

# Subordinate Šil'nikov bifurcations near some singularities of vector fields having low codimension

H. W. BROER AND G. VEGTER

*Mathematics Institute, Rijksuniversiteit Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands*

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*Abstract.* A specific singularity of a vector field on  $\mathbb{R}^3$  is considered, of codimension 2 in the dissipative case and of codimension 1 in the conservative case. In both contexts in generic unfoldings the existence is proved of subordinate Šil'nikov bifurcations, which have codimension 1. Special attention is paid to the  $C^\infty$ -flatness of this subordinate phenomenon.

## 1. Introduction

In this paper we study some aspects of a specific singularity of a vector field in dimension three. We consider this singularity both in the general dissipative context and in the world of vector fields preserving a given volume. In both cases the singularity has low codimension, namely two and one respectively. So generically in two, respectively one, parameter families of such vector fields this singularity will be met as a bifurcation.

This singularity was studied earlier by e.g. Takens [16] and Guckenheimer [7] in the dissipative case, while the conservative case was investigated by Broer [2], [3], [4]. In these studies it has emerged that, for a  $C^2$ -open class of unfoldings of the singularity, so staying within the appropriate category, from this central singularity there is a branching of two hyperbolic saddle points. The distance between these saddle points grows parabolically in the bifurcation parameters. Each of these saddle points has a pair of complex eigenvalues, corresponding to an invariant manifold of dimension two. The other eigenvalue then, of course, is real and it corresponds to a one dimensional invariant manifold.

Owing to the interplay of the two saddle points their invariant manifolds may exhibit homoclinic intersection. Such homoclinic intersections are not transversal: if it happens for one of the saddle points, then its one dimensional invariant manifold is totally contained in its two dimensional invariant manifold. Note, however, that this homoclinic phenomenon has codimension one.

The dynamics near such a homoclinic orbit has been analyzed by Šil'nikov [12], [13]. See also [1]. If the eigenvalues of the saddle point satisfy some open condition,

then this dynamical behaviour is very complicated: it involves shifts on an alphabet with infinitely many symbols: the dynamics contains at least infinitely many Smale horseshoes. In § 2, we present a brief description of this Šil'nikov bifurcation.

The main question of this paper concerns the occurrence of such Šil'nikov homoclinic intersections, appearing as subordinate or secondary bifurcations in the generic unfoldings of the central singularity referred to above. We shall show that for  $C^\infty$  unfoldings this occurrence is a 'flat phenomenon' in the sense that it can be completely annihilated by a flat perturbation. Compare [15]. This implies that there exists a dense set of such unfoldings which do not have Šil'nikov bifurcations near the central singularity. In the complement of this dense set we shall construct 'persistent' examples of unfoldings exhibiting subordinate Šil'nikov bifurcations in a codimension one set of parameter points, which accumulates at the point of central bifurcation, see § 4. Our construction implies that this Šil'nikov phenomenon also occurs densely. We use perturbation techniques as introduced by Broer & van Strien [5]. The 'persistence' of the examples has to be understood in a delicate way as may be guessed from the above. It can be expressed in terms of a strong Whitney topology. Note, however, that our construction includes another proof of [7, theorem 5], which states that the occurrence of Šil'nikov's bifurcation is also open in the weak topology, if only one does *not* require these occurrences to be close to the central singularity. (See § 5.)

We conclude this introduction with a few remarks:

(i) It is our conjecture that the results of this paper can also be obtained in the real analytic case, using perturbation theory as introduced in [17].

(ii) Note that the flatness of the described Šil'nikov phenomenon implies that it cannot be traced on any finite jet of the central bifurcation. This makes the phenomenon rather unwieldy for physical applications.

(iii) Sometimes the dynamics near Šil'nikov's bifurcation has been connected speculatively with the existence of a strange attractor. Mostly this is inspired by numerical work. See e.g. [1]. Since however this Šil'nikov bifurcation may occur both in dissipative and conservative systems, see below, and since in conservative systems there can be no attractors, it seems that the presence of dissipation induces a dynamics exceeding Šil'nikov's results, [13], concerning the shift. Compare recent work of Tangerman involving a dissipative bifurcation of the horseshoe map.

(iv) In our construction of the flat perturbations we do not need Mel'nikov functions to prove the existence of homoclinic orbits.

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## 2. The Šil'nikov bifurcation

In this section we describe briefly the Šil'nikov bifurcation for its own sake, partly quoting from Šil'nikov [12], [13], Arneodo, Couillet & Tresser [1] and others.

In  $\mathbb{R}^3$  consider a hyperbolic saddle point with eigenvalues  $\varepsilon \pm i\delta$  and  $-\gamma$ :  $\varepsilon$ ,  $\delta$  and  $\gamma$  being positive numbers. Assume that  $\varepsilon < \gamma$ . Consider the situation where the one dimensional invariant manifold of this saddle point, which corresponds to the

eigenvalue  $-\gamma$ , is a homoclinic orbit. (These conditions are modulo change of direction.)

Note that this phenomenon has codimension one. This means the following. Suppose that  $X^\mu$ ,  $\mu \in \mathbb{R}$ , is a one parameter family of vector fields on  $\mathbb{R}^3$  which for  $\mu = \mu_0$  possesses such a homoclinic orbit. Also assume that the homoclinic 'crossing' happens with positive speed as  $\mu$  passes through  $\mu_0$ . See figure 1.

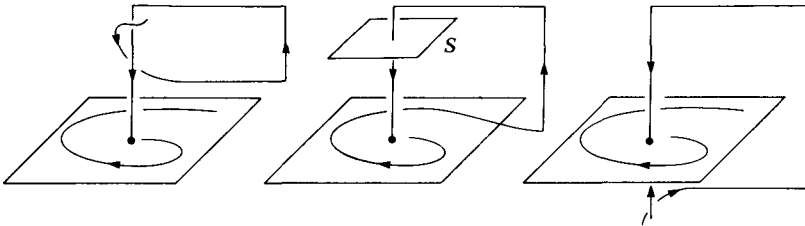


FIGURE 1. Šil'nikov's bifurcation.

Then any one parameter family  $Y^\mu$ ,  $\mu \in \mathbb{R}$ , which is close enough to  $X^\mu$  in the  $C^1$ -topology, also exhibits such a homoclinic 'crossing' for some  $\nu_0$  close to  $\mu_0$ . In other words: this Šil'nikov bifurcation as a one parameter family is persistent for  $C^1$ -small perturbations.

Observe that the above fact is true both for dissipative and for conservative systems. In the conservative case the condition  $\varepsilon < \gamma$  on the eigenvalues is automatically fulfilled, since then the trace of the linear part in the saddle point has to be zero.

