ON LINEAR PERTURBATION OF NON-LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. In the theory of the asymptotic solution or stability of ordinary differential equations most attention has been given to linear or nearly-linear cases. Investigations in this field, starting primarily with those of Kneser **(7)** on the equation $y'' + f(x)y = 0$, have by now mostly been summed up in results on the vector-matrix system $d\mathbf{y}/dx = \mathbf{Ay} + \mathbf{f}(\mathbf{y}, x)$, where $\mathbf{y}$ and $\mathbf{f}$ denote n-vectors of functions, and $\mathbf{A}$ an n-by-n matrix, frequently assumed constant. In the strictly linear case **(4; 8; 9)**, where $\mathbf{f}(\mathbf{y}, x) = \mathbf{B}(x)\mathbf{y}$, it is shown that with restrictions on $\mathbf{B}(x)$ as $x \to \infty$ all the solutions behave for large x as solutions of $d\mathbf{y}/dx = \mathbf{Ay}$. In the nearly-linear case however **(6; 10; 12)**, where we have restrictions on the magnitude of $||\mathbf{f}(\mathbf{y}, x)||/||\mathbf{y}||$ in some bounded $\mathbf{y}$-region, we may expect more than one type of solution; those for which $||\mathbf{y}(0)||$ is sufficiently small may be expected to behave asymptotically as solutions of $d\mathbf{y}/dx = \mathbf{Ay}$, while "larger" solutions may perhaps exhibit an entirely different behaviour.

In this paper I compare, in a special case, the single differential equations

1.1 $$y'' + y^{2n-1} = 0,$$

1.2 $$y'' + y^{2n-1} + h(x, y) = 0,$$

where the perturbing function $h(x, y)$ is in some sense small, and n is a positive integer. The cases in which $h(x, y)$ is, as a function of y, of the same or higher degree than y^{2n-1} are analogous to the above-mentioned linear and nearly-linear cases, respectively, and we may expect that all, or possibly only the "smaller," solutions of 1.2 will behave asymptotically as solutions of 1.1. In the non-linear case, $n > 1$, the possibility naturally presents itself that $h(x, y)$ might be of lower degree than y^{2n-1}, and here the situation may be expected to be just the opposite of that in the nearly-linear case, in that the "larger" solutions of 1.2 should behave as solutions of the unperturbed equation, while there may be (though there need not be) "smaller" solutions behaving differently.

My aim here is to make the latter considerations rigorous for the equation

1.3 $$y'' + y^{2n-1} + g(x)y = 0$$

where $n \geqslant 2$ and $g(x)$ is suitably smooth and is small for large x. Roughly speaking, the situation may be summarised by saying that 1.3 has under fairly general conditions solutions behaving as solutions of 1.1 in respect of magni-

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tude and oscillatory behaviour. It may in addition have non-oscillatory solutions which are $o(1)$ for large x, particularly it seems if $g(x)$ is negative and small, but not too small, for large x. I conclude by establishing some conditions under which solutions of 1.3 can be actually approximated to in terms of solutions of 1.1; this presents slightly greater difficulties than in the linear case, since in the non-linear case the amplitude of the oscillations affects their frequency.

The equation 1.3 has an additional interest in that it may be regarded as a canonical form by transformation of $y'' + f(x)y^{2n-1} = 0$. Equations of both forms have interest in astrophysics, and special cases have been studied by Fowler (**5**); oscillation criteria for the latter form have recently been given by me (**3**).

2. Classification of solutions. This, like most of the subsequent working, is based on an adaptation of the polar-coordinate method. I define the amplitude-variable r of a solution y of 1.3, not identically zero, by

$$r^{2n} = y^{2n} + n(y')^2, \qquad r > 0, \tag{2.1}$$

so that for 1.1 r would be constant. The phase-variable θ will be defined later.

I show, in §3, that $r(x)$ tends under fairly wide conditions to a constant value as $x \to \infty$. Solutions for which $r(\infty)$ exists and is positive I term of type I, those for which $r(x) \to 0$ of type II; these are respectively the oscillatory and the $o(1)$ solutions referred to in §1.

