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Part 5. Finance and econometrics

ASYMMETRIC COGARCH PROCESSES

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Abstract

Financial data are as a rule asymmetric, although most econometric models are symmetric. This applies also to continuous-time models for high-frequency and irregularly spaced data. We discuss some asymmetric versions of the continuous-time GARCH model, concentrating then on the GJR-COGARCH model. We calculate higher-order moments and extend the first-jump approximation. These results are prerequisites for moment estimation and pseudo maximum likelihood estimation of the GJR-COGARCH model parameters, respectively, which we derive in detail.

Keywords: APCOGARCH; asymmetric power COGARCH; COGARCH; first-jump approximation; continuous-time GARCH; GJR-GARCH; GJR-COGARCH; maximum-likelihood estimation; high-frequency data; method of moments; stochastic volatility

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1. Introduction

In 1982 Engle [7] suggested an autoregressive conditionally heteroscedastic (ARCH) model for the variance of United Kingdom inflation data. In this model the conditional variance was modelled as an autoregressive process of past variances. Bollerslev [4] enriched this model by an additional term of past squared volatilities resulting in the generalized ARCH (GARCH) model. Nowadays this is one of the most prominent econometric models as it captures relevant stylized facts of econometric data; it has the form

$$Y_n = \sigma_n \varepsilon_n, \qquad \sigma_n^2 = \theta + \sum_{i=1}^{q} \alpha_i Y_{n-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{n-j}^2, \qquad n \in \mathbb{N}, \tag{1.1}$$

for independent, identically distributed (i.i.d.) random variables $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\mathbb{E}[\varepsilon_n] = 0$, $\text{var}(\varepsilon_n) = 1$, and ε_n independent of $\mathcal{F}_{n-1}$, the σ-algebra generated by $\{Y_k : k \leq n - 1\}$. The parameters satisfy $\theta > 0$, $\alpha_i \geq 0$, and $\beta_i \geq 0$, with $\alpha_q, \beta_p > 0$.

In real data, however, there is an asymmetric response of the volatility, called the leverage effect, which says that stock returns are negatively correlated with changes in return volatility. More precisely, volatility tends to rise in response to bad news and fall in response to good news. This effect has also been investigated in empirical studies, which show the statistical significance of asymmetry in financial time series models (cf. [6, 10, 12, 20, 23, 24]).

As a consequence of their empirical findings, Ding *et al.* [6] introduced an asymmetric power GARCH (APGARCH) model defined via

$$Y_n = \varepsilon_n \sigma_n, \qquad \sigma_n^\delta = \theta + \sum_{i=1}^{q} \alpha_i h(Y_{n-i}) + \sum_{j=1}^{p} \beta_j \sigma_{n-j}^\delta, \qquad n \in \mathbb{N}, \tag{1.2}$$

for i.i.d. random variables $\{\varepsilon_n\}_{n \in \mathbb{N}}$ with $\mathbb{E}[\varepsilon_n] = 0$, $\mathrm{var}(\varepsilon_n) = 1$, and ε_n independent of $\mathcal{F}_{n-1}$. The asymmetry is introduced by $h(x) = (|x| - \gamma x)^\delta$, where $\delta > 0$ and $|\gamma| < 1$, and θ, α_i, and β_i are as for (1.1).

It was shown in [6] that the APGARCH model includes several important ARCH and GARCH models as special cases. In particular, if $\delta = 2$, it includes Engle's ARCH(p) [7], Bollerslev's GARCH(p, q) [4], and the Glosten–Jagannathan–Runkle (GJR) model [10], while, for $\delta = 1$, the threshold GARCH (TARCH) model (see [24, 27]) can be obtained; see [9] for further information on GARCH-type models.

With the advent of high-frequency data and irregularly spaced tick-by-tick data, continuous-time models came into the focus of econometrics. Nelson [22] derived a continuous-time GARCH model using diffusion approximations; it yields prices and volatilities in continuous time driven by Brownian motions. Consequently, Nelson's GARCH diffusion model cannot model jumps in prices and volatilities. However, it retains the heavy (Pareto) tails of the original GARCH model.

At the beginning of the new millennium empirical studies established stylized facts of high-frequency data, giving the important insight that prizes and volatilities exhibit jumps, including common jumps (cf. the excellent monographs [1] and [13] for insight and further references). In 2004 Klüppelberg *et al.* [15] suggested a continuous-time GARCH(1, 1) (COGARCH(1, 1)) model capturing the jump features of high-frequency data, and they proved properties like strict stationarity and second-order behaviour. Moment estimation for this COGARCH model works very well for high-frequency data, as demonstrated in [11], where also a simple leverage term has been added. Maller *et al.* [18] derived a first-jump approximation, which provides a sequence of GARCH processes converging in probability to the original COGARCH process in the Skorokhod topology. This allows for the use of existing software for maximum likelihood estimation for GARCH processes and is also applicable to nonequidistantly sampled data.

In this paper we discuss an asymmetric COGARCH(1, 1) model, which takes care of the observed leverage effect in a systematic way. The model is a continuous-time version of (1.2) with $\delta = 2$. We define the GJR-COGARCH model in Section 2, derive first properties, and present some results of simulation work. Sections 3 and 4 contain the estimation methods as well as their prerequisites. In particular, in Section 3 we calculate the moments of the asymmetric model and apply these to obtain explicit moment estimators. In Section 4 we extend the first-jump approximation from [18] to the asymmetric model and prove its convergence. This is then used to derive a pseudo maximum likelihood estimator for the parameters of the GJR-COGARCH process.