In order to describe the dynamics near the homoclinic orbit take a transversal section  $S$  as depicted in figure 1. On a part of  $S$  the Poincaré first return map is well defined. At the moment of bifurcation, i.e. of homoclinic intersection, this map possesses at least infinitely many horseshoes. In fact it is more complicated than that: for a symbolic dynamics on the non-wandering set one needs an alphabet of infinitely many symbols. In other words: the restriction of this Poincaré map to (a part of) its non-wandering set is conjugate to a subshift on an infinite alphabet. For the vector field this yields an Axiom A basic set, containing a countable dense set of hyperbolic closed orbits of saddle type.

### 3. The central bifurcation; normal forms

We now proceed to describe the singularity of a vector field on  $\mathbb{R}^3$  which plays the central rôle in our present study. We shall also present normal forms for unfoldings of this central singularity, both in the conservative and in the non-conservative contexts.

To this purpose consider on  $\mathbb{R}^3$  a vector field which has the origin as a singularity, where for some positive  $\alpha$  the eigenvalues of the linear part are  $0$ ,  $+\alpha$  and  $-\alpha$ . In Jordan normal form we may write for this linear part

$$(1) \quad \begin{cases} \dot{x} = -\alpha y \\ \dot{y} = \alpha x \\ \dot{z} = 0. \end{cases}$$

Evidently in the linear context such a singularity has codimension two, since it will be met only in generic families of linear systems, if they contain at least two parameters. Further observe that the linear system (1) has divergence zero: the corresponding matrix has trace zero. If one restricts to divergence zero linear systems, then this singularity has codimension one: it may occur already in generic one parameter families.

Note that the linear system (1) generates rotations around the  $z$ -axis. The idea now is to do the following: For any unfolding of the above singularity we successively make the higher order terms of the Taylor series invariant under these rotations, carrying out coordinate transformations. In the conservative case these transformations can be kept volume preserving. So the Taylor series achieves a normal form. Here we follow techniques which go back to Poincaré [10] and which were developed further by e.g. Takens [16] and Broer [3]. To be more precise:

**NORMAL FORM THEOREM.** *Let  $X = X^\mu(\xi)$  be a  $C^\infty$  family of vector fields, where  $\xi = (x, y, z)$  varies over  $\mathbb{R}^3$  and where  $\mu$  is a  $k$ -dimensional parameter. Suppose that for  $\mu = 0$  the vector field in  $\xi = 0$  has a singularity with first order term (1). Then, up to a  $C^\infty$ ,  $\mu$ -dependent change of coordinates, one may write  $X = \tilde{X} + p$ , where:*

(i) *in cylindrical coordinates  $r, \varphi$  and  $z$  the vector field  $\tilde{X} = \tilde{X}^\mu(\xi)$  has the form*

$$(2) \quad \begin{cases} \dot{\varphi} = f(r^2, z, \mu) \\ \dot{r} = rg(r^2, z, \mu) \\ \dot{z} = h(r^2, z, \mu), \end{cases}$$

*expressing rotational symmetry; also  $g(0, 0, 0) = h(0, 0, 0) = (\partial h / \partial z)(0, 0, 0) = 0$  and  $f(0, 0, 0) = \alpha$ , which means that the first order terms of (2) are (1);*

(ii)  *$p = p^\mu(\xi)$  is flat in  $(\xi, \mu) = (0; 0) \in \mathbb{R}^3 \times \mathbb{R}^k$ ;*

(iii) *in the case where  $X = X^\mu(\xi)$  has divergence zero, the change of coordinates may be chosen volume preserving, such that both  $\tilde{X} = \tilde{X}^\mu(\xi)$  and  $p = p^\mu(\xi)$  have divergence zero.*

In the dissipative case we shall take  $k = 2$ , in the conservative case  $k = 1$ . Also we shall assume that  $f(r^2, z, \mu) \equiv 1$  which can be obtained by a reparametrisation of the vector fields, since  $\alpha \neq 0$ . This is not an essential restriction. (For the divergence free case this is not completely trivial; see Broer [2].) We also note the following: apart from a formal procedure, which symmetrises the  $\infty$ -jet of unfoldings in an inductive process, the above theorem needs a Borel-like property which states that for all formal power series there exists representing germs; see Broer [3].

#### 4. Setting of the problem; main results

In this section we first specify a  $C^2$  open class of unfoldings of the central singularity, which may exhibit subordinate bifurcations of the Šil'nikov type. We do this by giving open conditions on the coefficients of the lower order terms in the normalised Taylor series. See (2).

(a) *The dissipative case.* We consider the normal form (2) with two parameters, to be denoted by  $\mu$  and  $\nu$ . In this case we may obtain as an extra consequence that

in (2)

$$\frac{\partial h}{\partial \nu}(0; 0) = \frac{\partial g}{\partial \mu}(0; 0) = 0.$$

As a first open condition we require that

$$\frac{\partial h}{\partial \mu}(0; 0) \neq 0 \neq \frac{\partial g}{\partial \nu}(0; 0).$$

A rescaling of the parameters then leads to

$$\frac{\partial h}{\partial \mu}(0; 0) = -1, \quad \frac{\partial g}{\partial \nu}(0; 0) = 1.$$

Now, if we truncate (2) at order two, we obtain

$$(3) \quad \begin{cases} \dot{\varphi} = 1 \\ \dot{r} = r(\nu - a_1 z) \\ \dot{z} = -\mu + b_1 r^2 + b_2 z^2 + b_3 \mu z + b_4 \nu z + b_5 \mu \nu, \end{cases}$$

where  $a_1, b_1, b_2, b_3, b_4$  and  $b_5$  are real constants. As a second open condition – to be explained below – we require that

$$0 < a_1 < 2b_2, \quad b_1 > 0 \quad \text{and} \quad b_4 > 0.$$

We now have

**LEMMA A.** *Under the above open conditions the symmetric vector field  $\tilde{X}^{\mu, \nu}$ , for sufficiently small  $\mu^2 + \nu^2$  and positive  $\mu$ , has the following properties:*

- (i)  $\tilde{X}^{\mu, \nu}$  possesses two hyperbolic saddle points  $s_{\pm}(\mu, \nu)$  lying on the  $z$ -axis with  $z$ -coordinates  $\pm\sqrt{\mu/b_2} + O((\mu^2 + \nu^2)^{1/2})$ ;
- (ii) the interval in-between  $s_{\pm}(\mu, \nu)$  on the  $z$ -axis is a saddle connection between the two saddle points;
- (iii) in the  $(\mu, \nu)$ -plane there exists a  $C^\infty$ -curve  $\Gamma$  of the form  $\nu = m\mu + O(\mu^{3/2})$  as  $\mu \downarrow 0$ , such that for  $(\mu, \nu) \in \Gamma$  the saddle points  $s_{\pm}(\mu, \nu)$  have coinciding invariant manifolds of dimension two. The constant  $m$  is completely determined by the 3-jet of  $\tilde{X}$  at  $(0; 0) \in \mathbb{R}^3 \times \mathbb{R}^2$ .

