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Injectivity of the Connecting Maps in AH Inductive Limit Systems

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Abstract. Let A be the inductive limit of a system

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots$$

with $A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$, where $X_{n,i}$ is a finite simplicial complex, and $P_{n,i}$ is a projection in $M_{[n,i]}(C(X_{n,i}))$. In this paper, we will prove that A can be written as another inductive limit

$$B_1 \xrightarrow{\psi_{1,2}} B_2 \xrightarrow{\psi_{2,3}} B_3 \longrightarrow \cdots$$

with $B_n = \bigoplus_{i=1}^{s_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}$, where $Y_{n,i}$ is a finite simplicial complex, and $Q_{n,i}$ is a projection in $M_{\{n,i\}}(C(Y_{n,i}))$, with the extra condition that all the maps $\psi_{n,n+1}$ are *injective*. (The result is trivial if one allows the spaces $Y_{n,i}$ to be arbitrary compact metrizable spaces.) This result is important for the classification of simple AH algebras. The special case that the spaces $X_{n,i}$ are graphs is due to the third author.

1 Introduction

An AH algebra A is the C^*-algebra inductive limit of a sequence

$$A_1 \xrightarrow{\phi_{1,2}} A_2 \xrightarrow{\phi_{2,3}} A_3 \longrightarrow \cdots$$

with $A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i}))P_{n,i}$, where $[n,i]$ and t_n are positive integers, $X_{n,i}$ are compact metrizable spaces, and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ are projections (see [Bl]). Let us write $A = \varinjlim(A_n, \phi_{n,m})$, where $\phi_{n,m} = \phi_{m-1,m} \circ \phi_{m-2,m-1} \circ \cdots \circ \phi_{n+1,n+2} \circ \phi_{n,n+1}$. As pointed out in [Bl], in such an inductive limit, one can always replace the compact metrizable spaces $X_{n,i}$ by finite simplicial complexes. Therefore, in the proof of any classification theorem for AH algebras, one may always assume that the spaces $X_{n,i}$ above are finite simplicial complexes.

Let $\phi\colon C(X) \to PM_n(C(Y))P$ be a unital homomorphism. Then for any $y \in Y$, there are $x_1, x_2, \ldots, x_k \in X$, where $k = \operatorname{rank}(P)$, and a unitary $u_y \in M_n(C(Y))$ such

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that

$$\phi(f)(y) = u_y \begin{pmatrix} f(x_1) & & & & & & \\ & f(x_2) & & & & & \\ & & \ddots & & & & \\ & & & f(x_k) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} u_y^*, \quad \text{for any } f \in C(X).$$

Let us call the set $\{x_1, x_2, \dots, x_k\}$, counting multiplicity, the spectrum of ϕ at y, and denote it by $\mathrm{SP}\phi_y$. The concept of $\mathrm{SP}\phi_y$ can be generalized to any homomorphism $\Phi\colon \bigoplus_i P_i M_{l_i}(C(X_i))P_i \to \bigoplus_j Q_j M_{k_j}(C(Y_j))Q_j$ (see [G1, 1.2.16]).

Let X be a compact metrizable space, and $\delta > 0$. A subspace $X_1 \subseteq X$ will be said to be δ-dense in X if for any $x \in X$, there is $x_1 \in X_1$ such that $d(x, x_1) < \delta$. In the study of simple AH algebras $A = \varinjlim(A_n, \phi_{n,m})$, the δ-density of $\mathrm{SP}(\phi_{n,m})_y$ in the space $X_{n,1} \coprod X_{n,2} \coprod \cdots \coprod X_{n,t_n}$ for any $y \in X_{m,1} \coprod X_{m,2} \coprod \cdots \coprod X_{m,t_m}$, for an arbitrary given small number δ, providing m is large enough (see [DNNP, Proposition 2.1] of and also see [P]), is very important as shown in [Ell1, Ell2, Li1, Li2, Li3, NT, G1, G2, EGL]. This δ-density holds only if all the homomorphisms $\phi_{n,m}$ are injective. Therefore, it is desirable to replace the homomorphisms $\phi_{n,m}$ in the inductive limit by injective homomorphisms. The following is a naive way to do this.

Suppose that $A = \varinjlim(A_n, \phi_{n,m})$. Let $B_n = \phi_{n,\infty}(A_n) \subset A$, and denote by $\psi_{n,m}$: $B_n \to B_m$ the inclusion map: $\phi_{n,\infty}(A_n) = \phi_{m,\infty}(\phi_{n,m}(A_n)) \hookrightarrow \phi_{m,\infty}(A_m)$. Then $\psi_{n,m}$ is injective and $A = \varinjlim(B_n, \psi_{n,m})$. Furthermore, since B_n is the quotient algebra of A_n modulo the ideal $\operatorname{Ker} \phi_{n,\infty}$, we have $B_n \cong \bigoplus_{i=1}^{s_n} Q_{n,i} M_{[n,i]}(C(Y_{n,i}))Q_{n,i}$, where $Y_{n,i}$ is a subspace of $X_{n,i}$ and the projection $Q_{n,i}$ is the restriction of the projection $P_{n,i}$ to $Y_{n,i}$.

Unfortunately, the spaces $Y_{n,i}$ are not simplicial complexes any more—they are just compact metrizable spaces. (Note that if the method in [Bl] is applied to change the compact metrizable spaces $Y_{n,i}$ by simplicial complexes, then the injectivity of the connecting homomorphisms will be lost again.) Let us point out that it is essential to use finite simplicial complexes (instead of compact metrizable spaces) in the proof of the main theorems in [G1, G2, EGL]. Among others, we only list two points below.

(1) A finite simplicial complex has finitely many path connected components and can be dealt with as a path connected space by separating the components. The path connectedness is used in two ways:
   (a) A theorem of Thomsen [T] (see [Li2, Theorem 2.1] for a version of the theorem used in [EGL]) is used in the proof of the Existence Theorem and this theorem holds only for path connected spaces.
   (b) The theorem about maximum spectral multiplicity of a homomorphism proved in [G1, §2] requires the space to be path connected.
(2) The K-theory of a finite simplicial complex is finitely generated, which is important for the construction of the intertwining at the level of K-theory and also important for the proof of the Uniqueness Theorem.

In this paper, we will prove that the finite simplicial complexes $X_{n,i}$ can be replaced by finite simplicial sub-complexes $Y_{n,i} \subset X_{n,i}$, and that the homomorphisms $\phi_{n,m}$ can be replaced by injective homomorphisms. (Note that the spaces $Y_{n,i}$ may not be connected, even if $X_{n,i}$ are assumed to be connected.)

The result in this paper is used in [G1, G2, EGL]. The special case that $\dim(X_{n,i}) = 1$ was due to the third author and played an important role in the classification of simple AH algebras with one-dimensional local spectra (see [Ell1, Ell2, Li1, Li2, Li3]). The proof given here is a modification of the proof of this special case.

2 The Proof of the Main Theorem

The following is the main theorem of this paper.

Theorem 2.1 *Let $A = \varinjlim(A_n, \phi_{n,m})$ be an inductive limit of*

$$A_n = \bigoplus_{i=1}^{t_n} P_{n,i} M_{[n,i]}(C(X_{n,i})) P_{n,i},$$

where $X_{n,i}$ is a connected finite simplicial complex, and $P_{n,i} \in M_{[n,i]}(C(X_{n,i}))$ is a projection. Then A can be written as the inductive limit of a system

$$\Big(B_n = \bigoplus_{i=1}^{t_n} Q_{n,i} M_{\{n,i\}}(C(Y_{n,i})) Q_{n,i}, \psi_{n,m} \Big),$$

where the spaces $Y_{n,i}$ are (not necessarily connected) finite simplicial complexes, and $Q_{n,i} \in M_{\{n,i\}}(C(Y_{n,i}))$ are projections, with the extra property that all connecting homomorphisms $\psi_{n,m}$ are injective. *Furthermore, the spaces $Y_{n,i}$ may be chosen such that* $\dim Y_{n,i} \leq \dim X_{n,i}$. *(In fact, the simplicial complexes $Y_{n,i}$ can be chosen to be subspaces of the spaces $X_{n,i}$.)*

Remark 2.2 In the above theorem, we do not assume the simplicity of A. Also, there is no restriction on the growth of $\dim X_{n,i}$.