2. GJR-COGARCH processes

Recall the GJR-GARCH(1, 1) model defined by

$$Y_n := \sigma_n \varepsilon_n, \qquad \sigma_n^2 = \theta + \alpha(|Y_{n-1}| - \gamma Y_{n-1})^2 + \beta \sigma_{n-1}^2, \qquad n \in \mathbb{N}, \qquad (2.1)$$

where $\theta \geq 0$, α, $\beta > 0$, $|\gamma| < 1$, and $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is an i.i.d. noise sequence with $\mathbb{E}[\varepsilon_0] = 0$ and $\mathrm{var}(\varepsilon_0) = 1$. Following the construction method of the COGARCH(1, 1) model in [15], and reparameterizing the model by defining $\eta = -\log \beta$ and $\varphi = \alpha/\beta$, we define a continuous-time GJR-GARCH (GJR-COGARCH) model as follows (cf. [17]):

$$\mathrm{d}G_t = \sigma_t \, \mathrm{d}L_t, \quad t \geq 0, \qquad G_0 = 0, \qquad (2.2)$$

$$\sigma_t^2 = \sigma_0^2 + \theta t - \eta \int_0^t \sigma_s^2 \, \mathrm{d}s + \varphi \sum_{0 < s \leq t} \sigma_s^2 h(\Delta L_s), \quad t \geq 0, \qquad \sigma_0^2 \geq 0. \qquad (2.3)$$

Here $h(x) = (|x| - \gamma x)^2$ with $|\gamma| < 1$, and $\theta, \eta, \varphi > 0$. The Lévy process L has Lévy measure $\nu_L \neq 0$, independent of σ_0^2. We choose L symmetric so that the asymmetry of the model originates only in γ. In particular, throughout this paper, we use $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1^2] = 1$. Note that, for a symmetric Lévy process, the sign of the chosen parameter γ is irrelevant for the resulting process because positive and negative jumps of the same size appear with the same probability. Hence, we assume from now on that $\gamma \in [0, 1)$.

Remark 2.1. Of course, asymmetry of a COGARCH process can also be achieved by choosing an asymmetric Lévy process as the driving process in the original symmetric COGARCH model. Replacing the term $h(\Delta L_s)$ for L with symmetric Lévy measure ν_L in (2.3) by ΔL_s^2 with asymmetric Lévy measure

$$\nu_a(\mathrm{d}x) = \nu_L(\mathrm{d}x)((1 - \gamma)\,\mathbf{1}_{\{x \geq 0\}} + (1 + \gamma)\,\mathbf{1}_{\{x < 0\}})$$

yields the same model. However, we prefer to have the asymmetry as a model parameter which we can estimate by standard statistical procedures.

The following lemma summarizes some properties of the GJR-COGARCH volatility which we will need later on. Analogous properties of the COGARCH volatility can be found in [15, Lemma 4.1] and [2, Proposition 3.2].

Lemma 2.1. (a) *The asymmetric GJR-COGARCH volatility* $\{\sigma_t^2\}_{t \geq 0}$ *is a generalized Ornstein–Uhlenbeck process with representation*

$$\sigma_t^2 = \left(\theta \int_0^t \mathrm{e}^{X_{s-}}\,\mathrm{d}s + \sigma_0^2\right)\mathrm{e}^{-X_t}, \qquad t \geq 0, \tag{2.4}$$

where X in (2.4) is a spectrally negative Lévy process defined by

$$X_t = t\eta - \sum_{0 < s \leq t} \log(1 + \varphi h(\Delta L_s)), \qquad t \geq 0,$$

whose Laplace exponent $\Psi(u) = \mathbb{E}[\mathrm{e}^{-uX_1}]$, $u \geq 0$, is given by

$$\Psi(u) = -\eta u + \int_{\mathbb{R}} ([1 + \varphi h(y)]^u - 1)\,\nu_L(\mathrm{d}y), \tag{2.5}$$

and $\Psi(u)$ is finite for $u > 0$ if and only if $\mathbb{E}[L^{2u}] < \infty$.

(b) *Provided the quantities are finite, the following identities hold:*

$$\Psi(1) = -\eta + \varphi(1 + \gamma^2)\int_{\mathbb{R}} y^2\,\nu_L(\mathrm{d}y) = -\eta + \varphi(1 + \gamma^2)\mathbb{E}[L_1^2], \tag{2.6}$$

$$\Psi(2) = 2\Psi(1) + \varphi^2(1 + 6\gamma^2 + \gamma^4)\int_{\mathbb{R}} y^4\,\nu_L(\mathrm{d}y). \tag{2.7}$$

(c) *The process* $\{\sigma_t^2\}_{t \geq 0}$ *is the unique solution of the stochastic differential equation*

$$\mathrm{d}\sigma_t^2 = \theta\,\mathrm{d}t + \sigma_{t-}^2\,\mathrm{d}U_t, \qquad t > 0, \tag{2.8}$$

with driving Lévy process

$$U_t = -X_t + \sum_{0 < s \leq t} (\mathrm{e}^{-\Delta X_s} - 1 + \Delta X_s) = -\eta t + \varphi \sum_{0 < s \leq t} h(\Delta L_s).$$

We shall work with the stationary solution of the GJR-COGARCH volatility, whose existence is guaranteed under certain conditions as given in the following proposition.

Proposition 2.1. (a) *The GJR-COGARCH volatility (2.4) has a stationary distribution if and only if the integral $\int_0^\infty e^{-X_{t-}}\, dt$ converges almost surely to a finite random variable. This is the case if and only if $\mathbb{E}[L_1^2] < \infty$ and*

$$\int_{\mathbb{R}} \log(1 + \varphi h(y))\, \nu_L(dy) < \eta. \tag{2.9}$$

(b) *The stationary distribution of the GJR-COGARCH volatility is uniquely determined by the law of*

$$\sigma_\infty^2 := \theta \int_0^\infty e^{-X_{t-}}\, dt. \tag{2.10}$$

(c) *Equation (2.9) holds if $\Psi(1) \le 0$, and then the corresponding symmetric COGARCH volatility (i.e. with $\gamma = 0$) has a stationary distribution.*

Proof. We use the generalized Ornstein–Uhlenbeck representation of $\{\sigma_t^2\}_{t\ge 0}$ as given in (2.4) and (2.8). Statement (a) is a consequence of [17, Theorem 3.1] or [16, Theorem 2.b], and (b) of [2, Theorem 2.1]. Statement (c) holds as $\log(1 + x) < x$ for positive x. The second part holds since (2.6) implies that $\Psi(1) > -\eta + \varphi\mathbb{E}[L_1]$. Now apply [16, Theorem 2.b]. ∎

In Figure 1 we show simulations of COGARCH and GJR-COGARCH processes, both driven by the same compound Poisson process with rate 1 and standard normal jumps. Although the sample paths of the symmetric and asymmetric COGARCH processes in the top row of Figure 1 look similar, the returns in the middle row already exhibit more pronounced downwards and less pronounced upwards peaks. This is due to the volatility process depicted in the bottom row, where the asymmetry in the jumps has rather dramatic consequences.

