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INDUCTION AND RESTRICTION OF π-PARTIAL CHARACTERS AND THEIR LIFTS

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ABSTRACT. Let G be a finite π-separable group, where π is a set of primes. The π-partial characters of G are the restrictions of the ordinary characters to the set of π-elements of G. Such an object is said to be irreducible if it is not the sum of two nonzero partial characters and the set of irreducible π-partial characters of G is denoted $\mathrm{I}_\pi(G)$. (If p is a prime and $\pi = p'$, then $\mathrm{I}_\pi(G)$ is exactly the set of irreducible Brauer characters at p.)

From their definition, it is obvious that each partial character $\varphi \in \mathrm{I}_\pi(G)$ can be "lifted" to an ordinary character $\chi \in \mathrm{Irr}(G)$. (This means that φ is the restriction of χ to the π-elements of G.) In fact, there is a known set of canonical lifts $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ for the irreducible π-partial characters. In this paper, it is proved that if $2 \notin \pi$, then there is an alternative set of canonical lifts (denoted $\mathrm{D}_\pi(G)$) that behaves better with respect to character induction.

An application of this theory to M-groups is presented. If G is an M-group and $S \subseteq G$ is a subnormal subgroup, consider a primitive character $\theta \in \mathrm{Irr}(S)$. It was known previously that if $|G : S|$ is odd, then θ must be linear. It is proved here without restriction on the index of S that $\theta(1)$ is a power of 2.

1. **Introduction.** This paper is intended to serve a double purpose. We obtain a few new results in the π-character theory of π-separable groups and we use some of these to study the character theory of subnormal subgroups of M-groups.

Consider M-groups first. Suppose that $S \lhd\lhd G$, where G is an M-group, and suppose $\gamma \in \mathrm{Irr}(S)$ is primitive. In [3], we showed that if S has odd index, then γ must be linear and it is a consequence of our work here that γ must also be linear in the case where S has odd order. In fact, we have more.

THEOREM A. *Let $S \lhd\lhd G$, where G is an M-group. If $\gamma \in \mathrm{Irr}(S)$ is primitive, then $\gamma(1)$ is a power of 2.*

We mention that the case of Theorem A where $S \lhd G$ is the main theorem of [1], but the result for subnormal subgroups does not follow from this. Our actual result here is an even stronger necessary condition for a group S to be subnormally embedded in an M-group, but unfortunately, our condition does not imply that the S must be an M-group, even when it is known to have odd order.

We turn now to π-character theory, with a brief review of definitions and basic results. (The reader may wish to consult the expository papers [8], [5] and [7] for more information and for proofs of some of the key theorems.)

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Fix a set π of prime numbers and let G be a π-separable group. The *π-partial characters* of G are the restrictions of the ordinary characters to the set G^0 of π-elements of G. (We write χ^0 to denote the restriction of $\chi \in \mathrm{Char}(G)$ to G^0.) A π-partial character is *irreducible* if it cannot be written as a sum of two nonzero π-partial characters and we write $\mathrm{I}_\pi(G)$ to denote the set of all irreducible π-partial characters of G. If $\chi \in \mathrm{Char}(G)$, it is immediate that we can write $\chi^0 = \sum a_\varphi \varphi$, where φ runs over $\mathrm{I}_\pi(G)$ and the coefficients a_φ are nonnegative integers. (In fact, these coefficients are uniquely determined by χ because $\mathrm{I}_\pi(G)$ is actually a basis for the complex vector space of class functions defined on G^0.) Given $\varphi \in \mathrm{I}_\pi(G)$, we can certainly write $\varphi = \chi^0$ for some character χ of G, and in this situation it is clear that χ must lie in $\mathrm{Irr}(G)$. In other words, each irreducible π-partial character of G can be *lifted* to an ordinary irreducible character.

It is possible to 'predetermine' a complete set of lifts for the irreducible π-partial characters of a π-separable group G. There is, in other words, a canonically defined subset $\mathrm{B}_\pi(G) \subseteq \mathrm{Irr}(G)$ such that the map $\chi \longmapsto \chi^0$ is a bijection from $\mathrm{B}_\pi(G)$ onto $\mathrm{I}_\pi(G)$. (The definition and principal properties of the set $\mathrm{B}_\pi(G)$ appear in [2] and an exposition of this material can be found in [7].)

The function $\mathrm{B}_\pi(\ )$, which assigns to each π-separable group G a particular set of irreducible characters of G, behaves well with respect to normal subgroups: if $\chi \in \mathrm{B}_\pi(G)$ and $N \lhd G$, then all irreducible constituents of the restriction χ_N lie in $\mathrm{B}_\pi(N)$. (As is explained in [8], this fact about the set-valued function $\mathrm{B}_\pi(\ )$ is crucial in establishing the analog of Clifford theory for the irreducible π-partial characters.)

There are some respects, however, in which the behavior of the function $\mathrm{B}_\pi(\ )$ is not under good control. Suppose, for example, that H is an arbitrary subgroup of G. If $\chi \in \mathrm{B}_\pi(G)$, then even if we know that χ_H is irreducible, we cannot always conclude that χ_H lies in $\mathrm{B}_\pi(H)$. Similarly, if we start with $\psi \in \mathrm{B}_\pi(H)$, where ψ^G is known to be irreducible, it does not follow that ψ^G lies in $\mathrm{B}_\pi(G)$.

In the case where G has odd order, it is an easy consequence of Lemma 3.1 of [6] that the restriction and induction conclusions of the previous paragraph actually are valid. In fact, an even stronger induction result holds: given that $|G|$ is odd and that $\psi^G = \chi \in \mathrm{Irr}(G)$, then $\psi \in \mathrm{B}_\pi(H)$ if and only if $\chi \in \mathrm{B}_\pi(G)$. (The additional information here, of course, is the 'if' assertion, which appears as Theorem 8.6 of [8].)

For groups of even order, one can use Lemma 3.3 of [6] to prove weak forms of the restriction and induction theorems for $\mathrm{B}_\pi(\ )$, but only when $2 \in \pi$. In the case where $2 \notin \pi$, on the other hand, strong replacements for these theorems are available, provided that we are willing to make certain changes. Specifically, we must replace $\mathrm{B}_\pi(\ )$ by a suitable alternative function $\mathrm{D}_\pi(\ )$ that also assigns a certain subset of $\mathrm{Irr}(G)$ to each π-separable group G and we must work with a twisted form of character induction called π-induction.

