Forum of Mathematics, Sigma (2021), **Vol. 9**:e12 1–40
doi:10.1017/fms.2021.7

CAMBRIDGE
UNIVERSITY PRESS

RESEARCH ARTICLE

Smoothness of solutions of a convolution equation of restricted type on the sphere

Diogo Oliveira e Silva[1] and René Quilodrán[2]

[1]School of Mathematics, University of Birmingham, Edgbaston, B15 2TT,England; E-mail: d.oliveiraesilva@bham.ac.uk.
[2]Santiago, Chile; E-mail: rquilodr@dim.uchile.cl.

Received: 3 June 2020; **Revised:** 15 December 2020; **Accepted:** 25 December 2020

2020 Mathematics Subject Classification: *Primary* – 49N60; *Secondary* – 35B38; 42B37

Abstract

Let $\mathbb{S}^{d-1}$ denote the unit sphere in Euclidean space $\mathbb{R}^d$, $d \geqslant 2$, equipped with surface measure σ_{d-1}. An instance of our main result concerns the regularity of solutions of the convolution equation

$$a \cdot (f\sigma_{d-1})^{*(q-1)}\big|_{\mathbb{S}^{d-1}} = f, \text{ a.e. on } \mathbb{S}^{d-1},$$

where $a \in C^\infty(\mathbb{S}^{d-1})$, $q \geqslant 2(d+1)/(d-1)$ is an integer, and the only a priori assumption is $f \in L^2(\mathbb{S}^{d-1})$. We prove that any such solution belongs to the class $C^\infty(\mathbb{S}^{d-1})$. In particular, we show that all critical points associated with the sharp form of the corresponding adjoint Fourier restriction inequality on $\mathbb{S}^{d-1}$ are C^∞-smooth. This extends previous work of Christ and Shao [4] to arbitrary dimensions and general even exponents and plays a key role in the companion paper [24].

1. Introduction

Sharp Fourier restriction theory has attracted a great deal of interest recently. In the particular case of the unit sphere equipped with surface measure $(\mathbb{S}^{d-1}, \sigma_{d-1})$, a natural starting point is that of the Tomas-Stein inequality,

$$\|\widehat{f\sigma}_{d-1}\|_{L^q(\mathbb{R}^d)} \leqslant \mathbf{T}_{d,q}\|f\|_{L^2(\mathbb{S}^{d-1})}, \tag{1.1}$$

which is known to hold [28, 29] with $\mathbf{T}_{d,q} < \infty$ provided that $d \geqslant 2$ and $q \geqslant q_d := 2\frac{d+1}{d-1}$; see (1.3) for the precise definition of the Fourier extension operator. Here $\mathbf{T}_{d,q}$ denotes the optimal constant given by

$$\mathbf{T}_{d,q} = \sup_{0\neq f\in L^2} \frac{\|\widehat{f\sigma}_{d-1}\|_{L^q(\mathbb{R}^d)}}{\|f\|_{L^2(\mathbb{S}^{d-1})}}. \tag{1.2}$$

By a *maximiser* of (1.1) we mean a nonzero, complex-valued function $f \in L^2(\mathbb{S}^{d-1})$ for which $\|\widehat{f\sigma}_{d-1}\|_{L^q(\mathbb{R}^d)} = \mathbf{T}_{d,q}\|f\|_{L^2(\mathbb{S}^{d-1})}$.

The existence of maximisers for the Tomas-Stein inequality (1.1) has been investigated in the works [3, 9, 13, 25], but the explicit form of the maximisers is only known in very few, special cases [1, 11]. Once maximisers are known to exist, it is natural to investigate their properties with methods from the calculus of variations. In the present article, we study the associated Euler-Lagrange equation and show that the corresponding critical points are C^∞-smooth whenever the exponent q is an even integer. Our motivation is twofold. On the one hand, our main result is used in the companion paper [24] to establish that constant functions are the unique real-valued maximisers for a number of new sharp instances of inequality (1.1) and to fully characterise all complex-valued maximisers. On the other hand, we extend the main results of Christ and Shao [4] to arbitrary dimensions and general even exponents.

Let $d \geqslant 2$ and $q \geqslant q_d$ be given. Consider the Fourier extension operator $\mathcal{E}(f) = \widehat{f\sigma}_{d-1}$, acting on functions $f : \mathbb{S}^{d-1} \to \mathbb{C}$ via

$$\widehat{f\sigma}_{d-1}(x) = \int_{\mathbb{S}^{d-1}} f(\omega) e^{-ix\cdot\omega}\, \mathrm{d}\sigma_{d-1}(\omega). \tag{1.3}$$

The operator $\mathcal{E}$ is bounded from $L^2(\mathbb{S}^{d-1})$ to $L^q(\mathbb{R}^d)$ in light of (1.1). Its adjoint equals the restriction operator $\mathcal{E}^*(g) = g^\vee|_{\mathbb{S}^{d-1}}$ and is bounded from $L^{q'}(\mathbb{R}^d)$ to $L^2(\mathbb{S}^{d-1})$; here, $q' = q/(q-1)$ denotes the conjugate Lebesgue exponent of q. Suppose that f maximises the functional $\Phi_{d,q}$ associated to (1.1),

$$\Phi_{d,q}(f) = \frac{\|\widehat{f\sigma}_{d-1}\|^q_{L^q(\mathbb{R}^d)}}{\|f\|^q_{L^2(\mathbb{S}^{d-1})}}, \tag{1.4}$$

and further assume f to be L^2-normalised, $\|f\|_{L^2(\mathbb{S}^{d-1})} = 1$. We can then estimate the operator norm of the extension operator as follows:

$$\begin{aligned}
\|\mathcal{E}\|^q_{L^2\to L^q} &= \|\mathcal{E}(f)\|^q_{L^q(\mathbb{R}^d)} = \langle |\mathcal{E}(f)|^{q-2}\mathcal{E}(f), \mathcal{E}(f)\rangle = \langle \mathcal{E}^*(|\mathcal{E}(f)|^{q-2}\mathcal{E}(f)), f\rangle_{L^2(\mathbb{S}^{d-1})} \\
&\leqslant \|\mathcal{E}^*(|\mathcal{E}(f)|^{q-2}\mathcal{E}(f))\|_{L^2(\mathbb{S}^{d-1})} \leqslant \|\mathcal{E}^*\|_{L^{q'}\to L^2} \||\mathcal{E}(f)|^{q-2}\mathcal{E}(f)\|_{L^{q'}(\mathbb{R}^d)} \\
&= \|\mathcal{E}^*\|_{L^{q'}\to L^2} \|\mathcal{E}(f)\|^{q-1}_{L^q(\mathbb{R}^d)} = \|\mathcal{E}\|^q_{L^2\to L^q},
\end{aligned} \tag{1.5}$$

where $\langle\cdot,\cdot\rangle$ denotes the $L^{q'} - L^q$ pairing in $\mathbb{R}^d$, and $\langle\cdot,\cdot\rangle_{L^2(\mathbb{S}^{d-1})}$ denotes the L^2 pairing in $\mathbb{S}^{d-1}$. In addition to easy algebraic manipulations, the first inequality in (1.5) amounts to an application of the Cauchy-Schwarz inequality, and the second inequality in (1.5) holds because the adjoint operator $\mathcal{E}^*$ is bounded from $L^{q'}$ to L^2. In the last identity, we also used the fact that the operator norms of $\mathcal{E}, \mathcal{E}^*$ coincide, $\|\mathcal{E}\|_{L^2\to L^q} = \|\mathcal{E}^*\|_{L^{q'}\to L^2}$. Because the first and last terms in the chain of inequalities (1.5) coincide, all inequalities are forced to be equalities. In particular, equality holds in the application of the Cauchy-Schwarz inequality, which in turn implies the existence of a constant μ, for which