*Proof.* In (2) omitting the angular component  $\dot{\varphi} = 1$  we obtain a reduction  $\bar{X} = \bar{X}^{\mu, \nu}(r, z)$  to the  $(r, z)$ -plane. In  $\bar{X}$  for  $\mu > 0$  we rescale the parameters, the coordinates and the time in the following way:

$$\delta = \sqrt{\mu}, \quad \varepsilon = \delta^{-1}\nu, \quad \bar{r} = \delta^{-1}r, \quad \bar{z} = \delta^{-1}z \quad \text{and} \quad \bar{t} = \delta t,$$

so obtaining a system  $\bar{X}^{\delta, \varepsilon}(\bar{r}, \bar{z})$

$$(4) \quad \begin{cases} \frac{d\bar{r}}{d\bar{t}} = \bar{r}(\varepsilon - a_1 \bar{z} + \delta(c_1 \bar{z}^2 + c_2 \bar{r}^2) + O(\delta^2 + \varepsilon^2)) \\ \frac{d\bar{z}}{d\bar{t}} = -1 + b_1 \bar{r}^2 + b_2 \bar{z}^2 + (b_3 \delta + b_4 \varepsilon) \bar{z} + \delta(d_1 \bar{r}^2 \bar{z} + d_2 \bar{z}^3) + O(\delta^2 + \varepsilon^2) \end{cases}$$

as  $\delta \downarrow 0, \varepsilon \rightarrow 0$  uniformly on compact neighbourhoods of  $0 \in \mathbb{R}^3$ . Note that the constants  $c_1, c_2, d_1$  and  $d_2$  are uniquely determined by the 3-jet of  $\tilde{X}$  at  $(0; 0) \in \mathbb{R}^3 \times \mathbb{R}^2$ .

Parts (i) and (ii) of the lemma follow easily from (4). To prove the remaining part, consider the function

$$H(\bar{r}, \bar{z}) = \bar{r}^\alpha (b_1 b_2 \bar{r}^2 + b_2(a_1 + b_1)\bar{z}^2 - (a_1 + b_1)), \quad \text{with } \alpha = 2b_2/a_1.$$

It is easy to see that  $H$  is a first integral of the vector field  $\bar{X}^{0,0}$ . Let  $U(\delta, \epsilon)$  denote the unstable manifold of the saddle point  $s_-(\delta, \epsilon)$  of  $\bar{X}^{\delta,\epsilon}$ , and  $S(\delta, \epsilon)$  the stable manifold of  $s_+(\delta, \epsilon)$  (see figure 2).

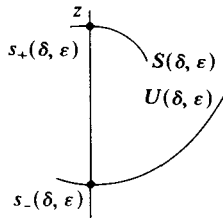


FIGURE 2

We denote the values of  $H$  at the points where  $S$  and  $U$  intersect the  $\bar{r}$ -axis for the first time by  $H(\delta, \epsilon; +)$  and  $H(\delta, \epsilon; -)$  respectively. The values of  $\delta$  and  $\epsilon$  for which  $S(\delta, \epsilon)$  and  $U(\delta, \epsilon)$  coincide are determined by the equation

$$(5) \quad H(\delta, \epsilon; +) - H(\delta, \epsilon; -) = 0$$

from which we can solve  $\epsilon$  as a function of  $\delta$  using the implicit function theorem (cf. the techniques used by Carr [6]). Note that  $\delta = \epsilon = 0$  satisfies the equation (5), since  $U(0, 0)$  and  $S(0, 0)$  coincide, constituting part of the ellipse

$$b_1 b_2 \bar{r}^2 + b_2(a_1 + b_1)\bar{z}^2 - (a_1 + b_1) = 0. \quad \square$$

*Remark.* The condition  $b_4 > 0$  imposed on the system (3) justifies the use of the implicit function theorem to solve equation (5).

Geometrically the lemma means that for  $(\mu, \nu) \in \Gamma$  the symmetric approximation  $\bar{X}^{\mu,\nu}$  of the unfolding  $X^{\mu,\nu}$  exhibits an invariant 2-sphere with the saddle points  $s_\pm(\mu, \nu)$  at its ‘north’ and ‘south’ poles.

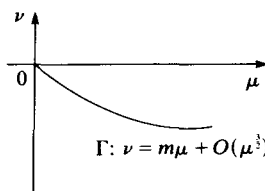


FIGURE 3

This ‘globe’ minus the opposite pole is the two dimensional invariant manifold of each of these two saddle points. Note that the size of this globe is of order  $\sqrt{\mu}$ .

Within the 3-disc bounded by the globe there is a connection of the saddle points  $s_+(\mu, \nu)$  and  $s_-(\mu, \nu)$  along the  $z$ -axis.

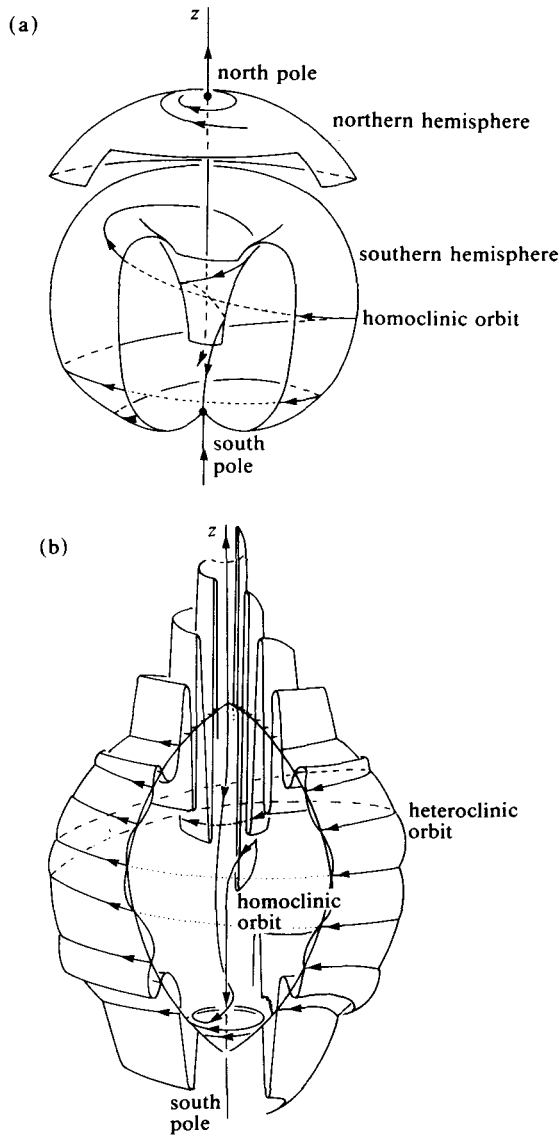


FIGURE 4. 'Cut open' perturbed globe with Šil'nikov homoclinic orbit.