3. Method of moments for GJR-COGARCH processes

3.1. Moments of GJR-COGARCH processes

In this subsection we describe the theoretical second-order structure of the returns of the integrated GJR-COGARCH process and its squared process. These results are the basis for the method of moment (MoM) estimation presented in Section 3.2.

In principle, as remarked in [17, Theorem 3.1 and Remark 2], but where no formulae were given, moments of a GJR-COGARCH process can be computed analogously to those of a COGARCH process, as done in [11] and [15]. Although the calculations are quite straightforward, they are tedious and lengthy. We restrict ourselves to presenting the explicit formulae in Propositions 3.1 and 3.2; full proofs for all formulae apart from (3.2) can be found in [19].

We start with the moments of the GJR-COGARCH volatility.

Proposition 3.1. *Let $\{\sigma_t^2\}_{t\ge 0}$ be the stationary GJR-COGARCH volatility process (2.4) with $\sigma_0^2 \overset{D}{=} \sigma_\infty^2$ as in (2.10). Let $\kappa \in \mathbb{N}$ be constant. Then $\mathbb{E}[\sigma_\infty^{2\kappa}] < \infty$ if and only if $\mathbb{E}[|L_1|^{2\kappa}] < \infty$ and $\Psi(\kappa) < 0$. In particular, with $\Psi(\cdot)$ as in (2.5),*

$$\mathbb{E}[\sigma_t^{2\kappa}] = \kappa!\, \theta^\kappa \prod_{l=1}^\kappa \frac{1}{-\Psi(l)}, \qquad t \ge 0,$$

$$\mathrm{cov}(\sigma_t^2, \sigma_{t+h}^2) = \theta^2 \left(\frac{2}{\Psi(1)\Psi(2)} - \frac{1}{\Psi(1)^2} \right) e^{h\Psi(1)}, \qquad t, h \ge 0.$$
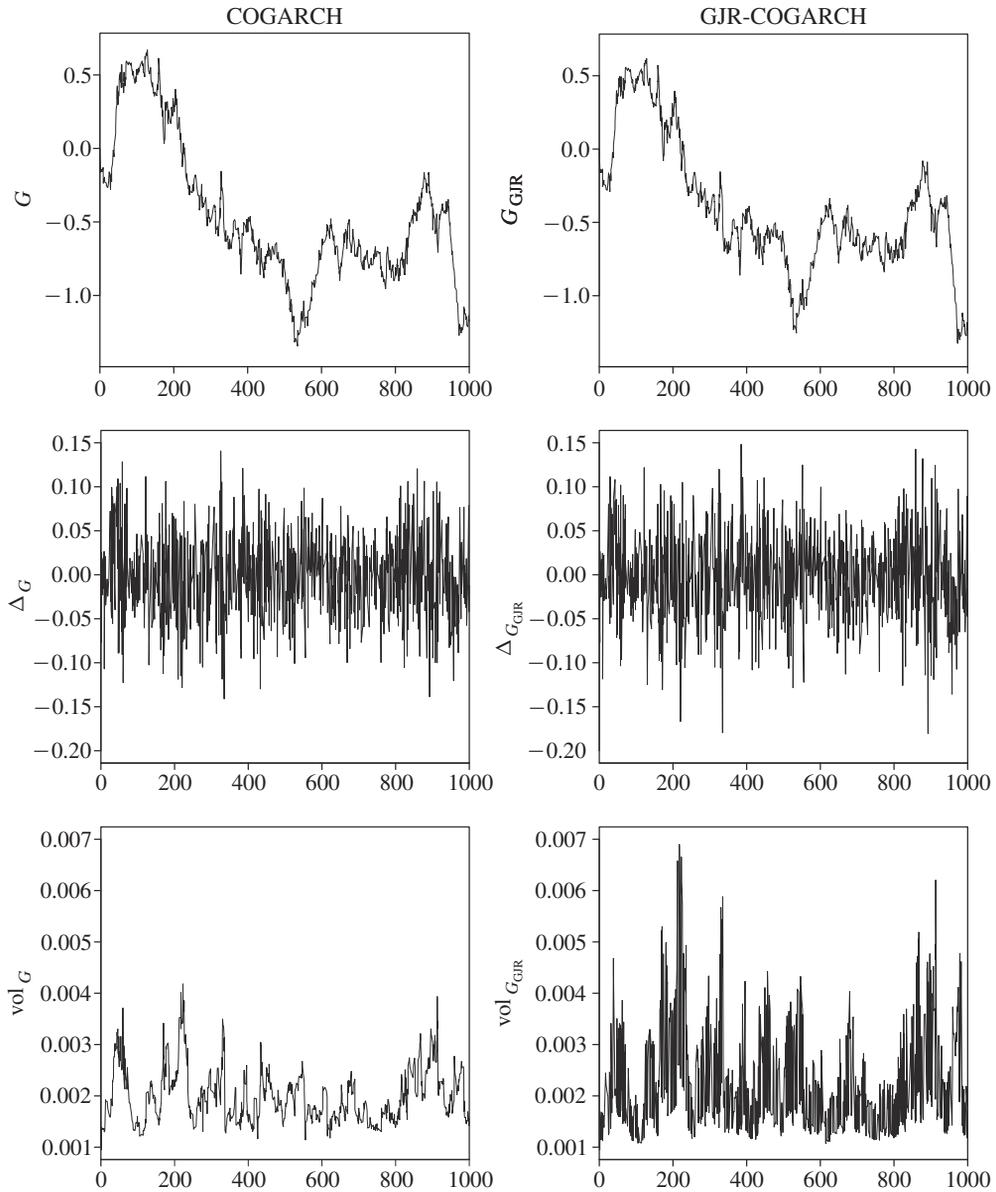
FIGURE 1: Simulations of COGARCH (*left*) and GJR-COGARCH (*right*) processes. *Top*: price process (integrated process). *Middle:* price differences for $\Delta = 1$. *Bottom:* volatilities. The values of the parameters are $\theta = 0.0001$, $\eta = -\log 0.9 = 0.04576$, $\varphi = \frac{1}{18} = 0.05556$, and $\gamma = 0.3$.

