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UNBOUNDED HERMITIAN OPERATORS ON KOLASKI SPACES

JAMES JAMISON AND RAENA KING
Department of Mathematical Sciences, The University of Memphis, Memphis, TN 38152, USA
e-mail: jjamison@memphis.edu; rbryant1@memphis.edu

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Abstract. We investigate strongly continuous one-parameter (C_0) groups of isometries acting on certain spaces of analytical functions which were introduced by Kolaski (C. J. Kolaski, Isometries of some smooth normed spaces of analytic functions, *Complex Var. Theory Appl.* **10**(2–3) (1988), 115–122). We characterize the generators of these groups of isometries and also the spectrum of the generators. We provide an example on the Bloch space of an unbounded hermitian operator with non-compact resolvent.

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1. Introduction. An operator A on a Banach space X is called hermitian if iA is the generator of a one-parameter (C_0) group of isometries. Uniformly continuous groups have bounded generators, while strongly continuous groups have unbounded generators (cf. [**3**]). The generators of uniformly continuous (C_0) groups are the bounded hermitian operators studied by Vidav [**12**]. Berkson and Porta in [**1**] showed that the bounded hermitian operators on H^p for $p \neq 2$ are just real multiples of the identity. In this paper we study one-parameter groups of surjective isometries on spaces, which were first introduced by Novinger and Oberlin [**10**] and studied later in more generality by Kolaski [**8**]. These spaces, which we call Kolaski spaces, consist of analytical functions on the disk with derivative in one of the classical spaces of analytical function spaces such as H^p. In this paper we determine the generators of certain (C_0) groups of isometries and describe their spectrum. We then provide an application of our results to the S^p spaces, which are special cases of a Kolaski space. Since the spaces studied are classical spaces of analytical functions on the disk, the well-known theorems on isometries of these spaces involve automorphisms of the disk. The groups of surjective isometries of these spaces are classified by the fixed point structure of the associated group of disk automorphisms.

In the first two sections we focus on the strongly continuous (C_0) groups of isometries on a general Kolaski space, determining the corresponding generator and its spectrum. In the third section we specify the Kolaski space by requiring that the derivative of the functions belong to a classical Hardy space. Using the results of Berkson and Porta [**1**], we give a more complete description of the spectrum of the generator. In the last section of the paper we provide an interesting example which shows that the fixed point structure of the groups of disk automorphisms can have a strong effect on the spectral properties of the generator.

2. (C_0)groups of Linear Operators on S_N. Kolaski [8] introduced a special class of Banach spaces of analytical functions as follows. Let $H(D)$ be the space of analytical functions on the unit disk such that $N(f) < \infty$, where $N : H_N \to [0, \infty)$ is a norm, and let S_N denote the space of functions such that $N(f') < \infty$. We will denote the norm of a function f on H_N with $\|f\|_{H_N}$ and the norm of a function g on S_N with $\|g\|_{S_N}$, where the norm on S_N is given by $\|g\|_{S_N} = |g(0)| + \|g'\|_{H_N}$. We recall that a Banach space is called smooth if its norm is weakly differentiable at every point except the origin (see [9]). Throughout this paper H_N will always be a smooth space.

Kolaski [8] characterized the surjective linear isometries of S_N as follows.

THEOREM 2.1. [8, Theorem 1]. *Let T be an isometry of S_N onto respectively (into) S_N. Then there is a linear isometry $\mathcal{T}$ of H_N onto respectively (into) H_N and a λ with $|\lambda| = 1$ such that*

$$Tf(z) = \lambda[f(0) + \int_0^z \mathcal{T}f'(\xi)d\xi]. \tag{1}$$

We use this theorem to study the one-parameter (C_0) groups on S_N and determine the respective generators. We follow standard terminology as used in [3, p. 614]. A family $\{T_t\}$ $t \in \mathbb{R}$ of bounded linear operators in a Banach space X is called a strongly continuous group if

(i) $T_{s+t} = T_s T_t$,
(ii) $T_0 = I$,
(iii) for each $x \in X$, the map $t \to T_t x$ is continuous as a function of $t \in \mathbb{R}$.

These groups are called (C_0) groups if $\lim_{t\to 0} T_t f = f$ for every $f \in X$. If $\{T_t\}$ is any (C_0) group of linear operators, then the generator A of $\{T_t\}$ is defined by

$$(*) \quad Af = \lim_{t\to 0} \frac{T_t f - f}{t},$$

where the domain of A, $D(A)$, is the set of all $f \in X$ for which the limit $(*)$ exists. The group $\{T_t\}$ is uniformly continuous if and only if its generator is a bounded linear operator A. If $\{T_t\}$ is strongly continuous, but not uniformly continuous, then its generator is an unbounded operator (cf. [3]).