Let us first discuss how to reduce the proof of Theorem 2.1 to the special case that the algebras A_n are direct sums of full matrix algebras (instead of their corner subalgebras) over $X_{n,i}$.

According to [G1, Lemma 1.3.3] (see [EG, 4.24]), there exists an inductive limit C^*-algebra $\tilde{A}$ of this special form, containing A as a corner sub-C^*-algebra (see [G1, §1.3]), and in fact such that

$$\tilde{A} = \varinjlim\Big(\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{[n,i]^\sim}(C(X_{n,i})), \tilde{\phi}_{n,m} \Big),$$

where, not only is A_n a corner subalgebra of $\tilde{A}_n$, but also $\phi_{n,m} = \tilde{\phi}_{n,m}|_{A_n}$. In particular, there is an increasing sequence of projections, $Q_1 \leq Q_2 \leq \cdots \leq Q_n \leq \cdots$, in $\tilde{A}$, such that

$$A = \overline{\bigcup_n Q_n \tilde{A} Q_n}.$$

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Suppose that Theorem 2.1 holds when all the algebras A_n are direct sums of full matrix algebras over $X_{n,i}$. Then, $\tilde{A}$ can be written as the inductive limit of a sequence

$$\Big(\tilde{B}_n = \bigoplus_{i=1}^{t_n} M_{\{n,i\}}(C(Y_{n,i})), \tilde{\psi}_{n,m}\Big)$$

where each $Y_{n,i}$ is finite simplicial complex of dimension at most that of $X_{n,i}$ (and, in fact, if chosen suitably, a subset of $X_{n,i}$), and each $\tilde{\psi}_{n,m}$ is injective.

Passing to a subsequence of $(\tilde{B}_n)$ (and, at the same time, of (A_n)), and replacing the increasing sequence of projections (Q_n) by a unitarily equivalent one, we may assume that $Q_n \in \tilde{B}_n$. Set $Q_n\tilde{B}_nQ_n = B_n$ and $\tilde{\psi}_{n,m}|_{B_n} = \psi_{n,m}$. Then each $\psi_{n,m}$ is injective and A is the inductive limit of the sequence $(B_n, \psi_{n,m})$. In other words, the conclusion of the theorem holds also in the general case.

Thanks to the above discussion, we may assume that each A_n is a direct sum of full matrix algebras, $A_n = \bigoplus M_{[n,i]}(C(X_{n,i}))$.

From the proof below, we will see that the algebras B_n are chosen to be

$$B_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(Y_{n,i})),$$

where $Y_{n,i} \subset X_{n,i}$. Furthermore, from the proof, we will see that

$$\mathrm{rank}(\phi^{i,j}_{n,m}(\mathbf{1}_{A_n^i})) = \mathrm{rank}(\psi^{i,j}_{n,m}(\mathbf{1}_{B_n^i})).$$

Therefore, if $(A_n, \phi_{n,m})$ has very slow dimension growth, then so also does $(B_n, \psi_{n,m})$. This result was used in [G1, G2] (and therefore also in [EGL]). (Recall that an inductive system $\varinjlim(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m})$ is said to satisfy the very slow dimension growth condition if for any summand $A_n^i = P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}$ of a fixed A_n,

$$\lim_{m\to+\infty} \max_j \left\{ \frac{(\dim X_{m,j})^3}{\mathrm{rank}\phi^{i,j}_{n,m}(\mathbf{1}_{A_n^i})} \right\} = 0,$$

where $\phi^{i,j}_{n,m}$ is the partial map of $\phi_{n,m}$ from A_n^i to A_m^j.)

The proof of this theorem is a modification of the proof of the same theorem for the case that the spaces $X_{n,i}$ are graphs (equivalently, finite simplicial complexes of dimension at most one), due to the third author [Li1].

2.3 For each $A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i}))$, we have

$$\phi_{n,\infty}(A_n) \cong \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(\tilde{X}_{n,i})),$$

where each space $\tilde{X}_{n,i}$ is a subspace of $X_{n,i}$. Replacing A_n by

$$\tilde{A}_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(\tilde{X}_{n,i})),$$

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we may write A as the inductive limit of the sequence $(\tilde{A}_n, \tilde{\phi}_{n,m})$, where $\tilde{\phi}_{n,m}$, the map induced by $\phi_{n,m}$, is injective. Since, in general, the spaces $\tilde{X}_{n,i}$ are no longer simplicial complexes, we now need to replace them by simplicial complexes.

2.4 Let X be a connected simplicial complex and let $\tilde{X} \subset X$ be a closed subset. (Later, we will let $X = X_{n,i}$ and $\tilde{X} = \tilde{X}_{n,i}$; see 2.3.) For each simplicial subdivision σ of X, we will define a finite simplicial complex Y and a surjection $\alpha\colon \tilde{X} \to Y$. ($Y$ may not be connected.) Note that such a simplicial complex Y and surjection $\alpha\colon \tilde{X} \to Y$ give rise to a subalgebra $M_k(C(Y)) \subset M_k(C(\tilde{X}))$ via $\alpha^*\colon M_k(C(Y)) \to M_k(C(\tilde{X}))$.

2.5 The simplicial complex Y will be constructed to be a subspace of X. In fact, it will be the union of two sets, one the underlying space of a sub-complex of (X, σ) and the other a finite subset of X.

Denote by Y_1 the underlying space of the sub-complex of (X, σ) consisting of all simplices $\Delta \in \sigma$ such that $\mathring{\Delta} \cap \tilde{X}$ is uncountable, together with the faces of these simplices. ($\mathring{\Delta}$ denotes the interior of Δ.)

The space Y is obtained from Y_1 by adding finitely many points as follows. First, add to Y_1 all the vertices of (X, σ) which are in $\tilde{X}$. Second, if Δ is a simplex of dimension ≥ 1 with the property that $\mathring{\Delta} \cap \tilde{X}$ is a non-empty set with at most countably many points, then by a standard argument of real analysis, there exist a point $x \in \mathring{\Delta} \cap \tilde{X}$ and a neighbourhood of x, $U_x \subset \mathring{\Delta}$, such that

$$U_x \cap \tilde{X} = \{x\}.$$

For each such simplex Δ, add such a point x to Y_1 (only one point for each such simplex and so only finitely many such points altogether). (Notice that such a simplex Δ (with at most countably many points in $\mathring{\Delta} \cap \tilde{X}$), may already be a subset of Y_1, since it is possible that there is another simplex $\Delta_1 \supset \Delta$ such that $\mathring{\Delta}_1 \cap \tilde{X}$ is uncountable, and hence $\Delta \subset \Delta_1 \subset Y_1$. In this case, adding the point x does not change the space Y_1 at all.)

Note that the new space $Y \subset X$ defined above is the union of the underlying space of a sub-complex of (X, σ) and a finite subset of X, and therefore has a simplicial structure. With this new space Y, for each simplex $\Delta \in \sigma$, there are the following three cases:

$$Y \cap \Delta = \Delta; \tag{1}$$

$$Y \cap \mathring{\Delta} = \varnothing; \tag{2}$$

$$Y \cap \mathring{\Delta} = \text{singleton}. \tag{3}$$

To define the surjective map $\alpha\colon \tilde{X} \to Y$, we will need the following two easy lemmas from real analysis.