Definitions and properties of π-induction and of the function $\mathrm{D}_\pi(\ )$ can be found in [4], and we review some of this material in Section 2, following this introduction. We prove here that $\mathrm{D}_\pi(G)$ (like $\mathrm{B}_\pi(G)$) is a set of lifts for the irreducible π-partial characters of G. In general, $\mathrm{D}_\pi(G)$ can be different from $\mathrm{B}_\pi(G)$ and we stress that $\mathrm{D}_\pi(G)$, unlike $\mathrm{B}_\pi(G)$, is defined only when $2 \notin \pi$. (We mention that the definition of $\mathrm{D}_\pi(G)$ was suggested by unpublished work of E. C. Dade.)

THEOREM B. *Let G be π-separable, where $2 \notin \pi$. Then the map $\chi \mapsto \chi^0$ defines a bijection from $\mathrm{D}_\pi(G)$ onto $\mathrm{I}_\pi(G)$.*

Using Theorem B, it is easy to deduce the following.

COROLLARY C. *If G has odd order and $2 \notin \pi$, then $\mathrm{D}_\pi(G) = \mathrm{B}_\pi(G)$.*

By Theorem 7.10 of [4], the function $\mathrm{D}_\pi(\ )$ (like $\mathrm{B}_\pi(\ )$) respects restriction to normal subgroups, but unlike $\mathrm{B}_\pi(\ )$, it is also reasonably well-behaved upon restriction to non-normal subgroups.

THEOREM D. *Let $H \subseteq G$, where G is π-separable and $2 \notin \pi$. If $\chi \in \mathrm{D}_\pi(G)$ and $\psi = \chi_H$ is irreducible, then $\psi \in \mathrm{D}_\pi(H)$.*

In view of Corollary C, we see that for odd-order groups, Theorem D implies the restriction theorem for $\mathrm{B}_\pi(\ )$, to which we referred earlier.

The induction theorem for the function $\mathrm{D}_\pi(\ )$ is known. It was first proved in unpublished work of Dade and it appears as Theorem 7.3 of [4]. For comparison with Theorem D and because together with Theorem B, we use it to prove our M-group theorem, we state this result below as Theorem E.

We must first give at least a cursory description of π-induction, however. (More detail can be found in Section 2, and still more appears in [4].) If $2 \notin \pi$ and G is π-separable, then π-induction from a subgroup $H \subseteq G$ is a certain map $\theta \mapsto \theta^{\pi G}$ from $\mathrm{Char}(H)$ to $\mathrm{Char}(G)$. Like the ordinary induced character θ^G, the π-induced character $\theta^{\pi G}$ has degree equal to $|G : H|\,\theta(1)$. More generally, $\theta^{\pi G}(x) = \theta^G(x)$ for all odd-order elements $x \in G$. (For odd-order groups, therefore, π-induction is the same as ordinary induction, and so by Corollary C, the induction theorem for $\mathrm{B}_\pi(\ )$ in odd-order groups (to which we referred earlier) is a consequence of Theorem E.)

THEOREM E. *Let $H \subseteq G$, where G is π-separable and $2 \notin \pi$. Suppose $\psi \in \mathrm{Irr}(H)$ and that $\chi = \psi^{\pi G}$ is irreducible. Then $\chi \in \mathrm{D}_\pi(G)$ iff $\psi \in \mathrm{D}_\pi(H)$.*

We mention that the 'if' part of Theorem E is immediate from the definition of the function $\mathrm{D}_\pi(\ )$ and the properties of π-induction; it is only the 'only if' part that requires work.

Since we are assuming that $2 \notin \pi$ and we know that θ^G and $\theta^{\pi G}$ agree on elements of odd order, we see that these characters agree on the set G^0 of π-elements of G. It follows that $(\theta^{\pi G})^0 = (\theta^G)^0$ for all characters θ of H, and so from the point of view of π-partial characters, π-induction is exactly the same as ordinary induction. It is induction of π-partial characters that we consider next.

Recall from [8] that a character $\chi \in \mathrm{Irr}(G)$ is *supermonomial* if whenever $\chi = \psi^G$, where ψ is a character of a subgroup of G, then ψ is monomial. Analogously, a partial character $\varphi \in \mathrm{I}_\pi(G)$ is *monomial* if it is induced from a linear π-partial character of a subgroup and φ is *supermonomial* if every π-partial character that induces it is monomial. Our principal supermonomiality result, used to prove Theorem A, is the following.

THEOREM F. *Let G be p-solvable, where p is an odd prime. Choose $\varphi \in \mathrm{I}_p(G)$ and let χ be the unique lift of φ in $\mathrm{D}_p(G)$. The following are then equivalent.*

(i) φ is monomial.

(ii) χ is monomial.

(iii) Both φ and χ are supermonomial.

The fact that monomial characters in $\mathrm{D}_p(G)$ must be supermonomial is a generalization of Theorem 10.2 of [8], which is the odd-order case of this result. (In [8], the theorem is stated for $\mathrm{B}_p(G)$, but we know that $\mathrm{B}_p(G) = \mathrm{D}_p(G)$ when $|G|$ is odd.)

Theorem A is not the only result about primitive characters of subgroups of M-groups that can be proved using techniques of this paper. Instead of discussing other theorems of this type here, we refer the reader to [9].

2. **Sign characters and twisted induction.** In this section we recall some definitions and facts from [4], and in particular, we review the definition of the subset $\mathrm{D}_\pi(G) \subseteq \mathrm{Irr}(G)$. The key to this is the π-*standard sign character* $\delta_{(G:H)}$, defined for each subgroup $H \subseteq G$, where G is π-separable and $2 \notin \pi$. This is a linear character of H that has values ± 1; it is determined by the action of H on the π-factors of an H-composition series for G. (We refer the reader to Section 2 of [4] for the full definition of $\delta_{(G:H)}$.) Among the properties of the π-standard sign character are the following, which are sufficient to determine it uniquely.

LEMMA 2.1. *Given $H \subseteq G$, where G is π-separable and $2 \notin \pi$, we have the following.*

(a) If $|G : H|$ is a π'-number, then $\delta_{(G:H)} = 1_H$.

(b) If $|G : H|$ is a π-number and H is a maximal subgroup, then $\delta_{(G:H)}$ is the permutation sign character of the action of H on the right cosets of H in G.

