

EVALUATION OF TWO CONVOLUTION SUMS
 INVOLVING THE SUM OF DIVISORS FUNCTION

MATHIEU LEMIRE AND KENNETH S. WILLIAMS

The convolution sums $\sum_{m < n/5} \sigma(m)\sigma(n - 5m)$ and $\sum_{m < n/7} \sigma(m)\sigma(n - 7m)$ are evaluated for all $n \in \mathbb{N}$.

1. INTRODUCTION.

Let \mathbb{N} denote the set of natural numbers. We set

$$\sigma_k(n) = \sum_{d|n} d^k, \quad n, k \in \mathbb{N},$$

where d runs through the positive integers dividing n . If $n \notin \mathbb{N}$ we set $\sigma_k(n) = 0$. We write $\sigma(n)$ for $\sigma_1(n)$. We define the convolution sum $W_k(n)$ by

$$(1.1) \quad W_k(n) := \sum_{m < n/k} \sigma(m)\sigma(n - km),$$

where m runs through the positive integers $< n/k$. The sum $W_k(n)$ has been evaluated for $k = 1, 2, 3, 4, 6, 8, 9$ and 16 for all $n \in \mathbb{N}$, see [4, 7] ($k = 1$), [7, 8, 9] ($k = 2, 3, 4$), [2] ($k = 6$), [11] ($k = 8$), [8, 9, 10, 12] ($k = 9$), [1] ($k = 16$). For $k = 5$ Melfi [8, 9] has shown that for $n \equiv 8 \pmod{16}, n \not\equiv 0 \pmod{5}$

$$(1.2) \quad W_5(n) = \frac{5}{312}\sigma_3(n) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n).$$

In this paper we make use of some recent results of Berndt, Chan, Sohn and Son [3] to evaluate $W_5(n)$ and $W_7(n)$ for all $n \in \mathbb{N}$. We prove

THEOREM 1. For all $n \in \mathbb{N}$

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{5}\right) - \frac{1}{130}c_5(n),$$

where the $c_5(n)$ ($n \in \mathbb{N}$) are integers defined by

$$(1.3) \quad q \prod_{n=1}^{\infty} (1 - q^n)^4 (1 - q^{5n})^4 = \sum_{n=1}^{\infty} c_5(n)q^n.$$

Received 5th October, 2005

Research of the second author was supported by Natural Sciences and Engineering Research Council of Canada grant A-7233.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9727/06 \$A2.00+0.00.

The first twenty values of $c_5(n)$ are $c_5(1) = 1, c_5(2) = -4, c_5(3) = 2, c_5(4) = 8, c_5(5) = -5, c_5(6) = -8, c_5(7) = 6, c_5(8) = 0, c_5(9) = -23, c_5(10) = 20, c_5(11) = 32, c_5(12) = 16, c_5(13) = -38, c_5(14) = -24, c_5(15) = -10, c_5(16) = -64, c_5(17) = 26, c_5(18) = 92, c_5(19) = 100, c_5(20) = -40.$

THEOREM 2. For all $n \in \mathbb{N}$

$$W_7(n) = \frac{1}{120}\sigma_3(n) + \frac{49}{120}\sigma_3\left(\frac{n}{7}\right) + \left(\frac{1}{24} - \frac{1}{28}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{7}\right) - \frac{1}{70}c_7(n),$$

where the $c_7(n)$ ($n \in \mathbb{N}$) are integers defined by

$$(1.4) \quad q \left(\prod_{n=1}^{\infty} (1 - q^n)^{16} (1 - q^{7n})^8 + 13q \prod_{n=1}^{\infty} (1 - q^n)^{12} (1 - q^{7n})^{12} + 49q^2 \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{7n})^{16} \right)^{1/3} = \sum_{n=1}^{\infty} c_7(n)q^n.$$

The first twenty values of $c_7(n)$ are $c_7(1) = 1, c_7(2) = -1, c_7(3) = -2, c_7(4) = -7, c_7(5) = 16, c_7(6) = 2, c_7(7) = -7, c_7(8) = 15, c_7(9) = -23, c_7(10) = -16, c_7(11) = -8, c_7(12) = 14, c_7(13) = 28, c_7(14) = 7, c_7(15) = -32, c_7(16) = 41, c_7(17) = 54, c_7(18) = 23, c_7(19) = -110, c_7(20) = -112.$

2. NOTATION.

Let $z = x + iy \in \mathbb{C}$ be such that $y > 0$. Set $q = e^{2\pi iz}$ so that $|q| = e^{-2\pi y} < 1$. The Dedekind eta function $\eta(z)$ is defined by

$$(2.1) \quad \eta(z) := e^{2\pi iz/24} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).$$