Lemma 2.6 *For any simplex Δ, if $T \subset \Delta$ is a closed subset such that $T \cap \mathring{\Delta}$ contains uncountably many points, then for any continuous map $\alpha\colon \partial\Delta \to \partial\Delta$, there is an extension $\tilde{\alpha}\colon \Delta \to \Delta$ such that*

$$\tilde{\alpha}|_{\partial\Delta} = \alpha \quad \text{and} \quad \tilde{\alpha}(T \cap \mathring{\Delta}) = \Delta.$$

Proof Since $T \cap \mathring{\Delta}$ has uncountably many points, there is a closed subset $T_1 \subset T \cap \mathring{\Delta}$ which is homeomorphic to the Cantor set. It is well known that there is a continuous surjection $\alpha_1\colon T_1 \to \Delta$ (a similar argument appeared in [Li1, 2.2.2]). Note that T_1 and $\partial\Delta$ are disjoint closed subsets of Δ, and so the map $\alpha_2\colon \partial\Delta \cup T_1 \to \Delta$, defined by

$$\alpha_2(y) = \begin{cases} \alpha(y) & \text{if } y \in \partial\Delta, \\ \alpha_1(y) & \text{if } y \in T_1, \end{cases}$$

is continuous. Choose $\tilde{\alpha}$ to be any continuous extension of α_2 to Δ. (Such an extension exists by the Tietze Extension Theorem.) ■

Lemma 2.7 *Suppose that X is a compact metric space, and $Y \subset X$ is a nonempty closed subspace such that the complement $X\backslash Y$ is countable. Then there is a continuous map $\alpha\colon X \to Y$ satisfying $\alpha|_Y = \mathrm{id}$.*

Proof Let $r\colon X \to \mathbb{R}^+$ be defined by

$$r(x) = \mathrm{dist}(x, Y) = \inf_{y\in Y} \mathrm{dist}(x, y).$$

Then r is continuous and, since X is compact, $r(X) \subset \mathbb{R}$ is a closed set. Since $r|_Y = 0$ and $X\backslash Y$ is a countable set, $r(X)$ is a countable closed subset of $\mathbb{R}^+$. Choose a sequence of positive numbers $\varepsilon_1 > \varepsilon_2 > \cdots > \varepsilon_i > \cdots$ satisfying $\sum_{i=1}^{+\infty} \varepsilon_i < +\infty$ and $\varepsilon_i \notin r(X)$ for all i. Then the sets

$$Y_{\varepsilon_i} = \{x \,;\, \mathrm{dist}(x, Y) < \varepsilon_i\}$$

are closed and open subsets of X. We shall define a sequence of continuous maps $\alpha_i\colon X \to Y_{\varepsilon_i} \subset X$, inductively. We shall then define the continuous map $\alpha\colon X \to X$ to be the limit of this sequence which, as we shall see, converges.

Choose any point $y_0 \in Y$. Define $\alpha_1\colon X \to X$ by

$$\alpha_1(x) = \begin{cases} x & \text{if } x \in Y_{\varepsilon_1} \\ y_0 & \text{if } x \notin Y_{\varepsilon_1}. \end{cases}$$

As the induction assumption, suppose that α_i is defined in such a way that $\alpha_i|_{Y_{\varepsilon_i}} = \mathrm{id}$ and $\alpha_i(X\backslash Y_{\varepsilon_i}) \subset Y$. Define α_{i+1} as follows. Since $Y_{\varepsilon_i}\backslash Y_{\varepsilon_{i+1}}$ is a countable closed and open subset of X, there are finitely many closed and open subsets $X_1, X_2, \ldots, X_n$ with

$$Y_{\varepsilon_i}\backslash Y_{\varepsilon_{i+1}} = X_1 \cup X_2 \cup \cdots \cup X_n \quad \text{and} \quad \mathrm{diameter}(X_j) < \varepsilon_i,$$

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for $j = 1, 2, \ldots, n$. Choose $y_1, y_2, \ldots, y_n \in Y$ such that $\operatorname{dist}(y_j, X_j) \le \varepsilon_i$. Define $\alpha_{i+1}\colon X \to Y_{\varepsilon_{i+1}}$ by

$$\alpha_{i+1}(x) = \begin{cases} x & \text{if } x \in Y_{\varepsilon_{i+1}} \\ y_j & \text{if } x \in X_j,\ j = 1, 2, \ldots, n \\ \alpha_i(x) & \text{if } x \notin Y_{\varepsilon_i}. \end{cases}$$

Then α_{i+1} is continuous. Since $\sum_{i=1}^{+\infty} \varepsilon_i < +\infty$, it is evident that (α_i) is Cauchy in the uniform metric. Denote by α the limit of (α_i). Then α is continuous, $\alpha(X) \subset Y$, and $\alpha|_Y = \mathrm{id}$, as desired. ∎

2.8 Set $X' = \tilde{X} \cup Y_1 = \tilde{X} \cup Y$, where Y_1 and Y are as defined in 2.5. We will define a map $\alpha'\colon X' \to Y$ such that $\alpha := \alpha'|_{\tilde{X}}\colon \tilde{X} \to Y$ is a surjection. For later use, we shall also ensure that $\alpha'(\Delta \cap X') \subset \Delta \cap Y$ for every simplex Δ of X. Then α' will be defined on $\Delta \cap X'$, inductively, for each simplex Δ with $X' \cap \mathring{\Delta} \neq \varnothing$. Namely, after we have the definition of α' on $\partial\Delta \cap X'$, we will extend the definition of α' to $\Delta \cap X'$.

Let $V(X)$ denote the collection of all vertices of (X, σ). From the construction of Y (see 2.5),

$$V(X) \cap X' = V(X) \cap Y \subset Y.$$

Define $\alpha'(x) = x$ for any $x \in V(X) \cap X'$.

Let us fix a simplex $\Delta \in \sigma$ with $X' \cap \mathring{\Delta} \neq \varnothing$. As the inductive assumption, we assume that α' is defined on $\partial\Delta \cap X'$ in such a way that

$$\alpha'(\partial\Delta \cap X') \subset \alpha(\partial\Delta \cap Y).$$

The definition of α' on $X' \cap \Delta$ will be broken up into the following three cases.

Case 1 *$X' \cap \Delta = \Delta$, and $\tilde{X} \cap \mathring{\Delta}$ contains at most countably many points. (This case occurs when there is a simplex $\Delta_1 \supset \Delta$ such that $\tilde{X} \cap \mathring{\Delta}_1$ contains uncountably many points, but $\tilde{X} \cap \mathring{\Delta}$ itself contains at most countably many points.) Choose any extension of $\alpha'|_{\partial\Delta}$ to Δ. (Note that $\Delta \subset Y$.)*

Case 2 *$X' \cap \Delta = \Delta$, and $\tilde{X} \cap \mathring{\Delta}$ contains uncountably many points. In this case, $X' \cap \Delta = Y \cap \Delta = \Delta$ and $X' \cap \partial\Delta = \partial\Delta$. By Lemma 2.6, we can extend $\alpha'|_{\partial\Delta}$ to Δ in such a way that $\alpha'(\tilde{X} \cap \mathring{\Delta}) = \Delta$.*

Case 3 *$X' \cap \Delta \neq \Delta$. In this case, $\tilde{X} \cap \mathring{\Delta}$ contains at most countably many points, and $Y \cap \mathring{\Delta} = \{y\}$ for some point $y \in \mathring{\Delta}$. From the way the point y is chosen in 2.5, we know that $(X' \cap \Delta)\backslash\{y\}$ is a closed and open subset of $X' \cap \Delta$.*

First, we assume that $X' \cap \partial\Delta \neq \varnothing$. Since $X' \cap \mathring{\Delta} = \tilde{X} \cap \mathring{\Delta}$ is a countable set, by Lemma 2.7, there is a continuous map

$$\beta\colon (X' \cap \Delta)\backslash\{y\} \longrightarrow X' \cap \partial\Delta$$

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satisfying

$$\beta|_{X'\cap\partial\Delta} = \mathrm{id}.$$

Define α' on $X' \cap \Delta$ by

$$\alpha'(x) = \begin{cases} y & \text{if } x = y \\ \alpha' \circ \beta(x) & \text{otherwise.} \end{cases}$$

(Note that α' is defined on $X' \cap \partial\Delta$.)