$$\mathcal{E}^*(|\mathcal{E}(f)|^{q-2}\mathcal{E}(f)) = \mu f$$

holds outside a set of zero σ_{d-1}-measure. Thus, we see that a maximiser of (1.1) necessarily satisfies

$$\left(|\widehat{f\sigma}_{d-1}|^{q-2}\widehat{f\sigma}_{d-1}\right)^\vee\Big|_{\mathbb{S}^{d-1}} = \lambda \|f\|^{q-2}_{L^2(\mathbb{S}^{d-1})} f, \quad \sigma_{d-1}\text{-a.e. on } \mathbb{S}^{d-1}, \tag{1.6}$$

for some $\lambda \in \mathbb{C}$. This is the Euler-Lagrange equation associated with the variational problem (1.2); see [2] for a more general statement. To determine the parameter $\lambda \in \mathbb{C}$, one simply multiplies both sides of (1.6) by $\bar{f}$ and integrates with respect to surface measure to check that $\lambda = \Phi_{d,q}(f)$. In particular, f is a maximiser of inequality (1.1) if and only if (1.6) holds with $\lambda = \mathbf{T}^q_{d,q}$.

General nonzero solutions of the Euler-Lagrange equation (1.6) are called *critical points* of the functional $\Phi_{d,q}$. As noted in [2], it follows at once that constant functions satisfy (1.6) for some $\lambda > 0$, simply because $|\widehat{\sigma}_{d-1}|^{q-2}\widehat{\sigma}_{d-1}$ is a radial function, the inverse Fourier transform of any radial function is radial and the restriction of any radial function on $\mathbb{R}^d$ to $\mathbb{S}^{d-1}$ is constant.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

If $q = 2n$ is an even integer, $n \in \mathbb{N}$, then the Tomas-Stein inequality (1.1) can be equivalently stated in convolution form via Plancherel's theorem as

$$\|(f\sigma_{d-1})^{*n}\|^2_{L^2(\mathbb{R}^d)} \leqslant (2\pi)^{-d}\mathbf{T}^{2n}_{d,2n}\|f\|^{2n}_{L^2(\mathbb{S}^{d-1})}, \tag{1.7}$$

where the n-fold convolution measure $(f\sigma_{d-1})^{*n}$ is recursively defined for integral values of $n \geqslant 2$ via

$$(f\sigma_{d-1})^{*2} = f\sigma_{d-1} * f\sigma_{d-1}, \text{ and } (f\sigma_{d-1})^{*(n+1)} = (f\sigma_{d-1})^{*n} * f\sigma_{d-1}. \tag{1.8}$$

The functional $\Phi_{d,2n}$ can then be rewritten as

$$\Phi_{d,2n}(f) = (2\pi)^d \frac{\|(f\sigma_{d-1})^{*n}\|^2_{L^2(\mathbb{R}^d)}}{\|f\|^{2n}_{L^2(\mathbb{S}^{d-1})}}, \tag{1.9}$$

and the Euler-Lagrange equation (1.6) translates to

$$\left((f\sigma_{d-1})^{*n} * (f_\star\sigma_{d-1})^{*(n-1)}\right)\Big|_{\mathbb{S}^{d-1}} = (2\pi)^{-d}\lambda\|f\|^{2n-2}_{L^2(\mathbb{S}^{d-1})} f, \quad \sigma_{d-1}\text{-a.e. on } \mathbb{S}^{d-1}, \tag{1.10}$$

where $f_\star$ denotes the *conjugate reflection* of f around the origin, defined via

$$f_\star(\omega) = \overline{f(-\omega)}, \qquad \text{for all } \omega \in \mathbb{S}^{d-1}.$$

A function $f : \mathbb{S}^{d-1} \to \mathbb{C}$ is said to be *antipodally symmetric* if $f = f_\star$, in which case basic properties of the Fourier transform imply that $\widehat{f\sigma}_{d-1}$ is real valued.

The convolution structure of equation (1.10) induces some extra regularity on its solutions, a phenomenon that turns out to hold in greater generality. To describe it precisely, consider the multilinear operator $\mathrm{M}\colon L^2(\mathbb{S}^{d-1})^{m+1} \to L^2(\mathbb{S}^{d-1})$,

$$\mathrm{M}(f_1, \ldots, f_{m+1}) = (f_1\sigma_{d-1} * \cdots * f_{m+1}\sigma_{d-1})\Big|_{\mathbb{S}^{d-1}}, \tag{1.11}$$

which is well defined for integral values of $m \geqslant 4$ if $d = 2$ and $m \geqslant 2$ if $d \geqslant 3$ in view of the chain of inequalities (1.5); see also [2, Prop. 2.4]. Further consider the conjugate reflection operator $R\colon L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, $R(f) = f_\star$. Given an integer $k \in \mathbb{N}_0$, the powers R^k are defined in the usual way via composition, with the understanding that $R^0 = \mathrm{Id}$. We are interested in solutions of the general equation

$$a \cdot \mathrm{M}(R^{k_1}(f), \ldots, R^{k_{m+1}}(f)) = \lambda f, \quad \sigma_{d-1}\text{-a.e. on } \mathbb{S}^{d-1}, \tag{1.12}$$

where $(k_1, \ldots, k_{m+1}) \in \{0,1\}^{m+1}$, $a \in C^\infty(\mathbb{S}^{d-1})$ and $\lambda \in \mathbb{C}$. The additional factor $a \in C^\infty(\mathbb{S}^{d-1})$ brings no further complications to the analysis but can be used to address the smoothness of critical points for weighted measures on $\mathbb{S}^{d-1}$ and, by an additional scaling argument, on ellipsoids.

Our main result concerns regularity properties of generic solutions of equation (1.12).

Theorem 1.1. *Let $d \geqslant 2$, and let m be an integer satisfying $m \geqslant 4$ if $d = 2$ and $m \geqslant 2$ if $d \geqslant 3$. Let $(k_1, \ldots, k_{m+1}) \in \{0,1\}^{m+1}$, $a \in C^\infty(\mathbb{S}^{d-1})$ and $\lambda \in \mathbb{C} \setminus \{0\}$. If $f \in L^2(\mathbb{S}^{d-1})$ is a complex-valued solution of equation (1.12), then $f \in C^\infty(\mathbb{S}^{d-1})$.*

The special case $(d, m) = (3, 2)$ of Theorem 1.1 implies [4, Theorem 1.1]. Thus, Theorem 1.1 extends [4, Theorem 1.1] to arbitrary dimensions and general even exponents. Interestingly, our proof of Theorem 1.1 bypasses the Banach fixed point argument from [4] and, as such, could be considered more elementary and of independent value. Moreover, the case $(d, m) = (2, 4)$ of Theorem 1.1 completes the proof of the main result in [26], where the following issue was detected: In [26, Proof of Prop. 3.6], the first (unnumbered) displayed equation on page 9 seems to be incorrect. We further believe that the argument in [26] cannot be repaired without studying the regularity of the fourfold convolution σ_1^{*4}, such as a Hölder-type estimate of the kind established in Subsection 4.3. The following result is an immediate consequence of Theorem 1.1 and is used in a crucial manner in the companion paper [24].

