THE FACTORS OF GRAPHS

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1. Introduction. A *graph* G consists of a non-null set V of objects called *vertices* together with a set E of objects called *edges*, the two sets having no common element. With each edge there are associated just two vertices, called its *ends*. Two or more edges may have the same pair of ends.

G is *finite* if both V and E are finite, and *infinite* otherwise.

The *degree* $d_G(a)$ of a vertex a of G is the number of edges of G which have a as an end. G is *locally finite* if the degree of each vertex of G is finite. Thus the locally finite graphs include the finite graphs as special cases.

A *subgraph* H of G is a graph contained in G. That is, the vertices and edges of H are vertices and edges of G, and an edge of H has the same ends in H as in G. A *restriction* of G is a subgraph of G which includes all the vertices of G.

A graph is said to be *regular of order* n if the degree of each of its vertices is n. An *n-factor* of a graph G is a restriction of G which is regular of order n.

The problem of finding conditions for the existence of an n-factor of a given graph has been studied by various authors [**3**; **4**; **5**]. It has been solved, in part, by Petersen for the case in which the given graph is regular. The author has given a necessary and sufficient condition that a given locally finite graph shall have a 1-factor [**6**; **7**]. In this paper we establish a necessary and sufficient condition that a given locally finite graph shall have an n-factor, where n is any positive integer. Actually we obtain a more general result. We suppose given a function f which associates with each vertex a of a given locally finite graph G a positive integer $f(a)$, and obtain a necessary and sufficient condition that G shall have a restriction H such that $d_H(a) = f(a)$ for each vertex a of G. The discussion is based on the method of alternating paths introduced by Petersen [**4**].

We also consider the problem of associating a non-negative integer with each edge of G so that for each vertex c of G the numbers assigned to the edges having c as an end sum to $f(c)$. We obtain a necessary and sufficient condition for the solubility of this problem.

My attention has been drawn to two other papers in which similar theories of factorization have been put forward. In one of these papers, Gallai [**2**] gives a valuable unified theory of factors and gives some new results on the factorization of regular graphs. He also claims to have obtained a necessary and sufficient condition for the existence of a 2-factor in a general locally finite graph, but leaves the discussion of this for another occasion. In the other paper Belck [**1**] establishes a necessary and sufficient condition for the existence of an n-factor in a general finite graph, where n is any positive integer. Prominent

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in his theory is the *hyper-n-prime* graph, a generalization of the *hyperprime* graph introduced in [**6**].

2. **Recalcitrance.** A *path* in a graph G is a finite sequence

$$P = (a_1, A_1, a_2, A_2, \ldots, A_{r-1}, a_r) \tag{1}$$

satisfying the following conditions:

(i) The members of P are alternately vertices and edges of G, the terms $a_1, a_2, \ldots, a_r$ being vertices.

(ii) If $1 \leqslant i < r$, then a_i and a_{i+1} are the two ends of A_i.

We say that P is a path *from* a_1 *to* a_r, and that its *length* is $r - 1$. We note that the terms of P need not be all distinct. We admit the case in which P has length 0. Then P has just one term, a vertex of G.

The vertices x and y of G are *connected in* G if a path from x to y in G exists. If this is so for each pair $\{x, y\}$ of vertices of G, then G is *connected*. The relation of being connected in G is evidently an equivalence relation. It therefore partitions G into a set $\{G_\alpha\}$ of connected graphs such that each edge or vertex of G belongs to some G_α and no two of the G_α have any edge or vertex in common. We call the graphs G_α the *components* of G.

If S is any proper subset of the set of vertices of a given graph G, we denote by $G(S)$ the subgraph of G obtained by suppressing the members of S and all edges of G having one or both ends in S.

Suppose now that G is locally finite and that S is a finite set of vertices of G.

If S does not include all the vertices of G the graph $G(S)$ is defined. Then if H is any finite component of $G(S)$ we denote the number of edges which have one end in S and the other a vertex of H by $v(H)$. We have

$$v(H) + \sum_{c \in H} d_G(c) \equiv 0 \pmod 2, \tag{2}$$

for the expression on the left is equal to twice the number of edges of G having an end which is a vertex of H. (We have used the symbol $c \in H$ to denote that c is a vertex of H.)

We denote by $K(G, S)$ the set of all finite components H of $G(S)$ which satisfy

$$v(H) + \sum_{c \in H} f(c) \equiv 1 \pmod 2. \tag{3}$$

If $K(G, S)$ is finite we denote the number of its elements by $k(G, S)$. If S includes all the vertices of G we write $k(G, S) = 0$. In either case we write

$$r(G, S) = k(G, S) + \sum_{c \in S} (f(c) - d_G(c)). \tag{4}$$

We call $r(G, S)$ the *recalcitrance* of G with respect to S. If $K(G, S)$ is infinite we say that $r(G, S)$ is infinite.