On the other hand, if $X' \cap \partial\Delta = \varnothing$, then we define $\alpha'(x) = y$ for any $x \in X' \cap \Delta = X' \cap \mathring{\Delta}$.

This ends the construction of the map α'.

Finally, we can define the map $\alpha\colon \tilde{X} \to Y$ by

$$\alpha = \alpha'|_{\tilde{X}}.$$

One can check the following properties of α from the construction:

(1) $\alpha\colon \tilde{X} \to Y$ is a surjection;
(2) $\alpha(\tilde{X} \cap \Delta) \subset Y \cap \Delta$ for all simplices Δ of X.

The property (2) is obvious. To check the property (1), we first note that for any point $x \in X$, there is a unique simplex $\Delta \in \sigma$ such that $x \in \mathring{\Delta}$. For any fixed point $y \in Y$, we need to verify that y is in the image of α. Let Δ be the unique simplex such that $y \in \mathring{\Delta}$. We must consider the following two cases.

Case a. $Y \cap \mathring{\Delta} = \{y\}$.

This case follows from the definition of α' in Case 3 above, if $\dim(\Delta) \geq 1$. The case $\dim(\Delta) = 0$ is a special case of Case b below.

Case b. $Y \cap \Delta = \Delta$.

If $\dim(\Delta) = 0$ and $\Delta = \{y\} \subset \tilde{X}$, then $y \in \mathrm{image}(\alpha)$, since $\alpha(y) = y$. Otherwise, there is a simplex $\Delta_1 \supset \Delta$ such that $\tilde{X} \cap \mathring{\Delta}_1$ contains uncountably many points. Therefore, from the definition of α' (for Δ_1) in Case 2 above, $\mathrm{image}(\alpha) \supset \mathrm{image}(\alpha'|_{\tilde{X}\cap\Delta_1}) = \Delta_1 \ni y$.

2.9 Suppose that X is a simplicial complex and $\tilde{X} \subset X$ is a closed subset. Let $G \subset M_n(C(\tilde{X}))$ be a finite set. For any $\varepsilon > 0$, there is an $\eta > 0$ such that if $\mathrm{dist}(x, x') < \eta$ then $\|g(x) - g(x')\| < \frac{\varepsilon}{2}$ for all $g \in G$. Choose a subdivision σ of X such that $\mathrm{diameter}(\Delta) < \eta$ for every simplex Δ of (X, σ). With respect to the subdivision σ, let us define the subspace $Y \subset X$ as in 2.5 and the surjection $\alpha\colon \tilde{X} \to Y$ as in 2.8.

Lemma 2.10 *Following the notation of* 2.9, *one has*

$$G \subset_{\varepsilon} M_n(C(Y))$$

(see [G1, 1.1.7(e)] *for the notation* $\subset_{\varepsilon}$*), where* $M_n(C(Y))$ *is regarded as a subalgebra of* $M_n(C(\tilde{X}))$ *by the inclusion* α^*.

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(This lemma is an analogue of of [Li1, Lemma 2.2.6]; the proof is also analogous.)

Proof Let us just sketch the argument. For any $g \in G$, define $\tilde{g} \in M_n(C(Y))$ as follows. For each vertex y of Y (including vertices in $V(X) \cap Y$ and discrete points of Y), there is at least one simplex Δ of X such that $\tilde{X} \cap \Delta \neq \varnothing$ and $y \in \Delta$. Choose any $y_1 \in \tilde{X} \cap \Delta$, and define $\tilde{g}(y) = g(y_1)$. Then extend $\tilde{g}$ to Y linearly on each simplex. It is easy to check that

$$\|g - \alpha^*(\tilde{g})\| < \varepsilon,$$

as in the proof of [Li1, Lemma 2.2.6]. ∎

Note that a sub-complex of $[0,1]$ is a union of finitely many intervals and finitely many single point spaces. As a corollary of the construction in 2.4–2.9, we have obtained the following result which will be used in [EGL].

Corollary 2.11 *Let $A = \bigoplus_{i=1}^t M_{l_i}(C(X_i))$, where each X_i is the interval $[0,1]$ or the single point space $\{pt\}$. Let $F \subset A$ be a finite subset and $\varepsilon > 0$. For any homomorphism $\phi\colon A \to B$, there is a C^*-algebra D which is a direct sum of matrix algebras over $C[0,1]$ or $\mathbb{C}$, and there are two homomorphisms $\phi_1\colon A \to D$ and $\phi_2\colon D \to B$ such that*

(1) $\|(\phi_2 \circ \phi_1)(f) - \phi(f)\| < \varepsilon$ *for any $f \in F$,*
(2) ϕ_1 *is surjective and* ϕ_2 *is injective.*

Proof Without loss of generality we may assume that $B = \phi(A)$, the image of A under the homomorphism ϕ. Then $B \cong \bigoplus_{i=1}^t M_{l_i}(C(\tilde{X}_i)), \tilde{X}_i \subset X_i$, and ϕ is the restriction map

$$\phi(f_1, f_2, \dots, f_t) = (f_1|_{\tilde{X}_1}, f_2|_{\tilde{X}_2}, \dots, f_t|_{\tilde{X}_t}) \text{ for all } (f_1, f_2, \dots, f_t) \in \bigoplus_{i=1}^t M_{l_i}(C(X_i)).$$

Obviously the proof can be reduced to the case of the single block $A = C([0,1])$ and $B = C(\tilde{X})$ with $\tilde{X} \subset [0,1]$. Since $C[0,1]$ is generated by the single function defined by $h(x) = x$ for $x \in [0,1]$, we can assume that $F = \{h\}$. Choose a subdivision σ of $[0,1]$ such that the length of each subinterval is smaller than ε. Apply 2.4–2.8 to $\tilde{X} \subseteq [0,1]$ and the subdivision σ to find a finite simplicial complex Y—a subspace of $[0,1]$ and a surjective map $\alpha\colon \tilde{X} \to Y$. Set $D = C(Y)$. Let $\phi_2\colon C(Y) \to C(\tilde{X})$ be defined by $\phi_2 = \alpha^*$. And let $\phi_1\colon C[0,1] \to C(Y)$ be defined as the restriction map $\phi_1(f) = f|_Y$ for any $f \in C[0,1]$. Evidently ϕ_1 is surjective. The injectivity of ϕ_2 follows from the surjectivity of α (see the property (1) of 2.8). It follows from the property (2) of 2.8 that

$$\|(\phi_2 \circ \phi_1)(h) - \phi(h)\| < \varepsilon.$$ ∎

Lemma 2.12 *Let $A = \bigoplus_{k=1}^q M_{l_k}(C(X_k))$, where X_k are connected simplicial complexes. Let $F \subset A$ be a finite set containing all matrix units e_{ij}^k of each block $A^k = M_{l_k}(C(X_k))$ and a set of generators of the centre $C(X_k)$ of each A^k.*

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For any $\varepsilon > 0$ and any positive integers n and l, there is a number $\delta > 0$ such that if $\phi\colon A \to M_n(C(\partial\Delta))$ (where Δ is an l-dimensional simplex) is a homomorphism satisfying

$$\|\phi(f)(t) - \phi(f)(t')\| < \delta \text{ for any } f \in F, t, t' \in \partial\Delta,$$

then there is a homomorphism $\psi\colon A \to M_n(C(\Delta))$ with the following properties:

(1) $i^*\psi = \phi$, *where $i^*\colon M_n(C(\Delta)) \to M_n(C(\partial\Delta))$ is induced by the inclusion $i\colon \partial\Delta \to \Delta$;*
(2) $\|\psi(f)(t) - \psi(f)(t')\| < \varepsilon$ *for any $f \in F, t, t' \in \Delta$.*