Corollary 1.2. *Let $d \geqslant 2$ and $q \geqslant 2\frac{d+1}{d-1}$ be an even integer. If $f \in L^2(\mathbb{S}^{d-1})$ is a critical point of the functional $\Phi_{d,q}$, then $f \in C^\infty(\mathbb{S}^{d-1})$. In particular, maximisers of $\Phi_{d,q}$ are C^∞-smooth.*

1.1. Outline

In Section 2 we recall some useful facts about the special orthogonal group and define the appropriate smoothness spaces on $\mathbb{S}^{d-1}$ on which our estimates will be based. In Section 3 we collect some simple properties of the multilinear operator M, defined in (1.11). A fundamental distinction arises, depending on whether or not the parameters (d, m) from Theorem 1.1 lie on the 'boundary' of the set of admissible values. In the latter case, there is an automatic uniform gain in the initial regularity, which leads to a quick proof of the smoothing property of M in the 'non-boundary' case; see Lemma 3.4. This is not possible if (d, m) lies on the boundary, because in that case the corresponding functional is essentially scale invariant. The analysis is then more delicate and relies on Hölder-type estimates for certain convolution operators, which are the subject of Section 4. In turn, these estimates are used in Section 5 to find a suitable replacement for Lemma 3.4 in the boundary case; see Lemma 5.2. The final section, Section 6, is devoted to the proof of Theorem 1.1. We proceed in two steps: firstly, we establish an initial 'kick' in the regularity of any solution of equation (1.12); secondly, we use a bootstrapping procedure to promote the initial gain in regularity to C^∞-smoothness.

1.2. Notation

The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given a set $E \subset \mathbb{R}^d$, its indicator function is denoted by $\mathbb{1}_E$, its Lebesgue measure by $|E|$ and its complement by $E^{\complement} = \mathbb{R}^d \setminus E$. Given $r > 0$, we let $B(x, r) \subset \mathbb{R}^d$ denote the closed ball of radius r centered at $x \in \mathbb{R}^d$ and abbreviate $B_r = B(0, r)$. We will continue to denote by $(f\sigma_{d-1})^{*k}$ the k-fold convolution measure, recursively defined in (1.8). We denote $\mathbf{1} : \mathbb{S}^{d-1} \to \mathbb{R}$ the function $\mathbf{1}(\omega) \equiv 1$ and the zero function by $\mathbf{0} : \mathbb{S}^{d-1} \to \mathbb{R}$, $\mathbf{0}(\omega) \equiv 0$. We use $X \lesssim Y$, $Y \gtrsim X$ or $X = O(Y)$ to denote the estimate $|X| \leqslant CY$ for an absolute constant C and $X \simeq Y$ to denote the estimates $X \lesssim Y \lesssim X$. We will often require the implied constant C in the above notation to depend on additional parameters, which we will indicate by subscripts (unless explicitly omitted); thus, for instance, $X \lesssim_j Y$ denotes an estimate of the form $|X| \leqslant C_j Y$ for some C_j depending on j.

2. Function spaces

The special orthogonal group $\mathrm{SO}(d)$ consists of all $d \times d$ orthogonal matrices of unit determinant and acts transitively on the unit sphere $\mathbb{S}^{d-1}$ in the natural way. This action extends to actions on functions $f : \mathbb{S}^{d-1} \to \mathbb{C}$ by $\Theta f = f \circ \Theta$ for $\Theta \in \mathrm{SO}(d)$ and on finite Borel measures μ on $\mathbb{R}^d$ by $\Theta(\mu)(E) = \mu(\Theta(E))$ for $E \subseteq \mathbb{R}^d$. This extension interacts well with convolutions, in the sense that $\Theta(\mu * \nu) = \Theta(\mu) * \Theta(\nu)$. In particular, for any $\Theta \in \mathrm{SO}(d)$,

$$\Theta(f_1\sigma_{d-1} * \cdots * f_k\sigma_{d-1}) = (\Theta f_1)\sigma_{d-1} * \cdots * (\Theta f_k)\sigma_{d-1}. \tag{2.1}$$

For further information on the special orthogonal group, see [17] and references therein.

Given $\alpha \in (0, 1)$, let $\Lambda_\alpha(\mathbb{R}^d)$ denote the space of Hölder continuous functions $f : \mathbb{R}^d \to \mathbb{C}$ of order α, with norm

$$\|f\|_{\Lambda_\alpha(\mathbb{R}^d)} = \|f\|_{C^0(\mathbb{R}^d)} + \sup_{x \neq x'} |x - x'|^{-\alpha} |f(x) - f(x')|. \tag{2.2}$$

Given $1 < \alpha \notin \mathbb{N}$, write $\alpha = k + \delta$, with $k \in \mathbb{N}$ and $\delta \in (0, 1)$. We then say that $f \in \Lambda_\alpha(\mathbb{R}^d)$ if f is k times continuously differentiable, $f \in C^k(\mathbb{R}^d)$ and all of the kth-order partial derivatives of f belong

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

to $\Lambda_\delta(\mathbb{R}^d)$. An equivalent definition of the space $\Lambda_\alpha(\mathbb{R}^d)$ via Littlewood-Paley projections is available, but we shall delay its precise formulation until the need arises in the proof of Proposition 3.1. Given $\alpha \in (0,1)$, the space of Hölder continuous functions $f : \mathbb{S}^{d-1} \to \mathbb{C}$ of order α, denoted $\Lambda_\alpha(\mathbb{S}^{d-1})$, is defined in a similar way to (2.2). We further consider the space $\mathrm{Lip}(\mathbb{S}^{d-1})$ of Lipschitz continuous functions $f : \mathbb{S}^{d-1} \to \mathbb{C}$, equipped with the norm

$$\|f\|_{\mathrm{Lip}(\mathbb{S}^{d-1})} = \|f\|_{C^0(\mathbb{S}^{d-1})} + \sup_{\omega\neq\omega'} |\omega-\omega'|^{-1}|f(\omega)-f(\omega')|.$$

By $H^s = H^s(\mathbb{S}^{d-1})$ we mean the usual Sobolev space of functions having $s \geqslant 0$ derivatives in $L^2(\mathbb{S}^{d-1})$, defined via spherical harmonic expansions – for example, as in [22, §1.7.3, Remark 7.6] – or by considering a smooth partition of unity and diffeomorphisms onto the unit ball in $\mathbb{R}^{d-1}$ together with the usual Sobolev norm on $\mathbb{R}^{d-1}$; we set $H^0 = L^2(\mathbb{S}^{d-1})$. If s is an integer, then the following norm is equivalent to any other norm for H^s:

$$\|f\|_{H^s} = \|f\|_{L^2(\mathbb{S}^{d-1})} + \sum_{1\leqslant i<j\leqslant d} \|X_{i,j}^s f\|_{L^2(\mathbb{S}^{d-1})}, \tag{2.3}$$

where the derivatives are given by

$$X_{i,j} = x_i\partial_j - x_j\partial_i = \frac{\partial}{\partial\theta_{i,j}},\ X_{i,j}^s = \frac{\partial^s}{\partial\theta_{i,j}^s}, \tag{2.4}$$

and $\theta_{i,j}$ denotes the angle in polar coordinates of the (x_i, x_j)-plane; see, for instance, [6, §4.5], and [8, Prop. 3.3].

We find it convenient to work with the function spaces $\mathcal{H}^s = \mathcal{H}^s(\mathbb{S}^{d-1})$, which for $d = 3$ were introduced in [4]. To extend the definition to general dimensions $d \geqslant 2$, recall (2.4), where we introduced the derivatives $X_{i,j} = \partial/\partial\theta_{i,j}$. We can equivalently view $X_{i,j}$ as the C^∞-vector field on $\mathbb{S}^{d-1}$ that generates rotations about the (x_i, x_j)-plane for each $1 \leqslant i < j \leqslant d$. In this way, for each $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{S}^{d-1}$, $\exp(tX_{i,j})(\nu)$ is obtained by rotating the vector (ν_i, ν_j) by t radians. We note that $\{X_{i,j} : 1 \leqslant i < j \leqslant d\}$ forms a basis for $\mathfrak{so}(d)$, the Lie algebra of $SO(d)$.