THEOREM I. *If G is finite, $r(G, S)$ is even or odd according as*

$$\sum_{c \in G} f(c)$$

is even or odd.

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Proof. By (2), (3), and (4),

$$r(G, S) \equiv \sum_{c \in G} d_G(c) + \sum_{c \in G} f(c) \pmod 2.$$

But the sum of the degrees of the vertices of G is even, since it is twice the number of edges of G. The theorem follows.

The locally finite graph G is *constricted* with respect to f if there exist disjoint finite sets S and T of vertices of G such that

$$\sum_{c \in T} f(c) < r(G(T), S). \tag{5}$$

As an example, G is constricted if it has a vertex a such that $d_G(a) < f(a)$. In this case (5) is satisfied if T is null and S has the single element a. Again, G is constricted if $r(G, S) > 0$ for any set S of vertices of G, for then (5) is satisfied with T null. So by Theorem I a finite graph G is constricted if the sum of the numbers $f(c)$, for all the vertices c of G, is odd. In this case (5) is satisfied if S and T are both null.

We define an *f-factor* of the given locally finite graph G as a restriction F of G such that $d_F(c) = f(c)$ for each vertex c of G. Similarly, a restriction F of a subgraph X of G is an f-factor of X if $d_F(c) = f(c)$ for each vertex c of X. A restriction F of a subgraph X of G is an *incomplete* f-factor of X if $d_F(c) \leqslant f(c)$ for each vertex c of X, and $d_F(c) = f(c)$ for all but a finite number of the vertices of X. The *deficiency* of such an incomplete f-factor is the sum

$$\sum (f(c) - d_F(c)),$$

taken over all vertices c of X for which $d_F(c) < f(c)$.

Our object in this paper is to show that G has no f-factor if and only if G is constricted with respect to f.

THEOREM II. *Let F be an incomplete f-factor of G, and let S be any finite set of vertices of G. Then the deficiency of F is not less than $r(G, S)$.*

Proof. If H is any member of $K(G, S)$, let $w(H)$ be the number of edges of F which have one end in S and the other a vertex of H. Analogously with (2) we have

$$\sum_{c \in H} d_F(c) \equiv w(H) \pmod 2. \tag{6}$$

Let P be the set of all elements H of $K(G, S)$ such that $d_F(c) = f(c)$ for each vertex c of H. Let Q be the set of all other members of $K(G, S)$. Let the numbers of members of P and Q be p and q respectively; q must be finite.

The sum of the numbers $f(c) - d_F(c)$ taken over all vertices of G not in S which satisfy $d_F(c) < f(c)$ is at least q.

If $H \in P$, then by (3) and (6), $v(H) \neq w(H)$. Hence at least p of the edges of G having just one end in S are not edges of F. It follows that

$$\sum_{c \in S} d_F(c) \leqslant \sum_{c \in S} d_G(c) - p,$$

$$\sum_{c \in S} (f(c) - d_F(c)) \geqslant p + \sum_{c \in S} (f(c) - d_G(c)).$$

Hence if D is the deficiency of F we have

$$D \geqslant p + q + \sum_{c \in S} (f(c) - d_G(c)) = r(G, S).$$

THEOREM III. *If G is constricted with respect to f, it has no f-factor.*

Proof. Suppose G is constricted. Then there are disjoint finite subsets S and T of the set of vertices of G such that (5) is satisfied. Assume G has an f-factor F. Then $F(T)$ is an incomplete f-factor of $G(T)$. Its deficiency D is equal to the number n of edges of F having one end in T and the other not in T. Hence, by Theorem II,

$$\sum_{c \in T} f(c) \geqslant n = D \geqslant r(G(T), S).$$

This contradicts the definition of S and T.

3. Alternating paths. An *f-subgraph* of G is a restriction J of G having the following properties:

(i) The number of edges of J is finite.

(ii) $d_J(c) \leqslant f(c)$ for each vertex c of G.

A vertex c of G is *deficient* in J if $d_J(c) < f(c)$.

Let us suppose that we are given an f-subgraph J of G and that a is a vertex of G which is deficient in J. Following a long-established tradition we refer to an edge of G as *blue* or *red* according as it is or is not an edge of J.

An *alternating path based on a* is a path P in G which satisfies the following conditions:

(i) The first term of P is a.

(ii) No edge of G occurs twice as a term of P.

(iii) If P has more than one term the edges of G which occur in P are alternately red and blue, the first one being red.

If P includes the subsequence (c, C, d) where C is an edge of G, we say that P *passes through c and then C*, or P *passes through C and then d*.