Proof Note that if δ is small enough, then the projections $\phi(e_{11}^k)$ $(k = 1, 2, \dots, q)$ are trivial in the sense of [G1, 1.2.1]. Therefore, $\phi(e_{11}^k)M_n(C(\partial\Delta))\phi(e_{11}^k) \cong M_{n_1}(C(\partial\Delta))$, where $n_1 = \operatorname{rank}(\phi(e_{11}^k))$. The lemma follows from the fact that the space

$$F^{n_1}X_k := \operatorname{Hom}(C(X_k), M_{n_1}(\mathbb{C}))_1$$

is locally contractible for each $n_1 \le n$ and $k \in \{1, 2, \dots, q\}$. ■

Lemma 2.13 *Let $A = \bigoplus_{k=1}^q M_{l_k}(C(Y_k))$, where Y_k are connected finite simplicial complexes. Let $F \subset A$ be a finite set containing all matrix units e_{ij}^k and a set of generators of the centre $C(Y_k)$ of every block $A^k = M_{l_k}(C(Y_k))$, $k \in \{1, 2, \dots, q\}$. Let $X_1, X_2, \dots, X_m$ be connected finite simplicial complexes and let $B = \bigoplus_{i=1}^m M_{n_i}(C(\tilde{X}_i))$, where each $\tilde{X}_i$ is a closed subset of the simplicial complex X_i. Suppose that $G \subset B$ is a finite subset and $\phi\colon A \to B$ is an injective homomorphism. For any $\varepsilon > 0$, there exists a subalgebra $B' = \bigoplus_{i=1}^m M_{n_i}(C(Z_i)) \subset B$, where each Z_i is a (possibly non-connected) finite simplicial complex, and there exists an injective homomorphism $\psi\colon A \to B'$ such that*

(1) $\|\phi(f) - \psi(f)\| < \varepsilon$ *for all $f \in F$;*
(2) $G \subset_\varepsilon B'$.

Proof Let $Y = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_q$. One can endow Y with a metric such that for any $t < 1$ and $y_0 \in Y$, the closed t-ball $\overline{B_t(y_0)}$ is a path connected simplicial complex (see [G1, 1.4.1]). Using the Peano curve, one can find a continuous surjective map from $[0, 1]$ onto $\overline{B_t(y_0)}$ for any $y_0 \in Y$ and $t < 1$. (We will use this fact later.)

Without loss of generality, we may assume $\varepsilon < 1$.

Let l denote the maximum of $\{\dim X_i\}$. Applying Lemma 2.12 to $\frac{\varepsilon}{4} > 0$, (each of) the positive integers n_i (corresponding to the integer n, the order of the matrices of $M_n(C(\Delta))$, in Lemma 2.12), and the integer l (corresponding to the integer l, the dimension of the simplex Δ, in Lemma 2.12), we can find a number ε_l, $0 < \varepsilon_l < \frac{\varepsilon}{4}$, as the number δ (works for all n_i) in Lemma 2.12. Then apply Lemma 2.12 to $\frac{\varepsilon_l}{4} > 0$, the positive integers n_i, and the integer $l - 1$, to find ε_{l-1}. In general, once we have ε_k, we apply 2.12 to the number $\frac{\varepsilon_k}{4} > 0$, the positive integers n_i, and the integer $k - 1$, to find a number ε_{k-1} $(0 < \varepsilon_{k-1} < \frac{\varepsilon_k}{4})$, as the number δ in Lemma 2.12. In this way, we obtain

$$\varepsilon > \varepsilon_l > \varepsilon_{l-1} > \cdots > \varepsilon_2 > \varepsilon_1 > 0.$$

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We further assume that the number $\varepsilon_1 \in (0, \frac{\varepsilon_2}{4})$ also satisfies that, if $\mathrm{dist}(y, y') < 2\varepsilon_1$, then

$$\|f(y) - f(y')\| < \frac{\varepsilon_2}{4} \text{ for all } f \in F.$$

For $\varepsilon_1 > 0$, there is a number $\eta > 0$ such that $\mathrm{dist}(x, x') < \eta$ implies that

$$\|\phi(f)(x) - \phi(f)(x')\| < \frac{\varepsilon_1}{4} \text{ for all } f \in F \subset A,$$

and

$$\|g(x) - g(x')\| < \frac{\varepsilon}{2} \text{ for all } g \in G.$$

(The latter is the condition in 2.9.) Furthermore, suppose that $\eta > 0$ satisfies the following condition. If $x, x' \in X = \tilde{X}_1 \coprod \tilde{X}_2 \coprod \cdots \coprod \tilde{X}_m$ are such that $\mathrm{dist}(x, x') < \eta$, then $\mathrm{SP}\phi_x$ and $\mathrm{SP}\phi_{x'}$ can be paired to be within $\frac{\varepsilon_1}{4}$, where $\mathrm{SP}\phi_x$ is a subset of $Y = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_q$. (See [EG, §1.4], or [G1, §1.2.2] for the definition; also see [G1, §1.2.12].)

Choose a subdivision for each X_i such that each simplex of the subdivision has diameter at most $\frac{\eta}{4}$. Using these subdivisions, one can construct spaces Z_i (as the space Y in 2.5) and surjective maps $\alpha_i \colon \tilde{X}_i \to Z_i$ (as in 2.8). Then $B' := \bigoplus_{i=1}^m M_{n_i}(C(Z_i))$ can be regarded as a subalgebra of $B = \bigoplus_{i=1}^m M_{n_i}(C(\tilde{X}_i))$ via α_i^*. By Lemma 2.10,

$$G \subset_\varepsilon B'.$$

We will define an injective homomorphism $\psi \colon A \to B'$ as follows.

Define ψ on each block $B'_i = M_{n_i}(C(Z_i))$ separately. For each $z \in Z_i$, one needs to define $\psi(f)(z)$. First, define it for each vertex $z \in V(Z_i)$, then define it for each 1-simplex, each 2-simplex and so on.

For each vertex $z \in V(Z_i)$, by the way Z_i is constructed (as in 2.8), there is a point $x \in \tilde{X}_i$ such that $\mathrm{dist}(z, x) < \frac{\eta}{4}$ and $\alpha_i(x) = z$. (Note that both Z_i and $\tilde{X}_i$ are subsets of X_i. The distance $\mathrm{dist}(z, x)$ is taken inside X_i.) Define

$$\psi(f)(z) = \phi(f)(x).$$

Since each simplex has diameter at most $\frac{\eta}{4}$, if z_1, z_2 are vertices of one simplex, then

$$\|\psi(f)(z_1) - \psi(f)(z_2)\| < \frac{\varepsilon_1}{4}.$$

Define ψ on each edge $[z_1, z_2]$ of Z_i as follows. Identify $[z_1, z_2]$ with $[0, 1]$. Then $\frac{z_1+z_2}{2}$ is well defined. For z_1, there are $y_1, y_2, \ldots, y_s \in Y_1 \coprod Y_2 \coprod \cdots \coprod Y_q$ and a unitary $u \in M_{n_i}(\mathbb{C})$ such that

$$\psi(f)(z_1) = u \begin{pmatrix} f(y_1) & & & & & & \\ & f(y_2) & & & & & \\ & & \ddots & & & & \\ & & & f(y_s) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} u^*$$

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for all $f \in \bigoplus_{k=1}^{q} M_{l_k}(C(Y_k)) = A$.

For each $y_j (1 \le j \le s)$, we can find a map

$$\beta_j \colon [z_1, \frac{z_1+z_2}{2}] \to Y = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_q$$

with the following properties:

(i) $\beta_j(z_1) = y_j$ and $\beta_j(\frac{z_1+z_2}{2}) = y_j$;
(ii) $\mathrm{Image}\beta_j = \overline{B_{\varepsilon_1}(y_j)}$ as a set.