Observe that the following quantity defines an equivalent norm on the space $\Lambda_\alpha(\mathbb{S}^{d-1})$, provided $\alpha \in (0,1)$:

$$\|f\|_{C^0(\mathbb{S}^{d-1})} + \max_{1\leqslant i<j\leqslant d} \sup_{\omega\in\mathbb{S}^{d-1}} \sup_{t\in\mathbb{R}} |t|^{-\alpha}|f(e^{tX_{i,j}}(\omega)) - f(\omega)|.$$

Given $s \in (0,1)$, the space $\mathcal{H}^s$ is defined as the set of all functions $f \in L^2(\mathbb{S}^{d-1})$ for which the norm

$$\|f\|_{\mathcal{H}^s} = \|f\|_{L^2(\mathbb{S}^{d-1})} + \sum_{1\leqslant i<j\leqslant d} \sup_{|t|\leqslant 1} |t|^{-s}\|f\circ e^{tX_{i,j}} - f\|_{L^2(\mathbb{S}^{d-1})} \tag{2.5}$$

is finite. We further set $\mathcal{H}^0 = L^2(\mathbb{S}^{d-1})$. Similar to the case of Euclidean space, the notion of weak differentiability of a function with respect to the vector field $X_{i,j}$ is made precise by the use of identity [24, Eq. (5.4)], which states that, for any complex-valued functions $f, g \in C^1(\mathbb{S}^{d-1})$,

$$\int_{\mathbb{S}^{d-1}} (X_{i,j}f)\,\overline{g}\,\mathrm{d}\sigma_{d-1} = -\int_{\mathbb{S}^{d-1}} f\,\overline{(X_{i,j}g)}\,\mathrm{d}\sigma_{d-1}. \tag{2.6}$$

In this way, we say that $f \in L^2(\mathbb{S}^{d-1})$ is weakly differentiable with respect to the vector field $X_{i,j}$ if there exists a function, denoted $X_{i,j}f$, that belongs to $L^1(\mathbb{S}^{d-1})$ and satisfies (2.6) for all $g \in C^\infty(\mathbb{S}^{d-1})$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

If $s = k + \alpha$, with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, then the space $\mathcal{H}^s$ consists of all functions $f \in L^2(\mathbb{S}^{d-1})$ for which the norm

$$\|f\|_{\mathcal{H}^s} = \|f\|_{L^2(\mathbb{S}^{d-1})} + \sum_Y \sum_{1\leqslant i<j\leqslant d} \sup_{|t|\leqslant 1} |t|^{-\alpha} \|Yf \circ e^{tX_{i,j}} - Yf\|_{L^2(\mathbb{S}^{d-1})} \tag{2.7}$$

is finite, where Y ranges over the finite set of all compositions $X_{i_1,j_1} \circ X_{i_2,j_2} \circ \cdots \circ X_{i_\ell,j_\ell}$ with $0 \leqslant \ell \leqslant k$ factors, and f itself is viewed as Yf where Y has zero factors. We implicitly assume the function f to be weakly differentiable with respect to the vector fields $\{X_{i,j}\}_{1\leqslant i<j\leqslant d}$ as many times as required by the definition of the norm.

The next result explores the relationship between the function spaces $\mathcal{H}^s$ and the usual Sobolev spaces H^t.

Lemma 2.1. *For every $0 \leqslant t < s$, $s \notin \mathbb{N}$, $\mathcal{H}^s$ is contained in the Sobolev space H^t, and*

$$\|f\|_{H^t} \leqslant C(s,t)\|f\|_{\mathcal{H}^s}, \tag{2.8}$$

for all $f \in \mathcal{H}^s$ and some constant $C(s,t) < \infty$.

Estimate (2.8) was noted in [4, Lemma 2.1] in the three-dimensional case $d = 3$ when $s < 1$. From Lemma 2.1 it follows at once that, given $s \in (1,\infty) \setminus \mathbb{N}$, $f \in \mathcal{H}^s$ and $X \in \{X_{i,j} : 1 \leqslant i < j \leqslant d\}$, then $\|Xf\|_{L^2(\mathbb{S}^{d-1})} \lesssim \|f\|_{\mathcal{H}^s}$ and therefore $Xf \in \mathcal{H}^{s-1}$. This observation will be useful in the sequel.

As a preliminary step towards the proof of Lemma 2.1, we recall the Euclidean Sobolev spaces $H^s(\mathbb{R}^d)$ and define the spaces $\mathcal{H}^s(\mathbb{R}^d)$ in analogy to the spherical ones, $\mathcal{H}^s$. Given $f : \mathbb{R}^d \to \mathbb{R}$ and $s = k + \alpha$ with $k \in \mathbb{N}_0$ and $\alpha \in (0, 1)$, we consider the norms

$$\|f\|^2_{H^s(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi, \tag{2.9}$$

$$\|f\|_{\mathcal{H}^s(\mathbb{R}^d)} := \|f\|_{L^2(\mathbb{R}^d)} + \sum_\ell \sup_{|x|\leqslant 1} |x|^{-\alpha} \|D^\ell f \circ \tau_x - D^\ell f\|_{L^2(\mathbb{R}^d)}, \tag{2.10}$$

where the sum in (2.10) runs over all multi-indices $\ell = (\ell_1, \ldots, \ell_d) \in \mathbb{N}^d$ satisfying $0 \leqslant |\ell| \leqslant k$, $D^\ell := \partial^{|\ell|}/\partial x_d^{\ell_d} \cdots \partial x_1^{\ell_1}$ denotes the partial derivative, $\tau_x : \mathbb{R}^d \to \mathbb{R}^d$, $y \mapsto x + y$ denotes translation by $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ and f itself is viewed as $D^0 f$. For $1 \leqslant i \leqslant d$, let e_i denote the ith canonical vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, with the 1 in the ith position. For every $s = k + \alpha$, $k \in \mathbb{N}_0$, $\alpha \in (0, 1)$, the following is an equivalent norm for $\mathcal{H}^s(\mathbb{R}^d)$, perhaps more reminiscent to that for $\mathbb{S}^{d-1}$ in (2.7) :

$$\|f\|_{L^2(\mathbb{R}^d)} + \sum_{0\leqslant|\ell|\leqslant k} \sum_{1\leqslant i\leqslant d} \sup_{|t|\leqslant 1} |t|^{-\alpha} \|D^\ell f \circ \tau_{te_i} - D^\ell f\|_{L^2(\mathbb{R}^d)}. \tag{2.11}$$

It is also worth observing that, by the triangle inequality and the translation invariance of the Lebesgue measure in $\mathbb{R}^d$, an equivalent norm to that in (2.10) or (2.11) is obtained by replacing $\sup_{|t|\leqslant 1}$ by $\sup_{|t|\leqslant \varepsilon}$ for any $\varepsilon > 0$. Likewise, by the triangle inequality and the $SO(d)$-invariance of the measure σ_{d-1} in $\mathbb{S}^{d-1}$, an equivalent norm for $\mathcal{H}^s$ is obtained from (2.7) by replacing $\sup_{|t|\leqslant 1}$ by $\sup_{|t|\leqslant \varepsilon}$ for any $\varepsilon > 0$.

