QUASILINEARITY OF SOME COMPOSITE FUNCTIONALS WITH APPLICATIONS

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Abstract

The quasilinearity of certain composite functionals defined on convex cones in linear spaces is investigated. Applications in refining the Jensen, Hölder, Minkowski and Schwarz inequalities are given.

1. Introduction

Let X be a linear space. A subset $C \subseteq X$ is called a *convex cone* in X provided the following conditions hold:

(i) $x, y \in C$ imply $x + y \in C$;

(ii) $x \in C, \alpha \geq 0$ imply $\alpha x \in C$.

A functional $h : C \to \mathbb{R}$ is said to be *superadditive (subadditive)* on C if:

(iii) $h(x + y) \geq (\leq) h(x) + h(y)$ for any $x, y \in C$,

and *nonnegative (strictly positive)* on C if it satisfies:

(iv) $h(x) \geq (>) 0$ for each $x \in C$.

The functional h is *s-positive homogeneous* on C, for a given $s > 0$, if:

(v) $h(\alpha x) = \alpha^s h(x)$ for any $\alpha \geq 0$ and $x \in C$.

In [3], the following result has been obtained.

THEOREM 1.1. *Let $x, y \in C$ and let $h : C \to \mathbb{R}$ be a nonnegative, superadditive and s-positive homogeneous functional on C. If $M \geq m \geq 0$ are such that $x - my$ and $My - x \in C$, then*

$$M^s h(y) \geq h(x) \geq m^s h(y). \tag{1.1}$$

Now, consider $v : C \to \mathbb{R}$ to be an *additive* and *strictly positive* functional on C which is also *positive homogeneous* on C. The following result concerning other bounds for a composite functional may be stated as well [3].

THEOREM 1.2. *Let $x, y \in C$, let h be strictly positive, superadditive and positive homogeneous on C and let v be an additive, strictly positive and positive homogeneous functional on C. If $M \geq m > 0$ are such that $x - my$ and $My - x \in C$, then*

$$\left[\frac{h(y)}{v(y)}\right]^{Mv(y)} \geq \left[\frac{h(x)}{v(x)}\right]^{v(x)} \geq \left[\frac{h(y)}{v(y)}\right]^{mv(y)}. \tag{1.2}$$

As shown in [3], the above results can be applied to obtain refinements of the Jensen, Hölder, Minkowski and Schwarz inequalities for weights satisfying certain conditions.

The main aim of the present paper is to study the quasilinearity properties of other composite functionals and to apply the obtained results to improving some classical inequalities as those mentioned above.

2. General results

The following result provides the quasilinearity property of a composite functional when one of the components is additive while the other is either superadditive or subadditive.

THEOREM 2.1. *Let C be a convex cone in the linear space X and let $v : C \to (0, \infty)$ be an additive functional on C. If $h : C \to [0, \infty)$ is a superadditive (subadditive) functional on C and $p \geq 1$ $(0 < p < 1)$, then the functional*

$$\Psi_p : C \to [0, \infty), \quad \Psi_p(x) = v^{1-1/p}(x)h(x) \tag{2.1}$$

is superadditive (subadditive) on C.

PROOF. First of all, we observe that the elementary inequality

$$(\alpha + \beta)^p \geq (\leq) \alpha^p + \beta^p \tag{2.2}$$

holds for any $\alpha, \beta \geq 0$ and $p \geq 1$ $(0 < p < 1)$.

Indeed, if we consider the function

$$f_p : [0, \infty) \to \mathbb{R}, \quad f_p(t) = (t + 1)^p - t^p$$

we have $f_p'(t) = p[(t + 1)^{p-1} - t^{p-1}]$. Observe that for $p > 1$ and $t > 0$ we have that $f_p'(t) > 0$, which shows that f_p is strictly increasing on the interval $[0, \infty)$. Now for $t = \alpha/\beta$ (where $\beta > 0$ and $\alpha \geq 0$) we have $f_p(t) > f_p(0)$ and consequently $((\alpha/\beta) + 1)^p - (\alpha/\beta)^p > 1$, which is the desired inequality (2.2).

For $p \in (0, 1)$ we have that f_p is strictly decreasing on $[0, \infty)$ which proves the second case in (2.2).

Now, if h is superadditive (subadditive) and $p \geq 1$ $(0 < p < 1)$, then we have by (2.2) that

$$h^p(x + y) \geq (\leq) [h(x) + h(y)]^p \geq (\leq) h^p(x) + h^p(y) \tag{2.3}$$

for all $x, y \in C$.