The observations upon which our estimation is based are the increments of the integrated GJR-COGARCH process $\{G_t\}_{t\geq 0}$. Hence, we set, for fixed $r > 0$,

$$G_t^{(r)} := G_{t+r} - G_t = \int_{(t,t+r]} \sigma_{s-} \, \mathrm{d}L_s, \qquad t \geq 0. \tag{3.1}$$

Obviously, this is a stationary process if the volatility $\{\sigma_t^2\}_{t\geq 0}$ is stationary.

Proposition 3.2. *Let $\{L_t\}_{t\geq 0}$ be a pure-jump Lévy process with $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1^2] = 1$. Assume that $\Psi(1) < 0$ for Ψ as in (2.5). Furthermore, let $\{\sigma_t^2\}_{t\geq 0}$ be the stationary GJR-COGARCH volatility (2.4) with $\sigma_0^2 \overset{\mathrm{D}}{=} \sigma_\infty^2$ as in (2.10). Then, for all $t \geq 0$ and $h \geq r > 0$,*

$$\mathbb{E}[G_t^{(r)}] = 0, \qquad \mathbb{E}[(G_t^{(r)})^2] = \frac{\theta r}{|\Psi(1)|}\mathbb{E}[L_1^2], \qquad \mathrm{cov}(G_t^{(r)}, G_{t+h}^{(r)}) = 0.$$

Assume further that $\mathbb{E}[L_1^4] < \infty$ and $\Psi(2) < 0$. Then $\mathbb{E}[(G_t^{(r)})^4] < \infty$ and if, additionally, $\int_{\mathbb{R}} y^3\, \nu_L(\mathrm{d}y) = 0$ then, for all $t \geq 0$ and $r > 0$,

$$\mathbb{E}[(G_t^{(r)})^4]$$
$$= \frac{6\theta^2 r}{|\Psi(1)|^2}\left(\frac{2\eta}{\varphi} - (1+\gamma^2)\mathbb{E}[L_1^2]\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\left(1 - \frac{1 - \mathrm{e}^{-r|\Psi(1)|}}{r|\Psi(1)|}\right)\mathbb{E}[L_1^2]$$
$$+ \frac{2\theta^2}{\varphi^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\frac{r}{1+6\gamma^2+\gamma^4} + \frac{3\theta^2}{|\Psi(1)|^2}(\mathbb{E}[L_1^2])^2 r^2, \tag{3.2}$$

while, for all $t \geq 0$ and $h \geq r > 0$,

$$\mathrm{cov}((G_t^{(r)})^2, (G_{t+h}^{(r)})^2) = \mathbb{E}[L_1^2]\frac{\theta^2}{|\Psi(1)|^3}\left(\frac{2\eta}{\varphi} - (1+\gamma^2)\mathbb{E}[L_1^2]\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)$$
$$\times (1 - \mathrm{e}^{-r|\Psi(1)|})(\mathrm{e}^{r|\Psi(1)|} - 1)\mathrm{e}^{-h|\Psi(1)|}$$
$$> 0. \tag{3.3}$$

Remarks 3.1. (a) Setting $\gamma = 0$, all moment expressions reduce to those of the symmetric COGARCH model in [11, Proposition 1].

(b) The asymmetry modelled by γ in (3.2) is also present in $\Psi(1)$ and $\Psi(2)$.

(c) Under the conditions of Proposition 3.2 and for fixed $r > 0$, the integrated GJR-COGARCH process $\{(G_{ri}^{(r)})^2\}_{i\in\mathbb{N}}$ has the autocorrelation structure of an ARMA(1,1) process (see, e.g. [5, Exercise 3.16]). For the COGARCH model, this was also noted in [11, Lemma 2.1]. Because of this, $\mathbb{E}[(G_t^{(r)})^4]$ cannot be deduced from (3.3), which holds only for $h \geq r$.

3.2. Method of moments

Our aim now is to estimate the model parameters $(\theta, \eta, \varphi, \gamma)$ from a sample of equally spaced returns over time intervals of length Δ. For $i \in \mathbb{N}$, we denote the stationary increment process of the integrated GJR-COGARCH process (cf. (3.1)) by

$$G_i := G_{i\Delta}^{(\Delta)} = G_{(i+1)\Delta} - G_{i\Delta}. \tag{3.4}$$

The following is the main result of this section and relates the moments of the observed increments of the integrated GJR-COGARCH process to its parameters.

Theorem 3.1. *Let L be a pure-jump Lévy process with finite fourth moment, $\mathbb{E}[L_1] = 0$, $\mathbb{E}[L_1^2] = 1$, and Lévy measure ν_L such that $\int x^3\, \nu_L(\mathrm{d}x) = 0$ and $S := \int x^4\, \nu_L(\mathrm{d}x)$ is known. Assume that $\Psi(2) < 0$. Let the stationary increment process of the integrated GJR-COGARCH process with parameters θ, η, φ, and γ be defined by (3.4). Let μ, Γ, k, and p be positive constants such that*

$$\mathbb{E}[G_i^2] = \mu, \qquad \mathrm{var}(G_i^2) = \Gamma, \qquad \mathrm{corr}(G_i^2, G_{i+h}^2) = k\mathrm{e}^{-hp\Delta}, \quad h \in \mathbb{N}.$$