While the type I solutions are all of the same character, the type II solutions, if they exist at all, need not form a homogeneous set. A possible subdivision would be according to the relative magnitude of the three terms on the left of 1.3, into type II(a) for which the first and third terms predominate, type II(b) for which the second and third terms predominate, and type II(c) for which all three terms are of the same order of magnitude. In this paper however I consider only type II solutions as a whole, irrespective of any subclassification.

These classifications may be illustrated in the case of

$$y'' + y^3 - ay/x^2 = 0, \tag{2.2}$$

where a is a real constant. For any a there is a two-parameter family of type I solutions, while type II solutions exist only for $a > 2$. If $a > 2$ we have precisely two solutions of type II(c), given by

$$y = \pm \sqrt{a-2}\, x^{-1},$$

and in addition a one-parameter family of solutions of type II(a) of the asymptotic form $y \sim cx^b$, where c is any non-zero constant and

$$b = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} + a}.$$

There are no solutions of a lower order of magnitude.

3. Restriction of solutions to types I and II. Before establishing properties of type I and type II solutions I give conditions under which the classification is exhaustive.

THEOREM 1. *Let $g(x)$ be expressible as the sum of $g_1(x)$ and $g_2(x)$, where $g_1(x)$ is continuous for $x \geqslant 0$ and $g_2(x)$ continuously differentiable for $x \geqslant 0$, and*

3.1 $$\int_0^\infty |g_1|\, dx < \infty, \quad \int_0^\infty |g_2'|\, dx < \infty, \quad g_2(\infty) = 0.$$

Then 1.3 *has solutions of type I and II at most. Those of type I will certainly exist, and will include all solutions with an initial lower bound of the form* $|y(0)| + |y'(0)| >$ const. > 0. *In particular, all solutions are bounded.*

It should be remarked that in the linear case the same conditions ensure the validity of certain asymptotic integration formulae (**1**; **9**; **11**).

I prove first that r is bounded, which will prove the last statement of the theorem, and will also preclude the eventuality of a solution becoming infinite for a finite x-value. If r were not bounded for some solution, we could for any sufficiently large $A > 0$ find $x_1, x_2 \geqslant 0$ with

3.2 $$r(x_1) = A, \quad r(x_2) = 2A, \quad A \leqslant r(x) \leqslant 2A \text{ for } x_1 \leqslant x \leqslant x_2.$$

Now from 1.3 we have

3.3 $$\begin{aligned}(r^{2n} + ng_2y^2)' &= -2nyy'g_1 + ng_2'\, y^2 \\ &= O(A^{n+1}|g_1| + A^2|g_2'|),\end{aligned}$$

in (x_1, x_2) and hence, taking A so large that $A^{2n-2} > 2n \max |g_2|$, we have

$$(d/dx) \log (r^{2n} + ng_2y^2) = O(A^{1-n}|g_1| + A^{2-2n}|g_2'|),$$

in (x_1, x_2). We integrate this over (x_1, x_2), getting

$$[\log (r^{2n} + ng_2y^2)]_{x_1}^{x_2} = O\left(A^{1-n}\int_{x_1}^{x_2} |g_1|\, dx + A^{2-2n}\int_{x_1}^{x_2} |g_2'|\, dx\right).$$

We now make A increase without limit, and therewith x_1, x_2 if necessary. Since $g_2 \to 0$, the left-hand side tends to $2n \log 2$. The right, however, tends to zero as $A \to \infty$, by 3.1. This contradiction shows the boundedness of r, and y, y' also.

To deduce that r tends to a finite limit, we remark that it now follows from 3.1 that the right of 3.3 is absolutely integrable over $(0, \infty)$. This shows that $(r^{2n} + ng_2 y^2)$ tends to a limit as $x \to \infty$, and hence r also, since $g_2 \to 0$. The solutions are therefore of types I and II at most.