We first consider (C_0) groups of surjective isometries on the space S_N and determine the generator of such a group.

PROPOSITION 2.2. *Let $\{T_t\}$ be a (C_0) group of surjective linear isometries on S_N. Then there is a one-parameter group of unimodular complex numbers $\{\lambda_t\}$ and a (C_0) group of surjective linear isometries $\{\mathcal{T}_t\}$ on H_N, such that*

$$T_t f(z) = \lambda_t \left[f(0) + \int_0^z \mathcal{T}_t f'(\xi)d\xi \right]. \tag{2}$$

Proof. From Kolaski's theorem (2.1), a one-parameter group of surjective linear isometries on S_N, T_t is given by $(T_t f)(z) = \lambda_t[f(0) + \int_0^z \mathcal{T}_t f'(\xi)d\xi]$, where for each t, $\mathcal{T}_t$ is a surjective linear isometry of H_N and λ_t is a unimodular constant. It can be shown

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by using constant functions that T_t induces a (C_0) group of unimodular scalars, λ_t, of the form $\lambda_t = e^{i\gamma t}$ and hence

$$T_t = e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_t f'(\xi) d\xi]. \tag{3}$$

The property that $T_0 = I$ gives $e^{i\gamma(0)}[f(0) + \int_0^z \mathcal{T}_0 f'(\xi)d\xi] = f(z)$. Differentiating this last equation we get $\mathcal{T}_0 = I$. Likewise, the property that $T_s(T_t) = T_{s+t}$ gives $e^{i\gamma s}e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_s(\mathcal{T}_t f)'(\xi)d\xi] = e^{i\gamma(s+t)}[f(0) + \int_0^z \mathcal{T}_{s+t} f'(\xi)d\xi]$. Again differentiating this last equation gives $\mathcal{T}_s(\mathcal{T}_t) = \mathcal{T}_{s+t}$. Since T_t is strongly continuous, and

$$||(T_t - I)f||_{S_N} = ||(\mathcal{T}_t - I)f'||_{H_N} + |(e^{i\gamma t} - 1)f(0)| \geq ||(\mathcal{T}_t - I)f'||_{H_N},$$

it follows that $\mathcal{T}_t$ is strongly continuous. □

Proposition 2.3. *Let $\mathcal{S}$ be the generator of the (C_0) group $\{T_t\}$ on S_N, let $\mathcal{R}$ be the generator of the (C_0) group of isometries $\{\mathcal{T}_t\}$ on H_N induced by $\{T_t\}$ and $e^{i\gamma t}$ is a (C_0) group of unimodular complex numbers. Then $D(\mathcal{S}) = \{f \in S_N : f' \in D(\mathcal{R})\}$. Further, $\mathcal{S}f(z) = i\gamma f(0) + \int_0^z \mathcal{R}f'(\xi)d\xi$.*

Proof. We will show that
$\lim_{t\to 0} \left\| \frac{\mathcal{T}_t f - f}{t} - \left(i\gamma f(0) + \int_0^z \mathcal{R}(f')(\xi)d\xi\right)\right\|_{S_N} = 0,$

$$\left\| \frac{T_t f - f}{t} - i\gamma f(0) - \int_0^z \mathcal{R}f'(\xi)d\xi \right\|_{S_N}$$

$$= \left\| \left[\frac{e^{i\gamma t}f(0) + e^{i\gamma t}\int_0^z \mathcal{T}_t f'(\xi)d(\xi) - f(z)}{t} \right] - i\gamma f(0) - \int_0^z \mathcal{R}f'(\xi)d\xi \right\|_{S_N}$$

$$= \left\| \left[\frac{e^{i\gamma t}f(0) + e^{i\gamma t}\int_0^z (\mathcal{T}_t f'(\xi) - f'(\xi))\, d\xi + e^{i\gamma t}f(z) - e^{i\gamma t}f(0) - f(z)}{t} \right] \right.$$
$$\left. - i\gamma f(0) - \int_0^z \mathcal{R}f'(\xi)d\xi \right\|_{S_N}$$

$$\leq \left\| e^{i\gamma t}\left[\frac{(\mathcal{T}_t f'(z) - f'(z))}{t} - \mathcal{R}f'(z) \right] \right\|_{H_N} + \left\| \mathcal{R}f'(z)(e^{i\gamma t} - 1) \right\|_{H_N}$$
$$+ \left| \left(\frac{(e^{i\gamma t} - 1)}{t} - i\gamma \right) f(0) \right|.$$

Clearly, the right side $\to 0$ as $t \to 0$ since $\mathcal{R}$ is the generator of the group $\{\mathcal{T}_t\}$ and $e^{i\gamma t} - 1 \to 0$ as $t \to 0$. □

