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HERMITE–HADAMARD TYPE INEQUALITIES FOR FUNCTIONS WHEN A POWER OF THE ABSOLUTE VALUE OF THE FIRST DERIVATIVE IS *P*-CONVEX

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Abstract

In this paper we extend some estimates of the right-hand side of a Hermite–Hadamard type inequality for functions whose derivatives' absolute values are *P*-convex. Applications to the trapezoidal formula and special means are introduced.

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1. Introduction

Let $I = [c, d]$ be an interval on the real line $\mathbb{R}$, let $f : I \to \mathbb{R}$ be a convex function and let $a, b \in [c, d]$, $a < b$. We consider the well-known Hermite–Hadamard inequality

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both inequalities hold in the reverse direction if f is concave (see [12]). The classical Hermite–Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. We note that the Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity, as follows easily from Jensen's inequality. The Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalisations has been found; see, for example, [4–7] and references therein.

Dragomir and Agarwal in [9] used the following lemma to prove Theorems 1.2 and 1.3.

Lemma 1.1. *The following equation holds true:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)\,dt.$$

Theorem 1.2. *Assume that* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is a differentiable function on* (a, b). *If* $|f'|$ *is convex on* $[a, b]$ *then the following inequality holds true:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.3. *Assume that* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is a differentiable function on* (a, b). *Assume that* $p \in \mathbb{R}$ *with* $p > 1$. *If* $|f'|^{p/(p-1)}$ *is convex on* $[a, b]$ *then the following inequality holds true:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \le \frac{b-a}{2(p+1)^{1/p}} \cdot \left(\frac{|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}}{2}\right)^{(p-1)/p}.\end{aligned}$$

In [12] Pečarić *et al.* proved the following theorem.

Theorem 1.4. *Let* $f : I \to \mathbb{R}$ *be a differentiable function on* I°, $a, b \in I^\circ$ *with* $a < b$. *If* $|f'|^q$ *is convex on* $[a, b]$ *for* $q \ge 1$, *then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{1/q}.$$

Recall that the function $f : [a, b] \to \mathbb{R}$ is said to be quasiconvex if, for every $x, y \in [a, b]$,

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\} \quad \text{for all } t \in [0, 1].$$

Ion in [10] presented some estimates of the right-hand side of a Hermite–Hadamard type inequality in which some quasiconvex functions are involved. The main results of [10] are given by the following theorems.

Theorem 1.5. *Assume that* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is a differentiable function on* (a, b). *If* $|f'|$ *is quasiconvex on* $[a, b]$ *then the following inequality holds true:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)\max\{|f'(a)|, |f'(b)|\}}{4}.$$

Theorem 1.6. *Assume that* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *is a differentiable function on* (a, b). *Assume that* $p \in \mathbb{R}$ *with* $p > 1$. *If* $|f'|^{p/(p-1)}$ *is quasiconvex on* $[a, b]$ *then the following inequality holds true:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ \le \frac{b-a}{2(p+1)^{1/p}}(\max\{|f'(a)|^{p/(p-1)}, |f'(b)|^{p/(p-1)}\})^{(p-1)/p}.\end{aligned}$$

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On the other hand, Dragomir *et al.* in [7] defined the following class of functions.

DEFINITION 1.7. Let $I \subseteq \mathbb{R}$ be an interval. The function $f : I \to \mathbb{R}$ is said to belong to the class $P(I)$ (or to be P-convex) if it is nonnegative and, for all $x, y \in I$ and $\lambda \in [0, 1]$, satisfies the inequality

$$f(\lambda x + (1 - \lambda)y) \le f(x) + f(y).$$

Note that $P(I)$ contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category; see [1, 8, 11, 13] and references therein.

The main purpose of this paper is to establish new estimations and refinements of the Hermite–Hadamard inequality (1.1) for functions whose derivatives in absolute value are P-convex. Applications to the trapezoidal formula and special means are introduced.

2. Hermite–Hadamard type inequality

In this section we generalise Theorems 1.4–1.6 with a P-convex function setting.

The next theorem gives a new result for the upper Hermite–Hadamard inequality for P-convex functions.