Proof of Lemma 2.1. We discuss the analogous Euclidean statement for the case of the sphere then follows by working in local coordinates. In fact, as already mentioned, the H^t-norm on $\mathbb{S}^{d-1}$ can be defined by considering a smooth partition of unity and diffeomorphisms onto the unit ball in $\mathbb{R}^{d-1}$ together with the usual Sobolev norm on $\mathbb{R}^{d-1}$, as in (2.9). In order to handle the $\mathcal{H}^s$-norm on $\mathbb{S}^{d-1}$, we observe that it is likewise amenable to the use of local coordinates: Given a smooth partition of unity $\{\varphi_i\}_{1\leqslant i\leqslant N}$ on $\mathbb{S}^{d-1}$, $f \in \mathcal{H}^s$ if and only if $\varphi_i f \in \mathcal{H}^s$ for every $1 \leqslant i \leqslant N$, and $\|f\|_{\mathcal{H}^s} \simeq \sum_{1\leqslant i\leqslant N} \|\varphi_i f\|_{\mathcal{H}^s}$. Let O_i denote the support of φ_i, which we may take to be connected and of small diameter if necessary, and let $\{(\Omega_i, \psi_i)\}_{1\leqslant i\leqslant N}$ denote a system of local coordinates for $\mathbb{S}^{d-1}$ subordinate to $\{O_i\}_{1\leqslant i\leqslant N}$; that is, Ω_i is open and connected, $\psi_i : \Omega_i \longrightarrow \mathrm{int}(B_1)$ is a diffeomorphism onto the open unit ball in $\mathbb{R}^{d-1}$ and $O_i \Subset \Omega_i$ is compactly contained in Ω_i. If $\varepsilon > 0$ is small enough, it then follows that $\|\varphi_i f\|_{L^2(\mathbb{S}^{d-1})} = \|\varphi_i f\|_{L^2(\Omega_i)}$ and, for every $|t| \leqslant \varepsilon$,

$$\|Y(\varphi_i f) \circ e^{tX_{k,l}} - Y(\varphi_i f)\|_{L^2(\mathbb{S}^{d-1})} = \|Y(\varphi_i f) \circ e^{tX_{k,l}} - Y(\varphi_i f)\|_{L^2(\Omega_i \cap e^{-tX_{k,l}}(\Omega_i))},$$

for every Y and (k, l) as in (2.10). In this way, in order to show that $\|\varphi_i f\|_{\mathcal{H}^s} \simeq \|(\varphi_i f) \circ \psi_i^{-1}\|_{\mathcal{H}^s(\mathbb{R}^{d-1})}$, one may appeal to the theory of differentiability along noncommuting vector fields, as developed in [20, §4]; see, in particular, Lemmas 4.1, 4.2 and Theorem 4.3 in [20].

In light of the previous paragraph, we can assume that in the Euclidean case the relevant supports are contained in the unit ball of $\mathbb{R}^d$. This will be useful later on in the argument. More precisely, the task is now to show that there exists $C(s, t) < \infty$ such that for every $f \in \mathcal{H}^s(\mathbb{R}^d)$ whose support is contained in the unit ball of $\mathbb{R}^d$, it holds that

$$\|f\|_{H^t(\mathbb{R}^d)} \leqslant C(s, t)\|f\|_{\mathcal{H}^s(\mathbb{R}^d)}, \tag{2.12}$$

whenever $0 \leqslant t < s \notin \mathbb{N}$.

We start by considering the case $0 < t < s < 1$ (the case $t = 0$ being trivial) and recalling the equivalent formulation of Sobolev spaces in terms of the Riesz potential. Fix $t \in (0, 1)$, and let $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ be given. From Plancherel's theorem, we have that

$$\int_{(\mathbb{R}^d)^2} \frac{|f(x+y) - f(y)|^2}{|x|^{d+2t}} \,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |e^{ix\cdot\xi} - 1|^2 |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi \frac{\mathrm{d}x}{|x|^{d+2t}} \tag{2.13}$$

$$= A_{t,d} \int_{\mathbb{R}^d} |\xi|^{2t} |\widehat{f}(\xi)|^2 \,\mathrm{d}\xi, \tag{2.14}$$

where we used the fact that the integral

$$I_{t,d}(\xi) := \int_{\mathbb{R}^d} \frac{|e^{ix\cdot\xi} - 1|^2}{|x|^{d+2t}} \,\mathrm{d}x$$

satisfies $I_{t,d}(\lambda\xi) = \lambda^{2t} I_{t,d}(\xi)$ for every $\lambda > 0$. The constant $A_{t,d}$ in (2.14) satisfies $A_{t,d} = I_{t,d}(\omega)$ for any $\omega \in \mathbb{S}^{d-1}$ and is finite as long as $t \in (0, 1)$. In turn, because $t \in (0, 1)$, we have that

$$(1 + |\xi|^2)^t \leqslant 1 + |\xi|^{2t} \leqslant 2(1 + |\xi|^2)^t,$$

for every $\xi \in \mathbb{R}^d$. In particular, the following two-sided estimate holds:

$$\|f\|^2_{H^t(\mathbb{R}^d)} \simeq_{t,d} \|f\|^2_{L^2(\mathbb{R}^d)} + \int_{(\mathbb{R}^d)^2} \frac{|f(x+y) - f(y)|^2}{|x|^{d+2t}} \,\mathrm{d}x\,\mathrm{d}y. \tag{2.15}$$

Given $s \in (t, 1)$, we use Hölder's inequality to estimate

$$\int_{(\mathbb{R}^d)^2} \frac{|f(x+y) - f(y)|^2}{|x|^{d+2t}} \,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_{|x|\leqslant 1}\int_{\mathbb{R}^d} \frac{|f(x+y) - f(y)|^2}{|x|^{d+2t}} \,\mathrm{d}y\,\mathrm{d}x + \int_{|x|>1}\int_{\mathbb{R}^d} \frac{|f(x+y) - f(y)|^2}{|x|^{d+2t}} \,\mathrm{d}y\,\mathrm{d}x$$
$$\leqslant \left(\sup_{|x|\leqslant 1} \int_{\mathbb{R}^d} \frac{|f(x+y) - f(y)|^2}{|x|^{2s}} \,\mathrm{d}y\right) \int_{|x|\leqslant 1} \frac{\mathrm{d}x}{|x|^{d-2(s-t)}}$$

$$+ 2\int_{|x|>1}\int_{\mathbb{R}^d}\frac{|f(x+y)|^2+|f(y)|^2}{|x|^{d+2t}}\,\mathrm{d}y\,\mathrm{d}x$$
$$\lesssim_d (s-t)^{-1}\|f\|^2_{\mathcal{H}^s(\mathbb{R}^d)} + t^{-1}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

In light of (2.15), this establishes (2.12) in the particular case when $0 < t < s < 1$.

We now consider the case when $s = k + \alpha$, with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. No generality is lost in assuming that $t \in (k, s)$ and specialising to the case $k = 1$, so that the desired conclusion would follow from the estimate $\|D^\ell f\|_{H^{t-1}(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{H}^s(\mathbb{R}^d)}$ for every $|\ell| \leqslant 1$. In this case, the previous argument applies to $D^\ell f$ *provided that* $D^\ell f \in L^2(\mathbb{R}^d)$ for any $|\ell| = 1$, with appropriately bounded $L^2(\mathbb{R}^d)$-norm, which we now verify in the special case when the support of f is contained in the unit ball of $\mathbb{R}^d$. As discussed above, this will suffice for our application to $\mathbb{S}^{d-1}$.