The condition  $0 < a_1 < 2b_2$  ensures that both saddle points have eigenvalues which satisfy the open condition necessary for the possibility of the Šil'nikov bifurcation. See § 2.

Of course for the symmetric family  $\tilde{X}$  there will be no homoclinic intersections, but the 'perturbation'  $X = \tilde{X} + p$  generically is not symmetric any more. In fact, according to the Kupka–Smale theorem, see e.g. [9], for generic vector fields  $X^{\mu,\nu}$  the invariant manifolds will be transversal. Slightly more subtle results hold in our situation with parameters; compare [14] or [8]. To be more precise: for generic 'perturbations'  $X = \tilde{X} + p$ :

- (i) the saddle points  $s_{\pm}(\mu, \nu)$  are persistent and for sufficiently small  $\mu^2 + \nu^2$  they remain hyperbolic and satisfying Šil'nikov's condition;
- (ii) the saddle connection along the  $z$ -axis breaks;
- (iii) in the complement of a codimension one set of parameter points the two dimensional invariant manifolds of the saddle points are transversal.

The transversality of the two dimensional invariant manifolds, described in (iii), is depicted in figures 4a and b. (Figure 4b is a slightly modified version of [2, fig. 21], also see [3, fig. 5]. Again we thank J. J. Duistermaat for his help in designing it.) In the first case the invariant manifolds have empty intersection, in the second they meet transversally along heteroclinic orbits.

Moreover these pictures correspond to a perturbation  $p = p^{\mu, \nu}(\xi)$  and a parameter point  $(\mu, \nu)$  such that  $X^{\mu, \nu} = \tilde{X}^{\mu, \nu} + p^{\mu, \nu}$  has a Šil'nikov homoclinic orbit. The existence of such perturbations  $p = p^{\mu, \nu}(\xi)$  is the content of our main result, the dissipative version of which is formulated now. The proof is postponed to the last section.

**THEOREM OF THE EXISTENCE OF 'ŠIL'NIKOV UNFOLDINGS': DISSIPATIVE CASE.** *Let  $\tilde{X} = \tilde{X}^{\mu, \nu}(\xi)$  be any symmetric family which satisfies the above open conditions. Also let  $\Gamma$  be as in lemma A. Then there exist flat perturbations  $p = p^{\mu, \nu}(\xi)$  such that along the curve  $\Gamma$  the family  $X = \tilde{X} + p$  possesses a sequence of Šil'nikov bifurcations. This sequence occurs at parameter points  $(\mu_l, \nu_l) \in \Gamma$ , which accumulate at  $(\mu, \nu) = (0, 0)$ . Moreover in the  $(\mu, \nu)$ -plane there exists an immersed  $C^\infty$  curve  $C$ , having possible components, such that*

- (i)  $C$  lies in a wedge-shaped neighbourhood  $W$  of  $\Gamma$  in  $\mathbb{R}^2 \setminus \{(0, 0)\}$ , of a width that is flat in  $\mu$ , as  $\mu \downarrow 0$ ;
- (ii) the boundary of  $C$  has empty intersection with  $W$ ;
- (iii) the curves  $\Gamma$  and  $C$  intersect transversally at the points  $(\mu_l, \nu_l)$ ,  $l \in \mathbb{N}$ ;
- (iv) if the parameter pair  $(\mu, \nu)$  moves transversally through the curve  $C$ , then  $X^{\mu, \nu}$  undergoes Šil'nikov's bifurcation.

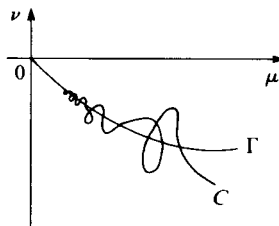


FIGURE 5

**Remark.** See figure 5. The existence of the curve  $C$  follows from the earlier part of the theorem using the same type of Kupka–Smale arguments as quoted before. These arguments concern transversality (implicit function theorem) in the sense that the homoclinic orbits cross the two dimensional invariant manifolds with positive speed. See § 2.



(b) *The conservative case:* In the second part of this section we consider the normal form (2) in the conservative case, i.e. with one real parameter, to be denoted by  $\mu$ . In the first place we require as a generic condition that

$$\frac{\partial h}{\partial \mu}(0; 0) \neq 0,$$

and a rescaling of the parameter again leads to

$$\frac{\partial h}{\partial \mu}(0; 0) = -1.$$

See above. Truncation of the normal form at order two then yields

$$(6) \quad \begin{cases} \dot{\phi} = 1 \\ \dot{r} = r(-a_1 z + a_2 \mu) \\ \dot{z} = -\mu + b_1 r^2 + a_1 z^2 - 2a_2 \mu z + b_2 \mu^2, \end{cases}$$

for real constants  $a_1, a_2, b_1$  and  $b_2$ . We now require the open condition

$$a_1 > 0 \quad \text{and} \quad b_1 > 0.$$

In this case we have (similar to lemma A):

**LEMMA B.** *Under the above open conditions the symmetric vector field  $\tilde{X}$ , for sufficiently small positive  $\mu$ , has the following properties:*

- (i)  $\tilde{X}^\mu$  possesses two hyperbolic saddle points  $s_\pm(\mu)$  lying on the  $z$ -axis, with  $z$ -coordinates  $\pm\sqrt{\mu/a_1} + O(|\mu|)$ ;
- (ii) along the  $z$ -axis there is a saddle connection between  $s_+(\mu)$  and  $s_-(\mu)$ ;
- (iii) the two dimensional invariant manifolds of  $s_+(\mu)$  and  $s_-(\mu)$  coincide (if one disregards the singularities  $s_\pm(\mu)$  themselves).

*Proof:* See Broer [2], [3]. In the conservative case the symmetric family  $\tilde{X}$  is integrable,  $\tilde{X}^\mu$  possesses the first integral

$$H^\mu(r, z) = -\frac{1}{2}r^2(-\mu + \frac{1}{2}b_1 r^2 + a_1 z^2 - 2a_2 \mu z + b_2 \mu^2) + \text{h.o.t.} \quad \square$$

Remarks, similar to the ones made after lemma A, apply here. Things are a bit simpler now, since there is only one parameter involved. Again there is the  $\tilde{X}^\mu$ -invariant globe, with size of order  $\sqrt{\mu}$ , in this case occurring for an open set of parameter values. The north and south pole saddle point are connected along the  $z$ -axis. We here recall the fact, mentioned in § 2, that the condition on the eigenvalues necessary for the possibility of Šil'nikov's bifurcation, is automatically fulfilled in this conservative (divergence zero) case.