Utilizing (2.3) and the additivity property of v, we have for any $x, y \in C$ that

$$
\frac{h^p(x+y)}{v(x+y)} \geq (\leq) \frac{h^p(x) + h^p(y)}{v(x) + v(y)}
$$

$$
= \frac{v(x) \cdot \frac{h^p(x)}{v(x)} + v(y) \cdot \frac{h^p(y)}{v(y)}}{v(x) + v(y)} \tag{2.4}
$$

$$
= \frac{v(x) \cdot \left[\frac{h(x)}{v^{1/p}(x)}\right]^p + v(y) \cdot \left[\frac{h(y)}{v^{1/p}(y)}\right]^p}{v(x) + v(y)} =: I.
$$

Since for $p \geq 1$ $(0 < p < 1)$ the power function $g(t) = t^p$ is convex (concave), then

$$
I \geq (\leq) \left[\frac{v(x) \cdot \frac{h(x)}{v^{1/p}(x)} + v(y) \cdot \frac{h(y)}{v^{1/p}(y)}}{v(x) + v(y)}\right]^p
$$

$$
= \left[\frac{h(x) v^{1-1/p}(x) + h(y) v^{1-1/p}(y)}{v(x+y)}\right]^p \tag{2.5}
$$

for any $x, y \in C$.

By combining (2.4) with (2.5) we get

$$
\frac{h^p(x+y)}{v(x+y)} \geq (\leq) \left[\frac{h(x) v^{1-1/p}(x) + h(y) v^{1-1/p}(y)}{v(x+y)}\right]^p
$$

which is equivalent to

$$
\frac{h(x+y)}{v^{1/p}(x+y)} \geq (\leq) \frac{h(x) v^{1-1/p}(x) + h(y) v^{1-1/p}(y)}{v(x+y)}
$$

that is, by multiplying by $v(x+y)$,

$$
\Psi_p(x+y) \geq (\leq) \Psi_p(x) + \Psi_p(y)
$$

for any $x, y \in C$ and the proof is complete. □

COROLLARY 2.2. *Assume that X, C and v are as in Theorem 2.1. If $h : C \to [0, \infty)$ is a superadditive (subadditive) functional on C and $p, q \geq 1$ $(0 < p, q < 1)$, then the two-parameter functional*

$$
\Psi_{p,q} : C \to [0, \infty), \quad \Psi_{p,q}(x) = v^{q(1-(1/p))}(x) h^q(x) \tag{2.6}
$$

is superadditive (subadditive) on C.

PROOF. Observe that $\Psi_{p,q}(x) = [\Psi_p(x)]^q$ for $x \in C$. Therefore, by Theorem 2.1 and the inequality (2.2), for $q \geq 1$ $(0 < q < 1)$ we have

$$
\Psi_{p,q}(x+y) = [\Psi_p(x+y)]^q \geq (\leq) [\Psi_p(x) + \Psi_p(y)]^q
$$

$$
\geq (\leq) [\Psi_p(x)]^q + [\Psi_p(y)]^q = \Psi_{p,q}(x) + \Psi_{p,q}(y)
$$

for any $x, y \in C$ and the statement is proved. □

REMARK 2.3. If we consider the functional $\psi_p(x) := v^{p-1}(x)h^p(x)$, then for $p \geq 1$ $(0 < p < 1)$ and $h : C \to [0, \infty)$ a superadditive (subadditive) functional on C, the functional ψ_p is also superadditive (subadditive) on C.

The following result provides upper and lower bounds for a value of the functional $\Psi_{p,q}$ in the case when the composite functionals are homogeneous.

COROLLARY 2.4. *Let* $x, y \in C$, $h : C \to \mathbb{R}$ *be a nonnegative, superadditive and s-positive homogeneous functional on* C *and* v *an additive, strictly positive and positive homogeneous functional on* C. *If* $p, q \geq 1$ *and* $M \geq m \geq 0$ *are such that* $x - my$ *and* $My - x \in C$, *then*

$$M^{sq+q(1-1/p)}\Psi_{p,q}(y) \geq \Psi_{p,q}(x) \geq m^{sq+q(1-1/p)}\Psi_{p,q}(y). \tag{2.7}$$

In particular,

$$M^{(s+1)p-1}\psi_p(y) \geq \psi_p(x) \geq m^{(s+1)p-1}\psi_p(y) \tag{2.8}$$

and

$$M^{s+1-1/p}\Psi_p(y) \geq \Psi_p(x) \geq m^{s+1-1/p}\Psi_p(y), \tag{2.9}$$

where ψ_p *and* Ψ_p *are defined as above.*

PROOF. The proof follows from Theorem 1.1 observing, from Corollary 2.2, that the functional $\Psi_{p,q}$ is superadditive and $[sq + q(1 - (1/p))]$-positive homogeneous, where $s > 0$ and $p, q \geq 1$. $\qquad\square$

The following result also holds.