This is possible because of the Peano curve we mentioned before. Define ψ on $[z_1, \frac{z_1+z_2}{2}]$ by

$$\psi(f)(z) = u \begin{pmatrix} f(\beta_1(z)) & & & & & & \\ & f(\beta_2(z)) & & & & & \\ & & \ddots & & & & \\ & & & f(\beta_s(z)) & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix} u^*$$

for each $z \in [z_1, \frac{z_1+z_2}{2}]$. Hence

$$\psi(f)(\frac{z_1+z_2}{2}) = \psi(f)(z_1) \quad \text{and} \quad \|\psi(f)(z_1) - \psi(f)(z)\| < \frac{\varepsilon_2}{4}$$

for all $f \in F$ and $z \in [z_1, \frac{z_1+z_2}{2}]$, since

$$\|\beta_j(z) - \beta_j(z_1)\| \le \varepsilon_1.$$

Recall that

$$\|\psi(f)(z_2) - \psi(f)(\frac{z_1+z_2}{2})\| = \|\psi(f)(z_2) - \psi(f)(z_1)\| < \frac{\varepsilon_1}{4}$$

for all $f \in F$. By Lemma 2.12 and the way ε_1 is chosen, one can extend the definition of ψ to $[\frac{z_1+z_2}{2}, z_2]$ so that it agrees with the definition of ψ at the endpoints $\frac{z_1+z_2}{2}$ and z_2, and

$$\|\psi(f)(z) - \psi(f)(\frac{z_1+z_2}{2})\| < \frac{\varepsilon_2}{4}$$

for all $f \in F$ and $z \in [\frac{z_1+z_2}{2}, z_2]$. Thus we obtain the definition of ψ on each edge $[z_1, z_2]$ with the property

$$\|\psi(f)(z) - \psi(f)(z_1)\| < \frac{\varepsilon_2}{4} \quad \text{for all } z \in [z_1, z_2] \text{ and } f \in F.$$

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Therefore, for each 2-simplex Δ, we have the definition of ψ on $\partial\Delta$ such that

$$\|\psi(f)(z) - \psi(f)(z')\| < \varepsilon_2$$

for all $f \in F$ and $z, z' \in \partial\Delta$. Apply Lemma 2.12 again to obtain the definition of ψ on each 2-simplex Δ such that

$$\|\psi(f)(z) - \psi(f)(z')\| < \frac{\varepsilon_3}{4} \text{ for all } z, z' \in \Delta.$$

Repeating this procedure, we can obtain the definition of ψ on the whole space Z_i such that, if z, z' are in the same simplex, then

$$\|\psi(f)(z) - \psi(f)(z')\| < \frac{\varepsilon}{4}.$$

Thus we have obtained the property (1) of the theorem.

To prove injectivity, we only need to verify

$$\mathrm{SP}(\psi) = \bigcup_{x \in \coprod_{i=1}^m Z_i} \mathrm{SP}\psi_x = Y.$$

The proof is the same as the corresponding part of the proof of [Li1, Theorem 2.2.10]. Namely, we use the fact that $\bigcup_{x \in \coprod_{i=1}^m \tilde{X}_i} \mathrm{SP}\phi_x = Y$ — a consequence of the injectivity of ϕ, the property (ii) of the maps β_j above, and the fact that $\mathrm{SP}\phi_x$ and $\mathrm{SP}\phi_{x'}$ can be paired to be within $\frac{\varepsilon_1}{4}$ whenever $\mathrm{dist}(x, x') < \eta$. (See [Li1, Theorem 2.2.10,] for details.) Note that one can prove

$$\bigcup_{x \in \coprod_{i=1}^m Z_i^{(1)}} \mathrm{SP}\phi_x = Y,$$

where $Z_i^{(1)}$ is the 1-skeleton of Z_i under the subdivision. ■

2.14 **The proof of Theorem 2.1** As pointed out in 2.2, we only need to prove the case for full matrix algebras. We will imitate [Li1, 2.2.12], with [Li1, 2.2.6 and 2.2.10] replaced by Lemma 2.10 and Lemma 2.13 above.

As in 2.3, let $\tilde{A}_n = \phi_{n,\infty}(A_n) \cong \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(\tilde{X}_{n,i}))$, where the spaces $\tilde{X}_{n,i}$ are closed subspaces of finite simplicial complexes $X_{n,i}$. Write $A = \varinjlim(\tilde{A}_n, \tilde{\phi}_{n,m})$, where the homomorphisms $\tilde{\phi}_{n,m}$ are induced by $\phi_{n,m}$ and they are injective.

Let $\varepsilon_n = \frac{1}{2^n}$. Let $\{x_i\}_{i=1}^{\infty}$ be a dense subset of A. We will construct an injective inductive limit $B_1 \to B_2 \to \cdots$ as follows.

Consider $G_1 = \{x_1\} \subset A$. There is an $\tilde{A}_{i_1}$ and a finite subset $\tilde{G}_1 \subset \tilde{A}_{i_1}$ such that $G_1 \subset_{\varepsilon_1/2} \tilde{G}_1$.

For $\tilde{G}_1 \subset \tilde{A}_{i_1}$, by Lemma 2.10 applied to each block of $\tilde{A}_{i_1}$, there exists a subalgebra $B_1 \subset \tilde{A}_{i_1}$ satisfying the following two conditions:

(1) B_1 is a finite direct sum of matrix algebras over finite simplicial complexes.

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(2) $\tilde{G}_1 \subset_{\varepsilon_1/2} B_1$. This gives us an injective homomorphism $B_1 \hookrightarrow \tilde{A}_{i_1}$.

Let $\{b_{1j}\}_{j=1}^{\infty}$ be a dense subset of B_1. Set $F_1 = \{b_{11}\} \subset B_1$ and $G_2 = \{x_1, x_2\} \subset A$. There exist $\tilde{A}_{i_2}$, $(i_2 > i_1)$ and a finite subset $\tilde{G}_2 \subset \tilde{A}_{i_2}$ such that $G_2 \subset_{\varepsilon_2/2} \tilde{G}_2$. By Lemma 2.13 applied to $F_1 \subset B_1$ (in place of $F \subset A$), $\tilde{G}_2 \subset \tilde{A}_{i_2}$ (in place of $G \subset B$), and the injective map $B_1 \hookrightarrow \tilde{A}_{i_1} \to \tilde{A}_{i_2}$ (in place of $\phi\colon A \to B$), there exist a subalgebra $B_2 \subset \tilde{A}_{i_2}$ (in place of $B' \subset B$), which is a direct sum of matrix algebras over finite simplicial complexes, and an injective homomorphism $\psi_{1,2}\colon B_1 \to B_2$ such that $\tilde{G}_2 \subset_{\varepsilon_2/2} B_2$ (see (2) of Lemma 2.13) and such that the diagram

$$\begin{tikzcd} \tilde{A}_{i_1} \arrow[r, "\tilde{\phi}_{i_1,i_2}"] & \tilde{A}_{i_2} \\ B_1 \arrow[u, hook] \arrow[r, "\psi_{1,2}"] & B_2 \arrow[u, hook] \end{tikzcd}$$

almost commutes on F_1 to within ε_1 (see Lemma 2.13(1)). Let $\{b_{2j}\}_{j=1}^{\infty}$ be a dense subset of B_2. Consider $F_2 = \{b_{21}, b_{22}\} \cup \{\psi_{1,2}(b_{11}), \psi_{1,2}(b_{12})\}$ and $G_3 = \{x_1, x_2, x_3\}$ in place of F_1 and G_2 respectively, and repeat the above construction to obtain $\tilde{A}_{i_3}, B_3 \subset \tilde{A}_{i_3}$ and an injective homomorphism $\psi_{2,3}\colon B_2 \to B_3$ (using ε_2 and ε_3 in place of ε_1 and ε_2, respectively).