THEOREM 2.1. *Let $f : I \to \mathbb{R}$ be a differentiable function on I° such that the function $|f'|$ is P-convex. Suppose that $a, b \in I$ with $a < b$ and $f' \in L_1[a, b]$. Then the following inequality holds:*

$$\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,dx\right| \le \frac{(b - a)(|f'(a)| + |f'(b)|)}{4}.$$

PROOF. By Lemma 1.1,

$$\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,dx = \frac{b - a}{2}\int_0^1 (1 - 2t)f'(ta + (1 - t)b)\,dt. \tag{2.1}$$

Since $|f'|$ is P-convex, by (2.1),

$$\begin{aligned}\left|\frac{f(a) + f(b)}{2} - \frac{1}{b - a}\int_a^b f(x)\,dx\right| &= \left|\frac{b - a}{2}\int_0^1 (1 - 2t)f'(ta + (1 - t)b)\,dt\right| \\ &\le \frac{(b - a)}{2}\int_0^1 |1 - 2t||f'(ta + (1 - t)b)|\,dt \\ &= \frac{(b - a)(|f'(a)| + |f'(b)|)}{4}.\end{aligned}$$

This completes the proof. □

The corresponding version for powers of the absolute value of the derivative is incorporated in the following result.

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THEOREM 2.2. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°. Assume that $p \in \mathbb{R}$, $p > 1$, is such that the function $|f'|^{p/(p-1)}$ is P-convex. Suppose that $a, b \in I$ with $a < b$ and $f' \in L_1[a, b]$. Then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)(|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)})^{(p-1)/p}}{2(p+1)^{1/p}}.$$

PROOF. Suppose that $a, b \in I$. By assumption, Hölder's inequality and Lemma 1.1,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|\,dt \\ &\le \frac{b-a}{2}\left(\int_0^1 |1-2t|^p\,dt\right)^{1/p}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \\ &= \frac{b-a}{2(p+1)^{1/p}}\left(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \\ &\le \frac{b-a}{2(p+1)^{1/p}}\left(\int_0^1 (|f'(a)|^q + |f'(a)|^q)\,dt\right)^{1/q} \\ &= \frac{b-a}{2(p+1)^{1/p}}(|f'(a)|^q + |f'(a)|^q)^{1/q},\end{aligned}$$

where $q := p/(p-1)$ and since $\int_0^1 |1-2t|^p\,dt = 1/(p+1)$. □

A more general inequality using Lemma 1.1 is as follows.

THEOREM 2.3. *Let $f : I \to \mathbb{R}$ be a differentiable function on I°. Assume that $q \in \mathbb{R}$, $q > 1$, is such that $|f'|^q$ is a P-convex function. Suppose that $a, b \in I$ with $a < b$ and $f' \in L_1[a, b]$. Then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{4}(|f'(a)|^q + |f'(a)|^q)^{1/q}.$$

PROOF. Suppose that $a, b \in I^\circ$. By the P-convexity of f and Lemma 1.1, and using the well-known power mean inequality,

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\le \frac{b-a}{2}\int_0^1 |1-2t||f'(ta+(1-t)b)|\,dt\end{aligned}$$

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$$\le \frac{b-a}{2}\Big(\int_0^1 |1-2t|^p\,dt\Big)^{1-1/q}\Big(\int_0^1 |1-2t||f'(ta+(1-t)b)|^q\,dt\Big)^{1/q}$$
$$= \frac{b-a}{4}\Big(\int_0^1 |f'(ta+(1-t)b)|^q\,dt\Big)^{1/q}$$
$$\le \frac{b-a}{4}(|f'(a)|^q + |f'(a)|^q)^{1/q}.$$

This completes the proof. □

3. An extension to functions of several variables

In this section some Hermite–Hadamard inequalities for functions of several variables on convex subsets of $\mathbb{R}^n$ will be given. First we introduce the notion of P-convexity for functions on a convex subset of $\mathbb{R}^n$.

Definition 3.1. The function $f : U \to \mathbb{R}$ is said to be P-convex on U if it is nonnegative and, for all $x, y \in U$ and $\lambda \in [0, 1]$, satisfies the inequality

$$f(\lambda x + (1-\lambda)y) \le f(x) + f(y).$$

The following proposition will be used throughout this section.