To complete the proof of the theorem, it will be sufficient to show that if $r(0)$ is sufficiently large, then $r(\infty) \neq 0$. Suppose then that $r(\infty) = 0$, and write $r(0) = B$; we show that this gives a contradiction if B is sufficiently large. We must be able to find x_3, x_4 such that

$$r(x_3) = B, \quad r(x_4) = \tfrac{1}{2}B, \quad \tfrac{1}{2}B \leqslant r(x) \leqslant B \text{ for } x_3 \leqslant x \leqslant x_4.$$

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If further we take B so large that $(\frac{1}{2}B)^{2n-2} > 2n \max |g_2|$, we have, by the above reasoning,

$$[\log (r^{2n} + ng_2y^2)]_{x_3}^{x_4} = O\left(B^{1-n}\int_{x_3}^{x_4} |g_1|\,dx + B^{2-2n}\int_{x_3}^{x_4} |g_2'|\,dx\right).$$

If now we make B increase without limit, the left-hand side will tend to $-(2n \log 2)$, and the right-hand side to zero, which constitutes the required contradiction.

In the rest of this paper I consider cases of 1.3 in which $g(x)$ is either monotonic and tending to zero, or else absolutely integrable over $(0, \infty)$. In both of these cases Theorem I shows that the solutions are at most of types I and II, and possibly of type I only.

4. Restriction of solutions to type I. I now give some simple sufficient conditions for the solutions to be of type I only. I prove first

THEOREM 2. *Let $g(x)$ be positive and continuously differentiable for $x \geqslant 0$, and tend monotonically to zero as $x \to \infty$. Then all solutions of* 1.3 *are of type I.*

Supposing if possible that for a certain solution of 1.3 we have $r \to 0$, we use the result that, by 1.3,

4.1 $$(r^{2n} g^{-1} + ny^2)' = -g'r^{2n} g^{-1},$$

which shows here that the function $(r^{2n} g^{-1} + ny^2)$ is non-decreasing. Since this function is positive for $y \not\equiv 0$, it follows that there is a positive constant C such that $(r^{2n} g^{-1} + ny^2) > C$ for all $x \geqslant 0$. Since $y \to 0$ we have, for all sufficiently large x, $r^{2n} g^{-1} > \frac{1}{2}C$, so that

4.2 $$r^{-1} = O(g^{-1/(2n)}).$$

Also from 1.3, or from 4.1, we have

$$\begin{aligned}(d/dx) \log (r^{2n} + ngy^2) &= ng'y^2(r^{2n} + ngy^2)^{-1}\\ &= O(g'y^2r^{-2n})\\ &= O(g'r^{2-2n})\\ &= O(g'g^{-1+1/n})\end{aligned}$$

using the fact that $g > 0$ and also 2.1, 4.2. Here the right-hand side is absolutely integrable over $(0, \infty)$, since g tends monotonically to zero. This proves that the function $\log (r^{2n} + ngy^2)$ tends to a finite limit as $x \to \infty$, thus contradicting the hypothesis that $r \to 0$.

The result just proved suggests that type II solutions may be expected when g is negative and small at ∞. That g must not be too small is shown by

THEOREM 3. *Let $g(x)$ be continuous for $x \geqslant 0$ and such that*

4.3 $$\int_0^\infty x|g(x)|\,dx < \infty.$$

Then 1.3 *has type I solutions only.*

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We re-write 1.3 in the form

4.4 $$y'' + g_1(x)y = 0,$$

where $g_1 = g + y^{2n-2}$. We show first that if y is a type II solution of 1.3, and 4.3 holds, then