Also here there is a Kupka–Smale theorem according to Robinson [11], implying that for a generic divergence zero vector field the invariant manifolds will be transversal. In our situation, with a parameter, it means that for a generic perturbation  $p = p^\mu(\xi)$  the vector fields  $X^\mu = \tilde{X}^\mu + p^\mu$  have transversal invariant manifolds, except for a discrete set of parameter values. Note that in this divergence zero situation only the case of figure 4b can occur!

We now formulate our main result for the conservative case:

**THEOREM OF THE EXISTENCE OF ‘ŠIL’NIKOV UNFOLDINGS’: CONSERVATIVE CASE.**  
*Given any symmetric family  $\tilde{X}$  which satisfies the above open conditions, there exist flat perturbations  $p = p^\mu(\xi)$  such that the family  $X = \tilde{X} + p$  possesses a sequence of Šil’nikov’s bifurcations, taking place for parameter values  $\mu_1 > \mu_2 > \dots$ , accumulating at  $\mu = 0$ .*

(c) We conclude this section by making some general remarks concerning the dynamics of the unfoldings discussed in the parts (a) and (b) of this section.

(i) Consider the saddle manifolds both for  $\tilde{X}$  and  $X = \tilde{X} + p$ , see figure 4. The  $X$ -invariant manifolds remain within a certain neighbourhood of the  $\tilde{X}$ -invariant manifolds and this neighbourhood collapses in a flat manner into the  $\tilde{X}$ -invariant manifolds, as the point of central bifurcation is approached. (The  $\tilde{X}$ -invariant globe, in turn, collapses parabolically into the central singularity.)

(ii) The 3-disc which is bounded by the  $\tilde{X}$ -invariant globe may contain quasi-periodic flow. In the conservative case it is even established that the 3-disc contains a vague attractor of Kolmogorov, see Broer [4]. For a definition of vague attractor see Abraham & Marsden [0].

(iii) The two dimensional  $X$ -invariant manifolds have an effect of uncertainty on integral curves which come close to them. If such an integral curve comes from the ‘outside’, then the  $X$ -invariant manifolds may capture it and pass it to the ‘inside’. Then, after an indeterminate amount of time, it may come out again, but also it may remain ‘inside’ for ever after (as the parameters move also). This last remark only applies to the case of figure 4b.

### 5. Denseness and persistence

We now present a description of the ‘genericity’ of our Šil’nikov unfoldings. The ‘persistence’ especially is problematic: we shall describe it in terms of a strong  $C^1$ -Whitney topology, or equivalently in terms of a specific topology on a space of germs.

The conservative and the non-conservative case will be treated simultaneously. We start with the central singularity as introduced in § 3. Let  $\mathcal{X}$  denote the set of all  $C^\infty k$ -parameter unfoldings of this singularity, situated at the origin of  $\mathbb{R}^3$ . As before we take  $k = 1$  in the conservative case and  $k = 2$  in the non-conservative case. In this section we consider a  $k$ -parameter family of vector fields on  $\mathbb{R}^3$  as a ‘vertical’ vector field on  $\mathbb{R}^3 \times \mathbb{R}^k$ . Now we fix a relatively compact neighbourhood  $K$  of  $(0; 0)$  in  $\mathbb{R}^3 \times \mathbb{R}^k$ . The set  $\mathcal{X}_K$  is defined as containing all restrictions of unfoldings in  $\mathcal{X}$  to the set  $K$ . Next let  $\mathcal{J}_K$  be the subset of  $\mathcal{X}_K$  defined by the open conditions on the 2-jet of the normal form, as specified in § 4(a), (b). Also we define  $\tilde{\mathcal{J}}_K$  as the subset of  $\mathcal{J}_K$  of which the elements have a *symmetrisable germ* at  $(0; 0) \in \mathbb{R}^3 \times \mathbb{R}^k$ , symmetrisable by an appropriate change of coordinates. Finally let  $\mathcal{S}$  denote the subset of  $\mathcal{J}_K$  where the unfoldings exhibit our Šil’nikov phenomenon: every neighbourhood of  $(0; 0)$  in  $\mathbb{R}^3 \times \mathbb{R}^k$  contains an infinite number of subordinate Šil’nikov bifurcations, occurring in a codimension one set of parameter points.

We now formulate the main result of this section:

TOPOLOGICAL THEOREM: CONSERVATIVE AND DISSIPATIVE CASE. (i)  $\mathcal{F}_K$  is a  $C^2$ -open subset of  $\mathcal{X}_K$ ;

(ii)  $\tilde{\mathcal{F}}_K$  is a  $C^\infty$ -dense subset of  $\mathcal{F}_K$ ;

(iii)  $\mathcal{S} \subset \mathcal{F}_K \setminus \tilde{\mathcal{F}}_K$  is a  $C^\infty$ -dense subset of  $\mathcal{F}_K$ ;

(iv) if the elements of  $\mathcal{X}$  are restricted to  $\mathbb{R}^3 \times (\mathbb{R}^k \setminus \{0\})$ , then correspondingly  $\mathcal{S}$  is an open subset of  $\mathcal{F}_K$  in the strong  $C^1$ -Whitney topology.

*Proof.* (i) is trivial.

(ii) is an immediate consequence of the normal form theorem.

(iii) follows from the normal form theorem and the existence theorems in § 4.

In order to prove (iv) we observe the following (conservative case only, the other case is analogous):

Let  $X = X^\mu(\xi)$  be an unfolding in  $\mathcal{F}_K \setminus \tilde{\mathcal{F}}_K$ , which for the parameter values  $\mu_1 > \mu_2 > \dots$ , accumulating at  $\mu = 0$ , undergoes Šil'nikov's bifurcation. For fixed  $l \in \mathbb{N}$  we consider the family  $X^\mu$ , as  $\mu$  varies near  $\mu_l$ . The  $C^1$ -persistence of the Šil'nikov bifurcation then yields a neighbourhood  $\mathcal{W}_l$  of  $X = X^\mu(\xi)$  in the set  $\mathcal{F}_K$ , such that all families  $Y \in \mathcal{W}_l$  undergo Šil'nikov bifurcation at some  $\nu_l$  near  $\mu_l$ . Now define

$$\mathcal{W} = \bigcap_{l=1}^{\infty} \mathcal{W}_l;$$

then for all  $Y \in \mathcal{W}$  one finds a Šil'nikov sequence  $\nu_1 > \nu_2 > \dots$ . The normal form theorem clearly implies that for all such  $Y$  the difference  $X - Y$  must be flat in  $(\xi, \mu) = (0; 0)$ : the neighbourhoods  $\mathcal{W}_l$  'decrease' as  $l$  increases. Now it is clear that if the domain of definition is restricted to  $K \cap \{\mu > 0\}$  the set  $\mathcal{W}$  is a strong  $C^1$ -neighbourhood of  $X$ . □