3. Spectrum of the generator of $\{T_t\}$. We recall that the spectrum of an operator $\mathcal{S}$ on a Banach space X, denoted by $\sigma(\mathcal{S})$, is the set of all complex numbers μ such that $\mathcal{S} - \mu I$ is non-invertible on X. The complement of this set is the resolvent and is denoted by $\rho(\mathcal{S})$. The point spectrum of $\mathcal{S}$, $\sigma_p(S)$, is the set of eigenvalues of $\mathcal{S}$. Hille and Phillips [**6**, p. 210] showed that if a generator of a one-parameter group has non-empty, compact resolvent, then it has pure point spectrum. We show how the spectrum of the generator of a (C_0) group on H_N is related to the generator of the (C_0) group on S_N. In this section, following is our main result:

THEOREM 3.1. *Let $\{T_t\}$ be a strongly continuous (C_0) group of isometries on S_N. Then*

$$T_t = e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_t f'(\xi)d\xi],$$

where $\{\mathcal{T}_t\}$ is a strongly continuous (C_0) group of isometries on H_N. Let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$. Let $\mathcal{S}$ be the generator of $\{T_t\}$. If $\mathcal{R}$ has compact resolvent, then $\mathcal{S}$ has compact resolvent.

We begin by showing the relationship between the point spectrums for the generators of groups on H_N and S_N.

PROPOSITION 3.2. *Let $\{T_t = e^{i\gamma t}[f(0) + \int_0^z \mathcal{T}_t f'(\xi)d\xi]\}$ be a strongly continuous (C_0) group of isometries on S_N with generator S, where $\{\mathcal{T}_t\}$ is a strongly continuous (C_0) group of isometries on H_N with generator $\mathcal{R}$. Then $\sigma_p(\mathcal{R}) \subsetneqq \sigma_p(\mathcal{S})$ or $\sigma_p(\mathcal{S})\backslash\{i\gamma\} \subsetneqq \sigma_p(\mathcal{R})$.*

Proof. Let μ be an eigenvalue of $\mathcal{R}$. So $\exists f_\mu \in H_N$ such that $\mathcal{R}f_\mu(z) = \mu f_\mu(z)$. Define the function $f(z) = \int_0^z f_\mu(\xi)d\xi$. Then $\mathcal{S}f(z) = i\gamma f(0) + \int_0^z \mathcal{R}f'(\xi)d\xi = \int_0^z \mathcal{R}f_\mu(\xi)d\xi = \mu \int_0^z f_\mu(\xi)d\xi = \mu f(z)$, so μ is an eigenvalue of $\mathcal{S}$.
Let μ be an eigenvalue of $\mathcal{S}$. So $\exists f_\mu \in S_N$ such that $\mathcal{S}f_\mu(z) = \mu f_\mu(z)$. We first suppose that f_μ is not constant. Then $i\gamma f_\mu(0) + \int_0^z \mathcal{R}f'(\xi)d\xi = \mu f_\mu$. Differentiation yields that $\mathcal{R}f'_\mu(z) = \mu f'_\mu$, so μ is an eigenvalue of $\mathcal{S}$. If f_μ is constant then $\mathcal{S}f_\mu = i\gamma f_\mu$ and so $\mu = i\gamma$. □

To discover properties of the generator of $\{T_t\}$, it is advantageous to consider S_N as the ℓ_1 direct sum of the spaces $\mathbb{C}$ and H_N. Define the map $V : S_N \to \mathbb{C} \oplus_1 H_N$ by $Vf(z) = (f(0), f')$. It is easy to check that S_N is isometric to $\mathbb{C} \oplus_1 H_N$, since the norm of $\mathbb{C} \oplus_1 H_N$ is given by $\|(\alpha, g)\|_* = |\alpha| + \|g\|_{H_N}$. Moreover, $V^{-1}(\alpha, g) = \alpha + \int_0^z g'(\xi)d\xi$.

PROPOSITION 3.3. *Let $\{T_t\}$ be a one-parameter (C_0) group of isometries on S_N. Define $V_t = VT_tV^{-1}$, with $Vf = (f(0), f')$. Then $\{V_t\}$ is a (C_0) group of isometries on $\mathbb{C} \oplus_1 H_N$.*