4.5 $$\int_0^\infty x|g_1(x)|\,dx < \infty.$$

In view of 4.3 it will be sufficient to prove that

4.6 $$\int_0^\infty xy^{2n-2}\,dx < \infty,$$

or again

4.7 $$\int_0^\infty xr^{2n-2}\,dx < \infty.$$

Now by 1.3 we have $(r^{2n})' = -2ngyy'$, and so, using 2.1,

4.8 $$r'r^{n-2} = O(g).$$

Integrating 4.8 over (x, ∞) we have, assuming that $r(\infty) = 0$,

4.9 $$r^{n-1} = O\left(\int_x^\infty |g|\,dx\right),$$

a result which will later be improved to 7.1. We have then

$$xr^{2n-2} = O\left\{x\left(\int_x^\infty |g|\,dx\right)^2\right\} = O\left(\int_x^\infty |g|\,dx\right),$$

using 4.3, so that 4.7 will be true if we have

$$\int_0^\infty dx \int_x^\infty |g(t)|\,dt < \infty.$$

This however is easily seen to follow from 4.3. We have therefore proved that a type II solution of 1.3 is also a solution of 4.4, where g_1 satisfies 4.5.

However 4.4 has, subject to 4.5, no solutions which are $o(1)$ as $x \to \infty$, having in fact, as is well known, two fundamental solutions of the asymptotic forms $y_1 \to 1$, $y_2 \sim x$, as $x \to \infty$. The existence of such a y_1 may be shown, for example, by transforming 4.4 to an integral equation and using successive approximation; we may then take $y_2 = y_1 \int^x y_1^{-2}\,dx$. This completes the proof of Theorem 3.

That the criterion 4.3 is fairly precise is shown by the example

$$y'' + y^{2n-1} - ayx^{-b} = 0,$$

for which it shows that there are no type II solutions for $b > 2$; in the case $b = 2$ they exist, as noted in §2, for $n = 2$ and $a > 2$.

5. The non-oscillation of type II solutions. Having considered the existence and magnitude (see 4.9) of type II solutions, I now give a simple sufficient criterion for their non-oscillatory character.

THEOREM 4. *Let $g(x)$ be negative and continuously differentiable for $x \geqslant 0$, and let it tend monotonically to zero as $x \to \infty$. Then type II solutions of* 1.3, *if any, have no zeros.*

The result $(r^{2n} + ngy^2)' = ng'y^2$ here shows that the function $(r^{2n} + ngy^2)$ is non-decreasing. At a zero of y this function would be positive, assuming $y \not\equiv 0$, and so could only tend to a positive limit. On the other hand for a type II solution we should have to have $(r^{2n} + ngy^2) \to 0$. This proves the theorem.

We can also deduce that for a type II solution we must have $r^{2n} + ngy^2 \leqslant 0$, showing that type II solutions satisfy in this case the bound $r^{2n-2} \leqslant n|g|$. While this result gives the correct order of magnitude of type II solutions in some cases, for 2.2 for example, a later result gives a precise numerical coefficient.

6. The phase-variable. In order to obtain sharper results, including asymptotic formulae for type I solutions, I introduce the phase-variable. I define by $\psi(\theta)$ the solution of

6.1 $$d^2\psi/d\theta^2 + n\psi^{2n-1} = 0,$$

with the initial conditions

$$\psi(0) = 1, \quad d\psi(\theta)/d\theta|_{\theta=0} = 0.$$

In the case $n = 1$ this reduces to the cosine function, for $n = 2$ to the lemniscate function. In the general case it is a periodic function of period $4K$, where

6.2 $$K = \int_0^1 (1 - \psi^{2n})^{-\frac{1}{2}} \, d\psi.$$

Using the abbreviation ψ_θ for $d\psi(\theta)/d\theta$ we have also

6.3 $$\psi_\theta^2 + \psi^{2n} = 1.$$

I now form the first-order differential equation satisfied by θ, the phase-variable defined for a solution y of 1.3 by

6.4 $$y = r\,\psi(\theta), \quad y' = r^n \, n^{-\frac{1}{2}} \, \psi_\theta(\theta), \qquad r > 0,$$

in agreement with 2.1. This definition leaves θ uncertain to the extent of an arbitrary multiple of $4K$, which need only be chosen so that θ is a continuous function of x.