In general, we can construct the diagram

$$\begin{tikzcd} \tilde{A}_{i_1} \arrow[r, "\tilde{\phi}_{i_1,i_2}"] & \tilde{A}_{i_2} \arrow[r, "\tilde{\phi}_{i_2,i_3}"] & \tilde{A}_{i_3} \arrow[r] & \cdots \tilde{A}_{i_k} \arrow[r] & \cdots \\ B_1 \arrow[u, hook] \arrow[r, "\psi_{1,2}"] & B_2 \arrow[u, hook] \arrow[r, "\psi_{2,3}"] & B_3 \arrow[u, hook] \arrow[r] & \cdots B_k \arrow[u, hook] \arrow[r] & \cdots \end{tikzcd}$$

with the following properties:

(i) The homomorphisms $\psi_{k,k+1}$ are injective;
(ii) For each k, $G_k := \{x_1, x_2, \dots, x_k\} \subset_{\varepsilon_k} \tilde{\phi}_{i_k,\infty}(B_k)$, where B_k is considered to be a sub-algebra of $\tilde{A}_{i_k}$;
(iii) The diagram

$$\begin{tikzcd} \tilde{A}_{i_k} \arrow[r, "\tilde{\phi}_{i_k,i_{k+1}}"] & \tilde{A}_{i_{k+1}} \\ B_k \arrow[u, hook] \arrow[r, "\psi_{k,k+1}"] & B_{k+1} \arrow[u, hook] \end{tikzcd}$$

almost commutes on $F_k := \{b_{ij} \;;\; 1 \le i \le k, 1 \le j \le k\}$ to within ε_k, where $\{b_{ij}\}_{j=1}^{\infty}$ is a dense subset of B_i. (Note for $i < k$, $b_{ij} \in B_i$ can be regarded as elements in B_k via the injective map $\psi_{i,k}$.)

Then, by [Ell3, 2.3; 2.4], the above diagram defines a homomorphism from $B = \varinjlim(B_n, \psi_{n,m})$ to $A = \varinjlim(\tilde{A}_n, \tilde{\phi}_{n,m})$. It is routine to check that the homomorphism is in fact an isomorphism. This ends the proof of Theorem 2.1. ∎

2.15 Let us briefly discuss the special case that A is a commutative C^*-algebra. That is, for a compact metrizable space X, how to write $A = C(X)$ as an inductive limit of C^*-algebras $C(X_n)$ with injective connecting maps, where X_n are finite simplicial complexes.

It is easy to see that the theorem for this special case follows if one can prove:

(1) For a finite simplicial complex X, a closed subset $\tilde{X} \subset X$ and a positive number $\varepsilon > 0$, there exist a finite simplicial complex $Y \subset X$ and a surjective map $f\colon \tilde{X} \to Y$ such that $\operatorname{dist}(x, f(x)) < \varepsilon$ for any $x \in \tilde{X}$;

(2) Under the assumption of (1), if we further assume that Z is a finite simplicial complex and $g\colon \tilde{X} \to Z$ is a surjective map, then the space Y and the map f can be chosen in such a way that there is a surjective map $g_1\colon Y \to Z$ with the property that $\operatorname{dist}(g_1(f(x)), g(x)) < \varepsilon$ for any $x \in \tilde{X}$.

The above statement (1) is Lemma 2.10 and statement (2) is Lemma 2.13 for the special case of commutative C^*-algebras. (To prove (1), there is a simpler construction. But the construction in 2.5–2.8 is important in the proof of (2) and Lemma 2.13 above.)

2.16 Suppose that $A = \varinjlim(A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i})), \phi_{n,m})$ is a simple inductive limit with $\sup_{n,i}(\dim(X_{n,i})) < \infty$, where the spaces $X_{n,i}$ are finite simplicial complexes. By Theorem 2.1 above, A can be written as the inductive limit of another sequence of the same form, but with injective connecting homomorphisms $\phi_{n,m}$. Then by [G1, Theorem 6.3, Remark 6.4], A can be written as

$$\varinjlim(B_n = \bigoplus_{i=1}^{s_n} M_{\{n,i\}}(C(Y_{n,i})), \psi_{n,m})$$

with $\psi_{n,m}$ injective, where the spaces $Y_{n,i}$ are from the list of spaces $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{II,k}$, $T_{III,k}$ and S^2.

Furthermore, by [G1, 6.5] and the discussion in 2.2 above, a simple inductive limit $A = \varinjlim\big(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}\big)$ can be written as $\varinjlim\big(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i}M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}, \psi_{n,m}\big)$ with $\psi_{n,m}$ injective, where the spaces $Y_{n,i}$ are from the list of spaces $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{II,k}$, $T_{III,k}$ and S^2.

Namely, we have the following theorem.

Theorem 2.17 *Suppose that*

$$A = \varinjlim\Big(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}\Big)$$

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is a simple inductive limit with $\sup_{n,i}\{\dim X_{n,i}\} < \infty$*, where the spaces* $X_{n,i}$ *are finite simplicial complexes. Then A can be written as*

$$B = \varinjlim\Big(B_n = \bigoplus_{i=1}^{s_n} Q_{n,i}M_{\{n,i\}}(C(Y_{n,i}))Q_{n,i}, \psi_{n,m}\Big)$$

with $\psi_{n,m}$ *injective, where* $Y_{n,i}$ *are from the list* $\{pt\}$, $[0,1]$, S^1, $T_{II,k}$, $T_{III,k}$*, and* S^2.

Combining with [G1, Lemma 6.2], we obtain the following corollary.

Corollary 2.18 *Let*

$$A = \varinjlim\Big(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}\Big)$$

be a simple inductive limit with $\phi_{n,m}$ *injective, where* $X_{n,i}$ *are spaces from the list* $\{pt\}$, $[0,1]$, S^1, $T_{II,k}$, $T_{III,k}$ *and* S^2. *Then there are a subsequence* $A_{k_1}, A_{k_2}, \dots, A_{k_n}, \dots$ *and homomorphisms* $\psi_{n,n+1}\colon A_{k_n} \to A_{k_{n+1}}$ *such that the following are true:*

(1) $A = \varinjlim(A_{k_n}, \psi_{n,m})$*, where* $\psi_{n,m} = \psi_{m-1,m} \circ \psi_{m-2,m-1} \circ \cdots \circ \psi_{n+1,n+2} \circ \psi_{n,n+1}$;
(2) *Each* $\psi_{n,m}^{i,j}$ *is injective if* $X_{k_m,j} \neq \{pt\}$;
(3) $KK(\psi_{n,m}) = KK(\phi_{k_n,k_m})$.

Proof The result is trivially true if there is a subsequence $k_1, k_2, \dots, k_n, \dots$ such that each A_{k_n} is a finite dimensional C^*-algebra, since we can choose $\psi_{n,n+1}\colon A_{k_n} \to A_{k_{n+1}}$ to be $\phi_{k_n,k_{n+1}}$. (For the chosen subsequence (k_n) and maps $\psi_{n,n+1}$, the condition (2) is trivially true, since $X_{k_m,j} = \{\mathrm{pt}\}$ for all m, j.) Without loss of generality we may assume that for any n, A_n is not of finite dimension. That is, for any n, there is at least one i such that $X_{n,i} \neq \{\mathrm{pt}\}$.

As explained in 2.2, we need only consider the case of full matrix algebras $A_n = \bigoplus_{i=1}^{t_n} M_{[n,i]}(C(X_{n,i}))$.