Let $\Pi(a)$ be the set of alternating paths based on a; $\Pi(a)$ is not null since it has one member whose only term is a.

Let C be an edge of G, with ends c and d. If no member of $\Pi(a)$ has C as a term, C is *acursal*. If some member of $\Pi(a)$ passes through C and then d, C is *describable to d* or *from c*. If C is describable to d but not to c, C is *unicursal to d* or *from c*. If C is describable both to c and to d, C is *bicursal*.

A vertex of G is *accessible from a* if it is a term of some member of $\Pi(a)$.

The vertex a is *singular* if no deficient vertex of G, other than a itself, is accessible from a.

THEOREM IV. *Only a finite number of vertices of G are accessible from a.*

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If b is a vertex of G accessible from a, then either $b = a$, or b is an end of a blue edge, or b is an end of an edge B whose other end is either a or an end of a blue edge. Since the number of blue edges and the degree of each vertex of G are finite, the theorem follows.

THEOREM V. *Let A and B be edges of G which are of different colours and have a common end x. Suppose A is unicursal to x. Then B is describable from x.*

There is a member P of $\Pi(a)$ which passes through A and then x. If B is not a term of P preceding A there is evidently a member of $\Pi(a)$ which agrees with P as far as A and continues $(x, B, \ldots)$. Then B is describable from x.

If B precedes A in P, either the theorem is satisfied or P passes through B and then x. In the latter case there is a member of $\Pi(a)$ which agrees with P as far as B and continues $(x, A, \ldots)$. Then A is not unicursal to x, contrary to hypothesis.

4. Bicursal components. Let us suppose that G has at least one bicursal edge. The bicursal edges of G, with their ends, define a subgraph of G. We refer to the components of this subgraph as the *bicursal components.*

THEOREM VI. *The bicursal components are finite graphs.*

This follows from Theorem IV, since the vertices of a bicursal component are all accessible from a and G is locally finite.

Let L denote any bicursal component. An *entrant* of L is any member of $\Pi(a)$ which has a vertex of L as a term. If P is an entrant of L we denote by $e(P)$ the vertex of L which occurs first as a term of P. We then say that P *enters* L *at* $e(P)$. A vertex of L at which some entrant of L enters L is an *entrance* of L.

Let P be an entrant of L. Let A be the first edge of G in P after the first occurrence of $e(P)$ which is not in L, if such an edge exists. The *section of P by L* is defined as follows. If the edge A exists, the section is the part of P extending from the first occurrence of $e(P)$ to the term immediately preceding A. Otherwise, the section is the part of P extending from the first occurrence of $e(P)$ to the last term of P. In either case the section is an alternating path based on $e(P)$ and having only edges and vertices of L as terms (except that its first edge may be blue).

If e is any entrance of L we denote by $\Delta(e)$ the set of sections by L of those members of $\Pi(a)$ which enter L at e.

Since the edges of L are not acursal, L has at least one entrance. If a is a vertex of L then a is an entrance of L.

In the following series of theorems (VII—XI)) we suppose that some entrance e of L is specified, with the proviso that e is a if a is a vertex of L.

THEOREM VII. *There exists an edge of L which is a term of some member of Δ (e).*

Proof. Suppose first that e is a. Any red edge of L having a as an end is clearly a term of a member of $\Delta(a)$. Suppose therefore that the edges of L

having a as an end are all blue. Each of these is describable from a, and no one is the first edge of a member of $\Pi(a)$. Hence some red edge C having a as an end is describable to a. But all red edges having a as an end are describable from a. Hence C is bicursal and therefore an edge of L, contrary to supposition.

Now consider the case in which a is not a vertex of L. Let P be an entrant of L such that $e(P) = e$. Let C be the edge of G which immediately precedes the first occurrence of e in P. Then C is unicursal to e. Any edge of L having e as an end and differing in colour from C is clearly a term of a member of $\Delta(e)$. Suppose therefore that the edges of L having e as an end all have the same colour as C. Since they are all describable from e, some edge E of G differing in colour from C is describable to e. But E is describable from e, by Theorem V. Hence E is bicursal and therefore an edge of L, contrary to supposition.

THEOREM VIII. *If A is an edge of L with ends x and y, and if some member P' of $\Delta(e)$ passes through x and then A, then some other member of $\Delta(e)$ passes through y and then A.*

Proof. Since A is bicursal there exists a member Q of $\Pi(a)$ which passes through y and then A. It may happen that every term of Q which precedes A is an edge or vertex of L. Then a is a vertex of L and therefore $e = a$ by the definition of e. Hence the section of Q by L is a member of $\Delta(e)$ which passes through y and then A.