*Remarks.* (i) Point (iv) in the above theorem points out in which way our Šil'nikov phenomenon is persistent. We shall now describe this persistence in an alternative way. To this purpose fix an  $\infty$ -jet  $u$  of an unfolding in  $\mathcal{F}_K$ , taken at  $(0; 0) \in \mathbb{R}^3 \times \mathbb{R}^k$ . Then define  $\mathcal{X}(u)$  to be the set of germs at  $(0; 0) \in \mathbb{R}^3 \times \mathbb{R}^k$  of all unfoldings in  $\mathcal{F}_K$ , having this fixed  $u$  as their  $\infty$ -jet. A natural topology on  $\mathcal{X}(u)$  can be given in the following way: For any germ  $\rho: (\mathbb{R}^3 \times \mathbb{R}^k, (0; 0)) \rightarrow (\mathbb{R}, 0)$  of a flat  $C^\infty$  function, which is positive except at  $(0; 0)$  and for any arbitrary  $X \in \mathcal{X}(u)$  define the set

$$\mathcal{V}_\rho(X) = \{ Y \in \mathcal{X}(u) : \| Y(x, \mu) - X(x, \mu) \| + \| DY(x, \mu) - DX(x, \mu) \| < \rho(x, \mu) \text{ as germs} \}.$$

Evidently the sets  $\mathcal{V}_\rho(X)$  define a system of neighbourhoods for a topology on  $\mathcal{X}(u)$ . Equivalent to point (iv) in the above theorem we now may say that the germs corresponding to the set  $\mathcal{S}$ , having the same  $\infty$ -jet  $u$ , constitute an open subset of  $\mathcal{X}(u)$ .

(ii) We conjecture that the property of not having such a Šil'nikov 'sequence' is just as well persistent in this same way.

(iii) As a consequence of the above theorem we see that  $\mathcal{F}_K$  is an open set of unfoldings, none of which is structurally stable: the elements in  $\tilde{\mathcal{F}}_K$  and in  $\mathcal{F}_K \setminus \tilde{\mathcal{F}}_K$  are topologically different and both sets are dense.

6. Existence proof

In this final section we shall prove the existence theorems from § 4, presenting appropriate flat perturbation terms  $p$  for a given symmetric unfolding  $\tilde{X}$ , such that  $X = \tilde{X} + p$  exhibits the Šil'nikov phenomenon.

Our main considerations are concerned with the case of figure 4b, which can occur both in the dissipative and the conservative context, see above. We construct flat terms which have divergence zero, since in that case the perturbation technique is most difficult. It may be emphasised, however, that this also provides a proof of the existence theorem in the dissipative situation!

At the end, moreover, we include a remark on the dissipative case which is depicted in figure 4a.

(a) As before a point  $\xi \in \mathbb{R}^3$  will be given Cartesian coordinates  $(x, y, z)$  or cylindrical coordinates  $(r, \varphi, z)$ , where as usual  $x = r \cos \varphi$  and  $y = r \sin \varphi$ . Following Robinson [11] we start with divergence zero perturbations  $P = P^\mu(\xi)$  of type

$$(7) \quad \begin{cases} \dot{x} = -\frac{\partial}{\partial y}(x\beta_\mu(\xi)) \\ \dot{y} = \frac{\partial}{\partial x}(x\beta_\mu(\xi)) \\ \dot{z} = 0, \end{cases}$$

where  $\beta = \beta_\mu(\xi)$  is a bump function. We control the perturbation by writing

$$p^\mu(\xi) = \delta(\mu)P^\mu(\xi),$$

where  $\delta = \delta(\mu)$  is a suitable function, flat for  $\mu = 0$ . Also compare Broer [2].

(b) For the sake of convenience we rescale the coordinates by  $\sqrt{\mu}$ , that is we introduce new variables  $\bar{x}, \bar{y}, \bar{z}, \bar{r}$  and  $\bar{\varphi}$  by

$$x = \bar{x}\sqrt{\mu}, \quad y = \bar{y}\sqrt{\mu}, \quad z = \bar{z}\sqrt{\mu},$$

which implies that

$$r = \bar{r}\sqrt{\mu} \quad \text{and} \quad \varphi = \bar{\varphi}.$$

In these rescaled coordinates the symmetric vector field  $\tilde{X}^\mu$  blows up to

$$(8) \quad \begin{cases} \dot{\bar{\varphi}} = 1 \\ \dot{\bar{r}} = -a_1\bar{r}\bar{z}\sqrt{\mu} + O(\mu) \\ \dot{\bar{z}} = (b_1\bar{r}^2 + a_1\bar{z}^2 - 1)\sqrt{\mu} + O(\mu), \end{cases}$$

as  $\mu \downarrow 0$ , uniformly on compact sets containing the origin. Compare (6). Note that now the invariant 3-disc under consideration has a size of order 1. In the limit for  $\mu \downarrow 0$  its boundary is the pure ellipsoid with equation

$$\frac{b_1}{2}\bar{r}^2 + a_1\bar{z}^2 = 1.$$

Also consider the perturbation (7) in the rescaled variables. Its blown up version becomes

$$(9) \quad \begin{cases} \dot{x} = -\frac{1}{\sqrt{\mu}} \frac{\partial}{\partial \bar{y}} (\bar{x}\beta_\mu(\bar{\xi}\sqrt{\mu})) \\ \dot{y} = \frac{1}{\sqrt{\mu}} \frac{\partial}{\partial \bar{x}} (\bar{x}\beta_\mu(\bar{\xi}\sqrt{\mu})) \\ \dot{z} = 0. \end{cases}$$

Observe that if  $p^\mu(\xi) = \delta(\mu)P^\mu(\xi)$  is flat for  $(\xi; \mu) = (0; 0)$ , then its blown up version is flat, uniformly on compact sets containing the origin, as  $\mu \downarrow 0$ .

(c) Our perturbation  $p = p^\mu(\xi)$ , which creates a Šil'nikov 'sequence', will be constructed in two steps. In the first step we perturb the two dimensional invariant manifolds near the plane  $\bar{z} = 0$ . In the second step we break the saddle connection along the  $\bar{z}$ -axis. Here we use the perturbation techniques from Broer & van Strien [5].

The first step consists of the construction of a divergence zero perturbation  $P_1^\mu$  with support in a flow-box of toroidal shape, centred at the circle  $\bar{x}^2 + \bar{y}^2 = 2/b_1$ ,  $\bar{z} = 0$ , which is near the equator of the  $\bar{X}^\mu$ -invariant globe and disjoint from the  $\bar{z}$ -axis.  $P_1^\mu$  can be made such that (for  $\mu > 0$  sufficiently small) each vector field in the family has, for example, two heteroclinic orbits, along which the two dimensional invariant manifolds of the north and south pole saddle points intersect transversally.