We have first

6.5 $$\begin{aligned}(y'y^{-n})' &= n^{-\frac{1}{2}}(\psi_\theta\,\psi^{-n})' \\ &= n^{-\frac{1}{2}}(\psi_{\theta\theta}\,\psi^{-n} - n\psi_\theta^2\,\psi^{-n-1})\,\theta' \\ &= -n^{+\frac{1}{2}}(\psi^{n-1} + \psi_\theta^2\,\psi^{-n-1})\,\theta' \\ &= -n^{+\frac{1}{2}}\,\psi^{-n-1}\,\theta',\end{aligned}$$

using 6.1 and 6.3. Also

$$
\begin{aligned}
(y'y^{-n})' &= y''y^{-n} - ny'^2 y^{-n-1} \\
&= -y^{n-1} - gy^{1-n} - ny'^2 y^{-n-1} \\
&= -r^{n-1}\psi^{n-1} - gr^{1-n}\psi^{1-n} - r^{n-1}\psi_\theta{}^2\psi^{-n-1} \\
&= -r^{n-1}\psi^{-n-1} - gr^{1-n}\psi^{1-n},
\end{aligned}
\tag{6.6}
$$

using 1.3 and 6.3. Combining 6.5, 6.6 we have

$$
\theta' = r^{n-1} n^{-\frac{1}{2}} + gr^{1-n}\psi^2, \tag{6.7}
$$

the required differential equation.

As regards the amplitude-variable r we have

$$
\begin{aligned}
(r^{2n})' &= (y^{2n} + ny'^2)' = -2ngyy' \\
&= -2n^{\frac{1}{2}} gr^{n+1}\psi\psi_\theta,
\end{aligned}
$$

and so

$$
r' = -n^{-\frac{1}{2}} r^{2-n} g\psi\psi_\theta. \tag{6.8}
$$

7. A bound for type II solutions. As a final result for type II solutions I give the precise form of the bound 4.9 for their magnitude.

THEOREM 5. *Let $g(x)$ be continuous and absolutely integrable over* $(0, \infty)$. *Then type II solutions of* 1.3, *if they exist, satisfy the bound*

$$
r^{n-1} \leqslant (n-1)(n+1)^{-(n+1)/(2n)} \int_x^\infty |g|\, dx. \tag{7.1}
$$

That the constant factor on the right of 7.1 cannot in general be reduced is shown by the example

$$
y'' + y^3 - 3yx^{-2} = 0, \quad n = 2, \quad y = x^{-1}, \quad r = 3^{\frac{1}{4}}x^{-1},
$$

for which the equality sign in 7.1 holds.

From 6.3 it may be deduced that

$$
|\psi\psi_\theta| \leqslant n^{\frac{1}{2}}(n+1)^{-(n+1)/(2n)}
$$

and so, by 6.8,

$$
(n-1)r^{n-2}|r'| \leqslant (n-1)(n+1)^{-(n+1)/(2n)}|g|.
$$

Integrating over (x, ∞) and putting $r(\infty) = 0$, for a type II solution, we have

$$
r^{n-1} \leqslant \int_x^\infty (n-1)\, r^{n-2}|r'|\, dx \leqslant (n-1)(n+1)^{-(n+1)/(2n)} \int_x^\infty |g|\, dx,
$$

the result stated.

8. The oscillations of type I solutions. I now pass to the investigation of type I solutions of 1.3, which are comparable to the non-trivial solutions of 1.1, at least in the respect that they have, by definition, an asymptotically constant amplitude. It remains to compare the two sets of solutions in respect of oscillatory properties. In this section I find a rough estimate for the density of the zeros of a type I solution of 1.3, in analogy to known results for the linear case.