THEOREM 2.5. *Let* C *be a convex cone in the linear space* X *and* $v : C \to (0, \infty)$ *an additive functional on* C. *If* $h : C \to (0, \infty)$ *is a superadditive functional on* C *and* $0 < p < 1$, *then the functional*

$$\Phi_p : C \to (0, \infty), \quad \Phi_p(x) = \frac{1}{v^{(1/p)-1}(x)h(x)} \tag{2.10}$$

is subadditive on C.

PROOF. Let $s := -p \in [-1, 0)$. For $s < 0$ we have the inequality

$$(\alpha + \beta)^s \leq \alpha^s + \beta^s \tag{2.11}$$

for any $\alpha, \beta > 0$.

Indeed, by the convexity of the function $f_s(t) = t^s$ on $(0, \infty)$ with $s < 0$, we have that

$$(\alpha + \beta)^s \leq 2^{s-1}(\alpha^s + \beta^s)$$

for any $\alpha, \beta > 0$ and since, obviously, $2^{s-1}(\alpha^s + \beta^s) \leq \alpha^s + \beta^s$, then (2.11) holds.

Taking into account the fact that h is superadditive, then by (2.11)

$$h^s(x + y) \leq [h(x) + h(y)]^s \leq h^s(x) + h^s(y) \tag{2.12}$$

for any $x, y \in C$.

Since v is additive, then by (2.11) we have that

$$
\begin{aligned}
\frac{h^s(x+y)}{v(x+y)} &\leq \frac{h^s(x)+h^s(y)}{v(x)+v(y)} \\
&= \frac{v(x) \cdot \left[\frac{h(x)}{v^{1/s}(x)}\right]^s + v(y) \cdot \left[\frac{h(y)}{v^{1/s}(y)}\right]^s}{v(x)+v(y)} \\
&= \frac{v(x) \cdot \left[\frac{v^{1/s}(x)}{h(x)}\right]^{-s} + v(y) \cdot \left[\frac{v^{1/s}(y)}{h(y)}\right]^{-s}}{v(x)+v(y)} =: J.
\end{aligned}
\tag{2.13}
$$

By the concavity of the function $g(t) = t^{-s}$ with $s \in [-1, 0)$ we also have

$$
J \leq \left[\frac{v(x) \cdot \frac{v^{1/s}(x)}{h(x)} + v(y) \cdot \frac{v^{1/s}(y)}{h(y)}}{v(x)+v(y)}\right]^{-s}.
\tag{2.14}
$$

Making use of (2.13) and (2.14) we get

$$
\frac{h^s(x+y)}{v(x+y)} \leq \left[\frac{v(x) \cdot \frac{v^{1/s}(x)}{h(x)} + v(y) \cdot \frac{v^{1/s}(y)}{h(y)}}{v(x)+v(y)}\right]^{-s}
$$

for any $x, y \in C$, which is equivalent to

$$
\frac{h^{-1}(x+y)}{v^{-1/s}(x+y)} \leq \frac{\frac{v^{1+1/s}(x)}{h(x)} + \frac{v^{1+1/s}(y)}{h(y)}}{v(x)+v(y)}
$$

and, since $v(x+y) = v(x) + v(y)$, to

$$
\frac{v^{1+1/s}(x+y)}{h(x+y)} \leq \frac{v^{1+1/s}(x)}{h(x)} + \frac{v^{1+1/s}(y)}{h(y)}
$$

for any $x, y \in C$.

This completes the proof.   □

The following result may also be stated.

COROLLARY 2.6. *Assume that X, C and v are as in Theorem 2.5. If $h : C \to (0, \infty)$ is a superadditive functional on C and $0 < p, q < 1$, then the functional*

$$
\Phi_{p,q} : C \to (0, \infty), \quad \Phi_{p,q}(x) = \frac{1}{v^{q((1/p)-1)}(x)h^q(x)}
\tag{2.15}
$$

is subadditive on C.

PROOF. Observe that $\Phi_{p,q}(x) = [\Phi_p(x)]^q$ for $x \in C$. Therefore, by Theorem 2.5 and the inequality (2.2) for $0 < q < 1$, we have that

$$
\begin{aligned}
\Phi_{p,q}(x+y) = [\Phi_p(x+y)]^q &\leq [\Phi_p(x) + \Phi_p(y)]^q \\
&\leq [\Phi_p(x)]^q + [\Phi_p(y)]^q = \Phi_{p,q}(x) + \Phi_{p,q}(y)
\end{aligned}
$$

for any $x, y \in C$ and the statement is proved.   □

REMARK 2.7. If we consider the functional

$$\varphi_p(x) := \frac{1}{v^{1-p}(x)h^p(x)}$$

then, for $0 < p < 1$ and $h : C \to (0, \infty)$ a superadditive functional on C, the functional φ_p is subadditive on C.