Proof. We will show that $\{V_t\}$ has the properties of a (C_0) group of isometries. Clearly, $\{V_t\}$ is an isometry for each t.
Also, $V_0 = VT_0V^{-1} = VIV^{-1} = I$ and $V_sV_t = (VT_sV^{-1})(VT_tV^{-1}) = VT_sT_tV^{-1} = V_{s+t}$.
Furthermore,

$$\begin{aligned}
\|V_t(\alpha, g) - (\alpha, g)\|_* &= \|VT_tV^{-1}(\alpha, g) - (\alpha, g)\|_* \\
&= \|VT_t(\alpha + \int_0^z g'(\xi)d\xi) - (\alpha, g)\|_* \\
&= \|V(\lambda_t(\alpha + \int_0^z \mathcal{T}_t g'(\xi)d\xi) - (\alpha, g)\|_* \\
&= \|(\lambda_t, \mathcal{T}_t g) - (\alpha, g)\|_*.
\end{aligned}$$

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Clearly, each term on the right side of this last equation goes to 0 as $t \to 0$. Hence, $\{V_t\}$ is strongly continuous and this completes the proof. □

For the remainder of this section, we will assume that H_N admits only trivial bounded hermitian operators. This is the case for the H^p spaces for $p \neq 2$. An application of a theorem by Fleming and Jamison [**4**, Theorem 2.5, p 174] gives that if the factor spaces of an ℓ_1 sum have have only trivial hermitian operators then the surjective isometries are diagonal. So $V_t = \begin{pmatrix} \lambda_t & 0 \\ 0 & \mathcal{T}_t \end{pmatrix}$.

PROPOSITION 3.4. *Let $\{T_t\}$ be a one-parameter (C_0) group of isometries on S_N and suppose that H_N admits only trivial hermitian operators. Then by Proposition (2.2), $\{T_t\}$ induces (C_0) groups $\{\lambda_t = e^{i\gamma t}\}$ and $\{\mathcal{T}_t\}$ on $\mathbb{C}$ and H_N respectively. Let γ be the generator of $\{\lambda_t\}$ and let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$. Then $\mathcal{G} = \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix}$ is the generator of the induced group $\{V_t\}$ on $\mathbb{C} \oplus_1 H_N$.*

Proof.

$$\begin{aligned}
\left\| \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix} \begin{pmatrix} \alpha \\ g \end{pmatrix} p - \frac{1}{t}\left(V_t \begin{pmatrix} \alpha \\ g \end{pmatrix} - \begin{pmatrix} \alpha \\ g \end{pmatrix} \right) \right\|_* &= \left\| \begin{pmatrix} \gamma\alpha \\ \mathcal{R}g \end{pmatrix} - \frac{1}{t}\left(\begin{pmatrix} \lambda_t & 0 \\ 0 & \mathcal{T}_t \end{pmatrix} \begin{pmatrix} \alpha \\ g \end{pmatrix} - \begin{pmatrix} \alpha \\ g \end{pmatrix} \right) \right\|_* \\
&= \left\| \begin{pmatrix} \gamma\alpha \\ \mathcal{R}g \end{pmatrix} - \frac{1}{t}\left(\begin{pmatrix} \lambda_t\alpha \\ \mathcal{T}_t g \end{pmatrix} - \begin{pmatrix} \alpha \\ g \end{pmatrix} \right) \right\|_* \\
&= \left\| \begin{pmatrix} \gamma\alpha \\ \mathcal{R}g \end{pmatrix} - \frac{1}{t} \begin{pmatrix} \lambda_t\alpha - \alpha \\ \mathcal{T}_t g - g \end{pmatrix} \right\|_* .
\end{aligned}$$

Thus, $\lim_{t\to 0} \| \begin{pmatrix} \gamma\alpha \\ \mathcal{R}g \end{pmatrix} - \frac{1}{t}(V_t\begin{pmatrix} \alpha \\ g \end{pmatrix} - \begin{pmatrix} \alpha \\ g \end{pmatrix})\|_* = \lim_{t\to 0}(|\gamma\alpha - \frac{\lambda_t\alpha - \alpha}{t}| + \|\mathcal{R}g - \frac{\mathcal{T}_t g - g}{t}\|_{H_N}) = 0$, since γ and $\mathcal{R}$ are generators of $\{\lambda_t\}$ and $\{\mathcal{T}_t\}$ respectively. This completes the proof. □

PROPOSITION 3.5. *Let $\{T_t\}$ be a one-parameter (C_0) group of isometries on S_N and $\{\lambda_t\}$ and $\{\mathcal{T}_t\}$ be the induced groups on $\mathbb{C}$ and H_N respectively. Let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$ and $\lambda_t = e^{i\gamma t}$. Let $\mathcal{G} = \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix}$ be the generator of the induced group $\{V_t\}$ on $\mathbb{C} \oplus_1 H_N$. If $\mu \neq \gamma$ is not an eigenvalue of $\mathcal{R}$ and $\mathcal{R} - \mu I$ is surjective, then $\mathcal{G} - \mu I$ is surjective.*