THEOREM 6. *Let $g(x)$ be continuous and tend to zero as $x \to \infty$. Then a type I solution of* 1.3, *if such exist, must be oscillatory as $x \to \infty$; if $N(x)$ denotes the number of its zeros in* $(0, x)$, *then*

8.1 $$N(x) \sim A^{n-1} x n^{-\frac{1}{2}} (2K)^{-1},$$

where K is given by 6.2, *and $A = r(\infty)$ for the solution in question.*

Before proceeding to the proof I remark that the condition $g(x) \to 0$ by itself may well be insufficient to ensure the existence of type I solutions; this certainly applies in the linear case, where the equation $y'' + y + g(x)y = 0$ with $g(x) \to 0$ can have solutions of asymptotically large and small amplitudes, without any of asymptotically finite positive amplitude. Examples of this phenomenon are given by

$$y'' + y(1 + x^{-a} \cos 2x) = 0 \qquad (0 < a \leqslant 1),$$

and by

$$y'' + y(1 + x^{-a} \cos x) = 0 \qquad (0 < a \leqslant \tfrac{1}{2}).$$

To prove 8.1 we integrate 6.7 over $(0, x)$, getting

8.2 $$\theta(x) - \theta(0) = \int_0^x r^{n-1} n^{-\frac{1}{2}} dx + \int_0^x g r^{1-n} \psi^2 dx$$

8.3 $$= I_1 + I_2.$$

Since $r \to A > 0$ we have

8.4 $$I_1 \sim A^{n-1} x n^{-\frac{1}{2}}, \quad I_2 = O\left(\int_0^x |g| \, dx\right) = o(x).$$

Furthermore, since zeros of ψ, and so of y, occur when θ is an odd multiple of K, we have

8.5 $$N(x) = \theta(x)/(2K) + O(1),$$

using also the fact that $\theta(x)$ is an increasing function of x in the neighbourhood of a zero of ψ. The results 8.3–8.5 yield the proof of 8.1 and so prove the theorem.

For similar arguments in the linear case and references to other work on the linear case I refer to my paper **(2)**.

9. Asymptotic solutions. Finally I prove an approximation formula for type I solutions of 1.3 in terms of solutions of 1.1. To simplify the argument I have imposed more severe restrictions than are actually necessary for the result. Some essential improvement in the argument would be required however to make the result of similar generality to that of Ascoli **(1**; see also **9** and **11)** for the linear case.

THEOREM 7. *Let $g(x)$ be continuously twice differentiable for $x \geqslant 0$ and tend with $g'(x)$ monotonically to zero as $x \to \infty$. Let also $\int_0^\infty g^2\,dx < \infty$. Then to each type I solution of* 1.3 *there correspond two constants A, B, with $A > 0$, such that as $x \to \infty$,*

9.1 $$y = A\psi\left(A^{n-1}\, x\, n^{-\frac{1}{2}} + c_n A^{1-n}\int_0^x g\,dx + B\right) + o(1),$$

where c_n is a constant dependent only on n, and $\psi(\theta)$ is as defined in §6. *A corresponding formula for y' may be obtained by formal differentiation.*

The result shows that the influence of the linear perturbation term becomes vanishingly small for solutions of large amplitude.

In 9.1 the constant A denotes $r(\infty)$, and in view of 6.4 it is only necessary to prove that

9.2 $$\theta(x) = A^{n-1}\, x\, n^{-\frac{1}{2}} + c_n A^{1-n}\int_0^x g\,dx + B + o(1),$$

as $x \to \infty$. By 8.2–3 it will be sufficient to prove that

9.3 $$I_1 = A^{n-1}\, x\, n^{-\frac{1}{2}} + B_1 + o(1),$$

9.4 $$I_2 = c_n A^{1-n}\int_0^x g\,dx + B_2 + o(1),$$

where B_1, B_2 are constants for the solution in question.

In order to approximate to θ it is first necessary to approximate to r; this difficulty does not arise in the linear case ($n = 1$), since the right of 8.2 is then independent of r. For a first approximation we use the result $(r^{2n}+ngy^2)' = ng'y^2$. Integrating over (x, ∞), we have

$$A^{2n} - r^{2n} - ngy^2 = O\left(\int_x^\infty |g'|\,dx\right) = O(g),$$

since g is monotonic. Since $|y| \leqslant r$ we deduce that

9.5 $$r = A + O(g).$$

Using now 9.5 to obtain the second approximation we have

9.6 $$(r^{2n} + ngy^2)' = ng'r^2\psi^2 = ng'A^2\psi^2 + O(gg').$$

Integrating 9.6 over (x, ∞) we get

9.7 $$A^{2n} - r^{2n} - ngy^2 = nA^2\int_x^\infty g'\psi^2\,dx + O(g^2),$$

using the fact that g is monotonic.