Set

$$M_1 := \Gamma - \frac{6k\Gamma}{E}(p\Delta - 1 + e^{-p\Delta}) - 2\mu^2, \qquad M_2 := 1 - \frac{\mu^2 S}{M_1 \Delta}, \qquad M_3 := \frac{k\Gamma p^2 S\Delta}{M_1 E},$$

where $E := (1 - e^{-p\Delta})(e^{p\Delta} - 1)$. *Then* $M_1, M_2, M_3 > 0$. *Furthermore, set*

$$\tilde{\gamma}_{1,2} := \frac{-M_3 - 4pS}{2pS - M_2} \pm \frac{\sqrt{8pSM_2^2 M_3 + 32p^2 S^2 M_2^2 + 2pSM_2 M_3^2 - 8pSM_2^3}}{M_2(2pS - M_2)} \in \mathbb{R}. \quad (3.5)$$

For $i = 1, 2$, *define additionally* $H_i := \tilde{\gamma}_i^2 + 4\tilde{\gamma}_i - 4$ *and*

$$M_4^i := \frac{p^2}{\tilde{\gamma}_i^2} + \frac{2k\Gamma p^3 \Delta}{\tilde{\gamma}_i M_1 E H_i},$$

and choose the unique $\tilde{\gamma} \in \{\tilde{\gamma}_i, i = 1, 2\}$ *such that* $M_4^i > 0$ *and*

$$\sqrt{M_4^i} H_i S \tilde{\gamma}_i = -M_2 \tilde{\gamma}_i^2 + M_3 \tilde{\gamma}_i + H_i Sp. \quad (3.6)$$

Then $\tilde{\gamma} \in [1, 2)$ *and the parameters* θ, η, φ, *and* γ *are uniquely determined by*

$$\theta = \frac{p\mu}{\Delta}, \quad \eta = p + \varphi\tilde{\gamma}, \quad \varphi = -\frac{p}{\tilde{\gamma}} + \sqrt{\frac{p^2}{\tilde{\gamma}^2} + \frac{2k\Gamma p^3 \Delta}{\tilde{\gamma} M_1 E(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)}}, \quad \gamma = \sqrt{\tilde{\gamma} - 1}.$$

Proof. It follows readily from Proposition 3.2 that

$$\mu = \frac{\theta\Delta}{|\Psi(1)|},$$

$$\Gamma = \frac{6\theta^2}{|\Psi(1)|^2}\left(\frac{2\eta}{\varphi} - (1 + \gamma^2)\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\left(1 - \frac{1 - e^{-|\Psi(1)|\Delta}}{|\Psi(1)|\Delta}\right)\Delta$$

$$+ \frac{2\theta^2}{\varphi^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)\frac{\Delta}{1 + 6\gamma^2 + \gamma^4} + \frac{2\theta^2}{|\Psi(1)|^2}\Delta^2$$

$$:= \theta^2\tilde{\Gamma}, \quad (3.7)$$

$$p = |\Psi(1)|,$$

$$k = \frac{\tilde{\Gamma}^{-1}}{|\Psi(1)|^3}\left(\frac{2\eta}{\varphi} - (1 + \gamma^2)\right)\left(\frac{2}{|\Psi(2)|} - \frac{1}{|\Psi(1)|}\right)(1 - e^{-p\Delta})(e^{p\Delta} - 1), \quad (3.8)$$

from which we immediately obtain the stated formula for θ. Furthermore, setting $\tilde{\gamma} := 1 + \gamma^2$ we obtain the formula for γ. Also, from (2.6) we observe that $p = |\Psi(1)| = \eta - \varphi\tilde{\gamma}$, which yields the given formula for η, while, by (2.7),

$$|\Psi(2)| = -\Psi(2) = -2\Psi(1) - \varphi^2(1 + 6\gamma^2 + \gamma^4)S = 2p - \varphi^2(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)S.$$

Replacing θ, γ, $|\Psi(1)|$, and $|\Psi(2)|$ in (3.7) and (3.8), we hence obtain

$$\Gamma = \frac{6\mu^2}{\Delta}\left(\frac{2\eta}{\varphi} - \tilde{\gamma}\right)\left(\frac{2}{2p - \varphi^2(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)S} - \frac{1}{p}\right)\left(1 - \frac{1 - e^{-p\Delta}}{p\Delta}\right)$$

$$+ \frac{2p^2\mu^2}{\varphi^2\Delta}\left(\frac{2}{2p - \varphi^2(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)S} - \frac{1}{p}\right)\frac{1}{\tilde{\gamma}^2 + 4\tilde{\gamma} - 4} + 2\mu^2,$$

$$k = \frac{\tilde{\Gamma}^{-1}}{p^3}\left(\frac{2\eta}{\varphi} - \tilde{\gamma}\right)\left(\frac{2}{2p - \varphi^2(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)S} - \frac{1}{p}\right)(1 - e^{-p\Delta})(e^{p\Delta} - 1). \quad (3.9)$$

Substituting from the second equation into the first yields

$$\Gamma = 6k\Gamma(p\Delta - 1 + e^{-p\Delta})E^{-1} + \frac{2k\Gamma p^3\Delta}{\varphi^2}\left(\frac{2\eta}{\varphi} - \tilde{\gamma}\right)^{-1}\frac{1}{(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)E} + 2\mu^2,$$

and, hence, replacing η also

$$M_1 := \Gamma - \frac{6k\Gamma}{E}(p\Delta - 1 + e^{-p\Delta}) - 2\mu^2 = \frac{2k\Gamma p^3\Delta}{\varphi}\frac{1}{(2p + \varphi\tilde{\gamma})(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)E},$$

i.e.

$$\varphi^2\tilde{\gamma}M_1 + 2\varphi p M_1 - \frac{2k\Gamma p^3\Delta}{(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)E} = 0. \tag{3.10}$$

Note that $M_1 > 0$ since inserting (3.7) and (3.8) into the definition of M_1 and using (2.7) we see that

$$M_1 = \frac{2p^2\mu^2}{\varphi^2\Delta^2}\left(\frac{2}{|\Psi(2)|} - \frac{1}{p}\right)\frac{\Delta}{1 + 6\gamma^2 + \gamma^4} = \frac{2\theta^2 S\Delta}{|\Psi(1)|\,|\Psi(2)|} > 0. \tag{3.11}$$

Hence, by (3.10), it follows that

$$\varphi = -\frac{p}{\tilde{\gamma}} \pm \sqrt{\frac{p^2}{\tilde{\gamma}^2} + \frac{2k\Gamma p^3\Delta}{\tilde{\gamma}M_1 E(\tilde{\gamma}^2 + 4\tilde{\gamma} - 4)}} =: -\frac{p}{\tilde{\gamma}} \pm \sqrt{M_4}.$$

Since M_1 and E are positive and $\tilde{\gamma} \geq 1$, we see that M_4 is also positive and, in particular, $\sqrt{M_4} > p/\tilde{\gamma}$. Since, by definition, $\varphi > 0$, this yields the given formula for φ in terms of p, μ, k, Γ, and $\tilde{\gamma}$.