Proof. Given $(\alpha, g) \in \mathbb{C} \oplus_1 H_N$, we want to show that there exists $(z, h) \in \mathbb{C} \oplus_1 H_N$ such that $(\mathcal{G} - \mu I)(z, h) = (\alpha, g)$ Let $(\alpha, g) \in \mathbb{C} \oplus_1 H_N$. Consider $(\mathcal{G} - \mu I)\begin{pmatrix} z \\ h \end{pmatrix} = \begin{pmatrix} \gamma - \mu & 0 \\ 0 & (\mathcal{R} - \mu I) \end{pmatrix}\begin{pmatrix} z \\ h \end{pmatrix} = \begin{pmatrix} (\gamma - \mu)z \\ (\mathcal{R} - \mu I)h \end{pmatrix} = \begin{pmatrix} \alpha \\ g \end{pmatrix}$. The surjectivity of $(\mathcal{R} - \mu I)$ and $\gamma \neq \mu$ implies that $\begin{pmatrix} \alpha \\ g \end{pmatrix}$ is in the range of $(\mathcal{G} - \mu I)$. This completes the proof. □

PROPOSITION 3.6. *Let $\{T_t\}$ be a one-parameter (C_0) group of isometries on S_N and $\{\lambda_t\}$ and $\{\mathcal{T}_t\}$ be the induced groups on $\mathbb{C}$ and H_N respectively. Let $\mathcal{R}$ be the generator of $\{\mathcal{T}_t\}$ and $\lambda_t = e^{i\gamma t}$. Let $\mathcal{G} = \begin{pmatrix} \gamma & 0 \\ 0 & \mathcal{R} \end{pmatrix}$ be the generator of the induced group $\{V_t\}$ on $\mathbb{C} \oplus_1 H_N$. If $\mathcal{R}$ has compact resolvent, then $\mathcal{G}$ has compact resolvent.*

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Proof. Let $(\mathcal{G}-\mu I)^{-1}\begin{pmatrix} z_n \\ g_n \end{pmatrix}$ be a bounded sequence in the resolvent of $\mathcal{G}$.

$$(\mathcal{G}-\mu I)^{-1}\begin{pmatrix} z_n \\ g_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\gamma-\mu} & 0 \\ 0 & (\mathcal{R}-\mu I)^{-1} \end{pmatrix}\begin{pmatrix} z_n \\ g_n \end{pmatrix} = \begin{pmatrix} \frac{z_n}{\gamma-\mu} \\ (\mathcal{R}-\mu I)^{-1} g_n \end{pmatrix}.$$

Since $\frac{z_n}{\gamma-\mu}$ is a bounded sequence of complex numbers, it has a convergent subsequence, say $\frac{z_{n_k}}{\gamma-\mu}$. Furthermore, since g_n is a bounded sequence of H_N functions and $(\mathcal{R}-\mu I)^{-1}$ is compact on H_N, $(\mathcal{R}-\mu I)^{-1} g_{n_k}$ is a bounded sequence, so it has a convergent subsequence, say $(\mathcal{R}-\mu I)^{-1} g_{n_{k_i}}$. Thus, $(\mathcal{G}-\mu I)^{-1}\begin{pmatrix} z_n \\ g_n \end{pmatrix}$ has a convergent subsequence, and the resolvent of $\mathcal{G}$ is compact. □

The main theorem is an immediate consequence of the preceding propositions.

4. Application to S^p spaces. In this section we consider a special example of a Kolaski space, namely S^p. This Banach space consists of analytical functions f on the disk with $f' \in H^p$. The norm is given $\|f\| = |f(0)| + \|f'\|_{H^p}$. The bounded hermitian operators on this space were classified by Hornor and Jamison in [**7**]. In this section we will determine the generators of a certain type of strongly continuous (C_0) groups of isometries and thereby characterize certain unbounded hermitian operators associated with these groups in terms of their action on the space. Berkson and Porta, cf. [**1**], determined the generator of the one-parameter (C_0) group of isometries on H^p and this result will be crucial to our work. Since H^p is a smooth space for $1 < p < \infty$, their result together with the previous results on the Kolaski spaces can be used to find the generator of the type (i) one-parameter (C_0) group of isometries on S^p.

In order to apply their result, we first introduce some notation and results. Let a one-parameter group of disk automorphisms be denoted by $\{\phi_t\}$ with

$$\phi_t(z) = \frac{a_t(z-b_t)}{1-\bar{b}_t z}, \tag{4}$$

where a_t and b_t are constants such that $|a_t| = 1$ and $|b_t| < 1$. A non-constant one-parameter (C_0) group of disk automorphisms $\{\phi_t\}$ on D is said to be of type (i) if the set of common fixed points in the extended complex plane of functions ϕ_t is a doubleton subset consisting of a point $\tau \in D$ and $\bar{\tau}^{-1}$ ($\bar{\tau}^{-1} = \infty$ if $\tau = 0$). We will call a one-parameter (C_0) group, $\{T_t\}$, on S^p of type (i) if the induced one-parameter (C_0) group of disk automorphisms is of type (i). To state our results we will quote the following result by Berkson and Porta [**1**] on the form of the one-parameter (C_0) group of disk automorphisms.