In the remaining case, let B be the last term of Q preceding A which is an edge of G but not an edge of L. Let b be the immediately succeeding term of Q. Then b is a vertex of L. Let C be the first edge of G in P' which succeeds B in Q but does not succeed A in Q. Such an edge exists since A is an edge both of P' and of Q. Let the ends of C be r and s. We may suppose that P' passes through r and then C.

Suppose Q passes through r and then C. Then there is a member of $\Delta(e)$ which agrees with P' as far as C and then continues with the terms of Q from C to A. This member of $\Delta(e)$ passes through y and then A.

Alternatively, suppose Q passes through s and then C. There is a member Q_1 of $\Pi(a)$ which enters L at e, then agrees with P' as far as C, and continues with the terms of Q in reverse order from C to b. Let D be the edge of Q_1 immediately preceding the first occurrence of e. If $B \neq D$ it follows that B is describable from b. But Q passes through B and then b. So B is bicursal and therefore an edge of L, contrary to its definition. We conclude that $B = D$ and therefore $b = e$. Hence there is a member of $\Pi(a)$ which agrees with Q_1 as far as B and agrees with Q from B to A. The section of this path by L is a member of $\Delta(e)$ which passes through y and then A.

THEOREM IX. *Let A be an edge of L which is a term of some member of $\Delta(e)$. Let x be an end of A distinct from e. Then there is an edge B of L which differs in colour from A, which has x as an end, and which is a term of some member of $\Delta(e)$.*

Proof. By Theorem VIII there is a member of $\Delta(e)$ which passes through x and then A. The last edge preceding A in this member of $\Delta(e)$ has the required properties.

THEOREM X. *If A is any edge of L and x is any end of A, then there is a member of $\Delta(e)$ which passes through x and then A.*

Proof. Let U be the set of all edges of L occurring as terms in the members of $\Delta(e)$; U is non-null, by Theorem VII. Let V be the set of all other edges of L.

Assume that V is non-null. Since L is connected there is a vertex z of L which is an end of a member B of U and a member C of V. If z is not e we may suppose that B and C differ in colour, by Theorem IX. By Theorem VIII there is a member of $\Delta(e)$ which passes through B and then z. C is not a term of this member of $\Delta(e)$. Hence there is a member of $\Delta(e)$ which agrees with this one as far as B and then continues with z and C. This contradicts the definition of C.

Suppose now that z is e. If B and C differ in colour we obtain a contradiction as before. We deduce that all the edges of L having e as an end have the same colour. If $e = a$ it follows from Theorem VII that e is an end of some red edge of L. Then C is red. Hence there is a member of $\Delta(e)$ which has C as its first edge, contrary to assumption. If e is not a it follows from Theorem VII that there is a member P of $\Pi(a)$ entering L at e in which the first occurrence of e is immediately succeeded by an edge of L. We may take this edge to be B. Since B and C have the same colour there is a member of $\Pi(a)$ which agrees with P as far as the first occurrence of e and then continues with C. Hence C is a member of U, contrary to assumption.

We conclude that V is null. The theorem now follows from Theorem VIII.

Let G_1 denote any subgraph of G. An edge A of G is said to *touch* G_1 if A is not an edge of G_1 and just one end, say x, of A is a vertex of G_1. Such an edge A is *unicursal to* or *from* G_1 if it is unicursal to or from x respectively.

THEOREM XI. *If a is a vertex of L then all edges of G which touch L are unicursal from L. If a is not a vertex of L then there exists just one edge of G which touches L and is unicursal to L, and all other edges of G which touch L are unicursal from L.*

Proof. Let A be an edge of G which touches L. Let x be the end of A which is a vertex of L. Assume that A is not unicursal from x. We recall that $a = e$ if a is a vertex of L.

If x is not e there is an edge C of L differing in colour from A and having x as an end, by Theorem IX. This is true also if $x = e = a$. For then A is blue since it is not unicursal from a and not bicursal, and Theorem VII shows that some red edge of L has a as an end. In either of these cases it follows from Theorem X that there is a member of $\Pi(a)$ which enters L at e, whose section by L passes through C and then x, and which continues from C with the terms x and A. But A is not bicursal since it is not an edge of L. Hence A is unicursal from x, contrary to assumption.

Now suppose that $x = e$ and e is not a. By Theorem VII, there is a member P of $\Pi(a)$ which enters L at e and in which the first occurrence of e is immediately succeeded by an edge C of L. Let the edge of G which immediately precedes the first occurrence of e in P be B. Clearly B touches L and is unicursal to L.