It remains to determine $\tilde{\gamma}$. For this, we return to (3.9) which leads via simple but lengthy algebra to

$$0 = \varphi^2\left(HSk\Gamma + \frac{\mu^2}{p^2\Delta^2}\tilde{\gamma}HSE\right) + \varphi\frac{2\mu^2}{p\Delta^2}HSE - 2pk\Gamma,$$

where $H := \tilde{\gamma}^2 + 4\tilde{\gamma} - 4$. Substituting our expression for φ gives

$$0 = -2pk\Gamma + 2\frac{HSE\mu^2}{p\Delta^2}\left(\sqrt{M_4} - \frac{p}{\tilde{\gamma}}\right) + \left(HSk\Gamma + \frac{\tilde{\gamma}HSE\mu^2}{p^2\Delta^2}v\right)\left(\sqrt{M_4} - \frac{p}{\tilde{\gamma}}\right)^2,$$

an equation which already determines $\tilde{\gamma}$. Further reordering, inserting the expression for M_4, and summarizing yields

$$\sqrt{M_4}\frac{HS}{\tilde{\gamma}} = -M_2 + HS\frac{p}{\tilde{\gamma}^2} + M_3\frac{1}{\tilde{\gamma}},$$

and, hence, (3.6). Taking squares on both sides and inserting the expression for M_4 now leads to a quadratic equation whose solutions are given by (3.5). Here the positivity of M_3 is obvious while the positivity of M_2 follows via (3.11) since

$$M_2 = 1 - \frac{\mu^2 S}{M_1\Delta} = 1 - \frac{\mu^2|\Psi(1)||\Psi(2)|}{2\theta^2\Delta^2} = 1 - \frac{|\Psi(2)|}{2|\Psi(1)|} = 1 - \frac{2p - \varphi^2 HS}{2p} = \frac{\varphi^2 HS}{2p} > 0.$$

In particular, when taken together with (3.11) and (3.9), this yields

$$M_3 = \frac{k\Gamma p^2 S\Delta}{M_1 E} = \frac{k\Gamma p^3|\Psi(2)|}{2\theta^2 E} = \left(\frac{2\eta}{\varphi} - \tilde{\gamma}\right)\left(1 - \frac{|\Psi(2)|}{2p}\right) = \left(\frac{2p}{\varphi} + \tilde{\gamma}\right)M_2 \geq M_2$$

since $\tilde{\gamma} \geq 1$. This implies that the expression under the square root in (3.5) is positive, since

the first term under the root has a larger absolute value than the fourth term. In particular, (3.5) leads to two real-valued solutions from which $\tilde{\gamma}$ can be determined via (3.6).

Remarks 3.2. (a) For the symmetric COGARCH model for which $\gamma = 0$, Theorem 3.1 reduces to [11, Theorem 1].

(b) Since the GJR-COGARCH volatility $\{\sigma_t^2\}_{t \geq 0}$ is a generalized Ornstein–Uhlenbeck process, it follows from [8, Proposition 3.4] that it is exponentially β-mixing. For strictly stationary $\{\sigma_t^2\}_{t \geq 0}$, this then implies that the return process $\{G_t^{(\Delta)}\}_{t \geq 0}$ as defined in (3.1) is ergodic. Then, by Birkhoff's ergodic theorem, strong consistency of the empirical moments and autocorrelation function follows. As shown in Theorem 3.1, the parameter vector $(\theta, \eta, \varphi, \gamma)$ is a continuous function of the first two moments of the GJR-COGARCH process and of the parameters p and k of the autocorrelation function. Hence, consistency of the moments implies consistency of the estimates for $(\theta, \eta, \varphi, \gamma)$ (cf. Remark 3.2, Theorem 3, and Corollary 1 of [11]).

(c) Prediction-based estimation methods for the COGARCH process, which involve even higher-order moments, are presented in [3].

(d) Finally, we discuss the choice of $S = \int x^4 \, \nu_L(\mathrm{d}x)$. In principle, there exist two possibilities: the first assumes that the driving Lévy process is known (as done in [3], where a simple variance gamma process was taken), and the second mimics pseudo maximum likelihood estimation (PMLE) and assumes normality of the increments (regardless of the true, but unknown driving process). In a simulation study of the symmetric COGARCH model, performed in [3], the MLEs based on the true variance gamma driving Lévy process showed a visible bias. The same effect has been observed and analysed for discrete-time heteroscedastic models in [26, Section 6.2.2], and exemplified for Laplace distributed noise in [26, Figure 6.2]. On the other hand, for discrete-time heteroscedastic models, PMLEs lead to consistent and asymptotically normal estimators; cf. [26, Chapter 5]. Based on this insight for discrete-time heteroscedastic models we vote for the second option and recommend a 'pseudo MoM' model setting $\mathbb{E}[(L_1)^4] = 3$ in (3.2) corresponding to the value for the normal distribution.

4. Pseudo MLE for GJR-COGARCH processes

In [18], the authors presented a first-jump approximation of the COGARCH(1, 1) process, and in [25] this approach was generalized to solutions of Lévy-driven stochastic differential equations. The results in [18] show explicitly how to construct a sequence of GARCH(1, 1) processes converging to the COGARCH(1, 1) process in probability in the Skorokhod topology. The benefit of this approximation is threefold. First, we obtain an alternative to the MoM estimation as we can perform pseudo maximum likelihood estimation (PMLE); second, it makes it possible to use GARCH software for the estimation of the COGARCH model parameters; and third, estimation can be based on tick-by-tick data observed on a nonequidistant grid.