Proposition 3.2. *Let $U \subseteq \mathbb{R}^n$ be a convex subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be a function. Then f is P-convex on U if and only if, for every $x, y \in U$, the function $\varphi : [0, 1] \to \mathbb{R}$, defined by*

$$\varphi(t) := f((1-t)x + ty),$$

is P-convex on I with $I = [0, 1]$.

Proof. Let $x, y \in U$ be fixed. Assume that the function φ is P-convex on I with $I = [0, 1]$. Suppose that $\lambda \in [0, 1]$. Then

$$f((1-\lambda)x + \lambda y) = \varphi(\lambda) = \varphi((1-\lambda)\cdot 0 + \lambda \cdot 1)$$
$$\le \varphi(0) + \varphi(1) = f(x) + f(y).$$

It follows that f is P-convex on U. Conversely, let f be P-convex on U. Fix $x, y \in U$ and $t_1, t_2 \in [0, 1]$. Set $z_1 := (1-t_1)x + t_1 y$ and $z_2 := (1-t_2)x + t_2 y$. Then, for every $\lambda \in [0, 1]$,

$$\begin{aligned}\varphi((1-\lambda)t_1 + \lambda t_2) &= f([1-(1-\lambda)t_1 - \lambda t_2]x + [(1-\lambda)t_1 - \lambda t_2]y)\\ &= f((1-\lambda)z_1 + \lambda z_2)\\ &\le f(z_1) + f(z_2)\\ &= \varphi(t_1) + \varphi(t_2).\end{aligned}$$

Therefore, φ is P-convex on I with $I = [0, 1]$. □

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The following theorem is a generalisation of [10, Proposition 1].

THEOREM 3.3. *Let $U \subseteq \mathbb{R}^n$ be an open convex subset of $\mathbb{R}^n$. Assume that $f : U \to \mathbb{R}$ is a differentiable P-convex function on U. Then, for every $x, y \in S$ and every $a, b \in [0, 1]$ with $a < b$, the following inequality holds:*

$$\begin{aligned} &\bigg|\frac{1}{2}\int_0^a f((1-s)x+sy)\,ds + \frac{1}{2}\int_0^b f((1-s)x+sy)\,ds \\ &\quad - \frac{1}{b-a}\int_a^b \Big(\int_0^s f((1-\theta)x+\theta y)\,d\theta\Big)\,ds\bigg| \qquad (3.1)\\ &\quad\leq \frac{b-a}{4}\max\{f((1-a)x+ay), f((1-b)x+by)\}. \end{aligned}$$

PROOF. Let $x, y \in S$ and $a, b \in (0, 1)$ with $a < b$. Since f is a P-convex function, by Proposition 3.2 the function $\varphi : [0, 1] \to \mathbb{R}^+$ defined by

$$\varphi(t) := f((1-t)x+ty)$$

is P-convex on I with $I = [0, 1]$. Define the function $\phi : [0, 1] \to \mathbb{R}^+$ by

$$\phi(t) := \int_0^t \varphi(s)\,ds = \int_0^t f((1-s)x+sy)\,ds.$$

Obviously, for every $t \in (0, 1)$,

$$\phi'(t) = \varphi(t) = f((1-t)x+ty) \geq 0.$$

Hence $|\phi'(t)| = \phi'(t)$. Applying Theorem 2.1 to the function ϕ implies that

$$\bigg|\frac{\phi(a)+\phi(b)}{2} - \frac{1}{b-a}\int_a^b \phi(s)\,ds\bigg| \leq \frac{(b-a)[\phi'(a)+\phi'(b)]}{4},$$

and we deduce that (3.1) holds. □

4. Applications to the trapezoidal formula

Assume that Δ is a division of the interval $[a, b]$ such that

$$\Delta : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

For a given function $f : [a, b] \to \mathbb{R}$ we consider the trapezoidal formula

$$T(f, \Delta) = \sum_{i=0}^{n-1} \frac{f(x_i) - f(x_{i+1})}{2}(x_{i+1} - x_i).$$

It is well known that if f is twice differentiable on (a, b) and

$$M = \sup_{x\in(a,b)} |f''(x)| < \infty,$$

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then

$$\int_a^b f(x)\,dx = T(f,\Delta) + E(f,d),$$

where the approximation error $E(f,d)$ of the integral $\int_a^b f(x)\,dx$ by $T(f,\Delta)$ satisfies

$$|E(f,d)| \le \frac{M}{12}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3. \tag{4.1}$$

Clearly, if the function f is not twice differentiable or the second derivative is not bounded on (a,b), then (4.1) does not hold true. In that context, the following result is important in order to obtain some estimates of $E(f,d)$.