(c) If $H \subseteq K \subseteq G$, then $\delta_{(G:H)} = (\delta_{(G:K)})_H \delta_{(K:H)}$.

PROOF. Parts (a) and (c) are respectively Theorem 2.5(c) and Theorem 2.5(b) of [4], while (b) is immediate from Corollary 2.9(a) of that paper and the fact that the sign character of an action is exactly the determinant of the associated permutation representation. ∎

Continuing to assume that G is π-separable, with $2 \notin \pi$, we recall from [4] that if $\theta \in \mathrm{Char}(H)$, then the π-*induced* character $\theta^{\pi G}$ of G is the character $(\delta\theta)^G$, where $\delta = \delta_{(G:H)}$. Since $\delta_{(G:H)}$ is trivial if $|G : H|$ is a π'-number, we see that π-induction from subgroups of π'-index agrees with ordinary induction. Also, for arbitrary subgroups $H \subseteq G$, if $x \in G$ has odd order, then $\theta^{\pi G}(x) = \theta^G(x)$. (This is because $\delta(y) = 1$ for every element y of H that is conjugate in G to x.)

In general, π-induction enjoys many of the familiar properties of ordinary induction. It is transitive, for example. In other words, if $H \subseteq K \subseteq G$ and $\theta \in \mathrm{Char}(H)$, then $(\theta^{\pi K})^{\pi G} = \theta^{\pi G}$. (This is an immediate consequence of Lemma 2.1 (c).) Somewhat more interesting is the situation where $G = XY$ for subgroups X and Y. If $\theta \in \mathrm{Char}(Y)$, it is a well-known consequence of Mackey's theorem that $(\theta^G)_X = (\theta_{X\cap Y})^X$. The analog of this formula for π-induction would be immediate if we knew that $(\delta_{(G:Y)})_{X\cap Y} = \delta_{(X:X\cap Y)}$, but in [4], this was proved only under the additional assumption that one of X or Y is the product of $X \cap Y$ with a normal subgroup of G. In fact, it is not very difficult to prove the general case. (In order to give a self-contained proof here, we do not assume the part of this result that appears in [4].) We require the following lemma, which, except for its dependence on the odd-order theorem, is entirely routine.

LEMMA 2.2. *Let G be π-separable, where $2 \notin \pi$, and suppose $\mathbf{O}_{\pi'}(G) = 1$. Then all core-free maximal subgroups of G are conjugate.*

PROOF. Let N be a minimal normal subgroup of G. Then N is a π-group, and thus is of odd order and hence is an abelian p-group for some prime $p \in \pi$. If X is a core-free maximal subgroup of G, then $N \not\subseteq X$ and thus $NX = G$ and $N \cap X = 1$. It follows that $\mathbf{C}_G(N) \cap X \triangleleft G$, and we conclude that $\mathbf{C}_G(N) = N$.

If $N = G$ then $X = 1$ and there is nothing to prove. We can thus assume that $N < G$ and we let K/N be a chief factor of G. We claim that K/N is a p'-group. This is certainly true if K/N is a π'-group, and so we can assume it is a π-group. It thus has odd order, and hence it is an abelian q-group for some prime q. But $q \neq p$; otherwise, K is a p-group and $N \subseteq \mathbf{Z}(K)$, which contradicts the fact that N is self-centralizing.

We see now that $X \cap K$ is a p-complement in K and that $X = \mathbf{N}_G(X \cap K)$. Since all p-complements in K are conjugate, their normalizers in G are also conjugate, and the result follows. ■

LEMMA 2.3. *Let $X, Y \subseteq G$, where G is π-separable and $2 \notin \pi$. Assume that $XY = G$ and write $D = X \cap Y$. Then*

(a) $(\delta_{(G:Y)})_D = \delta_{(X:D)}$.

(b) $(\delta_{(G:X)})_D = \delta_{(Y:D)}$.

(c) If $\theta \in \mathrm{Char}(Y)$, then $(\theta^{\pi G})_X = (\theta_D)^{\pi X}$.

PROOF. We observe first that (a) and (b) imply each other for each choice of subgroups X and Y with $XY = G$. To see this, we argue that since $\delta_{(Y:D)}$ is nonvanishing, the assertion of (a) is equivalent to the equation $\delta_{(X:D)}\delta_{(Y:D)} = (\delta_{(G:Y)})_D\delta_{(Y:D)}$. Since the right side of this equation is equal to $\delta_{(G:D)}$ by Lemma 2.1(a), it follows that (a) is true iff $\delta_{(G:D)} = \delta_{(X:D)}\delta_{(Y:D)}$, which is symmetric in X and Y. It follows that (b) is also equivalent to this equation, and hence (a) and (b) are equivalent to each other.

We now prove parts (a) and (b) by induction on the index $|G : D|$. If $X = G$, then $D = Y$ and (a) is trivially true, and it follows that (b) holds too. Similarly, (a) and (b) hold if $Y = G$, and thus we may thus assume that X and Y are each proper in G.

Suppose that $X < H < G$ for some subgroup H. Observe that $HY = G$ and that $XE = H$, where we have written $E = H \cap Y$. Since $H > X$, we see that $E > D$ and thus $|G : E| < |G : D|$. Also $|H : D| < |G : D|$ since $H < G$, and hence two applications of the inductive hypothesis yield that $\delta_{(X:D)} = (\delta_{(H:E)})_D = \left((\delta_{(G:Y)})_E\right)_D$. Thus (a) holds, and so (b) holds too, and similarly, both (a) and (b) hold if Y is not maximal in G.

We can now assume that each of X and Y is a maximal subgroup of G. In particular, since G is π-separable, each of the indices $|G : X|$ and $|G : Y|$ is either a π-number or a π'-number. Suppose that one of these indices (say $|G:Y|$) is a π'-number. Then also $|X:D|$ is a π'-number and hence $\delta_{(G:Y)} = 1_Y$ and $\delta_{(X:D)} = 1_D$ and assertion (a) follows. Since (a) and (b) both hold if either of the two indices $|G:X|$ or $|G:Y|$ is a π'-number, we can assume that both indices are π-numbers and hence $|G:D|$ is a π-number.