4.1. First-jump approximation of a GJR-COGARCH process

Recall the Skorokhod J_1-distance on the space $\mathbb{D}^d[0, T]$ of $\mathbb{R}^d$-valued, càdlàg functions, indexed by $[0, T] \subset \mathbb{R}_+$ and given by

$$\rho_d(U, V) = \inf_{\lambda \in \Lambda} \left\{ \sup_{0 \leq t \leq T} \|U_t - V_{\lambda(t)}\| + \sup_{0 \leq t \leq T} |\lambda(t) - t| \right\}$$

for two processes U and V in $\mathbb{D}^d[0, T]$, where Λ is the set of all increasing, continuous functions with $\lambda(0) = 0$ and $\lambda(T) = T$.

Now let $0 = t_0(n) < t_1(n) < \cdots < t_{N_n}(n) = T$ be a sequence of partitions of the time interval $[0, T]$ such that $\lim_{n\to\infty} N_n = \infty$ and $\Delta t(n) := \max_{i=1,\dots,N_n} \Delta t_i(n) \to 0$ as $n \to \infty$, where $\Delta t_i(n) := t_i(n) - t_{i-1}(n)$. For each $n \in \mathbb{N}$, we define the discrete-time processes $\{G_{i,n}\}_{i=1,\dots,N_n}$ and $\{\sigma_{i,n}^2\}_{i=1,\dots,N_n}$ recursively via

$$G_{i,n} = G_{i-1,n} + \sigma_{i-1,n}\sqrt{\Delta t_i(n)}\,\varepsilon_{i,n}, \qquad i = 1, 2, \dots, N_n, \tag{4.1}$$

$$\sigma_{i,n}^2 = \theta \Delta t_i(n) + (1 + [(1 - \gamma)^2 \, \mathbf{1}_{\{\varepsilon_{i-1,n} \geq 0\}} + (1 + \gamma)^2 \, \mathbf{1}_{\{\varepsilon_{i-1,n} < 0\}}] \varphi \Delta t_i(n) \varepsilon_{i-1,n}^2)$$
$$\times \, \mathrm{e}^{-\eta \Delta t_i(n)} \sigma_{i-1,n}^2, \tag{4.2}$$

with $G_{0,n} = G(0) = 0$. The innovation sequences $\{\varepsilon_{i,n}\}_{i=1,\dots,N_n}$ for $n \in \mathbb{N}$ are constructed via a first-jump approximation of the driving Lévy process L.

Observe that, by setting $Y_{i,n} := G_{i,n} - G_{i-1,n}$, (4.2) is equivalent to

$$\sigma_{i,n}^2 = \theta \Delta t_i(n) + \mathrm{e}^{-\eta \Delta t_i(n)} \sigma_{i-1,n}^2 + \varphi \mathrm{e}^{-\eta \Delta t_i(n)} (|Y_{i,n}| - \gamma Y_{i,n})^2.$$

Hence, (4.1) and (4.2) describe a recursion of a GJR-GARCH process. In particular, for equidistant time steps, a reparameterisation shows the equivalence of (4.1) and (4.2) to (2.1).

To construct the innovations $\{\varepsilon_{i,n}\}_{i=1,\dots,N_n}$ in (4.1), let $\{m(n)\}_{n\in\mathbb{N}}$ be a positive, decreasing sequence that converges to 0 and which is bounded above by 1. Assume that

$$\lim_{n\to\infty} \Delta t_i(n)[\nu_L(\{|x| \geq m(n)\})]^2 = 0,$$

and define, for all $n \in \mathbb{N}$,

$$\tau_{i,n} := \inf\{t: t_{i-1}(n) < t \leq t_i(n), \, |\Delta L_t| > m(n)\} \quad \text{for all } i = 1, \dots, N_n,$$

while $\tau_{i,n} := +\infty$ if L has no jump larger than $m(n)$ in the interval $(t_{i-1}(n), t_i(n)]$. Then define

$$\varepsilon_{i,n} = \frac{\mathbf{1}_{\{\tau_{i,n} < \infty\}} \Delta L_{\tau_{i,n}} - \mu_i(n)}{\xi_i(n)}, \qquad i = 1, 2, \dots, N_n,$$

where $\mu_i(n)$ and $\xi_i^2(n)$ denote the (finite) expectation and variance of the i.i.d. random variables $\{\mathbf{1}_{\{\tau_{i,n} < \infty\}} \Delta L_{\tau_{i,n}}\}_{i=1,\dots,N_n}$.

The discrete-time processes $\sigma_{\cdot,n}^2$ and $G_{\cdot,n}$ as in (4.1) can now be embedded in a continuous-time setting by taking $\{\sigma_n^2(t)\}_{t\geq 0}$ and $\{G_n(t)\}_{t\geq 0}$ as

$$\sigma_n^2(t) := \sigma_{i,n}^2 \quad \text{and} \quad G_n(t) := G_{i,n} \quad \text{for all } t \in [t_{i-1}(n), t_i(n)), \, 0 \leq t \leq T, \tag{4.3}$$

with $G_n(0) = 0$. This enables us to formulate the main result of this section.

Theorem 4.1. *Define the bivariate processes (σ^2, G) as in (2.3) and (2.2), and $\{(\sigma_n^2, G_n)\}_{n\in\mathbb{N}}$ as in (4.3). Then*

$$\lim_{n\to\infty} \rho_2((\sigma_n^2, G_n), (\sigma^2, G)) = 0 \quad \text{in probability.} \tag{4.4}$$

Proof. The long and technical proof of Theorem 4.1 can be carried out along the lines of the proof of [18, Theorem 2.1], replacing $(\Delta L_{\tau_{i,n}})^2$ there by $h(\Delta L_{\tau_{i,n}})$ with h as in (2.3); for details, see [19]. □

4.2. PMLE

In this section we extend the PMLE method for COGARCH processes from Maller *et al.* [18]. As in the MoM, we aim at estimation of the model parameters $(\theta, \eta, \varphi, \gamma)$ where, other

than for the MoM, we allow for unequally spaced returns as observations. The basic idea is to replace the unknown likelihood by a corresponding normal likelihood; in our case we assume that the increments of the integrated GJR-COGARCH process are normally distributed.