3. Applications for Jensen's inequality

Let C be a convex subset of the real linear space X and let $f : C \to \mathbb{R}$ be a convex mapping. Here we consider the following well-known form of *Jensen's discrete inequality*:

$$f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \le \frac{1}{P_I} \sum_{i \in I} p_i f(x_i), \tag{3.1}$$

where I denotes a finite subset of the set $\mathbb{N}$ of natural numbers, $x_i \in C$, $p_i \ge 0$ for $i \in I$ and $P_I := \sum_{i \in I} p_i > 0$.

Let us fix $I \in \mathcal{P}_f(\mathbb{N})$ (the class of finite parts of $\mathbb{N}$) to be a set consisting of two or more indices and $x_i \in C$ ($i \in I$). Now consider the functional $J : S_+(I) \to \mathbb{R}$ given by

$$J_I(\mathbf{p}) := \sum_{i \in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right) \ge 0, \tag{3.2}$$

where

$$S_+(I) := \{\mathbf{p} = (p_i)_{i \in I} \mid p_i \ge 0, i \in I \text{ and } P_I > 0\}$$

and f is convex on C.

We observe that $S_+(I)$ is a convex cone and the functional J_I is nonnegative and positive homogeneous on $S_+(I)$.

LEMMA 3.1 [5]. *The functional $J_I(\cdot)$ is a superadditive functional on $S_+(I)$.*

Define the functional

$$J_{p,q,I}(\mathbf{p}) := \left[P_I^{(p-1)/p} \sum_{i \in I} p_i f(x_i) - P_I^{(2p-1)/p} f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]^q \tag{3.3}$$

for $p > 1$ and $q \ge 1$.

The following proposition can be stated.

PROPOSITION 3.2. *The functional $J_{p,q,I}(\cdot)$ is superadditive on $S_+(I)$ for any $p > 1$ and $q \ge 1$.*

PROOF. Define $v(\mathbf{p}) = P_I$ and $h(\mathbf{p}) = J_I(\mathbf{p})$. Then, for $p > 1$ and $q \ge 1$,

$$\Psi_{p,q}(\mathbf{p}) = v^{q(1-(1/p))}(\mathbf{p})h^q(\mathbf{p}) = P_I^{q(1-(1/p))} J_I^q(\mathbf{p}) = J_{p,q,I}(\mathbf{p})$$

for any $\mathbf{p} \in S_+(I)$.

Since $v(\cdot)$ is additive and $J_I(\cdot)$ is superadditive on $S_+(I)$, on applying Corollary 2.2 we conclude that $J_{p,q,I}(\cdot)$ is also superadditive on $S_+(I)$. □

REMARK 3.3. We observe that, in particular, the functionals

$$J_{p,I}(\mathbf{p}) := P_I^{(p-1)/p} \sum_{i \in I} p_i f(x_i) - P_I^{(2p-1)/p} f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)$$

and

$$\tilde{J}_{p,I}(\mathbf{p}) := \left[P_I^{(p-1)/p} \sum_{i \in I} p_i f(x_i) - P_I^{(2p-1)/p} f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]^p$$

are superadditive on $S_+(I)$ for any $p > 1$.

The following result provides a refinement and a reverse for Jensen's inequality when bounds for the weights are known.

PROPOSITION 3.4. *If $\mathbf{p}, \mathbf{q} \in S_+(I)$ and $M \geq m \geq 0$ are such that $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, that is, $Mp_i \geq q_i \geq mp_i$ for each $i \in I$, then*

$$M^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\sum_{i \in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]$$

$$\geq \sum_{i \in I} q_i f(x_i) - Q_I f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i\right) \qquad (3.4)$$

$$\geq m^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\sum_{i \in I} p_i f(x_i) - P_I f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right],$$

for any $p > 1$.

PROOF. Applying Corollary 2.4 for the functional $\Psi_{p,q}(p) = J_{p,q,I}(\mathbf{p})$ and $s = 1$,

$$M^{q(2-1/p)}\left[P_I^{(1-1/p)} \sum_{i \in I} p_i f(x_i) - P_I^{(2-1/p)} f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]^q$$

$$\geq \left[Q_I^{(1-1/p)} \sum_{i \in I} q_i f(x_i) - Q_I^{(2-1/p)} f\left(\frac{1}{Q_I} \sum_{i \in I} q_i x_i\right)\right]^q \qquad (3.5)$$

$$\geq m^{q(2-1/p)}\left[P_I^{(1-1/p)} \sum_{i \in I} p_i f(x_i) - P_I^{(2-1/p)} f\left(\frac{1}{P_I} \sum_{i \in I} p_i x_i\right)\right]^q.$$

Taking the power $1/q > 0$ in the inequality (3.5) we deduce the desired result (3.4). □

The above Proposition 3.4 can be utilized to obtain various inequalities generated by the appropriate choices of the convex function f.