THEOREM 4.1. [**1**, Theorem 1.10]. *Let $\{\phi_t\}$, $t \in \mathbb{R}$ be a non-constant one-parameter group of disk automorphisms of D such that $\phi_t(z)$ is continuous in t for each $z \in D$. If $\{\phi_t\}$ is of type (i), then there are uniquely determined (by $\{\phi_t\}$) constants τ and c, the former in D and the latter real and non-zero such that the parameters a_t and b_t of equation (4) are given for all $t \in \mathbb{R}$ by*

$$a_t = \frac{|\tau|^2 - e^{ict}}{|\tau|^2 e^{ict} - 1}; \qquad b_t = \frac{\tau(e^{ict} - 1)}{e^{ict} - |\tau|^2}. \tag{5}$$

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The constant τ is, in fact, the unique element of D which is left fixed by all ϕ_t for $t \in \mathbb{R}$. The constant c satisfies and is determined by the equation $e^{ict}z = (\Psi \circ \phi_t \circ \Psi)(z)$ for $t \in \mathbb{R}$ and $z \in D$ and Ψ a disk automorphism given by

$$\Psi(z) = \frac{z - \tau}{\bar{\tau} z - 1}. \tag{6}$$

Conversely, if τ is any element of D and c is any non-zero real number, the parameters a_t and b_t, defined for each $t \in \mathbb{R}$ as in (5) have moduli satisfying the requirements in (4), and defined by the formula in (4) a one-parameter group, $\{\phi_t\}$ of type (i).

Associated with every continuous one-parameter group of disk automorphisms there is the so-called invariance polynomial. It was shown in [**1**] that the zeroes of this polynomial are precisely the set of fixed points of ϕ_t in the finite plane.

Corollary 4.2. [**1**, Corollary 1.13]. *If $\{\phi_t\}$ is a group of type (i), then its invariance polynomial can be written in the form*

$$R(z) = \frac{-ic(\bar{\tau} z^2 - (1 + |\tau|^2)z + \tau)}{1 - |\tau|^2} \tag{7}$$

for all $z \in \mathbb{C}$, where τ is the unique point of D fixed by every ϕ_t and c is the unique constant as in Theorem 4.1.

We also recall the form of the surjective isometries on H^p for $1 \le p < \infty,\ p \neq 2$, which were determined by Forelli [**5**].

Proposition 4.3. [**5**, Theorem 2]. *If A is a linear isometry of H^p onto H^p, $1 \le p \le \infty$, $p \neq 2$, then there is a disk automorphism ϕ and $\lambda \in \mathbb{C}$ such that*

$$(Af)(z) = \lambda[\phi'(z)]^{1/p} f(\phi(z)) \tag{8}$$

for all $f \in H^p$ and $z \in D$.
On the other hand, if $1 \le p \le \infty$ and λ is a unimodular constant and ϕ is a disk automorphism, then (8) defines a linear isometry of H^p onto H^p.

Let $\{T_t\}$ be a one-parameter (C_0) group of isometries induced by a group of disk automorphisms of type (i) and define a hermitian operator A to be of type (i) if it is the generator of a group of isometries where the associated group of disk automorphisms is of type (i). Berkson and Porta [**1**] defined $\mathcal{R}$ with the domain, $\mathcal{D}(\mathcal{R})$, to be all functions $f \in H^p$ such that the function

$$\mathcal{R} = Rf' + (1/p)R'f \tag{9}$$

is in H^p, where R is the invariance polynomial given in (7) and juxtaposition indicates multiplication. They proved the following theorem.

Theorem 4.4. [**1**, Theorem 3.1]. *If A is a hermitian operator of type (i) in H^p, $1 \le p < \infty, p \neq 2$, then:*
(1) there is a unique real number β such that $A = \beta I - i\mathcal{R}$, where $\mathcal{R}$ is as in (9);
(2) the eigenvalues of A are precisely the real numbers $\sigma_n = c(n + 1/p) + \beta, n = 0, 1, 2, \ldots;$

(3) for each n, the eigenmanifold of A corresponding to σ_n is the one-dimensional span of the function $f_{n,\tau}(z)$, where for each $z \in D$

$$f_{n,\tau}(z) = \begin{cases} (z-\tau)^n/(z-\bar{\tau}^{-1})^{n+(2/p)} & ,\text{if } \tau \neq 0 \\ z^n & ,\text{if } \tau = 0 \end{cases};$$

(4) A has compact resolvent, and hence pure point spectrum.