Assume that we are given observations G_{t_i} of the integrated GJR-COGARCH process as in (2.2) and (2.3) at fixed (nonrandom) times $0 = t_0 < t_1 < \cdots < t_N = T$. Denote the observed returns by $Y_i := G_{t_i} - G_{t_{i-1}}$ and the time steps by $\Delta t_i := t_i - t_{i-1}$ for $i = 1, \ldots, N$. We also assume $\{Y_i\}$ to be a stationary process (in discrete time). Then

$$Y_i = \int_{t_{i-1}}^{t_i} \sigma_{s-} \, \mathrm{d}L_s$$

for a Lévy process L with $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1^2] = 1$. Denote by $\mathcal{F}_{t_{i-1}}$ the σ-algebra generated by $\{Y_k \colon k \leq i-1\}$. Then the returns Y_i are conditionally independent of $Y_{i-1}, Y_{i-2}, \ldots$, given $\mathcal{F}_{t_{i-1}}$, since $\{\sigma_t^2\}_{t \geq 0}$ is a Markov process (see, e.g. [2, Lemma 3.3]). In particular, by independence of the Lévy increments, we have

$$\mathbb{E}[Y_i \mid \mathcal{F}_{t_{i-1}}] = \mathbb{E}[Y_i] = \mathbb{E}[G_{t_i} - G_{t_{i-1}}] = 0,$$

while one deduces similarly as in the proof of Equation (5.4) of [15],

$$\rho_i^2 := \mathbb{E}[Y_i^2 \mid \mathcal{F}_{t_{i-1}}] = \mathbb{E}[L_1^2]\left[(\sigma_{t_{i-1}}^2 - \mathbb{E}[\sigma_0^2]) \frac{\mathrm{e}^{-\Delta t_i \Psi(1)} - 1}{-\Psi(1)} + \mathbb{E}[\sigma_0^2]\Delta t_i \right], \tag{4.5}$$

with $\Psi(1)$ as in (2.6). Moreover, noting the stationarity assumption and Proposition 3.1, set

$$\mathbb{E}[\sigma_0^2] = \frac{\theta}{-\Psi(1)} = \frac{\theta}{\eta - \varphi(1 + \gamma^2)}.$$

Inserting this into (4.5) yields

$$\rho_i^2 = \left(\sigma_{t_{i-1}}^2 - \frac{\theta}{\eta - \varphi(1 + \gamma^2)} \right) \frac{\mathrm{e}^{\Delta t_i [\eta - \varphi(1 + \gamma^2)]} - 1}{\eta - \varphi(1 + \gamma^2)} + \frac{\theta}{\eta - \varphi(1 + \gamma^2)} \Delta t_i. \tag{4.6}$$

To apply PMLE, we assume that the returns Y_i are conditionally normally distributed with expectation 0 and variance ρ_i^2 given in (4.6). The occurring sequence $\{\sigma_{t_i}^2\}_{i=1,\ldots,N_n}$ can be iterated starting from $\sigma_0 := \mathbb{E}[\sigma_0] = \theta/(\eta - \varphi(1 + \gamma^2))$ and using the observations $Y_0, \ldots, Y_N$ via the first-jump approximation model (4.1) and (4.2), i.e.

$$Y_i = \sigma_{i-1}\sqrt{\Delta t_i}\, \varepsilon_{i-1} \qquad \sigma_{t_i}^2 = \theta \Delta t_i + \mathrm{e}^{-\eta \Delta t_i}\sigma_{t_{i-1}}^2 + \varphi \mathrm{e}^{-\eta \Delta t_i}(|Y_{i-1}| - \gamma Y_{i-1})^2,$$

which is of the form (2.1) with parameters $\theta \Delta t_i$, $\alpha = \varphi \mathrm{e}^{-\eta \Delta t_i}$, and $\beta = \mathrm{e}^{-\eta \Delta t_i}$.

Then we obtain as the pseudo maximum likelihood the function

$$\mathcal{L}_N = \mathcal{L}_N(\theta, \varphi, \eta, \gamma)$$

$$= \log\left(\prod_{i=1}^{N} \frac{1}{\rho_i \sqrt{2\pi}} \exp\left\{ -\frac{1}{2}\left(\frac{Y_i}{\rho_i} \right)^2 \right\} \right)$$

$$= -\frac{1}{2}\sum_{i=1}^{N} \log(\rho_i^2) - \frac{1}{2}N \log(2\pi) - \frac{1}{2}\sum_{i=1}^{N} \frac{Y_i^2}{\rho_i^2}.$$

We can now use standard algorithms to obtain the PMLEs as

$$\text{argmin}_{\theta,\varphi,\eta,\gamma} \sum_{i=1}^{N}\left(\frac{Y_i^2}{\rho_i^2} + \log(\rho_i^2)\right),$$

or special algorithms designed for GARCH and GJR-GARCH models (cf. [4, 6, 23] or [26, Chapter 5]).

Remark 4.1. Consistency and asymptotic normality under certain regularity conditions have been proved for the GARCH and asymmetric GARCH models in [21], and for the symmetric COGARCH model in [14]. Consequently, such results are also expected to hold for the GJR-COGARCH model.

5. Conclusion

Extending the COGARCH(1, 1) model to an asymmetric GRJ-COGARCH(1, 1) model allowed us to capture the observed asymmetry in financial data. Under stationarity conditions, we calculated up to four moments and the covariance function of the squared returns of the integrated process. Matching the analytical and empirical moments for the GJR-COGARCH model is by no means standard and involved the complex calculations given in Section 3.2. This method needs equidistant data.

We also derived the first-jump approximation of the GRJ-COGARCH model and proved convergence in probability in the Skorokhod topology. The PMLE based on normality of the returns is derived in Section 4.2 and is the basis for the use of algorithms developed for the discrete-time GJR-GARCH model. This method has the advantage that it also applies to irregularly spaced data.

Acknowledgements

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