I now write $\psi^2(\theta) = c_n + \phi(\theta)$, where

$$c_n = (4K)^{-1}\int_0^{4K}\psi^2(\theta)\,d\theta,$$

and

$$\Phi(\theta) = \int_0^\theta \phi(\theta)\, d\theta,$$

so that $\Phi(\theta)$ is a periodic and so bounded function of θ. We have then

$$ngy^2 = ngA^2\psi^2 + O(g^2) = ngA^2c_n + ngA^2\phi + O(g^2),$$

and also

$$\int_x^\infty g'\psi^2\, dx = -c_n g + \int_x^\infty g'\phi\, dx.$$

Using these in 9.7 we obtain

9.8 $$A^{2n} - r^{2n} - ngA^2\phi = nA^2\int_x^\infty g'\phi\, dx + O(g^2).$$

To estimate the integral on the right of 9.8 we use the fact that

$$1 = \theta' A^{1-n} n^{-\frac{1}{2}} + O(g),$$

which follows from 6.7 and 9.5. From this we deduce that

$$\begin{aligned}\int_x^\infty g'\phi\, dx &= \int_x^\infty g'\phi\,\theta' dx . A^{1-n} n^{-\frac{1}{2}} + O\left(\int_x^\infty |gg'|\, dx\right)\\ &= [g'\Phi]_x^\infty A^{1-n} n^{-\frac{1}{2}} - \int_x^\infty g''\Phi\, dx . A^{1-n} n^{-\frac{1}{2}} + O(g^2)\\ &= O(g') + O(g^2).\end{aligned}$$

From 9.8 we therefore have

$$A^{2n} - r^{2n} - ngA^2\phi = O(g') + O(g^2),$$

and so, finally, we have the required second approximation

9.9 $$r = A - \tfrac{1}{2}g\phi\, A^{3-2n} + O(g') + O(g^2).$$

We pass to proving 9.3, 9.4 and so completing the proof of Theorem 7. As regards 9.3 we have

$$\begin{aligned}I_1 &= \int_0^x r^{n-1} n^{-\frac{1}{2}}\, dx\\ &= \int_0^x (A^{n-1} - \tfrac{1}{2}(n-1)\, g\phi A^{1-n})\, dx + \int_0^x O(g')\, dx + \int_0^x O(g^2)\, dx.\end{aligned}$$

Here the last two integrals are by hypothesis absolutely convergent taken over $(0, \infty)$. The term involving $g\phi$ is treated in the same way as the term involving $g'\phi$ in 9.8. We have

$$\begin{aligned}\int_0^x g\phi\, dx &= A^{1-n} n^{-\frac{1}{2}} \int_0^x g\phi\theta'\, dx + \int_0^x O(g^2)\, dx\\ &= A^{1-n} n^{-\frac{1}{2}}\left\{[g\Phi]_0^x - \int_0^x g'\Phi\, dx\right\} + \int_0^x O(g^2)\, dx,\end{aligned}$$

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and here both the integrals and the integrated term are asymptotic to constants as $x \to \infty$.

As regards 9.4 we have, using 9.5,

$$I_2 = \int_0^x g r^{1-n} \psi^2 \, dx = A^{1-n} \int_0^x g\psi^2 \, dx + \int_0^x O(g^2) \, dx$$
$$= A^{1-n} c_n \int_0^x g \, dx + A^{1-n} \int_0^x g\phi \, dx + \int_0^x O(g^2) \, dx.$$

Here the last integral on the right tends to a constant by hypothesis, while the second integral on the right has just been shown to do so. This justifies 9.4, and so completes the proof of Theorem 7.

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