For any given A_n, finite set $F \subseteq A_n$, and $\varepsilon > 0$, by [G1, Lemma 6.2]] with $C = A_n$ and $\phi = \mathrm{id}\colon C \to A_n$, there are an A_m and a homomorphism $\psi\colon A_n \to A_m$ such that for any i, j, either the partial map $\psi^{i,j}\colon A_n^i \to A_m^j$ is injective or $\psi^{i,j}(A_n^i)$, the image of A_n^i under $\psi^{i,j}$, is a finite dimensional algebra, and such that $\|\phi_{n,m}(f) - \psi(f)\| < \varepsilon$. Since A_n is a direct sum of matrix algebras over the special spaces $\{\mathrm{pt}\}$, $[0,1]$, S^1, $T_{II,k}$, $T_{III,k}$, and S^2, by [G1, 5.13, 5.16, 5.17] (see [DL, Proposition 2.9] also), the element $KK(\phi) \in KK(A_n, A_m)$ of a homomorphism $\phi\colon A_n \to A_m$ is completely determined by the image of a finite set of projections $\mathcal{P}$ from $A_n \otimes M_\bullet(C(W_k \times S^1))$, $k = 2, 3, 4, \dots$, under the homomorphisms $\phi, \phi \otimes \mathrm{id}$ (see 5.16 and 5.17 for notations). (Note that the case of $X_{n,i} = S^1$ is not discussed in [G1]. But it can be dealt with in the same way.) Therefore, if F generates A_n as a C^*-algebra and $\varepsilon > 0$ is sufficiently small, then $\|\phi_{n,m}(f) - \psi(f)\| < \varepsilon$ for all $f \in F$ implies that $KK(\phi_{n,m}) = KK(\psi)$. (Also see [R, Proposition 5.4]. Note that the spaces $X_{n,i}$ are finite simplicial complexes. Hence $KL(A_n, A_m) = KK(A_n, A_m)$.) From the proof of

[G1, Lemma 6.2], we know that $\psi^{i,j}$ is injective if $X_{m,j} \neq \{\mathrm{pt}\}$. (On lines 6–7 of page 443 of [G1], the author assumed $X_{m,j} \neq \{\mathrm{pt}\}$ and then constructed the map ψ to be injective.)

Based on the above fact, passing to a subsequence, it is routine to construct an approximate intertwining

$$\begin{tikzcd}
A_{k_1} \arrow[r, "\phi_{k_1,k_2}"] \arrow[d, leftrightarrow, "\mathrm{id}"'] & A_{k_2} \arrow[r, "\phi_{k_2,k_3}"] \arrow[d, leftrightarrow, "\mathrm{id}"'] & A_{k_3} \arrow[r, "\phi_{k_3,k_4}"] \arrow[d, leftrightarrow, "\mathrm{id}"'] & \cdots A \\
 & & & \cdots \\
A_{k_1} \arrow[r, "\psi_{k_1,k_2}"] & A_{k_2} \arrow[r, "\psi_{k_2,k_3}"] & A_{k_3} \arrow[r, "\psi_{k_3,k_4}"] & \cdots A
\end{tikzcd}$$

of the maps $\phi_{k_n,k_{n+1}}$ with the maps $\psi_{n,n+1}$, which therefore give rise to the same limit C^*-algebra $A = \varinjlim(A_{k_n}, \psi_{n,m})$ (see [Ell3, §2] and [EG, §1.1]), such that $\psi_{n,n+1}$ satisfy the desired conditions $KK(\psi_{n,n+1}) = KK(\phi_{k_n,k_{n+1}})$ and $\psi_{n,m}^{i,j}$ is injective if $X_{k_{n+1},j} \neq \{\mathrm{pt}\}$. The only thing left to verify is the condition (2) for $\psi_{n,m}$ with $m > n+1$. From the assumption at the beginning of the proof, there is an index $i(n)$ for each k_n such that $X_{k_n,i(n)} = \mathrm{SP}(A_{k_n}^{i(n)}) \neq \{\mathrm{pt}\}$. For any pair of blocks A_n^i, A_m^j $(m \geq n+1)$, consider the homomorphism

$$\alpha := \psi_{m-1,m}^{i(m-1),j} \circ \psi_{m-2,m-1}^{i(m-2),i(m-1)} \circ \cdots \circ \psi_{n+1,n+2}^{i(n+1),i(n+2)} \circ \psi_{n,n+1}^{i,i(n+1)}.$$

The map α is injective if $X_{m,j} \neq \{\mathrm{pt}\}$. Note that α is the part of the map $\psi_{n,m}^{i,j}$ obtained by cutting down by the projection $\alpha(\mathbf{1}_{A_n^i})$. Therefore the map $\psi_{n,m}^{i,j}$ itself is also injective if $X_{m,j} \neq \{\mathrm{pt}\}$. ∎

Applying Corollary 2.11, we can strengthen [G1, Corollary 6.12] to the following result.

Corollary 2.19 *Suppose that* $A = \varinjlim\big(A_n = \bigoplus_{i=1}^{t_n} P_{n,i}M_{[n,i]}(C(X_{n,i}))P_{n,i}, \phi_{n,m}\big)$ *is a simple inductive limit* C^**-algebra. Suppose that each of the spaces* $X_{n,i}$ *belongs to the list* $\{pt\}, [0,1], S^1, T_{II,k}, T_{III,k}$ *and* S^2. *Suppose that all the connecting maps* $\phi_{n,m}$ *are injective. For any* $F \subset A_n$ *and* $\varepsilon > 0$, *if* m *is large enough, then there are two mutually orthogonal projections* $P, Q \in A_m$ *and two homomorphisms* $\phi\colon A_n \to PA_mP$ *and* $\psi\colon A_n \to QA_mQ$ *such that*

(1) $\|\phi_{n,m}(f) - (\phi \oplus \psi)(f)\| < \varepsilon$ *for all* $f \in F$;
(2) $\phi(F)$ *is weakly approximately constant to within* ε *and* $SPV(\phi) < \varepsilon$;
(3) ψ *factors through* D*—a direct sum of matrix algebras over* $C[0,1]$ *or* $\mathbb{C}$*—as*

$$A_n \xrightarrow{\psi_1} D \xrightarrow{\psi_2} QA_mQ$$

with ψ_2 *injective.*

Furthermore, if for some i, j, *the partial map* $\phi_{n,m}^{i,j}\colon A_n^i \to A_m^j$ *is homotopic to a homomorphism with finite dimensional image, then the part* ϕ *of the decomposition*

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$\phi \oplus \psi$ corresponding to this partial map can be chosen to be zero (or, equivalently, $\phi_{n,m}^{i,j}$ itself is close to a homomorphism ψ factoring through D — a matrix algebra over $C[0,1]$ or $\mathbb{C}$ as in (3) *above).*

Proof Apply [G1, Corollary 6.12] to the finite set F and $\frac{\varepsilon}{2}$ (instead of ε). Let $\psi\colon A_n \to QA_mQ$ be as in [G1, Corollary 6.12]], factoring through C — a direct sum of matrix algebras over $C[0,1]$ — as

$$\psi\colon A_n \xrightarrow{\psi_1} C \xrightarrow{\psi_2} QA_mQ.$$

Apply Corollary 2.11 to $\psi_2\colon C \to QA_mQ$ (in place of $\phi\colon A \to B$), $\psi_1(F) \subset C$ (in place of $F \subset A$), and $\frac{\varepsilon}{2}$ (in place of ε) to obtain D and homomorphisms

$$C \xrightarrow{\phi_1} D \xrightarrow{\phi_2} QA_mQ$$

with ϕ_2 injective, such that

$$\|(\phi_2 \circ \phi_1)(f) - \psi_2(f)\| < \frac{\varepsilon}{2} \quad \text{for all } f \in \psi_1(F).$$

Finally let new$\psi_1 = \phi_1 \circ (\text{old}\psi_1)$, new$\psi_2 = \phi_2$ and new$\psi = (\text{new}\psi_2) \circ (\text{new}\psi_1)$ to finish the proof. ■

Corollaries 2.18 and 2.19 will be used in the proof of the classification theorem for simple AH algebras in [EGL].

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