $g \in G$ and $e, f \in E$. That $M\theta$ is cofull follows from the fact that $G_1 = \{1\}$ which shows that $G\theta = \overline{G}$ (using the notation defined after Lemma 1).

To show the converse it suffices, since $\mathscr{C}$ is clearly closed under isomorphisms, to show that $N \in \mathscr{C}$ for every cofull submonoid N of $\mathcal{C}(G_M)$. Since N is a factorizable inverse monoid by Lemma 4, it remains only to show that the map

$$\psi_N : E_N \to \mathcal{S}(G_N)$$

is an embedding. Now $E_N = \mathcal{S}(G) \cap N$, and $G_N = \overline{G}$, since N is cofull. Furthermore, if $H \in E_N$, then $H\psi_N = \overline{H}$. It follows that ψ_N is an embedding since $\overline{H} \vee \overline{K} = \overline{H \vee K}$ for any subgroups $H, K \in \mathcal{S}(G)$. $\square$

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Remark 6. The sufficiency of Theorem 5 can also be deduced from [5, Theorem 3.2] (although the reader should note that the injectivity of ψ_M is not part of the definition of the class $\mathcal{C}$ used in [5]), or alternatively from [10, Theorems 3.1.5 and 4.2.3].

As a corollary, we have the following characterization for finite monoids.

Theorem 7. *A finite monoid M embeds in $\mathcal{C}(G_M)$ if and only if $M \in \mathscr{C}$.*

Proof. Write $G = G_M$ and let $\Psi : M \to \mathcal{C}(G)$ be an embedding. Now $G\Psi$ is a section of G by Lemma 1(iii) and, since $G\Psi \cong G$ is finite, we must have $G\Psi = G/\{1\} = G_{\mathcal{C}(G)}$. Thus Ψ is cofull, and we are done by Theorem 5. □

We now apply Theorem 7 to provide a short proof of Theorem 2(ii).

Corollary 8. *Let X be a finite nonempty set. Then $\mathcal{I}_X$ does not embed in $\mathcal{C}(\mathcal{G}_X)$.*

Proof. Put $G = \mathcal{G}_X = G_{\mathcal{I}_X}$ and, for $A \subseteq X$, denote by id_A the identity map on A so that $E_{\mathcal{I}_X} = \{\mathrm{id}_A \mid A \subseteq X\}$. For each $A \subseteq X$, we have $G_{\mathrm{id}_A} = \{\pi \in G \mid a\pi = a \text{ (for all } a \in A)\}$. We will denote this subgroup, the pointwise stabilizer of A, by $\mathrm{Stab}(A)$. Now if $x \in X$, then $\mathrm{Stab}(X) = \mathrm{Stab}(X \setminus \{x\}) = \{\mathrm{id}_X\}$ so that $\psi_{\mathcal{I}_X}$ is not injective, and we are done by Theorem 7. □

Remark 9. If X is any (possibly infinite) set with $|X| \geq 2$, then the map $\psi_{\mathcal{I}_X}$ is not even a homomorphism since if $x, y \in X$ with $x \neq y$ then, writing $A = X \setminus \{x\}$ and $B = X \setminus \{y\}$,

$$G_{\mathrm{id}_A} \vee G_{\mathrm{id}_B} = \mathrm{Stab}(A) \vee \mathrm{Stab}(B) = \mathrm{Stab}(A) = \mathrm{Stab}(B) = \{\mathrm{id}_X\},$$

while the transposition which interchanges x and y is in

$$\mathrm{Stab}(X \setminus \{x, y\}) = \mathrm{Stab}(A \cap B) = G_{\mathrm{id}_{A\cap B}} = G_{\mathrm{id}_A \circ \mathrm{id}_B}.$$

2. Other embeddings

In this final section we consider examples of factorizable inverse monoids M which embed in $\mathcal{C}(G_M)$ but do not belong to $\mathscr{C}$. These monoids are necessarily infinite and, of course, the embeddings are not cofull.

Example 1. Let X be an infinite set. Then the symmetric inverse semigroup $\mathcal{I}_X$ does not belong to $\mathscr{C}$ as noted in Remark 9. On the other hand, $\mathcal{I}_X$ does embed in the coset monoid of the symmetric group $\mathcal{G}_X = G_{\mathcal{I}_X}$ by Theorem 2(i).

Now $\mathcal{I}_X$ (indeed $E_{\mathcal{I}_X}$) is uncountable for any infinite set X. Our second example is a countable factorizable inverse monoid M for which $|E_M| = 3$ and $\mathrm{rank}(G_M) = 1$. (For a group G, $\mathrm{rank}(G)$ denotes the minimal cardinality of a set which generates G as a group.)