Suppose that A and B are distinct. If A differs in colour from B it is describable from $x = e$, by Theorem V. If A and B have the same colour this differs from that of C. By Theorem X there is a member Q of $\Pi(a)$ which enters L at e, and whose section by L passes through C and then e. It is clear that A and B cannot both precede C in Q. Hence there is a member Q' of $\Pi(a)$ which agrees with Q as far as C and then continues with e and one of the edges A and B. Actually, it continues with e and A since B is unicursal to e. Hence if A and B are distinct, A is unicursal from x.

This completes the proof of the theorem.

5. Bicursal units. Let T be the set of all vertices of G which are ends of bicursal edges. Let T' be the set of all edges of G having both ends in T. Then T and T' define a subgraph G' of G. We refer to the components of G' as *bicursal units*. Evidently a bicursal component having a given vertex b is a subgraph of the bicursal unit having the vertex b. By Theorem IV the bicursal units are finite graphs.

THEOREM XII. *Let M be any bicursal unit. If a is a vertex of M then all edges of G which touch M are unicursal from M. If a is not a vertex of M then there exists just one edge of G which touches M and is unicursal to M, and all other edges of G which touch M are unicursal from M.*

Proof. Since some edges of M are bicursal there exists a member P of $\Pi(a)$ having a vertex of M as a term. Let e be the first vertex of M to occur in P. If a is not a vertex of M there is an edge E of G which immediately precedes the first occurrence of e in P. Then E touches M and is unicursal to e and M. We denote the bicursal component of which e is a vertex by L. If instead a is a vertex of M we denote the bicursal component of which a is a vertex by L.

A subgraph L' of M which is a bicursal component distinct from L is *supplied from* L if there exists a sequence $(L_1, L_2, \ldots, L_t)$ of bicursal components and a sequence $(A_1, A_2, \ldots, A_{t-1})$ of edges of M such that

(i) $L_1 = L$ and $L_t = L'$,

(ii) the L_i are subgraphs of M,

(iii) for each integer i in the range $1 \leqslant i < t$, A_i is unicursal from L_i and to L_{i+1}.

We can show that any subgraph of M which is a bicursal component distinct from L is supplied from L. For suppose it is not. Then since M is connected there is an edge B of M with ends b and c belonging to bicursal components L' amd L'', where L' is L or is supplied from L, and L'' is not L and is not supplied from L. Now B is not bicursal by the definition of a bicursal component, and is not acursal, by Theorem XI. It is not unicursal to L'', since L'' is not supplied from

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L. Hence B is unicursal to L'. But this is contrary to Theorem XI since L' is either L or is supplied from L.

The Theorem now follows by the application of Theorem XI to each of the bicursal components which are subgraphs of M.

If a is not a vertex of the bicursal unit M, we call the edge of G which touches M and is unicursal to M the *entrance-edge* of M. We classify such bicursal units as *red-entrant* and *blue-entrant* according as their entrance-edges are red or blue. A bicursal unit having a as a vertex is *a-entrant.*

6. Singular vertices. In this section we suppose that a is a singular vertex. We denote the numbers of red-entrant and blue-entrant bicursal units by k_r and k_b respectively. These numbers are finite, by Theorem IV.

Let U denote the set of all vertices of G which are not vertices of G'. Thus no bicursal edge has an end in U. Let V be the set of all members of U to which some red edge is unicursal. Let W be the set of all members of U from which some red edge is unicursal or to which some blue edge is unicursal. Clearly,

$$a \notin V. \tag{9}$$

Suppose $c \in V$. Any blue edge of G having c as an end is unicursal from c, by Theorem V. Hence, by (9), no red edge of G can be unicursal from c. There are just $f(c)$ blue edges of G which have c as an end and are therefore unicursal from c since c is accessible from, but distinct from, the singular vertex a.

Now suppose $i \in W$. If some red edge is unicursal from i then either $a = i$ or there is a blue edge unicursal to i. If $a = i$ or there is a blue edge unicursal to i, then each red edge having i as an end is unicursal from i, by Theorem V and the definition of $\Pi(a)$. Hence any red edge having i as an end is unicursal from i. Consequently no blue edge of G can be unicursal from i.

It is clear from these results that V and W are disjoint sets. By Theorem IV they are finite sets.

If $i \in W$, let $y(i)$ be the number of red edges of G unicursal from i which are entrance-edges of red-entrant bicursal units. Let $z(i)$ be the number of blue edges of G which are unicursal to i from members of V.

Let H denote the graph $G(V)$. If $i \in W$, any red edge unicursal from i is unicursal to a vertex p distinct from a. For no red edge is unicursal to a. So by Theorem V, p is either a vertex of G' or a member of V. Hence in the graph H, the number of edges having i as an end is $y(i) + (d_J(i) - z(i))$. Thus we have

$$z(i) = y(i) + (d_J(i) - d_H(i)). \tag{10}$$

By the definition of a bicursal unit the entrance-edge of any bicursal unit which is not a-entrant is unicursal either from a member of V or from a member of W.