We now combine this theorem of Berkson and Porta [**1**] along with one of the main results of this paper, Corollary (3.1), to find the generator and spectrum of the associated one-parameter (C_0) group of isometries on S^p.

COROLLARY 4.5. *Let $\mathcal{S}$ be the generator of type (i) of a one-parameter (C_0) group of isometries on S^p. Then $\mathcal{D}(\mathcal{S}) = \{f \in S^p : f' \in \mathcal{D}(\mathcal{R})\}$, where $\mathcal{R}$ is given in (9) and*

$$\mathcal{S}f(z) = \gamma f(0) + \int_0^z \beta f'(\xi) - (\mathcal{R}f')(\xi)d\xi = (\gamma-\beta)f(0) + \beta f(z) - \int_0^z (\mathcal{R}f')(\xi)d\xi,$$

where γ and β are generators of one-parameter (C_0) groups of complex numbers and $\mathcal{R}$ is as given in (9). Furthermore, $\mathcal{S}$ has compact resolvent, thus pure point spectrum, so

$$\sigma(\mathcal{S}) = \{\sigma_n = c(n + (1/p)) + \beta : n = 0, 1, 2, ...\},$$

where the eigenmanifolds corresponding to each σ_n are spanned by the functions

$$g_{n,\tau} = \begin{cases} \int_0^z (\xi-\tau)^n/(\xi-\bar{\tau}^{-1})^{n+(2/p)}d\xi & \text{if } \tau = 0 \\ (n+1)^{-1}z^{n+1} & \text{if } \tau \neq 0 \end{cases}.$$

Proof. The proof is a direct application of our results from Sections 1 and 2 for S_N and H_N by considering $H_N = H^p$. □

REMARK 4.6. The first two terms in $\mathcal{S}$ gives the form of the bounded hermitian operators on S^p, cf. [**7**].

5. An example in the Bloch Space. In the previous section we assumed that H_N is a smooth Banach space. We now consider an example of a (C_0) group of isometries on a non-smooth Banach space. Rather than considering the associated Kolaski space, we directly focus on the S_N group, since the (C_0) group properties on H_N are reflected in S_N. We give an example of a one-parameter (C_0) group of isometries in which the properties of the generator in the preceding sections do not hold and in fact the resolvent is not compact. The space we wish to consider is the Bloch space and is defined as follows:

$$\mathcal{B} = \{f : \text{f is analytical on } D, f(0) = 0, \text{and } \ sup_{|z|<1}|f'(z)|(1-|z|^2) = N(f) < \infty\}.$$

Let $\mathcal{B}_0$ denote the closed subspace of $\mathcal{B}$ spanned by all polynomials. Cima and Wogen [**2**] characterized the surjective isometries of $\mathcal{B}_0$ as follows.

LEMMA 5.1. [**2**, Corollary 3]. *If $S : \mathcal{B}_0 \to \mathcal{B}_0$ is a surjective isometry, then there is a disk automorphism ϕ of D and λ such that $|\lambda| = 1$ so that*

$$Sf(z) = \lambda[f(\phi(z)) - f(\phi(0))] \qquad \forall f \in \mathcal{B}_0.$$

We show an example of a (C_0) one-parameter group in the case where ϕ_t is as in equation (4) with $a_t = 1$ and $b_t = (1 - e^{-t})(1 + e^t)^{-1}$. Hence,

$$\phi_t(z) = \frac{(1 + e^{-t})z + (1 - e^{-t})}{(1 - e^{-t})z + (1 + e^{-t})}. \tag{10}$$

In this case the generator of ϕ_t is given by $R = (1 - z^2)/2$ and $\frac{\partial}{\partial t}|_{t=0}\phi_t' = R' = -z$. Furthermore, ϕ_t is a one-parameter (C_0) group of disk automorphisms.

For each $t \in \mathbb{R}$, $T_t f(z) = \lambda_t[f(\phi_t(z)) - f(\phi_t(0))]$. With the given ϕ_t, if $\{T_t\}$ is a one-parameter (C_0) group of isometries on $\mathcal{B}_0$, then it is easy to show that $\{\lambda_t\}$ is a (C_0) group of unimodular complex numbers.