Example 2. Let $G = \langle x \rangle$ be the infinite cyclic group generated by x, and let G^y be the semigroup obtained by adjoining a zero y to G. Let $M = (G^y)^z$ be the semigroup

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obtained by adjoining a new zero z to G^y. It is easy to check that M is a factorizable inverse monoid with $G_M = G$ and $E_M = \{1, y, z\}$. We also have $G_y = G_z = G$ so that $M \notin \mathscr{C}$. Now define

$$\Psi : M \to \mathcal{C}(G) : \begin{cases} x \mapsto \{x^2\}, \\ y \mapsto \langle x^2 \rangle, \\ z \mapsto G. \end{cases}$$

Then one may easily check that Ψ is an embedding.

Our final example is also countable, but in this case we have $|E_M| = \mathrm{rank}(G_M) = 2$.

EXAMPLE 3. Let $G = \langle x, y \rangle$ be the free group freely generated by $\{x, y\}$. Define a homomorphism

$$\varphi : G \to G : x \mapsto x^2, \quad y \mapsto y^2,$$

and put $H = \langle x^2, y^2 \rangle$, the image of φ. Let $B = G/N$ where N is the normal closure in G of $\{xyxy^{-1}x^{-1}y^{-1}\}$. So B has the presentation $\langle x, y \mid xyx = yxy \rangle$ and is isomorphic to the braid group on three strings; see [1]. It is well known that Nx^2 and Ny^2 generate a free subgroup of B of rank two; see for example [3]. It follows that $N \cap H = \{1\}$.

Now let $E = \{0, 1\}$, which we consider as a semilattice under multiplication, and put $M = E \times G$. So M is a factorizable inverse monoid with $E_M = (E, 1) \cong E$ and $G_M = (1, G) \cong G$. We see that $M \notin \mathscr{C}$ since $G_{(1,1)} = G_{(0,1)} = \{(1, 1)\}$. Now define

$$\Psi : M \to \mathcal{C}(G) : \begin{cases} (1, g) \mapsto \{g\varphi\} & \text{for all } g \in G, \\ (0, g) \mapsto N(g\varphi) & \text{for all } g \in G. \end{cases}$$

Then Ψ is a homomorphism since N is normal in G and φ is a homomorphism. To show that Ψ is injective, suppose that $e_1, e_2 \in E$ and $g_1, g_2 \in G$ are such that $(e_1, g_1)\Psi = (e_2, g_2)\Psi$. Then we clearly must have $e_1 = e_2$. Suppose first that $e_1 = e_2 = 1$. Then

$$\{g_1\varphi\} = (e_1, g_1)\Psi = (e_2, g_2)\Psi = \{g_2\varphi\}.$$

It then follows that $g_1 = g_2$, since φ is injective, and so $(e_1, g_1) = (e_2, g_2)$. Finally, suppose that $e_1 = e_2 = 0$. Then

$$N(g_1\varphi) = (e_1, g_1)\Psi = (e_2, g_2)\Psi = N(g_2\varphi),$$

from which it follows that $(g_1g_2^{-1})\varphi = (g_1\varphi)(g_2\varphi)^{-1} \in N$. But then $(g_1g_2^{-1})\varphi = 1$ since $N \cap H = \{1\}$, and so $g_1g_2^{-1} = 1$ since φ is injective, whence $g_1 = g_2$ and $(e_1, g_1) = (e_2, g_2)$. This completes the proof that Ψ is injective.

While the monoids M considered in Examples 2 and 3 had different values of $|E_M|$ and $\mathrm{rank}(G_M)$, they shared the property that $|E_M| + \mathrm{rank}(G_M) = 4$. This number turns out to be minimal among all such examples, as the next proposition demonstrates.

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PROPOSITION 10. *Suppose that $M \notin \mathscr{C}$ is a factorizable inverse monoid and that there exists an embedding $\Psi : M \to \mathcal{C}(G_M)$. Then $|E_M| + \operatorname{rank}(G_M) \geq 4$.*

PROOF. Write $E = E_M$ and $G = G_M$ and suppose that $|E| + \operatorname{rank}(G) \leq 3$. Since $M \notin \mathscr{C}$, we have $|E| \geq 2$ and, since Ψ is injective, we have $\operatorname{rank}(G) \geq 1$. It then follows that $|E| = 2$ and $\operatorname{rank}(G) = 1$. Write $E = \{1, e\}$ where 1 is the identity of M. Since $M \notin \mathscr{C}$ we must have $G_e = \{1\}$. Since M is infinite, G must be an infinite cyclic group generated by an element x (say) and, since Ψ is an embedding, we have $x\Psi = \{x^i\}$ and $e\Psi = \langle x^j \rangle$ for some $i, j \in \mathbb{Z} \setminus \{0\}$. But then $ex^j \neq e$ since $G_e = \{1\}$, yet

$$(ex^j)\Psi = (e\Psi) * (x^j\Psi) = \langle x^j \rangle * \{x^{ij}\} = \langle x^j \rangle x^{ij} = \langle x^j \rangle = e\Psi,$$

contradicting the injectivity of Ψ. This completes the proof. □

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