Fix a multi-index $\ell \in \mathbb{N}^d$, $|\ell| = 1$, and write $D := D^\ell$. Let $g \in C_0^\infty(\mathbb{R}^d)$ be such that $\operatorname{supp}(g) \subset B_1$. Then $(Dg)(y - t\ell) = -\frac{\mathrm{d}}{\mathrm{d}t}(g(y - t\ell)) = D(g(\cdot - t\ell))(y)$. By Fubini's theorem and the definition of weak derivative of f, it follows that

$$0 = \int_{-2}^{2}\int_{\mathbb{R}^d}\overline{(Dg)}(y-t\ell)f(y)\,\mathrm{d}y\,\mathrm{d}t = -\int_{-2}^{2}\int_{\mathbb{R}^d}\overline{g}(y)(Df)(y+t\ell)\,\mathrm{d}y\,\mathrm{d}t.$$

As a consequence,

$$\int_{\mathbb{R}^d} Df(y)\overline{g}(y)\,\mathrm{d}y = \frac{1}{4}\int_{-2}^{2}\int_{\mathbb{R}^d}\big(Df(y) - Df(y+t\ell)\big)\overline{g}(y)\,\mathrm{d}y\,\mathrm{d}t.$$

By adding and subtracting appropriate terms, the triangle and Cauchy-Schwarz inequalities and the invariance of the Lebesgue measure in $\mathbb{R}^d$ with respect to translations together imply

$$\begin{aligned}\left|\int_{\mathbb{R}^d} Df(y)\overline{g}(y)\,\mathrm{d}y\right| &\leqslant \frac{1}{4}\int_{-2}^{2}\|Df\circ\tau_{t\ell} - Df\circ\tau_{\frac{t}{2}\ell}\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\\ &\quad + \frac{1}{4}\int_{-2}^{2}\|Df\circ\tau_{\frac{t}{2}\ell} - Df\|_{L^2(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\\ &\lesssim \int_0^1 \|Df\circ\tau_{t\ell} - Df\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\,\|g\|_{L^2(\mathbb{R}^d)}\\ &\leqslant \left(\int_0^1 t^\alpha\,\mathrm{d}t\right)\|f\|_{\mathcal{H}^{1+\alpha}(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}.\end{aligned}\tag{2.16}$$

Consequently, $Df \in L^2(\mathbb{R}^d)$ and $\|Df\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{\mathcal{H}^s(\mathbb{R}^d)}$. This concludes the proof of the lemma. □

Remark 2.2. For our purposes later on, it will suffice to invoke the following simpler consequence of Lemma 2.1: For any $0 < s \notin \mathbb{N}$, there is a continuous embedding $\mathcal{H}^s \subseteq H^{\lfloor s\rfloor}$. We now provide a short proof of this fact that is intrinsic to the sphere. The case $s \in (0, 1)$ is clear because then $H^{\lfloor s\rfloor} = L^2(\mathbb{S}^{d-1})$. For $s > 1$, $s \notin \mathbb{N}$ and $f \in \mathcal{H}^s$ it suffices to show that $\|Xf\|_{L^2(\mathbb{S}^{d-1})} \lesssim \|f\|_{\mathcal{H}^s}$, where X ranges over the finite set of all compositions $X_{i_1,j_1}\circ X_{i_2,j_2}\circ\cdots\circ X_{i_\ell,j_\ell}$ with $1 \leqslant \ell \leqslant \lfloor s\rfloor$ factors.[1] As noted in the course of the proof of Lemma 2.1, we may specialise to the case $\ell = 1$ because the general case follows in the same way. We then simply note that, for any $g \in C^\infty(\mathbb{S}^{d-1})$, $X \in \{X_{i,j} : 1 \leqslant i < j \leqslant d\}$ and $\omega \in \mathbb{S}^{d-1}$, it holds that $(Xg)(e^{tX}\omega) = (X(g\circ e^{tX}))(\omega) = \frac{d}{dt}(g(e^{tX}\omega))$, so that $\int_0^{2\pi}(Xg)(e^{tX}\omega)\,\mathrm{d}t = 0$, and the desired estimate,

$$\left|\int_{\mathbb{S}^{d-1}} Xf(\omega)\overline{g}(\omega)\,\mathrm{d}\sigma_{d-1}(\omega)\right| \lesssim \|f\|_{\mathcal{H}^s}\|g\|_{L^2(\mathbb{S}^{d-1})},$$

[1] As stated in (2.3) and explained in the references thereafter, it suffices to consider the case $\ell = \lfloor s\rfloor$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

follows in the same way as (2.16). See also [5, Cor. 7] for a discussion of this embedding using an equivalent definition[2] of $\mathcal{H}^s$.

3. Preliminary inequalities

We establish some linear and multilinear inequalities that will be used to analyse the solutions of equation (1.12). Our first result translates into a modest amount of control over the regularity of convolution measures in a number of situations of interest.

Proposition 3.1. *Given integers $d, m \geqslant 2$, set $\alpha = \frac{1}{2}(d-1)(m-2)-1$. Let $\{f_j\}_{j=1}^m \subset C^\infty(\mathbb{S}^{d-1})$. If $\alpha > 0$, then $f_1\sigma_{d-1} * \cdots * f_m\sigma_{d-1} \in \Lambda_\alpha(\mathbb{R}^d)$.*

The proof of Proposition 3.1 is based on the classical Littlewood-Paley characterisation of the Hölder spaces $\Lambda_\alpha(\mathbb{R}^d)$; see [14, §6.3] and [28, Ch. VI, §5].

Proof of Proposition 3.1. Consider a smooth partition of unity in $\mathbb{R}^d$. More precisely, fix $\eta \geqslant 0$, a nonnegative, decreasing and radial C^∞-function of compact support, defined on $\mathbb{R}^d$, with the properties that $\eta(x) = 1$ for $|x| \leqslant 1$ and $\eta(x) = 0$ for $|x| \geqslant 2$. Together with η, define another function δ by $\delta(x) := \eta(x) - \eta(2x) \geqslant 0$. For each integer $j \geqslant 1$, consider the function $\varphi_j := \delta(2^{-j}\cdot)$, which is supported on the spherical shell $\{x \in \mathbb{R}^d : 2^{j-1} \leqslant |x| \leqslant 2^{j+1}\}$, and let $\varphi_0 = \eta$, so that

$$\sum_{j=0}^\infty \varphi_j(x) = 1, \quad \text{for every } x \in \mathbb{R}^d.$$

For $\alpha > 0$, a function $G : \mathbb{R}^d \to \mathbb{C}$ belongs to $\Lambda_\alpha(\mathbb{R}^d)$ if and only if

$$\sup_{j\in\mathbb{N}_0} 2^{j\alpha} \|(\widehat{G}\varphi_j)^\vee\|_{L^\infty(\mathbb{R}^d)} < \infty. \tag{3.1}$$

Moreover, the expression on the left-hand side of (3.1) produces a norm that is equivalent to any other norm for $\Lambda_\alpha(\mathbb{R}^d)$; see [14, Theorem 6.3.7]. The Hausdorff-Young inequality implies that estimate (3.1) is fulfilled if

$$\int_{\mathbb{R}^d} |\widehat{G}(x)\varphi_j(x)|\,\mathrm{d}x \lesssim 2^{-j\alpha}, \quad j = 0, 1, 2, \ldots,$$

for some implicit constant that does not depend on j. Now, the Fourier transform of $F := f_1\sigma_{d-1} * \cdots * f_m\sigma_{d-1}$ is given by $\widehat{F} = \prod_{k=1}^m \widehat{f_k\sigma}_{d-1}$, which leads to the analysis of the integrals

$$\int_{B_2} \prod_{k=1}^m \big|\widehat{f_k\sigma}_{d-1}(x)\big|\,\mathrm{d}x, \quad \int_{B_{2^{j+1}}\setminus B_{2^{j-1}}} \prod_{k=1}^m \big|\widehat{f_k\sigma}_{d-1}(x)\big|\,\mathrm{d}x, \quad j = 1, 2, \ldots$$