(1)   If $f : X \to \mathbb{R}$, $f(x) = \|x\|^r$, $r \geq 1$, $p > 1$ where $(X, \|\cdot\|)$ is a normed linear space, then we can state the inequality

$$M^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\sum_{i \in I} p_i \|x_i\|^r - P_I^{1-r}\left\|\sum_{i \in I} p_i x_i\right\|^r\right]$$

$$\geq \sum_{i \in I} q_i \|x_i\|^r - Q_I^{1-r}\left\|\sum_{i \in I} q_i x_i\right\|^r \qquad (3.6)$$

$$\geq m^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\sum_{i \in I} p_i \|x_i\|^r - P_I^{1-r}\left\|\sum_{i \in I} p_i x_i\right\|^r\right]$$

and, in particular,

$$M^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\sum_{i \in I} p_i \|x_i\| - \left\|\sum_{i \in I} p_i x_i\right\|\right]$$

$$\geq \sum_{i \in I} q_i \|x_i\| - \left\|\sum_{i \in I} q_i x_i\right\| \qquad (3.7)$$

$$\geq m^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\sum_{i \in I} p_i \|x_i\| - \left\|\sum_{i \in I} p_i x_i\right\|\right]$$

for $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{p}, \mathbf{q} \in S_+(I)$ with $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$ and $M \geq m > 0$, for any vectors $x_i \in X$, $i \in I$.

(2)   For $x_i > 0$ and $p_i \geq 0$ $(i \in \mathbb{N})$ so that $P_I > 0$, let us denote

$$A(I, \mathbf{p}, x) := \frac{1}{P_I} \sum_{i \in I} p_i x_i, \quad G(I, \mathbf{p}, x) := \left(\prod_{i \in I}(x_i)^{p_i}\right)^{1/P_I},$$

to be the *weighted arithmetic* and *geometric means*, respectively.

Applying the above Proposition 3.4 for the convex function $f(x) = -\ln x$, $x \in (0, \infty)$, we can state the inequality

$$\left[\frac{A(I, \mathbf{p}, x)}{G(I, \mathbf{p}, x)}\right]^{M^{(2p-1)/p}(P_I/Q_I)^{(2p-1)/p}} \geq \frac{A(I, \mathbf{q}, x)}{G(I, \mathbf{q}, x)}$$

$$\geq \left[\frac{A(I, \mathbf{p}, x)}{G(I, \mathbf{p}, x)}\right]^{m^{(2p-1)/p}(P_I/Q_I)^{(2p-1)/p}} \qquad (3.8)$$

for $p > 1$, $I \in \mathcal{P}_f(\mathbb{N})$ and $\mathbf{p}, \mathbf{q} \in S_+(I)$ with $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$ and $M \geq m > 0$, for any $x_i > 0$, $i \in I$.

It is well known that if $f : C \to \mathbb{R}$ is a strictly convex function and not all vectors $x_i \in C$, $i \in I$ are equal between them, then a strict inequality holds in Jensen's result (3.1). Therefore, we can consider the functional

$$R_{p,q,I}(\mathbf{p}) := \frac{1}{v^{q((1/p)-1)}(x)h^q(x)} \qquad (3.9)$$

where, as above, $v(\mathbf{p}) = P_I$, $h(\mathbf{p}) = J_I(\mathbf{p})$ and $0 < p, q < 1$.

In a more explicit way we then have

$$R_{p,q,I}(\mathbf{p}) = \frac{1}{P_I^{q/p}\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^q}. \tag{3.10}$$

We can also consider the particular functional

$$R_{p,I}(\mathbf{p}) = \frac{1}{P_I\left[\frac{1}{P_I}\sum_{i\in I} p_i f(x_i) - f\left(\frac{1}{P_I}\sum_{i\in I} p_i x_i\right)\right]^p}. \tag{3.11}$$

By utilizing Corollary 2.6, we can state the following result.

PROPOSITION 3.5. *If $f : C \to \mathbb{R}$ is a strictly convex function and not all vectors $x_i \in C$, $i \in I$ are equal between them, then the functional $R_{p,q,I}$ is subadditive on $S_+(I)$ for any p, q with $0 < p, q < 1$.*

As a simple example, we can consider the functional

$$\lambda_{p,q,I}(\mathbf{p}) := \frac{1}{P_I^{q/p}\left[\ln\left(\frac{A(I,\mathbf{p},x)}{G(I,\mathbf{p},x)}\right)\right]^p}, \tag{3.12}$$

which is therefore subadditive on $S_+(I)$ for any p, q with $0 < p, q < 1$ and any $x_i > 0$, $i \in I$ that are not all equal between them.