Following Ramanujan (see [3, p. 82]) we set

$$(2.2) \quad f(-q) := e^{-2\pi iz/24} \eta(z) = \prod_{n=1}^{\infty} (1 - e^{2\pi inz}) = \prod_{n=1}^{\infty} (1 - q^n).$$

It is convenient to set

$$(2.3) \quad A := f(-q), \quad B := f(-q^5), \quad C := f(-q^7).$$

We define $c_5(n)$ and $c_7(n)$ by (1.3) and (1.4) respectively. It is clear that $c_5(n) \in \mathbb{Z}$. We shall show that $c_7(n) \in \mathbb{Z}$ in the proof of Theorem 2. From (1.3), (1.4), (2.2) and (2.3), we obtain

$$(2.4) \quad qA^4B^4 = \sum_{n=1}^{\infty} c_5(n)q^n.$$

$$(2.5) \quad qA^3B^3 \left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2 \frac{C^7}{A} \right)^{1/3} = \sum_{n=1}^{\infty} c_7(n)q^n.$$

The Eisenstein series $P(q)$ and $Q(q)$ (see [3, p. 81]) are defined by

$$(2.6) \quad P(q) := 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n$$

and

$$(2.7) \quad Q(q) := 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n.$$

3. PROOF OF THEOREM 1.

We begin with a classical result.

LEMMA 3.1.

$$P(q)^2 = 1 + \sum_{n=1}^{\infty} (240\sigma_3(n) - 288n\sigma(n))q^n.$$

PROOF: See for example Glaisher [5, 6]. □

LEMMA 3.2.

$$P(q^5)^2 = 1 + \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{5}\right) - \frac{288}{5}n\sigma\left(\frac{n}{5}\right) \right) q^n.$$

PROOF: Letting $q \rightarrow q^5$ in Lemma 3.1 we obtain the result. □

LEMMA 3.3.

$$(P(q) - 5P(q^5))^2 = 16\frac{A^{10}}{B^2} + 352qA^4B^4 + 2000q^2\frac{B^{10}}{A^2}.$$

PROOF: This follows from [3, (3.21), (3.34)] and (2.3). □

LEMMA 3.4.

$$(a) \quad Q(q) = \frac{A^{10}}{B^2} + 250qA^4B^4 + 3125q^2\frac{B^{10}}{A^2}.$$

$$(b) \quad Q(q^5) = \frac{A^{10}}{B^2} + 10qA^4B^4 + 5q^2\frac{B^{10}}{A^2}.$$

PROOF: This follows from [3, Theorem 3.1, p. 85] and (2.3). □

LEMMA 3.5.

$$(a) \quad \frac{A^{10}}{B^2} = 1 + \sum_{n=1}^{\infty} \left(-\frac{5}{13}\sigma_3(n) + \frac{3125}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{125}{13}c_5(n) \right) q^n.$$

$$(b) \quad q^2\frac{B^{10}}{A^2} = \sum_{n=1}^{\infty} \left(\frac{1}{13}\sigma_3(n) - \frac{1}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{1}{13}c_5(n) \right) q^n.$$

PROOF: From Lemma 3.4 we obtain

$$(3.1) \quad \frac{A^{10}}{B^2} = -\frac{1}{624}Q(q) + \frac{625}{624}Q(q^5) - \frac{125}{13}qA^4B^4$$

and

$$(3.2) \quad q^2 \frac{B^{10}}{A^2} = \frac{1}{3120}Q(q) - \frac{1}{3120}Q(q^5) - \frac{1}{13}qA^4B^4.$$

Letting $q \rightarrow q^5$ in (2.7), we obtain

$$(3.3) \quad Q(q^5) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3\left(\frac{n}{5}\right)q^n.$$

Using (2.4), (2.7) and (3.3) in (3.1) and (3.2), we obtain the asserted results. □

LEMMA 3.6.

$$(P(q) - 5P(q^5))^2 = 16 + \sum_{n=1}^{\infty} \left(\frac{1920}{13}\sigma_3(n) + \frac{4800}{13}\sigma_3\left(\frac{n}{5}\right) + \frac{576}{13}c_5(n) \right) q^n.$$

PROOF: This follows from Lemma 3.3, Lemma 3.5 and (2.4). □

LEMMA 3.7.