THEOREM 5.2. *Let $\{\phi_t\}$ be given as in equation (10) and β be the generator of $\{\lambda_t\}$. Then $\mathcal{G}f(z) = i\beta f(z) + Rf'(z)$ is the generator of $\{T_t\}$, where $\mathcal{R} = (1 - z^2)/2$.*

Proof. We show that $\lim_{t\to 0} \|\frac{T_t f - f}{t} - \mathcal{G}(f)\| = 0$ for every $f \in \mathcal{D}(\mathcal{G})$.

$$\begin{aligned}
&\left\| \frac{T_t f - f}{t} - \mathcal{G}(f) \right\| \\
&\quad = \left\| \frac{e^{i\beta t}[f \circ \phi_t - f \circ \phi_t(0)] - f}{t} - (i\beta f + (1 - z^2)f'/2) \right\| \\
&\quad = \sup_{|z|<1}(1 - |z|^2)\left[\left\| \frac{e^{i\beta t}(f' \circ \phi_t)(\phi_t') - f'}{t} - i\beta f' - (1 - z^2)f''/2 - (-zf') \right\| \right] \\
&\quad \le \sup_{|z|<1}(1 - |z|^2)\left(\left\| \frac{e^{i\beta t}(f' \circ \phi_t)(\phi_t') - e^{i\beta t}\phi_t' f'}{t} - (1 - z^2)f''/2 \right\| \right. \\
&\qquad \left. + \left\| \frac{e^{i\beta t}\phi_t' f' - \phi_t' f'}{t} - i\beta f' \right\| + \left\| \frac{\phi_t' f' - f'}{t} - (-zf') \right\| \right) \\
&\quad = \sup_{|z|<1}(1 - |z|^2)\left\{ \left\| \frac{e^{i\beta t}\phi_t'(f' \circ \phi_t - f' \circ \phi_0)}{\phi_t - \phi_0} \cdot \frac{\phi_t - \phi_0}{t} - (1 - z^2)f''/2 \right\| \right. \\
&\qquad \left. + \left\| \frac{\phi_t' f'(e^{i\beta t} - 1)}{t} - i\beta f' \right\| + \left\| \frac{\phi_t' - \phi_0'}{t} - (-z)'f' \right\| \right\}.
\end{aligned}$$

Each term of the previous equation goes to 0 as $t \to 0$. Thus, the proof is complete. □

We now find the point spectrum of $\mathcal{G}$.

COROLLARY 5.3. *Let $\mathcal{G}$ be the generator of the one-parameter (C_0) group of isometries $\{T_t\}$ on $\mathcal{B}_0$ in the previous theorem induced by the group of disk automorphisms in* (10). *Then $\sigma_p(\mathcal{G}) = \emptyset$.*

Proof. We find values of α for which the differential equation $\mathcal{G}f = \alpha f$ has a solution. Then,

$$(1 - z^2)f'/2 + i\beta f = \alpha f.$$

After rearranging and integrating, the equation becomes

$$\int (f'/f)dz = 2(\alpha - i\beta) \int (1 - z^2)^{-1} dz.$$

Hence, $log f = (\alpha - i\beta)(log(1 + z) - log(1 - z)) + C$. Whence, $f(z) = (\frac{1+z}{1-z})^{\alpha - i\beta} e^C$. However, $f(0) \neq 0$, so the solution is not in $\mathcal{B}_0$. Therefore, $\sigma_p(\mathcal{G}) = \emptyset$. □

To find a representation for $(\mathcal{G} - \lambda I)^{-1}$, we solve the differential equation $(\mathcal{G} - \lambda I)f(z) = g(z)$ for g by using an integrating factor of $[(1 + z)(1 - z)^{-1}]^{i\beta - \lambda}$. Thus,

$$(\mathcal{G} - \lambda I)^{-1} g(z) = [(1 - z)(1 + z)^{-1}]^{i\beta - \lambda} \int_0^z 2(1 - w^2)^{-1}[(1 + w)(1 - w)^{-1}]^{i\beta - \lambda} g(w)dw. \tag{11}$$

Before giving our last result, we recall that a closed operator with non-empty resolvent set and compact resolvent always has only eigenvalues in its spectrum [**6**, Theorem 56.14.2].

Corollary 5.4. *Let $\mathcal{G}$ be the generator of a one-parameter* (C_0) *group of isometries on $\mathcal{B}_0$ induced by the group of disk automorphisms in* (10). *Then the resolvent of $\mathcal{G}$ is not compact.*

Proof. To prove this, we only need to show that the resolvent of $\mathcal{G}$ in non-empty. Palmer notes that if $i\mathcal{G}$ is the infinitesimal generator of a one-parameter group of isometries, then $\sigma(\mathcal{G}) \subset \mathbb{R}$ (see [**11**, p. 387]). Thus, the resolvent of $\mathcal{G}$ is non-empty. □

We leave open the question concerning the form of the unbounded hermitian operators on $\mathcal{B}_0$ induced by other groups of disk automorphisms.

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