4. Applications for Hölder's and Minkowski's inequalities

Let $(X, \|\cdot\|)$ be a normed space and $I \in \mathcal{P}_f(\mathbb{N})$. We define

$$E(I) := \{x = (x_j)_{j\in I} \mid x_j \in X, \, j \in I\}$$

and

$$\mathbb{K}(I) := \{\lambda = (\lambda_j)_{j\in I} \mid \lambda_j \in \mathbb{K}, \, j \in I\}.$$

We consider the functional

$$H_I(\mathbf{p}, \lambda, x; \alpha, \beta) := \left(\sum_{j\in I} p_j |\lambda_j|^\alpha\right)^{1/\alpha}\left(\sum_{j\in I} p_j \|x_j\|^\beta\right)^{1/\beta} - \left\|\sum_{j\in I} p_j \lambda_j x_j\right\|.$$

for $\alpha, \beta > 1$, where $1/\alpha + 1/\beta = 1$.

The following result has been proved in [3].

LEMMA 4.1. *For any $\mathbf{p}, \mathbf{q} \in S_+(I)$,*

$$H_I(\mathbf{p} + \mathbf{q}, \lambda, x; \alpha, \beta) \geq H_I(\mathbf{p}, \lambda, x; \alpha, \beta) + H_I(\mathbf{q}, \lambda, x; \alpha, \beta), \tag{4.1}$$

where $x \in E(I)$, $\lambda \in \mathbb{K}(I)$ and $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$.

REMARK 4.2. The same result can be stated if $(B, \|\cdot\|)$ is a normed algebra and the functional H is defined by

$$H_I(\mathbf{p}, \lambda, x; \alpha, \beta) := \left(\sum_{i \in I} p_i \|x_i\|^\alpha\right)^{1/\alpha} \left(\sum_{i \in I} p_i \|y_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i \in I} p_i x_i y_i\right\|,$$

where $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I} \subset B$, $\mathbf{p} \in S_+(I)$ and $\alpha, \beta > 1$ with $1/\alpha + 1/\beta = 1$.

Define the following functional on $S_+(I)$:

$$H_{p,q,I}(\mathbf{p}, \lambda, x; \alpha, \beta) := P_I^{q(1-(1/p))} H_I^q(\mathbf{p}, \lambda, x; \alpha, \beta). \tag{4.2}$$

Since $H(\cdot, \lambda, x; \alpha, \beta)$ is positive homogeneous, on utilizing Corollary 2.4, we can state the following result.

PROPOSITION 4.3. *If* $\mathbf{p}, \mathbf{q} \in S_+(I)$ *and* $M \geq m \geq 0$ *with* $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, *then*

$$M^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\left(\sum_{i \in I} p_i |\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i \in I} p_i \|x_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i \in I} p_i \lambda_i x_i\right\|\right]$$

$$\geq \left(\sum_{i \in I} q_i |\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i \in I} q_i \|x_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i \in I} q_i \lambda_i x_i\right\|$$

$$\geq m^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left[\left(\sum_{i \in I} p_i |\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i \in I} p_i \|x_i\|^\beta\right)^{1/\beta}\right. \tag{4.3}$$

$$\left. - \left\|\sum_{i \in I} p_i \lambda_i x_i\right\|\right],$$

for $x \in E(I)$, $\lambda \in \mathbb{K}(I)$, $p > 1$ *and* $\alpha, \beta > 1$, *where* $1/\alpha + 1/\beta = 1$.

PROOF. By Corollary 2.4 applied to the functional $\Psi_{p,q}(\mathbf{p}) = H_{p,q,I}(\mathbf{p}, \lambda, x; \alpha, \beta)$,

$$M^{q(2-1/p)} P_I^{q(1-1/p)}\left[\left(\sum_{i \in I} p_i |\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i \in I} p_i \|x_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i \in I} p_i \lambda_i x_i\right\|\right]^q$$

$$\geq Q_I^{q(1-1/p)}\left[\left(\sum_{i \in I} q_i |\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i \in I} q_i \|x_i\|^\beta\right)^{1/\beta} - \left\|\sum_{i \in I} q_i \lambda_i x_i\right\|\right]^q$$

$$\geq m^{q(2-1/p)} P_I^{q(1-1/p)}\left[\left(\sum_{i \in I} p_i |\lambda_i|^\alpha\right)^{1/\alpha}\left(\sum_{i \in I} p_i \|x_i\|^\beta\right)^{1/\beta}\right. \tag{4.4}$$

$$\left. - \left\|\sum_{i \in I} p_i \lambda_i x_i\right\|\right]^q.$$

Taking the power $1/q > 0$ in the inequality (4.4) we deduce the desired result (4.3). □

We define the functional

$$M_I(\mathbf{p}, x, y; \alpha) = \left[\left(\sum_{i \in I} p_i \|x_i\|^\alpha\right)^{1/\alpha} + \left(\sum_{i \in I} p_i \|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha$$
$$- \sum_{i \in I} p_i \|x_i + y_i\|^\alpha, \tag{4.5}$$

where $\mathbf{p} \in S_+(I)$, $\alpha \geq 1$ and $x, y \in E(I)$.