Finally we make the perturbation flat at  $(0; 0) \in \mathbb{R}^3 \times \mathbb{R}$  by choosing a flat function  $\delta_1 = \delta_1(\mu)$ , which is positive for  $\mu > 0$ , and taking the first perturbation to be  $p_1^\mu = \delta_1(\mu) \cdot P_1^\mu$ . Observe that this perturbation does not affect the saddle connection along the  $\bar{z}$ -axis.

For the second perturbation  $p_2^\mu(\bar{\xi}) = \delta_2(\mu)P_2^\mu(\bar{\xi})$  we choose

$$(10) \quad \beta_\mu(\bar{\xi}\sqrt{\mu}) = \gamma\left(\frac{1}{\sqrt{\mu}}\sqrt{\bar{x}^2 + \bar{y}^2}\right) \cdot \gamma\left(\frac{1}{\mu^2}\left(\bar{z}\sqrt{\mu} + \frac{1}{2}\sqrt{\frac{1}{a_1}}\right)\right),$$

where  $\gamma$  is a bump function, which is  $C^\infty$  and nowhere negative, such that  $\text{supp}(\gamma) = [-2, 2]$  and  $\gamma(s) \equiv 1$  for  $-1 \leq s \leq 1$  (compare Broer & van Strien [5]).

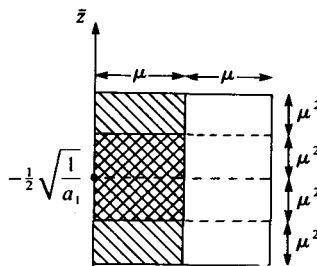


FIGURE 6

In figure 6 we visualise the support of  $\beta_\mu$ , in rescaled coordinates and modulo revolution around the  $\bar{z}$ -axis. Note that in the doubly shaded area of figure 6 by

(9) we have for  $P_2^\mu$ :

$$\begin{cases} \dot{x} = 0 \\ \dot{y} = (1/\sqrt{\mu})\beta_\mu(\bar{\xi}\sqrt{\mu}) \\ \dot{z} = 0, \end{cases}$$

since in this region the function  $\beta_\mu$  depends only on  $\bar{z}$ .

The flat function  $\delta_2 = \delta_2(\mu)$  will be chosen later on; here we specify only that  $\delta_2$  must be positive for  $\mu > 0$ .

(d) We proceed by formulating some properties of the perturbed vector field  $X^\mu = \tilde{X}^\mu + p_1^\mu + p_2^\mu$ ,  $\mu$  positive and small. Our interest is in homoclinic intersections connected with the south pole saddle point. Its one dimensional stable invariant manifold is denoted by  $W_\mu^s$ , its two dimensional unstable invariant manifold by  $W_\mu^u$ .

First recall the facts that for sufficiently small, positive  $\mu$  we have

$$\text{supp}(p_1^\mu) \cap \text{supp}(p_2^\mu) = \emptyset \quad \text{and} \quad \text{supp}(p_2^\mu) \cap \{\bar{z} = 0\} = \emptyset.$$

The question now is what is the effect of the perturbation  $p_1^\mu$  concerning the shape of the set

$$(11) \quad W_\mu^u \cap \{\bar{z} = 0\}.$$

In particular we are interested in the part of this where the integral curves leaving the southern hemisphere hit the plane  $\bar{z} = 0$  for the first time. We denote this subset of (11) by  $L_\mu$ .

Let  $\bar{r}$  and  $\bar{\varphi} = \varphi$  be rescaled polar coordinates as introduced in part (b) of this section. Assume that the angle  $\varphi$  has been lifted to  $\mathbb{R}$ . The following then states that asymptotically, as  $\mu \downarrow 0$ , the set  $L_\mu$  is a logarithmic spiral:

**LEMMA C.** *There exists a positive flat function  $\delta_3 = \delta_3(\mu)$  such that for  $\mu > 0$  and  $\mu$  sufficiently small the set  $L_\mu$  satisfies the equation*

$$\bar{r} = \bar{r}(\varphi, \mu) \quad \text{where} \quad \bar{r}(\varphi, \mu) \simeq \delta_3(\mu) e^{-\varphi\sqrt{\mu a_1}}$$

as  $\varphi \rightarrow \infty$ .

*Remark.* The symbol  $\simeq$  here means that the expressions on both sides are of the same order as  $\varphi \rightarrow \infty$ .

*Proof of lemma C:* Let  $t \mapsto \gamma^\mu(t)$  be one of the heteroclinic orbits of  $X^\mu$ , obtained by the perturbation  $p_1^\mu$ . See part (c) of this section. At its intersection with the plane  $\bar{z} = 0$  we take a one dimensional transversal section  $\sigma_\mu$ , which is totally contained in  $W_\mu^u$ . Recall that  $\dim(W_\mu^u) = 2$ . Suppose that  $\sigma_\mu$  is chosen depending smoothly on  $\mu$ . Now consider

$$\Sigma_\mu = \bigcup_{t \geq 0} X_t^\mu(\sigma_\mu),$$

where  $X_t^\mu$  denotes the time  $t$  evolution of the vector field  $X^\mu$ . Clearly

$$L_\mu \subset \Sigma_\mu \cap \{\bar{z} = 0\};$$

$L_\mu$  is obtained from  $\Sigma_\mu \cap \{\bar{z} = 0\}$  by taking ‘first’ intersections.

Next apply the  $\lambda$ -lemma, see e.g. [9], to the north pole saddle point. It follows that the set  $\Sigma_\mu$ , being a cylinder on the curve  $t \mapsto \gamma^\mu(t)$ , intersects a transversal

section contained in the plane  $\bar{z} = \sqrt{1/a_1} - \varepsilon$  along a logarithmic spiral with the 'same' asymptotic behaviour as the integral curve  $\gamma^\mu$ . Consider the linear part of the expression (8) near the north pole saddle in order to obtain this asymptotic behaviour. This same behaviour translates to the plane  $\bar{z} = 0$ , as can be seen e.g. by using the blown up version of the first integral  $H_\mu$  from the proof of lemma B. In fact the integral curves are contained in level surfaces of  $H_\mu$ , which are asymptotically parallel to the  $\bar{z}$ -axis. □

The effects of the perturbation  $p_2^\mu$  were studied by Broer & van Strien [5]. Consider the integral curve of  $X^\mu$  that is contained in the stable manifold  $W_\mu^s$  of the south pole saddle point. Let  $R(\mu)$  be the  $\bar{r}$ -coordinate of the point where this integral curve enters  $\text{supp}(p_2^\mu)$ , the last time before its disappearance into the south pole saddle point. Then:

LEMMA D (see [5]).

$$R(\mu) \approx \delta_2(\mu)\sqrt{\mu} \quad \text{as } \mu \downarrow 0.$$

(e) We now finish our proof by constructing a 'Šil'nikov sequence'. In particular we shall indicate how to choose the flat function  $\delta_2 = \delta_2(\mu)$  once the perturbation  $p_1^\mu = \delta_1(\mu) \cdot P_1^\mu$  has been chosen. So we assume that  $p_1^\mu$  is fixed in the way described above. This provides us with a flat function  $\delta_3 = \delta_3(\mu)$ , as in lemma C. The choice of the supports of  $p_1^\mu$  and  $p_2^\mu$  ensures the latter perturbation does not affect the logarithmic spirals  $\bigcup_{\mu>0} L_\mu$  created by the former. We define  $(\bar{r}(\mu), \varphi(\mu))$  to be the point where the stable manifold  $W_\mu^s$  of the south pole saddle point of the vector field

$$X^\mu = \tilde{X}^\mu + p_1^\mu + p_2^\mu$$

hits the plane  $\bar{z} = 0$  for the last time. A sufficient condition for the existence of a homoclinic orbit of this vector field  $X^\mu$  is

$$(12) \quad (\bar{r}(\mu), \varphi(\mu)) \in L_\mu.$$

So we have to choose  $\delta_2 = \delta_2(\mu)$  in such a way that the corresponding curve  $\mu \rightarrow (\bar{r}(\mu), \varphi(\mu), \mu)$  hits the surface  $\bigcup_{\mu>0} L_\mu$  infinitely often as  $\mu \downarrow 0$ . See figure 7.

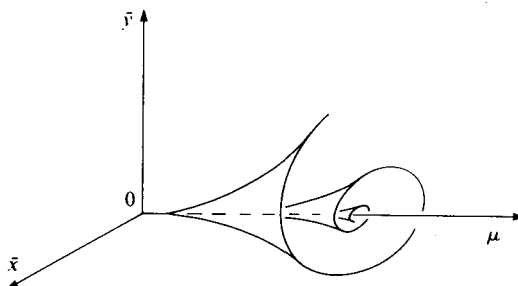


FIGURE 7.  $\bigcup_{\mu>0} L_\mu$ .

Analytically condition (12) reads: There exists a non-negative integer  $l$  such that

$$(13) \quad \bar{r}(\varphi(\mu) + 2l\pi, \mu) = \bar{r}(\mu).$$

Here  $\bar{r} = \bar{r}(\varphi, \mu)$  determines the logarithmic spirals  $L_\mu$ , see lemma C.

In other words, we have to ensure that our choice of  $\delta_2 = \delta_2(\mu)$  gives rise to sequences  $\{\mu_j\}$  of parameter values and  $\{l_j\}$  of positive integers, such that  $\mu_j$  is a solution of (13) with  $l = l_j$ , such that  $\mu_j \downarrow 0$  as  $j \rightarrow \infty$ . We distinguish two main situations (other cases are also possible):

- (i) The sequence  $\{l_j\}$  is unbounded; and
- (ii) the sequence  $\{l_j\}$  is bounded.

In case (i) the distance  $\bar{r}(\mu)$  of the ‘arrowhead’  $(\bar{r}(\mu), \varphi(\mu))$  and the  $\bar{z}$ -axis shrinks faster than any characteristic distance of the logarithmic spiral  $L_\mu$ , as  $\mu \downarrow 0$ , while in case (ii) this approach to zero has approximately the same speed.

Before we decide how to effect the situations (i) and (ii) we first observe the following. In view of lemma D we write

$$(14) \quad \bar{r}(\mu) \approx \delta_2(\mu)\sqrt{\mu} \quad \text{as } \mu \downarrow 0.$$

Moreover we note that the perturbation  $p_2$  does not cause large oscillations of the arrowhead  $(\bar{r}(\mu), \varphi(\mu))$ , in fact

$$(15) \quad \varphi(\mu) = o(1) \quad \text{as } \mu \downarrow 0.$$

Now, using (14) and (15) together with lemma C easily shows that case (i) will occur if, for example, we take

$$(16) \quad \delta_2(\mu) = \frac{1}{\sqrt{\mu}} \delta_3(\mu) e^{-1/\sqrt{\mu}}.$$

In order to bring about case (ii) we restrict the angle  $\varphi$  to a bounded interval, and we obtain

$$(17) \quad \bar{r}(\varphi, \mu) \approx \delta_3(\mu) \quad \text{as } \mu \downarrow 0.$$

Then, in view of (14), (15) and (17) it is obvious that we can take  $\delta_2 = \delta_2(\mu)$  such that, for example

$$(18) \quad \bar{r}(\mu) = \bar{r}(\varphi(\mu), \mu) \cdot (1 + \sin(1/\mu)) \quad \text{as } \mu \downarrow 0.$$

In this case we find ourselves in case (ii) with  $l_j \equiv 0$  and  $\mu_j = 1/j\pi$ .

The homoclinic intersections, occurring for parameter values  $\mu_1 > \mu_2 > \dots$ , with  $\lim_{j \rightarrow \infty} \mu_j = 0$ , do not necessarily have to be ‘generic Šil’nikov bifurcations’ yet, as  $\mu$  moves through the subsequent  $\mu_j$ . This, however, can be arranged by carrying out countably many small perturbations near the  $\mu_j$ , with disjoint supports and with a flat sum. Compare [15] and the transversality arguments used above. This completes our construction concerning the case of figure 4b.

*Remark on the case of figure 4a.* In the dissipative case we may, for example, take the perturbation  $p_1^{\mu, \nu}$  to be rotationally symmetric with respect to the  $\bar{z}$ -axis, in such a way that the two dimensional invariant manifolds behave as indicated in figure 4a. In that case the unstable manifold of the south pole saddle point of the vector fields  $\tilde{X}^{\mu, \nu} + p_1^{\mu, \nu}$  intersects the plane  $\bar{z} = 0$ , near the  $\bar{z}$ -axis, in finitely many circles. These circles collapse into the point  $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 0)$  as  $(\mu, \nu) \rightarrow (0, 0)$ , their radii being of flat order in  $\mu^2 + \nu^2$ . Compare lemma C.



Now we take the perturbation  $p_1^{\mu, \nu} = \delta_2(\mu, \nu)P_2^{\mu, \nu}$  similar to the one described in case (ii) above. In this way we create a Šil'nikov sequence along the curve  $\Gamma$  (see figure 5). This sequence can be made 'generic' as described before, thus creating the curve  $C$  of the dissipative existence theorem.

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