$$P(q)P(q^5) = 1 + \sum_{n=1}^{\infty} \left(\frac{120}{13}\sigma_3(n) + \frac{3000}{13}\sigma_3\left(\frac{n}{5}\right) - \frac{144}{5}n\sigma(n) - 144n\sigma\left(\frac{n}{5}\right) - \frac{288}{65}c_5(n) \right) q^n.$$

PROOF: This follows from Lemmas 3.1, 3.2, 3.6 and the identity

$$P(q)P(q^5) = \frac{1}{10}P(q)^2 + \frac{5}{2}P(q^5)^2 - \frac{1}{10}(P(q) - 5P(q^5))^2.$$
□

PROOF OF THEOREM 1. Letting $q \rightarrow q^5$ in (2.6), we obtain

$$(3.4) \quad P(q^5) = 1 - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{5}\right)q^n.$$

Next

$$\begin{aligned} \sum_{n=1}^{\infty} W_5(n)q^n &= \sum_{n=1}^{\infty} \left(\sum_{m < n/5} \sigma(m)\sigma(n - 5m) \right) q^n \\ &= \left(\sum_{l=1}^{\infty} \sigma(l)q^l \right) \left(\sum_{m=1}^{\infty} \sigma(m)q^{5m} \right) \\ &= \left(\frac{1 - P(q)}{24} \right) \left(\frac{1 - P(q^5)}{24} \right) \\ &= \frac{1}{576} - \frac{1}{576}P(q) - \frac{1}{576}P(q^5) + \frac{1}{576}P(q)P(q^5). \end{aligned}$$

Appealing to (2.6), (3.1) and Lemma 3.7, and equating coefficients of q^n ($n \in \mathbb{N}$), we obtain the required result. \square

Since $c_5(mn) = c_5(m)c_5(n)$ for all $m, n \in \mathbb{N}$ with $(m, n) = 1$ and $c_5(8) = 0$ (see [9]), we deduce that $c_5(n) = 0$ for all $n \in \mathbb{N}$ with $n \equiv 8 \pmod{16}$. Hence for $n \equiv 8 \pmod{16}$ Theorem 1 gives

$$W_5(n) = \frac{5}{312}\sigma_3(n) + \frac{125}{312}\sigma_3\left(\frac{n}{5}\right) + \left(\frac{1}{24} - \frac{1}{20}n\right)\sigma(n) + \left(\frac{1}{24} - \frac{1}{4}n\right)\sigma\left(\frac{n}{5}\right).$$

Melfi's result (1.2) is the special case of this result when $n \not\equiv 0 \pmod{5}$.

We show that the evaluation

$$(3.5) \quad c_5(5^k) = (-1)^k 5^k, \quad k \in \mathbb{N} \cup \{0\},$$

is a consequence of Theorem 1. Let $n \in \mathbb{N}$. The elementary identities

$$(3.6) \quad \sigma(5n) = 6\sigma(n) - 5\sigma(n/5)$$

and

$$(3.7) \quad \sigma_3(5n) = 126\sigma_3(n) - 125\sigma_3(n/5)$$

are easily proved. Thus

$$\begin{aligned} W_5(5n) &= \sum_{m < n} \sigma(m)\sigma(5n - 5m) \\ &= 6 \sum_{m < n} \sigma(m)\sigma(n - m) - 5 \sum_{m < n} \sigma(m)\sigma((n - m)/5). \end{aligned}$$

It is a classical result that

$$\sum_{m < n} \sigma(m)\sigma(n - m) = \frac{5}{12}\sigma_3(n) + \left(\frac{1}{12} - \frac{n}{2}\right)\sigma(n),$$

see for example [7]. Also

$$\sum_{m < n} \sigma(m)\sigma((n - m)/5) = \sum_{t < n/5} \sigma(n - 5t)\sigma(t) = W_5(n).$$

Hence

$$W_5(5n) + 5W_5(n) = \frac{5}{2}\sigma_3(n) + \left(\frac{1}{2} - 3n\right)\sigma(n).$$

Appealing to Theorem 1, (3.6) and (3.7), we obtain

$$c_5(5n) = -5c_5(n), \quad n \in \mathbb{N},$$

from which (3.5) follows.

4. PROOF OF THEOREM 2.

It is convenient to set

$$(4.1) \quad H := \left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A} \right)^{1/3},$$

where A and C are defined in (2.3).