The following result concerning the superadditivity of the functional $M_I(\cdot, x, y; \alpha)$ holds [3].

LEMMA 4.4. *For any* $\mathbf{p}, \mathbf{q} \in S_+(I)$,

$$M_I(\mathbf{p} + \mathbf{q}, x, y; \alpha) \geq M_I(\mathbf{p}, x, y; \alpha) + M_I(\mathbf{q}, x, y; \alpha),$$

where $x, y \in E(I)$ *and* $\alpha \geq 1$.

Since the functional $M_I(\cdot, x, y; \alpha)$ is positive homogeneous on $S_+(I)$, then on utilizing Corollary 2.4, we can state the following proposition.

PROPOSITION 4.5. *If* $\mathbf{p}, \mathbf{q} \in S_+(I)$ *and* $M \geq m \geq 0$ *with* $M\mathbf{p} \geq \mathbf{q} \geq m\mathbf{p}$, *then*

$$M^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left\{\left[\left(\sum_{i \in I} p_i \|x_i\|^\alpha\right)^{1/\alpha} + \left(\sum_{i \in I} p_i \|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha\right.$$
$$\left. - \sum_{i \in I} p_i \|x_i + y_i\|^\alpha\right\}$$
$$\geq \left[\left(\sum_{i \in I} q_i \|x_i\|^\alpha\right)^{1/\alpha} + \left(\sum_{i \in I} q_i \|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha - \sum_{i \in I} q_i \|x_i + y_i\|^\alpha \tag{4.6}$$
$$\geq m^{(2p-1)/p}\left(\frac{P_I}{Q_I}\right)^{(p-1)/p}\left\{\left[\left(\sum_{i \in I} p_i \|x_i\|^\alpha\right)^{1/\alpha} + \left(\sum_{i \in I} p_i \|y_i\|^\alpha\right)^{1/\alpha}\right]^\alpha\right.$$
$$\left. - \sum_{i \in I} p_i \|x_i + y_i\|^\alpha\right\},$$

where $x, y \in E(I)$, $p > 1$ *and* $\alpha \geq 1$.

5. Applications for Schwarz's inequality

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X or, for simplicity, nonnegative forms on X, that is, the mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions:

(i) $\langle x, x \rangle \geq 0$ for all $x \in X$;
(ii) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$;
(iii) $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for all $x, y \in X$.

If $\langle \cdot, \cdot \rangle \in \mathcal{H}(X)$, then the functional $\| \cdot \| = \langle \cdot, \cdot \rangle^{1/2}$ is a semi-norm on X and the following version of Schwarz's inequality holds:

$$\|x\|\|y\| \geq |\langle x, y \rangle| \tag{5.1}$$

for each $x, y \in H$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$. Also, we can introduce on $\mathcal{H}(X)$ the following *binary relation* [4]:

$$\langle \cdot, \cdot \rangle_2 \geq \langle \cdot, \cdot \rangle_1 \text{ if and only if } \|x\|_2 \geq \|x\|_1 \text{ for any } x \in H. \tag{5.2}$$

This is an *order relation* on $\mathcal{H}(X)$ (see [4]).

Consider the functional [4]

$$\sigma : \mathcal{H}(X) \times X^2 \to \mathbb{R}_+, \quad \sigma(\langle \cdot, \cdot \rangle; x, y) := \|x\|\|y\| - |\langle x, y \rangle|,$$

which is closely related to the Schwarz inequality in (5.1).

LEMMA 5.1 [4]. *The functional $\sigma(\cdot; x, y)$ is nonnegative, superadditive and positive homogeneous on $\mathcal{H}(X)$.*

Now, if we consider the composite functional

$$\begin{aligned} \Psi_{p,q,e}(\langle \cdot, \cdot \rangle, x, y) &= \langle e, e \rangle^{q(1-(1/p))} \sigma^q(\langle \cdot, \cdot \rangle; x, y) \\ &= \|e\|^{2q(1-(1/p))} [\|x\|\|y\| - |\langle x, y \rangle|]^q \end{aligned} \tag{5.3}$$

where $x, y, z \in X$, and $p, q \geq 1$, then, by Corollary 2.2, we can state the following result.

PROPOSITION 5.2. *The functional $\Psi_{p,q,e}(\cdot, x, y)$ is superadditive and monotonic nondecreasing on $\mathcal{H}(X)$.*

The following proposition concerning some inequalities for equivalent norms generated by inner products is of interest for applications.