Now let $M = \mathrm{core}_G(D)$ and suppose that $M < \mathrm{core}_G(Y)$. Writing $N = \mathrm{core}_G(Y)$, we see that $N \not\subseteq X$ and thus $NX = G$ since X is maximal in G. In this situation, $Y = ND$ and we claim that D is a maximal subgroup of X. To see this, suppose $D < U < X$. Then $YU = NU$ is a subgroup and $Y < YU < G$ since $YU \cap X = U$, and this contradicts the maximality of Y in G.

Since D is maximal and of π-index in X, we know from Lemma 2.1(b) that $\delta_{(X:D)}$ is the permutation sign character of the action of D on the right cosets of D in X. This action, however, is permutation isomorphic to the action of D on the right cosets of Y in G. (Intersection of a coset of Y with X gives the desired isomorphism.) Since $\delta_{(G:Y)}$ is the permutation sign character of the action of Y on the right cosets of Y in G, we see that $(\delta_{(G:Y)})_D$ is the permutation sign character of the action of D on these cosets. In this situation, we have $(\delta_{(G:Y)})_D = \delta_{(X:D)}$ and assertions (a) and (b) follow.

We can now assume that M is the core of Y in G and similarly, it is the core of X. Working in G/M, therefore, we see that X/M and Y/M are core-free maximal subgroups. They have π-index, and so they contain $\mathbf{O}_{\pi'}(G/M)$, which is therefore trivial. It follows from Lemma 2.2 that X/M and Y/M are conjugate, and thus X and Y are conjugate in G. This is impossible, however, since $XY = G$, and this contradiction completes the proof of (a) and (b).

The proof of (c) is now immediate because

$$(\theta^{\pi G})_X = \left((\delta_{(G:Y)}\theta)^G\right)_X = \left((\delta_{(G:Y)}\theta)_D\right)^X = (\delta_{(X:D)}\theta_D)^X = (\theta_D)^{\pi X}$$

where the penultimate equality follows from (a). ■

We turn now to the definition of the set $\mathrm{D}_\pi(G)$. If G is π-separable, we recall (for comparison) that $\mathrm{B}_\pi(G)$ is a superset of the set $\mathcal{X}_\pi(G)$ of π-special characters. In fact, the π-special characters of G are exactly those members of $\mathrm{B}_\pi(G)$ that have π-degree and every member of $\mathrm{B}_\pi(G)$ is induced from an appropriate π-special character of a subgroup of G. (See [7] for an exposition of this.) Given that $2 \notin \pi$, the set $\mathrm{D}_\pi(G)$ is also a superset of $\mathcal{X}_\pi(G)$. By the definition given in [4], it consists of those irreducible characters of G that can be obtained via π-induction from π-special characters of subgroups of G. In other words, $\chi \in \mathrm{Irr}(G)$ lies in $\mathrm{D}_\pi(G)$ precisely when there exists a π-special character θ of some subgroup $U \subseteq G$ such that $\theta^{\pi G} = \chi$.

Note that by the transitivity of π-induction, it is clear that if $H \subseteq G$ and $\psi \in \mathrm{D}_\pi(H)$ with $\psi^{\pi G} = \chi \in \mathrm{Irr}(G)$, then $\chi \in \mathrm{D}_\pi(G)$. This proves the 'if' part of Theorem E. (Theorem E is fully proved in [4], and we say no more about it here.)

We can now prove Theorem D.

PROOF OF THEOREM D. We are given $\chi \in \mathrm{D}_\pi(G)$ such that $\psi = \chi_H$ is irreducible and our task is to show that $\psi \in \mathrm{D}_\pi(H)$. By definition of $\mathrm{D}_\pi(G)$, we can write $\chi = \theta^{\pi G}$ for some π-special character θ of some subgroup $U \subseteq G$. Since χ is induced from a character of U and it restricts irreducibly to H, it follows that $UH = G$. By Lemma 2.3(c), we have $\psi = \chi_H = (\theta_{U\cap H})^{\pi H}$, and we know that this is irreducible. It follows that $\theta_{U\cap H}$ is irreducible, and thus it is π-special by Theorem A of [4]. By definition, therefore, ψ lies in $\mathrm{D}_\pi(H)$, as required. ∎

By Corollary 7.2 of [4], the π-special characters of G are exactly those members of $\mathrm{D}_\pi(G)$ that have π-degree. For the remaining members of $\mathrm{D}_\pi(G)$ we shall need the following technical result, which is part of Lemma 7.7 of [4]. We quote it here without proof.

LEMMA 2.4. *Let G be π-separable, where $2 \notin \pi$, and suppose that $\chi \in \mathrm{D}_\pi(G)$. If $\chi(1)$ is not a π-number, then there exists $N \lhd G$ such that χ_N has π-special irreducible constituents that are not invariant in G.* ∎

3. **Lifting π-partial characters.** In this section we prove Theorem B and its corollary.

PROOF OF THEOREM B. Given $\varphi \in \mathrm{I}_\pi(G)$, we must produce a character $\chi \in \mathrm{D}_\pi(G)$ such that $\chi^0 = \varphi$. There exists $\psi \in \mathrm{B}_\pi(G)$ with $\psi^0 = \varphi$ and by definition of the set $\mathrm{B}_\pi(G)$ in [2] (or see the exposition in [7]), there exists a π-special character γ of some subgroup $W \subseteq G$ such that $\gamma^G = \psi$. (As explained in [7], we can take the pair (W, γ) to be a 'nucleus' for ψ.)

Now let $\chi = \gamma^{\pi G} = (\delta\gamma)^G$, where $\delta = \delta_{(G:W)}$ is a certain linear character of W such that $\delta^2 = 1_W$. If $x \in G$ is a π-element, then since $2 \notin \pi$, we know that x has odd order, and it follows that

$$\chi(x) = \gamma^{\pi G}(x) = \gamma^G(x) = \psi(x) = \varphi(x)$$

and thus $\chi^0 = \varphi$. Since φ is an irreducible π-partial character, it follows that $\chi \in \mathrm{Irr}(G)$ and hence by definition, $\chi \in \mathrm{D}_\pi(G)$.

