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MONOTONIC NORMS IN ORDERED BANACH SPACES

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Abstract

Let B be an ordered Banach space with ordered Banach dual space. Let N denote the canonical half-norm. We give an alternative proof of the following theorem of Robinson and Yamamuro: the norm on B is α-monotone ($\alpha \geq 1$) if and only if for each f in B^* there exists $g \in B^*$ with $g \geq 0$, f and $||g|| \leq \alpha N(f)$. We also establish a dual result characterizing α-monotonicity of B^*.

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Let $(B, B_+, ||\cdot||)$ be an ordered Banach space with closed cone (thus $a \leq b$ in B means $b - a \in B_+$). Let B^* denote the Banach dual space, ordered by the dual cone $B^*_+ = \{f \in B^*\colon f(a) \geq 0 \text{ for all } a \geq 0\}$. Let N denote the canonical half-norms in B or B^*. Thus

$$\begin{aligned} N(a) &:= \inf\{||b||\colon a \leq b \text{ in } B\} \qquad (a \in B) \\ &= \sup\{f(a)\colon f \in B^*_+,\ ||f|| \leq 1\} \end{aligned}$$

and

$$\begin{aligned} N(f) &:= \inf\{||g||\colon f \leq g \text{ in } B^*\} \qquad (f \in B^*) \\ &= \sup\{f(a)\colon a \in B_+,\ ||a|| \leq 1\}. \end{aligned}$$

The definition of N is given by W. Arendt, P. R. Chernoff and T. Kato in [1], and the characterizations of N are due to D. W. Robinson and S. Yamamuro [6, Theorems 2.1 and 3.5]. Theorem 1 below is essentially known:

(a) It was proved by Robinson and Yamamuro [6, Theorem 3.8] in the special case when $\alpha = 1$.

(b) The equivalence (i) $\leftrightarrow$ (iv) was proved also in [2, Proposition 5]. Our proof given below seems to be more direct.

THEOREM 1. *For $\alpha \geq 1$ the following conditions are equivalent:*

(i) *The norm is α-monotone on B, that is, $0 \leq a \leq b$ in $B \rightarrow ||a|| \leq \alpha||b||$.*

(ii) $||a|| \leq \alpha N(a) + (\alpha+1)N(-a)$ *for all a in B.*

(iii) *For each f in B^* there exists $g \in B^*$ such that $0, f \leq g$ and $||g|| \leq \alpha N(f)$.*

(iv) *For each f in B^* there exists $g \in B^*$ such that $f, 0 \leq g$ and $||g|| \leq \alpha||f||$.*

The verification for (i) $\rightarrow$ (ii) is as in (1) $\rightarrow$ (2) of Theorem 3.8 in [6]. Conversely, suppose (ii) holds and $0 \leq a \leq b$. Then $N(a) \leq ||b||$ and $N(-a) = 0$ by definition of N, so

$$||a|| \leq \alpha N(a) + (\alpha+1)N(-a) \leq \alpha N(a) + 0 \leq \alpha||b||.$$

Therefore (i) $\leftrightarrow$ (ii). That (iii) $\rightarrow$ (iv) is trivial. Assuming (iv), let $f \in B^*$ with $||f|| \leq 1$ and take g as in (iv). Then, whenever $0 \leq a \leq b$ in B, one has

$$f(a) \leq g(a) \leq g(b) \leq \alpha||f||\,||b|| \leq \alpha||b||$$

and it follows from the Hahn-Banach theorem that $||a|| \leq \alpha||b||$. Thus (iv) $\rightarrow$ (i), and it remains to prove (i) $\rightarrow$ (iii). We do this as in [4, Theorem 9.6]. Let $f \in B^*$. We define

$$q(a) = \sup\{f(b)\colon 0 \leq b \leq a\} \qquad (a \in B_+),$$
$$p(a) = \alpha N(f)||a|| \qquad (a \in B).$$

Then q is superlinear, p is sublinear and $q \leq p$ on B_+. By Bonsall's generalization of the Hahn-Banach theorem (see [4, Theorem 1.15]), there exists a linear functional g on$B$ such that $q \leq g$ on B_+ and $g \leq p$ on B. Then, for all $a \in B_+$,

$$0, f(a) \leq q(a) \leq g(a)$$

that is, $0, f \leq g$. Also, $g \leq p = \alpha N(f)||\cdot||$ on B so $||g|| \leq \alpha N(f)$, proving (iii).

REMARK. We have not used the completeness of B, that is, Theorem 1 is valid for ordered normed spaces. However, for the following dual result of Theorem 1, the completeness will be essential. (Again, the equivalence (i) $\leftrightarrow$ (iv) was known in the special case when $\alpha = 1$, see [2, Proposition 6].)

THEOREM 2. *For $\alpha \geq 1$ the following conditions are equivalent:* (i) *The norm is α-monotone on B^*.*

(ii) $||f|| \leq \alpha N(f) + (\alpha+1)N(-f)$.

(iii) *For each a in B and $\varepsilon > 0$ there exists $b \in B$ with $0, a \leq b$ and $||b|| \leq \alpha N(a) + \varepsilon$.*

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(iv) *For each a in B and $\varepsilon > 0$ there exists $b \in B$ with* $0, a \leq b$ *and* $||b|| \leq \alpha||a|| + \varepsilon$.

PROOF. We need only prove (i) $\rightarrow$ (iii) as the other implications can be proved as in Theorem 1, or follow from Theorem 1 (applied to B^* instead of B). Let $a \in B$. Define

$$q(f) = \sup\{g(a) : 0 \leq g \leq f \text{ in } B^*\} \qquad (f \in B^*_+),$$
$$p(f) = \alpha N(a)||f|| \qquad (f \in B^*).$$

Then, as in the proof of [4, Theorem 9.7], we can verify that p and $-q$ are lower semi-continuous sublinear functionals (respectively on B^* and B^*_+) under the w^*-topology. By a result, dual to Bonsall's theorem (see [3, Theorem 3] or [4, Corollary 2.9]), for any $\varepsilon > 0$, there exists b in B such that $q(f) \leq f(b)$ and $g(b) \leq p(g) + \varepsilon||g||$ for all $f \in B^*_+$, and $g \in B^*$. Then $g(b) \leq \alpha N(a)||g|| + \varepsilon||g||$ and it follows from the Hahn-Banach theorem that $||b|| \leq \alpha N(a) + \varepsilon$. Also, for all $f \in B^*_+$, $0, f(a) \leq q(f) \leq f(b)$; since B_+ is assumed to be closed it follows that 0, $a \leq b$.

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