PROPOSITION 5.3. *Let $M \geq m > 0$ and let $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ be two inner products on X such that $M\|x\|_1 \geq \|x\|_2 \geq m\|x\|_1$ for each $x \in X$. If $e \in X$, $e \neq 0$ and $p \geq 1$, then*

$$M^{2(2p-1)/p} \left(\frac{\|e\|_1}{\|e\|_2} \right)^{2(p-1)/p} (\|x\|_1\|y\|_1 - |\langle x, y \rangle_1|) \geq \|x\|_2\|y\|_2 - |\langle x, y \rangle_2|$$
$$\geq m_1^{2(2p-1)/p} \left(\frac{\|e\|_1}{\|e\|_2} \right)^{2(p-1)/p} (\|x\|_1\|y\|_1 - |\langle x, y \rangle_1|), \tag{5.4}$$

for any $x, y \in H$.

PROOF. On applying Corollary 2.4, we deduce that

$$
M^{2q(2-(1/p))} \|e\|_1^{2q(1-1/p)} [\|x\|_1 \|y\|_1 - |\langle x, y \rangle_1|]^q
$$
$$
\geq \|e\|_2^{2q(1-1/p)} [\|x\|_2 \|y\|_2 - |\langle x, y \rangle_2|]^q \tag{5.5}
$$
$$
\geq m^{2q(2-1/p)} \|e\|_1^{2q(1-1/p)} [\|x\|_1 \|y\|_1 - |\langle x, y \rangle_1|]^q
$$

for x, y fixed in X.

Taking the power $1/q > 0$ in the inequality (5.5), we deduce the desired result (5.4). $\square$

The above result can be used to obtain some inequalities for positive definite operators.

COROLLARY 5.4. *Assume that $A : H \to H$ is a self-adjoint linear operator on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ satisfying the property that there exist $\Gamma \geq \gamma > 0$ such that $\Gamma I \geq A \geq \gamma I$ in the operator order (that is, $\Gamma \|x\|^2 \geq \langle Ax, x \rangle \geq \gamma \|x\|^2$ for any $x \in H$). Then for $p \geq 1$ we have the inequality*

$$
\frac{\Gamma^{(2p-1)/p}}{\langle Ae, e \rangle^{(p-1)/p}} (\|x\| \|y\| - |\langle x, y \rangle|) \geq \langle Ax, x \rangle^{1/2} \langle Ay, y \rangle^{1/2} - |\langle Ax, y \rangle|
$$
$$
\geq \frac{\gamma^{(2p-1)/p}}{\langle Ae, e \rangle^{(p-1)/p}} (\|x\| \|y\| - |\langle x, y \rangle|), \tag{5.6}
$$

for any x, $y \in H$ and $e \in H$ with $\|e\| = 1$.

The following result for invertible operators also holds.

COROLLARY 5.5. *Assume that $A : H \to H$ is an invertible linear operator on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$. Then for $p \geq 1$ and $e \in H$ with $\|e\| = 1$ we have the inequality*

$$
\frac{\|A\|^{2((2p-1)/p)}}{\|Ae\|^{2((p-1)/p)}} (\|x\| \|y\| - |\langle x, y \rangle|) \geq \|Ax\| \|Ay\| - |\langle Ax, Ay \rangle|
$$
$$
\geq \frac{\|A^{-1}\|^{-2((2p-1)/p)}}{\|Ae\|^{2((p-1)/p)}} (\|x\| \|y\| - |\langle x, y \rangle|), \tag{5.7}
$$

for any x, $y \in H$.

PROOF. Since A is invertible,

$$
\|A\| \|x\| \geq \|Ax\| \geq \|A^{-1}\|^{-1} \|x\|
$$

for any $x \in H$. Applying Proposition 5.3 we deduce the desired result (5.7). $\square$

REMARK 5.6. Similar results can be stated if one uses the following nonnegative, superadditive and *positive* homogeneous functionals on $\mathcal{H}(X)$ (see [2, pp. 8–15]):

$$\sigma_r(\langle \cdot, \cdot \rangle; x, y) := \|x\| \|y\| - \mathrm{Re} \langle x, y \rangle,$$
$$\delta(\langle \cdot, \cdot \rangle; x, y) := \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2,$$
$$\delta_r(\langle \cdot, \cdot \rangle; x, y) := \|x\|^2 \|y\|^2 - (\mathrm{Re} \langle x, y \rangle)^2,$$
$$\gamma(\langle \cdot, \cdot \rangle; x, y) := \frac{\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2}{\|y\|^2},$$

where, in the definition of γ, $\langle \cdot, \cdot \rangle$ is an inner product and y is not zero, and

$$\beta(\langle \cdot, \cdot \rangle; x, y) := (\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2)^{\frac{1}{2}},$$

for each $x, y \in X$.

The details are left to the interested reader.

REMARK 5.7. For other examples of superadditive (subadditive) functionals that can provide interesting inequalities similar to the ones outlined above, we refer to [1, 6–9].

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