To complete the proof, we show by induction on $|G|$ that the map $\chi \longmapsto \chi^0$ is an injection from $\mathrm{D}_\pi(G)$ into $\mathrm{I}_\pi(G)$. Let $\chi \in \mathrm{D}_\pi(G)$ and write $\varphi = \chi^0$. We must prove that φ is irreducible and that if also $\xi \in \mathrm{D}_\pi(G)$ with $\xi^0 = \varphi$, then $\xi = \chi$. We suppose first that $\chi(1)$ is a π-number. Then χ is π-special, and hence $\chi \in \mathrm{B}_\pi(G)$. Also, since $\xi^0 = \varphi$, we have $\xi(1) = \chi(1)$, and thus ξ is also π-special and lies in $\mathrm{B}_\pi(G)$. But we know that restriction to π-elements defines a bijection from $\mathrm{B}_\pi(G)$ to $\mathrm{I}_\pi(G)$, and we deduce that $\chi = \xi$ and that φ is irreducible, as desired.

We can now assume that $\chi(1)$ is not a π-number. By Lemma 2.4, there exists a subgroup $N \lhd G$ such that an irreducible constituent θ of χ_N is π-special and is not invariant in G. Let $T < G$ be the stabilizer of θ in G and observe that T is also the stabilizer of θ^0, which is an irreducible π-partial character of N. (This is because $\theta \in \mathrm{B}_\pi(N)$ and restriction to π-elements defines a bijection from $\mathrm{B}_\pi(N)$ to $\mathrm{I}_\pi(N)$.)

We have $\chi = \alpha^G$ for some uniquely determined Clifford correspondent $\alpha \in \mathrm{Irr}(T|\theta)$. If we write $\delta = \delta_{(G:T)}$ and $\psi = \delta\alpha$, we see that $\psi^{\pi G} = \alpha^G = \chi$, and since $\chi \in \mathrm{D}_\pi(G)$, we deduce from Theorem E that $\psi \in \mathrm{D}_\pi(T)$. Clearly, $(\psi^0)^G = \varphi$ and ψ^0 lies over $\theta^0 \in \mathrm{I}_\pi(N)$.

By the inductive hypothesis, we know that $\psi^0 \in \mathrm{I}_\pi(T)$, and since T is the stabilizer of θ^0 in G, the Clifford correspondence for π-partial characters guarantees that $\varphi = (\psi^0)^G$ is irreducible, as desired. (See Theorem 5.3 of [8].)

Given $\xi \in \mathrm{D}_\pi(G)$ with $\xi^0 = \varphi$, we know that θ^0 is a constituent of $\varphi_N = (\xi_N)^0$, and so θ^0 is a constituent of μ^0, where μ is some (ordinary) irreducible constituent of ξ_N. But the irreducible constituents of ξ_N all lie in $\mathrm{D}_\pi(N)$. (As we mentioned earlier, this is Theorem 7.10 of [4].) However, $N \subseteq T < G$ and the inductive hypothesis guarantees that restriction to π-elements defines a bijection from $\mathrm{D}_\pi(N)$ to $\mathrm{I}_\pi(N)$. Since both θ and μ lie in $\mathrm{D}_\pi(N)$ and θ^0 is known to be a constituent of μ^0, it follows that $\theta = \mu$, and thus θ lies under ξ and we let $\beta \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of ξ with respect to θ. Thus $\beta^G = \xi$ and we have $\eta^{\pi G} = \xi$, where $\eta = \delta\beta$. Reasoning as in the previous paragraph, we deduce that $\eta \in \mathrm{D}_\pi(T)$ and that η^0 is the Clifford correspondent of φ with respect to θ^0. It follows that $\eta^0 = \psi^0$, and thus $\eta = \psi$ by the inductive hypothesis applied to T. Thus $\xi = \eta^{\pi G} = \psi^{\pi G} = \chi$, and the proof is complete. ∎

PROOF OF COROLLARY C. Given that $|G|$ is odd, we claim that $\mathrm{B}_\pi(G) \subseteq \mathrm{D}_\pi(G)$. If $\chi \in \mathrm{B}_\pi(G)$, we know that $\chi = \gamma^G$, where γ is a π-special character of some subgroup $W \subseteq G$. But $\delta_{(G:W)} = 1_W$ since $|W|$ is odd; thus $\chi = \gamma^{\pi G}$, and hence $\chi \in \mathrm{D}_\pi(G)$. Thus $\mathrm{B}_\pi(G) \subseteq \mathrm{D}_\pi(G)$ as claimed. But by Theorem B, we have $|\mathrm{D}_\pi(G)| = |\mathrm{I}_\pi(G)| = |\mathrm{B}_\pi(G)|$, and hence $\mathrm{B}_\pi(G) = \mathrm{D}_\pi(G)$, as desired. ∎

4. **Supermonomial partial characters.** In this section, we prove Theorem F. The principal nontrivial fact on which we rely is that for p-solvable groups with $p \neq 2$, monomial irreducible characters of p-power degree are necessarily supermonomial. This result, which is discussed in [8] and proved there as Theorem 10.1, is a relatively easy consequence of a deep result of E. C. Dade. The argument we use here to obtain Theorem F from Theorem 10.1 is really the same as the proof of Theorem 10.2 in [8], which is essentially the odd-order case of Theorem F.

We need the following easy consequence of Theorem B.

LEMMA 4.1. *Let $H \subseteq G$, where G is π-separable and $2 \notin \pi$. Let $\chi \in \mathrm{D}_\pi(G)$ and suppose that $\chi^0 = \alpha^G$, where $\alpha \in \mathrm{I}_\pi(H)$. Then $\chi = \eta^G$ for some character $\eta \in \mathrm{Irr}(H)$ such that $\eta^0 = \alpha$.*

PROOF. To find η, first choose $\psi \in \mathrm{D}_\pi(H)$ with $\psi^0 = \alpha$. (This is possible by Theorem B). Now $(\psi^{\pi G})^0 = \alpha^G = \chi^0$ is irreducible, and thus $\psi^{\pi G}$ lies in $\mathrm{D}_\pi(G)$ by the easy part of Theorem E. Since also $\chi \in \mathrm{D}_\pi(G)$ and the map $\xi \longmapsto \xi^0$ is injective on $\mathrm{D}_\pi(G)$, we deduce that $\psi^{\pi G} = \chi$. We can now take $\eta = \delta_{(G:H)}\psi$ to complete the proof. ∎

PROOF OF THEOREM F. We are given $\varphi \in \mathrm{I}_p(G)$, where G is p-solvable and $p \neq 2$. It is clear that if any lift of φ in $\mathrm{Irr}(G)$ is monomial, then φ must be monomial. We may assume, therefore, that φ is monomial, and our task is to show that φ is supermonomial and that its lift in $\mathrm{D}_p(G)$ is also supermonomial.