LEMMA 4.1.

$$P(q^7)^2 = 1 + \sum_{n=1}^{\infty} \left(240\sigma_3\left(\frac{n}{7}\right) - \frac{288}{7}n\sigma\left(\frac{n}{7}\right) \right) q^n.$$

PROOF: Letting $q \rightarrow q^7$ in Lemma 3.1, we obtain the result. □

LEMMA 4.2.

$$(P(q) - 7P(q^7))^2 = 36H^4.$$

PROOF: From [3, Lemma 6.2, p. 109] we obtain

$$P(q) - 7P(q^7) = -6\left(\frac{A^7}{C}\right)^{2/3} \left(1 + 13q\frac{C^4}{A^4} + 49q^2\frac{C^8}{A^8} \right)^{2/3}.$$

Squaring both sides we obtain

$$\begin{aligned} (P(q) - 7P(q^7))^2 &= 36\left(\frac{A^7}{C}\right)^{4/3} \left(1 + 13q\frac{C^4}{A^4} + 49q^2\frac{C^8}{A^8} \right)^{4/3} \\ &= 36\left(\frac{A^7}{C} + 13qA^3C^3 + 49q^2\frac{C^7}{A}\right)^{4/3} \\ &= 36H^4. \end{aligned}$$
□

LEMMA 4.3.

$$(a) \quad Q(q) = \left(\frac{A^7}{C} + 245qA^3B^3 + 2401q^2\frac{C^7}{A} \right) H.$$

$$(b) \quad Q(q^7) = \left(\frac{A^7}{C} + 5qA^3B^3 + q^2\frac{C^7}{A} \right) H.$$

PROOF: This follows from [3, Theorem 5.1, p. 100]. □

LEMMA 4.4.

$$qA^3B^3H = \sum_{n=1}^{\infty} c_7(n)q^n.$$

PROOF: This follows from (2.5) and (4.1). □

LEMMA 4.5.

$$(a) \quad \frac{A^7}{C}H = 1 + \sum_{n=1}^{\infty} \left(-\frac{1}{10}\sigma_3(n) + \frac{2401}{10}\sigma_3\left(\frac{n}{7}\right) - \frac{49}{10}c_7(n) \right) q^n.$$

$$(b) \quad q^2\frac{C^7}{A}H = \sum_{n=1}^{\infty} \left(\frac{1}{10}\sigma_3(n) - \frac{1}{10}\sigma_3\left(\frac{n}{7}\right) - \frac{1}{10}c_7(n) \right) q^n.$$

PROOF: From Lemma 4.3 we obtain

$$(4.2) \quad \frac{A^7}{C} = -\frac{1}{2400}Q(q) + \frac{2401}{2400}Q(q^7) - \frac{49}{10}qA^3C^3H$$

and

$$(4.3) \quad q^2\frac{C^7}{A}H = \frac{1}{2400}Q(q) - \frac{1}{2400}Q(q^7) - \frac{1}{10}qA^3C^3H.$$

Letting $q \rightarrow q^7$ in (2.7) we obtain

$$(4.4) \quad Q(q^7) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3\left(\frac{n}{7}\right)q^n.$$

Using (2.7), (4.4) and Lemma 4.3 in (4.2) and (4.3), we obtain the asserted results. \square

LEMMA 4.6.

$$(P(q) - 7P(q^7))^2 = 36 + \sum_{n=1}^{\infty} \left(\frac{864}{5}\sigma_3(n) + \frac{42336}{5}\sigma_3\left(\frac{n}{7}\right) + \frac{576}{5}c_7(n) \right) q^n.$$

PROOF: This follows from Lemmas 4.2, 4.5 and 4.6. \square

LEMMA 4.7.

$$P(q)P(q^7) = 1 + \sum_{n=1}^{\infty} \left(\frac{24}{5}\sigma_3(n) + \frac{1176}{5}\sigma_3\left(\frac{n}{7}\right) - \frac{144}{7}n\sigma(n) - 144n\sigma\left(\frac{n}{5}\right) - \frac{288}{35}c_7(n) \right) q^n.$$

PROOF: This follows from Lemmas 3.1, 4.1, 4.7 and the identity

$$P(q)P(q^7) = \frac{1}{14}P(q)^2 + \frac{49}{14}P(q^7)^2 - \frac{1}{14}(P(q) - 7P(q^7))^2. \quad \square$$

PROOF OF THEOREM 2. Letting $q \rightarrow q^7$ in (2.6), we obtain

$$(4.5) \quad P(q^7) = 1 - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{7}\right)q^n.$$

Next

$$\begin{aligned} \sum_{n=1}^{\infty} W_7(n)q^n &= \sum_{n=1}^{\infty} \left(\sum_{m < \frac{n}{7}} \sigma(m)\sigma(n - 7m) \right) q^n \\ &= \left(\sum_{l=1}^{\infty} \sigma(l)q^l \right) \left(\sum_{m=1}^{\infty} \sigma(m)q^{7m} \right) \\ &= \left(\frac{1 - P(q)}{24} \right) \left(\frac{1 - P(q^7)}{24} \right) \\ &= \frac{1}{576} - \frac{1}{576}P(q) - \frac{1}{576}P(q^7) + \frac{1}{576}P(q)P(q^7). \end{aligned}$$

Appealing to (2.6), (4.5) and Lemma 4.7, and equating coefficients of q^n , we obtain the formula for $W_7(n)$ in Theorem 2.