THEOREM 4.1. *Let $f : I \to \mathbb{R}$ be a differentiable function such that $|f'|$ is P-convex. Suppose that $a, b \in I$ with $a < b$ and $f' \in L_1[a,b]$. Then, for every division Δ of the interval $[a,b]$,*

$$\left|T(f,\Delta) - \int_a^b f(x)\,dx\right| = \frac{|f'(a)| + |f'(b)|}{2}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2. \tag{4.2}$$

PROOF. Applying Theorem 2.1 on the subinterval $[x_i, x_i]$ $(i = 0, \ldots, n-1)$ of the division Δ and adding from $i = 0$ to $i = n-1$, we deduce that

$$\left|T(f,\Delta) - \int_a^b f(x)\,dx\right| = \frac{1}{4}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2(f'(x_i) + f(x'_{i+1})). \tag{4.3}$$

On the other hand, for every $x_i \in [a,b]$ there exists $\alpha_i \in [0,1]$ such that

$$x_i = \alpha_i a + (1-\alpha_i)b.$$

By the P-convexity of $|f'|$,

$$|f'(x_i)| \le |f'(a)| + |f'(b)|.$$

Thus

$$|f'(x_i) + f(x'_{i+1})| \le 2(|f'(a)| + |f'(b)|). \tag{4.4}$$

Therefore, combining relations (4.3) and (4.4) implies that (4.2) holds true and the proof is complete. □

5. Applications to special means

We now give applications of our theorems to some special means of real numbers. We consider the following means for arbitrary real numbers α, β $(\alpha \ne \beta)$.

(1) Arithmetic mean:

$$A(\alpha,\beta)=\frac{\alpha+\beta}{2},\quad \alpha,\beta\in\mathbb{R}.$$

(2) Logarithmic mean:

$$L(\alpha,\beta)=\frac{\alpha-\beta}{\ln|\alpha|-\ln|\beta|},\quad |\alpha|\neq|\beta|,\ \alpha,\beta\neq 0,\ \alpha,\beta\in\mathbb{R}.$$

(3) Generalised logarithmic mean:

$$L_n(\alpha,\beta)=\Big(\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\Big)^{1/n},\quad n\in\mathbb{N},\ \alpha,\beta\in\mathbb{R},\ \alpha\neq\beta.$$

Using the results of Section 2, we have the following propositions.

Proposition 5.1. *Let $a,b\in\mathbb{R}$, $a<b$, and $n\in\mathbb{N}$, $n\geq 2$. Then*

$$|L_n^n(a,b)-A(a^n,b^n)|\leq\frac{n(b-a)}{4}(|a|^{n-1}+|b|^{n-1}).$$

Proof. The assertion follows from Theorem 2.1 applied to the function $f(x)=x^n$, $x\in\mathbb{R}$, because $|f'|$ is P-convex. □

Proposition 5.2. *Let $a,b\in\mathbb{R}$, where $a<b$ and $0\notin[a,b]$. Then*

$$|L^{-1}(a,b)-A(a^{-1},b^{-1})|\leq\frac{b-a}{4}(|a|^{-2}+|b|^{-2}).$$

Proof. The assertion follows from Theorem 2.1 applied to the function $f(x)=1/x$, $x\in[a,b]$, because $|f'|$ is P-convex. □

Proposition 5.3. *Let $a,b\in\mathbb{R}$, $a<b$, and $n\in\mathbb{N}$, $n\geq 2$. Then, for all $p>1$,*

$$|L_n^n(a,b)-A(a^n,b^n)|\leq\frac{(b-a)(|a|^{-2p/p-1}+|b|^{-2p/p-1})^{(p-1)/p}}{2(p+1)^{1/p}}.$$

Proof. The assertion follows from Theorem 2.2 applied to the function $f(x)=1/x$, $x\in\mathbb{R}$, because for all $p>1$ the function $|f'|^{p/(p-1)}$ is P-convex. □

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