Write $\varphi = \alpha^G$, where α is a linear p-partial character of some subgroup $A \subseteq G$, and assume that also $\varphi = \beta^G$, where $\beta \in \mathrm{I}_p(B)$ for some subgroup $B \subseteq G$. We must show that β is monomial and we consider first, the case where $\varphi(1)$ is a power of p.

We have $\chi \in \mathrm{D}_p(G)$ with $\chi^0 = \varphi$. Since $\alpha^G = \varphi = \beta^G$, it follows by Lemma 4.1 that we can find $\psi \in \mathrm{Irr}(A)$ and $\eta \in \mathrm{Irr}(B)$ such that $\psi^G = \chi = \eta^G$, where $\psi^0 = \alpha$ and $\eta^0 = \beta$. In particular, $\psi(1) = \alpha(1) = 1$, and hence χ is a monomial character. Since χ has p-power degree, it follows by Theorem 10.1 of [8] that it is supermonomial, and hence η is monomial. We deduce that $\beta = \eta^0$ is monomial, as required.

In the general case, where $\varphi(1)$ may not be a p-power, we can assume that β is primitive. By Corollary 5.5 of [8], we know that $\beta(1)$ is a power of p and, of course $\alpha(1) = 1$ is also a power of p. In other words, in the language of Section 6 of [8], each of the pairs (A, α) and (B, β) is a p-inducing pair belonging to φ, and thus each of these pairs is a node of the graph $\mathcal{G}(\varphi)$. By Theorem 6.2 of [8], therefore, if we replace the pair (A, α) by a suitable conjugate pair, we can assume that the two nodes (A, α) and (B, β) lie in the same connected component of this graph.

Recall that by the definition of the graph, nodes (U, μ) and (V, ν) are joined in $\mathcal{G}(\varphi)$ if either $U < V$ and $\mu^V = \nu$ or $V < U$ and $\nu^U = \mu$. We claim that whenever nodes (U, μ) and (V, ν) are joined, if either of the p-partial characters μ or ν is monomial, then the other is monomial too. To see this, assume that μ is monomial. In the case where $U < V$ and $\mu^V = \nu$, it is clear that ν is monomial, and so we need consider only the situation where $V < U$ and $\nu^U = \mu$. To prove that ν is monomial in this case, it suffices to show that the monomial p-partial character μ of U is actually supermonomial. But $\mu(1)$ is a p-power (since (U, μ) was assumed to be a p-inducing pair for φ) and this is the case of the theorem we have already proved. Thus ν is monomial, as claimed.

Recall that the pairs (A, α) and (B, β) lie in the same component of $\mathcal{G}(\varphi)$. Since α is linear, it is certainly monomial, and thus by the result of the previous paragraph, β is monomial too, as required.

Finally, we must prove that χ is supermonomial, where $\chi \in \mathrm{D}_p(G)$ and $\chi^0 = \varphi$. Assuming that $\chi = \psi^G$, where $\psi \in \mathrm{Irr}(H)$ and $H \subseteq G$, we need to show that ψ is monomial. If we write $\beta = \psi^0$, we see that $\beta^G = \varphi$ and since φ is supermonomial, we know that β is monomial.

Now write $\eta = \delta_{(G:H)}\psi$, so that $\eta^{pG} = \chi$ and $\eta^0 = \psi^0 = \beta$. Then $\eta \in \mathrm{D}_p(H)$ by Theorem E, and since β is monomial, it follows by Lemma 4.1 that η is monomial. We can thus write $\eta = \lambda^H$ for some linear character $\lambda \in \mathrm{Irr}(K)$, where $K \subseteq H$, and we see that $\psi = \big((\delta_{(G:H)})_K\lambda\big)^H$ is monomial, as desired. ∎

5. **One-prime M-groups.** Fix an odd prime p. We shall say that a p-solvable group G is a D_pM-*group* if every character $\chi \in \mathrm{D}_p(G)$ is monomial. Of course, an M-group is automatically a D_pM-group for every odd prime p. (Note also that every group of order not divisible by p is a D_pM-group since $\mathrm{D}_p(G)$ consists just of the principal character in this case.)

One of the principal outstanding conjectures about M-groups is that normal (and hence also subnormal) subgroups of odd-order M-groups must also be M-groups. We prove the corresponding assertion for D_pM-groups, and from this we obtain Theorem A of the introduction.

THEOREM 5.1. *If $N \lll G$, where G is a D_pM-group, then N is also a D_pM-group.*

It follows that subnormal subgroups of M-groups must be D_pM-groups for all odd primes p. This imposes a severe restriction on the collection of groups that can be subnormally embedded in an M-group, but it does not settle the normal-subgroup conjecture since there do exist odd-order groups that are D_pM-groups for all odd primes p, and yet are not M-groups.

Theorem 5.1 is an easy consequence of the following stronger result.

THEOREM 5.2. *Let $N \lll G$, where G is p-solvable for some odd prime p. If $\chi \in \mathrm{D}_p(G)$ is monomial, then every irreducible constituent of χ_N is monomial.*

PROOF. Since all irreducible constituents of χ_S lie in $\mathrm{D}_p(S)$ for all subnormal subgroups S of G, it suffices to prove the result in the case where N is a maximal normal subgroup of G. We assume this, therefore, and we let ψ be an irreducible constituent of χ_N.

Let T be the stabilizer in G of ψ and let $\eta \in \mathrm{Irr}(T \mid \psi)$ be the Clifford correspondent of χ with respect to ψ. Then $\eta^G = \chi$, and by Theorem F, χ is supermonomial (since it is monomial). It follows that η is monomial and we can write $\eta = \lambda^G$ for some linear character λ of some subgroup of $H \subseteq T$. If we can show that $\eta_N = \psi$, it will follow that $\psi = (\lambda_{H\cap N})^N$, as required.