A well-known stationary phase argument applied to each $f_k \in C^\infty(\mathbb{S}^{d-1})$ yields the following decay estimate:

$$|\widehat{f_k\sigma}_{d-1}(x)| \lesssim (1+|x|)^{-\frac{d-1}{2}}, \text{ for every } x \in \mathbb{R}^d,$$

where the implicit constant depends only on the dimension d and the function f_k; see [28, Chapter VIII, §3.1]. Using polar coordinates, it is then direct to check that

$$\int_{B_{2^{j+1}}\setminus B_{2^{j-1}}} \prod_{k=1}^m \big|\widehat{f_k\sigma}_{d-1}(x)\big|\,\mathrm{d}x \lesssim 2^{jd}2^{-\frac{jm(d-1)}{2}} = 2^{-j((d-1)(\frac{m}{2}-1)-1)},$$

[2]We comment on various equivalent definitions of the space $\mathcal{H}^s$ in Subsection 6.2.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

for every $j \in \mathbb{N}$. The desired conclusion follows from this and from the observation that $\widehat{F}$ defines a continuous function on $\mathbb{R}^d$ and is thus bounded on the ball $B_2 \subset \mathbb{R}^d$. □

Remark 3.2. We find it convenient to consider the 'universe' of admissible parameters

$$\mathfrak{U} = \{(d,m) \in \mathbb{N}^2 \colon d = 2 \text{ and } m \geqslant 4, \text{ or } d \geqslant 3 \text{ and } m \geqslant 2\},$$

together with its 'boundary'

$$\partial\mathfrak{U} = \{(2,4),(3,3)\} \cup \{(d,2) \colon d \geqslant 3\}. \tag{3.2}$$

Note that the set $\mathfrak{U}$ encapsulates the hypotheses on d, m imposed by Theorem 1.1. On the other hand, with the exception of $(d,m) = (3,3)$, the set $\partial\mathfrak{U}$ contains precisely those values (d,m) for which m is the smallest even integer such that $\mathbf{T}_{d,m+2} < \infty$, and therefore the corresponding inequality (1.1) holds. As the upcoming sections will reveal, the analysis simplifies considerably if $(d,m) \in \mathfrak{U} \setminus \partial\mathfrak{U}$, which is the reason for treating the boundary set $\partial\mathfrak{U}$ separately. As a first instance of this phenomenon, note that, given $(d,m) \in \mathfrak{U}$, we have that $(d,m) \notin \partial\mathfrak{U}$ if and only if $\frac{1}{2}(d-1)(m-2) - 1 > 0$. These are precisely the cases covered by Proposition 3.1. See also the comments following Lemma 3.4 and Remark 6.4.

Recall the operator $\mathrm{M} \colon L^2(\mathbb{S}^{d-1})^{m+1} \to L^2(\mathbb{S}^{d-1})$, which was defined in (1.11) as

$$\mathrm{M}(f_1,\ldots,f_{m+1}) = (f_1\sigma_{d-1} * \cdots * f_{m+1}\sigma_{d-1})\Big|_{\mathbb{S}^{d-1}}.$$

Lemma 3.3. *The operator* M *defined in* (1.11) *satisfies the following properties:*

(i) M *is an* $(m+1)$*-linear operator.*
(ii) M *is symmetric in the sense that, given any permutation* τ *of* $\{1,2,\ldots,m+1\}$,

$$\mathrm{M}(f_1,\ldots,f_{m+1}) = \mathrm{M}(f_{\tau(1)},\ldots,f_{\tau(m+1)}). \tag{3.3}$$

(iii) *For any* $\Theta \in SO(d)$*, the following identities hold:*

$$\mathrm{M}(f_1,\ldots,f_{m+1}) \circ \Theta = \mathrm{M}(f_1 \circ \Theta,\ldots,f_{m+1} \circ \Theta). \tag{3.4}$$

$$\begin{aligned}(\Theta - I)\mathrm{M}(f_1,\ldots,f_{m+1}) &= \sum_{j=1}^{m+1} \mathrm{M}(f_1,\ldots,f_{j-1},(\Theta-I)f_j,\Theta f_{j+1},\ldots,\Theta f_{m+1}) \\ &= \mathrm{M}((\Theta-I)f_1,\Theta f_2,\ldots,\Theta f_{m+1}) \\ &\quad + \mathrm{M}(f_1,(\Theta-I)f_2,\ldots,\Theta f_{m+1}) \\ &\quad\ \vdots \\ &\quad + \mathrm{M}(f_1,f_2,\ldots,(\Theta-I)f_{m+1}).\end{aligned} \tag{3.5}$$

(iv) *For any* $s \geqslant 0$*, there exists* $A_s < \infty$ *such that if* $\{f_j\}_{j=1}^{m+1} \subset H^s$*, then*

$$\|\mathrm{M}(f_1,\ldots,f_{m+1})\|_{H^s} \leqslant A_s \prod_{j=1}^{m+1} \|f_j\|_{H^s}. \tag{3.6}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

(v) *If* $X = X_{i,j}$ [3] *for some* $1 \leqslant i < j \leqslant d$ *and* $\{f_k\}_{k=1}^{m+1} \subset H^1$, *then*

$$XM(f_1, \dots, f_{m+1}) = \sum_{k=1}^{m+1} M(f_1, \dots, f_{k-1}, Xf_k, f_{k+1}, \dots, f_{m+1}). \tag{3.7}$$

(vi) *For any* $0 < s \notin \mathbb{Z}$, *there exists* $C_s < \infty$ *such that, if* $\{f_j\}_{j=1}^{m+1} \subset \mathcal{H}^s$, *then*

$$\|M(f_1, \dots, f_{m+1})\|_{\mathcal{H}^s} \leqslant C_s \prod_{j=1}^{m+1} \|f_j\|_{\mathcal{H}^s}. \tag{3.8}$$

We record the basic L^2-estimate, which coincides with the case $s = 0$ of (3.6):

$$\|M(f_1, \dots, f_{m+1})\|_{L^2(\mathbb{S}^{d-1})} \lesssim \prod_{j=1}^{m+1} \|f_j\|_{L^2(\mathbb{S}^{d-1})}. \tag{3.9}$$

Proof of Lemma 3.3. We prove estimate (3.8) only, the rest being direct from the definitions or simple to verify; in particular, the proof of (iv) is analogous to that of [4, Lemma 2.2]. Let us first assume that $s \in (0, 1)$. Given $\{f_k\}_{k=1}^{m+1} \subset \mathcal{H}^s$, set $g := M(f_1, \dots f_{m+1})$. Let $\Theta = e^{tX} \in SO(d)$, where $X = X_{i,j}$ for some $1 \leqslant i < j \leqslant d$. In light of (3.5), we then have that

$$\Theta g - g = \sum_{k=1}^{m+1} M(f_1, \dots, f_{k-1}, (\Theta - I)f_k, \Theta f_{k+1}, \dots, \Theta f_{m+1}). \tag{3.10}$$