Let λ be the number of blue edges of G unicursal from a member of V to a member of W. It is equal to the total number of blue edges unicursal from members of V less the number of the entrance-edges of the blue-entrant bicursal units. The latter number is k_b, by Theorem XII. But λ is also equal to the sum

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of the numbers $z(i)$ taken over all $i \in W$. The corresponding sum of the $y(i)$ is k_r, by Theorem XII. Hence we have

$$\sum_{c \in V} f(c) = k_b + k_r + \sum_{i \in W} (d_J(i) - d_H(i)). \tag{11}$$

The bicursal units, if any, are connected finite graphs. By Theorem XII, they are components of $(G(V))(W) = H(W)$.

Let M be any bicursal unit. Write $q(M) = 0$ or 1 according as M is or is not a-entrant. Let $u(M)$ be the number of blue edges of G which touch M and let $v(M)$ be the number of edges of G which touch M and have an end in W.

Using Theorem XII we readily obtain the following results: if M is blue-entrant $q(M) = 1$ and $u(M) = v(M) + 1$, if M is red-entrant $q(M) = 1$ and $u(M) = v(M) - 1$, and if M is a-entrant $q(M) = 0$ and $u(M) = v(M)$. In each case we have

$$u(M) \equiv v(M) + q(M) \pmod 2. \tag{12}$$

The sum of $u(M)$ and the degrees in J of the vertices of M is even, since it is twice the number of blue edges of G having vertices of M as ends. Moreover, if c is a vertex of M, we have $d_J(c) = f(c)$ unless $c = a$; and a is a vertex of M if and only if $q(M) = 0$. It follows from (12) that

$$v(M) + \sum_{c \in M} f(c) + (q(M) + 1)(d_J(a) - f(a)) + q(M) \equiv 0 \pmod 2. \tag{13}$$

Referring to the definitions of §2 we see that M is a member of $K(G(V), W)$ if and only if $(q(M) + 1)\ (d_J(a) - f(a)) + q(M) \equiv 1 \pmod 2$. Hence M is not a member of $K(G(V), W)$ if and only if M is a-entrant ($q(M) = 0$) and the deficiency $f(a) - d_J(a)$ of a in J is even.

THEOREM XIII. *If G is not constricted there exists an a-entrant bicursal unit, and the deficiency of a in J is even.*

Proof. Suppose, first, that a is a member of U but not of W. Then no red edge of G has a as an end. Hence $d_G(a) = d_J(a) < f(a)$. But then G is constricted, contrary to hypothesis, for (5) is satisfied if we take T to be null and S to have the single element a. We conclude that either $a \in W$ or there exists an a-entrant bicursal unit.

Applying formula (4) we have

$$r(G(V), W) = k(G(V), W) + \sum_{i \in W} (f(i) - d_H(i)). \tag{14}$$

If $a \in W$ then $k(G(V), W) \geqslant k_b + k_r$, and $f(a) > d_J(a)$. If there exists an a-entrant bicursal unit and the deficiency of a in J is odd we have

$$k(G(V), W) \geqslant k_b + k_r + 1.$$

Here we have used the results proved above concerning the membership of bicursal units in $K(G(V), W)$. In each of these cases it follows that the expression on the right of (11) is less than $r(G(V), W)$. Then G is constricted, contrary to hypothesis. The theorem follows.

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7. Augmentation. In this section we no longer assume that the deficient vertex a is singular.

Suppose P is a member of $\Pi(a)$ which has more than one term, and whose last term is a vertex i of G deficient in J. To *transform J by P* is to replace J by a restriction K of G, defined as follows. The edges of K consist of the blue edges of G which are not terms of P, together with the red edges of G which are terms of P.

We say the f-subgraph J is *augmentable* at the deficient vertex a if there is an f-subgraph K of G satisfying the following conditions:

(i) $d_K(a) > d_J(a)$.

(ii) If $d_J(c) = f(c)$, then $d_K(c) = f(c)$.

Suppose a is not singular. Then there is a member P of $\Pi(a)$ whose last term is a deficient vertex i of G distinct from a. Let K be the restriction of G obtained by transforming J by P. By the definition of $\Pi(a)$ we have

$$d_K(a) = d_J(a) + 1, \quad d_K(i) = d_J(i) \pm 1,$$

and $d_K(c) = d_J(c)$ if c is not a or i. Hence K is an f-subgraph of G, and J is augmentable at a.