We know that $\eta_N = e\psi$, where e is the multiplicity of ψ as a constituent of χ_N. Our task, therefore, is to show that this multiplicity is equal to 1. If $|G : N| = p$, this is clear, and thus we can assume that $|G : N|$ is not divisible by p. Write $\varphi = \chi^0$ and $\theta = \psi^0$, and note that both φ and θ are irreducible p-partial characters. Also, the multiplicity of θ as a constituent of φ_N is 1 since p does not divide $|G : N|$. (See, for example, Corollary 5.1 of [8].) It follows that the multiplicity of ψ as a constituent of χ_N cannot exceed 1, and the proof is complete. ∎

PROOF OF THEOREM 5.1. Let $\psi \in \mathrm{D}_p(N)$. To show that ψ is monomial, it suffices by Theorem 5.2 to find a character $\chi \in \mathrm{D}_p(G)$ that lies over ψ. Now $\theta = \psi^0$ lies in $\mathrm{I}_p(N)$ and it is easy to see that there exists $\varphi \in \mathrm{I}_p(G)$ such that θ is a constituent of φ_N. (This is Lemma 4.3 of [8].) By Theorem B, choose $\chi \in \mathrm{D}_p(G)$ such that $\chi^0 = \varphi$ and observe that since N is subnormal, all irreducible constituents of χ_N lie in $\mathrm{D}_p(N)$. Since θ is a constituent of $(\chi_N)^0$, it follows by Theorem B that ψ must be a constituent of χ_N, and the proof is complete. ∎

Recall that a primitive irreducible character of a π-separable group can be factored as a product of a π-special character and a π'-special character. (This theorem was proved first in [1], and it is discussed in [7], where it appears as Theorem 2.4.) We can use this factorization of primitive characters to prove the following extension of Theorem A.

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COROLLARY 5.3. *Let $N \lll G$, where G is a D_pM-group for some odd prime p. If $\psi \in \mathrm{Irr}(N)$ is primitive, then $\psi(1)$ is not divisible by p.*

PROOF. Factor $\psi = \alpha\beta$, where α is p-special and β is p'-special. Since α is p-special, it lies in $\mathrm{D}_p(N)$, and hence it is monomial by Theorem 5.1. But α is a factor of a primitive character, and thus α is primitive, and hence is linear. We conclude that $\psi(1) = \beta(1)$ is not divisible by p, as required. ∎

PROOF OF THEOREM A. We are given that $N \lll G$, where G is an M-group, and thus N is a D_pM-group for all odd primes p. By Corollary 5.3, the degree of a primitive irreducible character of N can be divisible by no odd prime, and thus it is a power of 2. ∎

6. **Examples and further remarks.** We mentioned that in general, the function $\mathrm{B}_\pi(\ )$ does not respect restriction to arbitrary subgroups. If $|G|$ is odd, however, and $\chi \in \mathrm{B}_\pi(G)$, then by Corollary 8.4 of [8], there is always a constituent of χ_H in $\mathrm{B}_\pi(H)$ for subgroups $H \subseteq G$. In particular, if we know that χ_H is irreducible, then it lies in $\mathrm{B}_\pi(H)$. (This is the result that Theorem D generalizes.) If $|G|$ is even, however, then it is possible to have $\chi \in \mathrm{B}_\pi(G)$ with χ_H irreducible, but with $\chi_H \notin \mathrm{B}_\pi(H)$. This can even happen if χ is π-special, as is shown in Example 8.2 of [4]. If χ is π-special and $2 \notin \pi$, however, no such example can occur. We have been unable to decide the following, however.

QUESTION 6.1. Let G be π-separable, where $2 \notin \pi$, and suppose $\chi \in \mathrm{B}_\pi(G)$. Suppose $H \subseteq G$ and that χ_H is irreducible. Must we have $\chi_H \in \mathrm{B}_\pi(H)$ in this case?

Of course, if it were always true that $\mathrm{B}_\pi(G) = \mathrm{D}_\pi(G)$ for π-separable groups G with $2 \notin \pi$, then Question 6.1 would have an affirmative answer, by Theorem D. In fact, however, $\mathrm{B}_\pi(G)$ and $\mathrm{D}_\pi(G)$ are different, in general.

EXAMPLE 6.2. Let $\pi = \{3\}$. Then there exists a π-separable group G for which $\mathrm{B}_\pi(G) \neq \mathrm{D}_\pi(G)$.

PROOF. Let G be the semidirect product of the symmetric group on four symbols, acting on an elementary abelian group V of order 3^4, where the symmetric group permutes a basis $\{v_1, v_2, v_3, v_4\}$ for V in its natural action. Let λ be the unique linear character of V such that $\lambda(v_1) = \lambda(v_2) = 1$, $\lambda(v_3) = \omega$ and $\lambda(v_4) = \overline{\omega}$, where ω is a fixed primitive cube root of unity.

The stabilizer of λ in S_4 is exactly the transposition $t = (1, 2)$, and thus V has index 2 in the stabilizer T of λ in G. We see that λ has exactly two extensions α and β to T, and these are distinguished by the values $\alpha(t) = 1$ and $\beta(t) = -1$. Write $\alpha^G = \chi$ and $\beta^G = \psi$ and observe that by the Clifford correspondence, these are distinct irreducible characters of G.

We see that $\alpha^0 = \beta^0$, and thus $\chi^0 = \psi^0 \in \mathrm{I}_\pi(G)$ by the Clifford correspondence for partial characters. Since $\chi \neq \psi$ and each is a lift of the same π-partial character, it suffices to show that $\chi \in \mathrm{B}_\pi(G)$ and $\psi \in \mathrm{D}_\pi(G)$.

First, we consider χ. It is easy to check that the pair (V, λ) is maximal among π-factored subnormal pairs in G. (See Section 4 of [7] for the relevant definitions.) Since

T is the stabilizer of this pair and $\alpha \in \mathrm{Irr}(T)$ is π-special and lies over λ, we deduce that $\chi = \alpha^G$ lies in $\mathrm{B}_\pi(G)$, as claimed.