We now show that the $c_7(n)$ are integers. It is easy to check that for all $n \in \mathbb{N}$ we have

$$\sigma_3(n) \equiv \sigma(n) \pmod{3}$$

and

$$\sigma_3(n) \equiv (2n - 1)\sigma(n) \pmod{4}.$$

Hence

$$\begin{aligned} 7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7) \\ \equiv \sigma_3(n) + \sigma_3(n/7) - \sigma(n) - \sigma(n/7) \equiv 0 \pmod{3} \end{aligned}$$

and

$$\begin{aligned} 7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7) \\ \equiv -\sigma_3(n) - \sigma_3(n/7) - (1 + 2n)\sigma(n) - (1 + 2n)\sigma(n/7) \equiv 0 \pmod{4} \end{aligned}$$

so

$$7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7) \equiv 0 \pmod{12}.$$

Thus

$$e(n) := \frac{7\sigma_3(n) + 343\sigma_3(n/7) + (35 - 30n)\sigma(n) + (35 - 210n)\sigma(n/7)}{12} \in \mathbb{Z}.$$

Hence

$$c_7(n) = e(n) - 70W_7(n) \in \mathbb{Z}. \quad \square$$

Similarly to the proof of (3.5) we can use Theorem 2 to show that

$$(4.6) \quad c_7(7^k) = (-1)^k 7^k, \quad k \in \mathbb{N} \cup \{0\}.$$

REFERENCES

- [1] A. Alaca, S. Alaca and K.S. Williams, ‘The convolution sum $\sum_{m < n/16} \sigma(m)\sigma(n - 16m)$ ’, *Canad. Math. Bull.* (to appear).
- [2] S. Alaca and K.S. Williams, ‘The convolution sums $\sum_{l+6m=n} \sigma(l)\sigma(m)$ and $\sum_{2l+3m=n} \sigma(l)\sigma(m)$ ’, (submitted).
- [3] B.C. Berndt, H.H. Chan, J. Sohn and S.H. Son, ‘Eisenstein series in Ramanujan’s lost notebook’, *Ramanujan J.* 4 (2000), 81–114.
- [4] M. Besge, ‘Extrait d’une lettre de M. Besge à M. Liouville’, *J. Math. Pures Appl.* 7 (1862), 256.
- [5] J.W.L. Glaisher, ‘On the square of the series in which the coefficients are the sums of the divisors of the exponents’, *Mess. Math.* 14 (1885), 156–163.

- [6] J.W.L. Glaisher, *Mathematical Papers 1883-1885* (Cambridge, 1885).
- [7] J.G. Huard, Z.M. Ou, B.K. Spearman and K.S. Williams, 'Elementary evaluation of certain convolution sums involving divisor functions', in *Number Theory for the Millenium II* (A.K. Peters, Natick, MA, 2002), pp. 229–274.
- [8] G. Melfi, *Some problems in elementary number theory and modular forms*, Ph.D thesis (University of Pisa, 1998).
- [9] G. Melfi, 'On some modular identities', in *Number Theory* (de Gruyter, Berlin, 1998), pp. 371–382.
- [10] K.S. Williams, 'A cubic transformation formula for ${}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1, z\right)$ and some arithmetic convolution formulae', *Math. Proc. Cambridge Philos. Soc.* **137** (2004), 519–539.
- [11] K.S. Williams, 'The convolution sum $\sum_{m < n/8} \sigma(m)\sigma(n-8m)$ ', *Pacific J. Math.* (to appear).
- [12] K.S. Williams, 'The convolution sum $\sum_{m < n/9} \sigma(m)\sigma(n-9m)$ ', *Int. J. Number Theory* **1** (2005), 193–205.

School of Mathematics and Statistics
 Carleton University
 Ottawa, Ontario K1S 5B6
 Canada
 e-mail: mathieul@math.carleton.ca

Centre for Research in Algebra
 and Number Theory
 School of Mathematics and Statistics
 Carleton University
 Ottawa, Ontario K1S 5B6
 Canada
 e-mail: williams@math.carleton.ca