Suppose next that a is singular and that G is not constricted. The deficiency of a in J is at least 2, by Theorem XIII. Also by Theorem XIII, a is the entrance of a bicursal unit M_0. By Theorem VII there is a red edge A of M_0 having a as an end. Since A is bicursal there is a member P of $\Pi(a)$ including at least two edges, whose last term is a and whose last edge is A. Let K be the restriction of G obtained by transforming J by P. By the definition of $\Pi(a)$ we have

$$d_K(a) = d_J(a) + 2 \leqslant f(a),$$

and $d_K(c) = d_J(c)$ if c is not a. Hence K is an f-subgraph of G, and J is augmentable at a.

Thus we have the following

THEOREM XIV. *Let J be any f-subgraph of G, and let a be any vertex of G which is deficient in J. Then either G is constricted with respect to f or J is augmentable at a.*

8. Condition for an f-factor.

THEOREM XV. *G has no f-factor if and only if it is constricted with respect to f.*

Proof. Suppose first that the locally finite graph G is constricted with respect to f. Then G has no f-factor, by Theorem III.

Suppose next that G is not constricted with respect to f. Let J_0 be the restriction of G which has no edges. Then J_0 is an f-subgraph of G.

If a vertex a of G is deficient in a given f-subgraph J of G, we can, by Theorem XIV, replace J by an f-subgraph K in which the degree of a is increased and no vertex of G which is not deficient in J is deficient in K. By repeating this process sufficiently often we can obtain an f-subgraph K' of G in which a and those vertices of G not deficient in J are not deficient.

It follows that if S is any finite set of vertices of G, we can, by the above process, build from J_0 an f-subgraph J of G in which no member of S is deficient.

The theorem follows at once in the case in which G is finite. Then we can take S to be the set of all vertices of G, and the corresponding f-subgraph J must be an f-factor of G.

If G is infinite and connected we use the following non-constructive argument. (I have replaced my original proof by a shorter one for which I am indebted to the referee.)

Let x be any vertex of G. The number of paths in G whose first term is x and which have just $2n + 1$ terms, where n is any given non-negative integer, is finite since G is locally finite. Hence the set of paths in G having x as first term is denumerable. Since G is connected it follows that the set of vertices of G is denumerable, say $\{a_1, a_2, \dots\}$. By the foregoing argument, to every positive integer n there is an f-subgraph J_n such that

$$d_{J_n}(a_r) = f(a_r), \qquad r \leqslant n.$$

The set of edges of G is at most denumerable, say equal to $\{A_1, A_2, \dots\}$. Put $F_n(s) = 1$ if A_s is an edge of J_n and $F_n(s) = 0$ otherwise. Then by the diagonal process, there is an increasing sequence $n_1, n_2, \dots$. such that

$$\lim_{k\to\infty} F_{n_k}(s) = F(s)$$

exists for all s. Let J be the restriction of G whose edges are those A_s for which $F(s) = 1$. Then $d_J(a_r) = f(a_r)$ for all r, and J is an f-factor of G.

Lastly, we must consider the case in which G is infinite and not connected. We can show that no component of G is constricted with respect to f. For if this is not so, there is a component G_α of G such that for some disjoint finite subsets S and T of the set of vertices of G,

$$\sum_{c\in T} f(c) < k(G_\alpha(T), S) + \sum_{c\in S} (f(c) - d_{G_\alpha(T)}(c)). \tag{15}$$

Clearly each component of $G_\alpha(T)$ is a component of $G(T)$. Hence (15) holds with $G_\alpha(T)$ replaced by $G(T)$, so that

$$\sum_{c\in T} f(c) < r(G(T), S).$$

Thus G is constricted with respect to f, contrary to hypothesis.

Since the theorem has been proved for connected graphs it follows that each component of G has an f-factor. Hence (assuming the multiplicative axiom) there is a set Z of f-factors of components of G which contains just one f-factor of each component of G. The restriction of G whose edges are the edges of the members of Z is an f-factor of G.

9. n-factors. A necessary and sufficient condition for the existence of an n-factor of G, where n is a given positive integer, can be obtained by applying Theorem XV to the special case in which the value of $f(c)$ is n for each vertex c of G.

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It is convenient to denote the number of elements of a finite set U by $\alpha(U)$. We then obtain the following

THEOREM XVI. *G has no n-factor if and only if there exist disjoint finite sets S and T of vertices of G such that*

$$n\alpha(T) < k(G(T), S) - \sum_{c \in S} (d_{G(T)}(c) - n). \tag{16}$$

Here $k(G(T), S)$ is the number of finite components H of $(G(T))(S) = G(S \cup T)$ for which n times the number of vertices differs in parity from the number of edges of G which have one end in S and the other end a vertex of H.