To see that $\psi \in \mathrm{D}_\pi(G)$, it suffices to show that $\alpha^{\pi G} = \psi$, and for this purpose, we check that $\alpha\delta_{(G:T)} = \beta$. It suffices, therefore, to verify that $\delta_{(G:T)}(t) = -1$. To compute $\delta_{(G:T)}$, we choose a subgroup P such that $T \subseteq P \subseteq G$, where $|G : P| = 3$. Then $|P : T| = 4$, and hence $\delta_{(P:T)} = 1_T$ and $\delta_{(G:T)}(t) = \delta_{(G:P)}(t)$ by Lemma 2.1(a) and (c). Furthermore, since P is maximal in G, we see by Lemma 2.1(b) that $\delta_{(G:P)}$ is the permutation sign character of the action of P on the three right cosets of P in G. Since t is not in the core of P, the action of t is nontrivial, and thus it is an odd permutation. It follows that $\delta_{(G:P)}(t) = -1$, as desired. ∎

It is amusing to note that in general, the symmetric difference of the sets $\mathrm{B}_\pi(G)$ and $\mathrm{D}_\pi(G)$ consists of characters with degrees that are neither π-numbers nor π'-numbers. That a character with π-degree in either set must be in both is a consequence of the fact that such a character must be π-special. On the other hand, if $\chi \in \mathrm{B}_\pi(G)$, then we know that $\chi = \gamma^G$ for some π-special character γ of a subgroup $W \subseteq G$. If χ has π'-degree, then W must have π'-index, and thus $\delta_{(G:W)}$ is trivial and $\chi \in \mathrm{D}_\pi(G)$. Our assertion now follows (using Theorem B).

Finally, we consider the class of odd-order groups that can occur as normal (or subnormal) subgroups of M-groups. We know by Theorem 5.1 that such a group must be a D_pM-group for every odd prime p. If it were true that only M-groups had this property, this would prove the conjecture that odd-order normal subgroups of M-groups must be M-groups. Unfortunately, we have the following.

EXAMPLE 6.3. There exists an odd-order group G of that is a D_pM-group for every odd prime p, but that is not an M-group.

In fact, it is relatively easy to construct groups that are D_pM-groups for all odd primes.

LEMMA 6.4. *Suppose that $N \lhd G$, where N is a normal abelian Sylow q-subgroup of G and G/N is a D_pM-group for all odd primes p. Then G is a D_pM-group for all odd primes p.*

PROOF. Let $\chi \in \mathrm{D}_p(G)$. Then the irreducible constituents of χ_N lie in $\mathrm{D}_p(N)$, and so if $p \neq q$, these constituents are trivial and $N \subseteq \ker\chi$. In this case, we can view χ as a character of G/N, and it is easy to see that $\chi \in \mathrm{D}_p(G/N)$. (We omit the details of this routine argument.) If $p \neq q$, therefore, χ is monomial as a character of G/N and hence also as a character of G.

We can suppose now that $p = q$ so that $\chi = \theta^{qG}$ for some q-special character θ of some subgroup $H \subseteq G$. Since θ is q-special, however, it restricts irreducibly to a Sylow q-subgroup of H, which is abelian. It follows that θ is linear and thus $\chi = (\delta_{(G:H)}\theta)^G$ is monomial, as required. ∎

PROOF OF EXAMPLE 6.3. Choose any odd-order M-group H that has a subgroup K that is not an M-group. (Examples of this abound.) Next, choose an odd prime q not dividing $|H|$ and let N be an elementary abelian q-group of order $q^{|H:K|}$. Let H act on N

by permuting a basis so that K is the full stabilizer of one of the basis vectors and let G be the semi-direct product of N by H.

Since G/N is an M-group, it is certainly a D_pM-group for all odd primes p. By Lemma 6.4, therefore, G is a D_pM-group for all odd primes p, and it suffices to show that G is not an M-group.

There exists a linear character λ of N whose stabilizer in H is exactly K, and so its stabilizer in G is $T = NK$. Since λ extends to T and T/N is not an M-group, it is easy to see that T has a non-monomial irreducible character θ lying over λ. Also, $\chi = \theta^G$ is irreducible by the Clifford correspondence, and we claim that χ is not monomial.

Suppose that $\chi = \alpha^G$ for some linear character α of a subgroup X of G. We see that $|G : X| = \chi(1)$ is not divisible by q, and hence $N \subseteq X$. Since α_N lies under χ, it is G-conjugate to λ, and so we can replace the pair (X, α) by a conjugate pair and suppose that $\alpha_N = \lambda$. Then $X \subseteq T$ and $\alpha^T \in \mathrm{Irr}(T)$ lies over λ and under χ. It follows that $\alpha^T = \theta$, and this is a contradiction since θ is not monomial. ∎

We close with something only marginally related to the topic of this paper. In Theorem E of [4], it was shown that if $H \subseteq G$, where G is π-separable and $2 \notin \pi$, and if $\psi \in \mathrm{Irr}(H)$ is π-special and extends to G, then in fact, ψ has a π-special extension to G. It was asked in [4] whether or not this result might remain valid without the assumption that $2 \notin \pi$. In fact, the answer is 'yes', and even more is true, both in the case where $2 \in \pi$ and where $2 \notin \pi$.

COROLLARY 6.5. *Let $H \subseteq G$, where G is π-separable, and suppose that $\theta \in \mathrm{I}_\pi(H)$ extends to $\varphi \in \mathrm{I}_\pi(G)$.*

(a) If $2 \in \pi$, then the lift of θ in $\mathrm{B}_\pi(H)$ extends to the lift of φ in $\mathrm{B}_\pi(G)$.

(b) If $2 \notin \pi$, then the lift of θ in $\mathrm{D}_\pi(H)$ extends to the lift of φ in $\mathrm{D}_\pi(G)$.

PROOF. Suppose first that $2 \in \pi$ and let $\chi \in \mathrm{B}_\pi(G)$ with $\chi^0 = \varphi$. Write $\psi = \chi_H$ and observe that $\psi^0 = \theta$, so that ψ is irreducible. By results of [6], there is a 'magic' field automorphism that fixes χ, and so it also fixes $\psi = \chi_H$. As $(\psi)^0 = \theta$ is irreducible, Lemma 3.3 of [6] guarantees that $\psi \in \mathrm{B}_\pi(H)$, as required.

Assuming now that $2 \notin \pi$, let $\chi \in \mathrm{D}_\pi(G)$ lift φ and again write $\psi = \chi_H$. Then $\psi^0 = \theta$, and so ψ is irreducible, and it follows that $\psi \in \mathrm{D}_\pi(H)$ by Theorem D. ∎

COROLLARY 6.6. *Let H be π-separable and suppose $\theta \in \mathrm{I}_\pi(H)$. Then θ has a lift $\psi \in \mathrm{Irr}(H)$ such that whenever $H \subseteq G$, where G is π-separable, ψ extends to a character of G iff θ extends to a π-partial character of G.*

PROOF. If $2 \in \pi$, take $\psi \in \mathrm{B}_\pi(H)$ and otherwise, take $\psi \in \mathrm{D}_\pi(H)$. ∎

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