A necessary and sufficient condition for the existence of a 1-factor of a given locally finite graph G has been given in previous papers [**6**; **7**]. It is simpler in form than the expression obtained by writing $n = 1$ in (16). In the next section, this simpler formula is deduced from Theorem XV. The argument suggests no analogous simplification in the case $n > 1$.

10. An allied problem. Suppose that we are given a locally finite graph G, and a function f which associates with each vertex c of G a positive integer $f(c)$. We consider the problem of associating with each edge A of G a non-negative integer $h(A)$ so that for each vertex c of G the sum of the numbers $h(A)$, taken over all edges A of G having c as an end, is $f(c)$. If such a set of non-negative integers $h(A)$ exists we say that G is *f-soluble*.

We note that if $f(c) = 1$ for each vertex of G, then G is f-soluble if and only if it has a 1-factor.

Let T be any finite set of vertices of G. We denote by $S(T)$ the set of all vertices c of G having the following properties:

(i) c is not an element of T.

(ii) Each edge of G having c as an end has its other end in T.

If T does not include every vertex of G we denote by $k(T)$ the number of finite components H of $G(T)$ having the following properties:

(i) H has more than one vertex.

(ii) The sum of the numbers $f(a)$, taken over all vertices of H, is odd.

If T is the set of all vertices of G we write $k(T) = 0$.

THEOREM XVII. *G is not f-soluble if and only if there exists a finite set T of vertices of G such that*

$$\sum_{c \in T} f(c) < k(T) + \sum_{c \in S(T)} f(c). \tag{17}$$

Proof. By adjoining new edges to G we can obtain a graph G' having the following properties:

(i) The vertices of G' are the vertices of G.

(ii) Two vertices are joined by an edge in G' if and only if they are joined by an edge in G.

(iii) If two vertices a and b are joined by an edge in G', the number of distinct edges of G' which join them is finite and not less than $d_G(a) + f(a)$.

Clearly G' is locally finite.

If S and T are disjoint finite sets of vertices of G such that S is contained in $S(T)$ it follows from the definition of G' that

$$k(G'(T), S) = k(G(T), S). \tag{18}$$

It is clear that G is f-soluble if and only if G' has an f-factor. Hence, by Theorem XV, G is not f-soluble if and only if there exist disjoint finite sets S and T of vertices of G such that

$$\sum_{c \in T} f(c) < k(G'(T), S) - \sum_{c \in S} (d_{G'(T)}(c) - f(c)). \tag{19}$$

Suppose first that (17) is satisfied for some finite T. If $S(T)$ is not finite then all but a finite number of its elements have degree 0 in G, since G is locally finite. Hence G is not f-soluble since it has a vertex of degree 0. If $S(T)$ is finite it follows from (17) and (18) that (19) will be satisfied if we put $S = S(T)$. Hence G is not f-soluble.

Conversely, suppose that G is not f-soluble. Then (19) is satisfied for some disjoint finite sets S and T. If possible let a be any member of S not in $S(T)$. Consider the effect of replacing S by $S' = S - \{a\}$. Clearly the replacement diminishes

$$\sum_{c \in S} (d_{G'(T)}(c) - f(c))$$

by $d_{G'(T)}(a) - f(a)$, that is, by at least $d_G(a)$, from (iii).

The replacement diminishes $k(G'(T), S)$ by not more than $d_{G(T)}(a)$, the maximum number of finite components of $G'(S \cup T)$ joined to a by an edge of G'. But $d_{G(T)}(a) \leqslant d_G(a)$. Hence, if a is not an element of $S(T)$, formula (19) remains valid when S is replaced by S'.

If S' has an element not in $S(T)$ we repeat the argument with S' replacing S, and so on. Since S is finite we find, eventually,

$$\sum_{c \in T} f(c) < k(G'(T), U) + \sum_{c \in U} f(c), \tag{20}$$

where U is the intersection of S and $S(T)$. But, by (18), $k(G'(T), U)$ is equal to $k(T)$ plus the number of components of $G(T \cup U)$ which consist of a single vertex, the value of f for this vertex being odd. Hence

$$k(G'(T), U) \leqslant k(T) + \sum_{c \in S(T) - U} f(c). \tag{21}$$

Now (20) and (21) imply (17). This completes the proof of the theorem.

If $f(c) = 1$ for each vertex c of G it is clear that G is f-soluble if and only if it has a 1-factor. Applying Theorem XVII to this case we find that G has no 1-factor if and only if there exists a finite set T of vertices of G such that

$$\alpha(T) < h_u(T),$$

where $h_u(T)$ is the number of finite components of $G(T)$ having an odd number of vertices. This is the simple criterion for the existence of a 1-factor mentioned in §9.

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