By (3.9), the first summand on the right-hand side of (3.10) satisfies

$$\|M((\Theta - I)f_1, f_2, \dots, f_{m+1})\|_{L^2(\mathbb{S}^{d-1})} \lesssim \|\Theta f_1 - f_1\|_{L^2(\mathbb{S}^{d-1})} \prod_{\ell=2}^{m+1} \|f_\ell\|_{L^2(\mathbb{S}^{d-1})},$$

and similarly for the other m summands. It follows that

$$\begin{aligned}\sup_{|t|\leqslant 1} |t|^{-s} \|g \circ e^{tX} - g\|_{L^2(\mathbb{S}^{d-1})} &\lesssim \sum_{k=1}^{m+1} \sup_{|t|\leqslant 1} |t|^{-s} \|e^{tX} f_k - f_k\|_{L^2(\mathbb{S}^{d-1})} \prod_{\ell:\,\ell\neq k} \|f_\ell\|_{L^2(\mathbb{S}^{d-1})} \\ &\leqslant \sum_{k=1}^{m+1} \|f_k\|_{\mathcal{H}^s} \prod_{\ell:\,\ell\neq k} \|f_\ell\|_{L^2(\mathbb{S}^{d-1})} \leqslant \prod_{k=1}^{m+1} \|f_k\|_{\mathcal{H}^s}.\end{aligned}$$

Because this holds whenever X is any of the vector fields $\{X_{i,j}\}_{1\leqslant i<j\leqslant d}$, estimate (3.8) as follows, settling (vi) in the special case when $s \in (0, 1)$. Now suppose that $s = k + \alpha$, with $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. Let $1 \leqslant \ell \leqslant k$ and consider a composition Y with ℓ factors as in (2.7). Remark 2.2 and estimate (3.6) imply that $g \in H^k$. In light of (3.7), we then see that Yg can be written as a sum of terms of the form $M(Y_1 f_1, \dots, Y_{m+1} f_{m+1})$, where $Y_1, \dots, Y_{m+1}$ are compositions of $i_1, \dots, i_{m+1}$ vector fields $X_{i,j}$, and $\sum_{j=1}^{m+1} i_j = \ell$. Note that $Y_j f_j \in \mathcal{H}^\alpha$ for all such vector fields, and $\|Y_j f_j\|_{\mathcal{H}^\alpha} \leqslant \|f_j\|_{\mathcal{H}^s}$. Expanding $(\Theta - I)Yg$ as in (3.10), we find in the same way as before that

$$\sup_{|t|\leqslant 1} |t|^{-\alpha} \|Yg \circ e^{tX} - Yg\|_{L^2(\mathbb{S}^{d-1})} \lesssim \prod_{j=1}^{m+1} \|f_j\|_{\mathcal{H}^s}.$$

This implies the desired $\mathcal{H}^s$-bound for the function g and concludes the proof of the lemma. □

[3] Recall the definition (2.4) of $X_{i,j} = \frac{\partial}{\partial \theta_{i,j}}$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

The following result details a sense in which M can be viewed as a smoothing operator but requires $(d,m) \notin \partial\mathfrak{U}$.

Lemma 3.4. *Given* $(d,m) \in \mathfrak{U} \setminus \partial\mathfrak{U}$, *set* $\alpha_{d,m} = \frac{1}{2}(d-1)(m-2)-1$. *If* $\alpha \in (0,1)$ *is such that* $\alpha \leqslant \alpha_{d,m}$, $\{\varphi_j\}_{j=1}^m \subset C^\infty(\mathbb{S}^{d-1})$ *and* $g \in L^2(\mathbb{S}^{d-1})$, *then* $\mathrm{M}(\varphi_1,\ldots,\varphi_m,g) \in \mathcal{H}^\alpha$. *Moreover, the following estimate holds:*

$$\|\mathrm{M}(\varphi_1,\ldots,\varphi_m,g)\|_{\mathcal{H}^\alpha} \lesssim \left(\prod_{j=1}^m \|\varphi_j\|_{L^2(\mathbb{S}^{d-1})} + \|\varphi_1\sigma_{d-1} * \cdots * \varphi_m\sigma_{d-1}\|_{\Lambda_{\alpha_{d,m}}(\mathbb{R}^d)}\right)\|g\|_{L^2(\mathbb{S}^{d-1})}. \quad (3.11)$$

It is natural to wonder whether a similar gain in regularity holds in the case when $(d,m) \in \partial\mathfrak{U}$. The (affirmative) answer is more subtle, and we postpone the discussion until Section 5; see Lemma 5.2.

Proof of Lemma 3.4. Recall that $\alpha_{d,m} > 0$ because $(d,m) \in \mathfrak{U} \setminus \partial\mathfrak{U}$. It then follows from Proposition 3.1 that $\varphi_1\sigma_{d-1} * \cdots * \varphi_m\sigma_{d-1} \in \Lambda_{\alpha_{d,m}}(\mathbb{R}^d)$. For notational convenience, we shall only consider the special case when $\varphi_j = \varphi$, for all j. Given $\Theta \in \mathrm{SO}(d)$ and $\omega \in \mathbb{S}^{d-1}$, estimate

$$|\mathrm{M}(\varphi,\ldots,\varphi,g)\circ\Theta(\omega) - \mathrm{M}(\varphi,\ldots,\varphi,g)(\omega)| \leqslant \int_{\mathbb{S}^{d-1}} \Big|(\varphi\sigma_{d-1})^{*m}(\Theta\omega-\eta) - (\varphi\sigma_{d-1})^{*m}(\omega-\eta)\Big| |g(\eta)|\,\mathrm{d}\sigma_{d-1}(\eta).$$

If $\alpha \in (0,1)$ is such that $\alpha \leqslant \alpha_{d,m}$, then $(\varphi\sigma_{d-1})^{*m} \in \Lambda_\alpha(\mathbb{R}^d)$ and, consequently,

$$\begin{aligned}|\mathrm{M}(\varphi,\ldots,\varphi,g)\circ\Theta(\omega) - \mathrm{M}(\varphi,\ldots,\varphi,g)(\omega)| &\leqslant |(\Theta-I)\omega|^\alpha \|(\varphi\sigma_{d-1})^{*m}\|_{\Lambda_\alpha(\mathbb{R}^d)}\|g\|_{L^1(\mathbb{S}^{d-1})}\\ &\lesssim |\Theta-I|^\alpha \|(\varphi\sigma_{d-1})^{*m}\|_{\Lambda_\alpha(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{S}^{d-1})}.\end{aligned}$$

Letting $\Theta = e^{tX}$ for some $X \in \{X_{i,j}\}_{1\leqslant i<j\leqslant d}$ and integrating the square of both sides of the latter estimate, we obtain

$$\sup_{|t|\leqslant 1} |t|^{-\alpha}\|\mathrm{M}(\varphi,\ldots,\varphi,g)\circ\Theta - \mathrm{M}(\varphi,\ldots,\varphi,g)\|_{L^2(\mathbb{S}^{d-1})} \lesssim \sup_{|t|\leqslant 1} |t|^{-\alpha}|e^{tX}-I|^\alpha \|(\varphi\sigma_{d-1})^{*m}\|_{\Lambda_\alpha(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{S}^{d-1})}.$$

In turn, this and the basic L^2-estimate (3.9) together imply

$$\|\mathrm{M}(\varphi,\ldots,\varphi,g)\|_{\mathcal{H}^\alpha} \lesssim (\|\varphi\|_{L^2(\mathbb{S}^{d-1})}^m + \|(\varphi\sigma_{d-1})^{*m}\|_{\Lambda_\alpha(\mathbb{R}^d)})\|g\|_{L^2(\mathbb{S}^{d-1})}.$$

To obtain (3.11), simply rerun the argument with the φ_js in place of φ. This completes the proof of the lemma. □

4. Hölder regularity

In this section, we prove Hölder-type estimates for certain convolution measures, which will pave the way towards finding a suitable replacement for Lemma 3.4 in the case when $(d,m) \in \partial\mathfrak{U}$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

4.1. Twofold convolutions

The purpose of this subsection is to generalise [4, Lemma 2.3] to arbitrary dimensions $d \geqslant 2$. Though for the most part the analysis follows similar lines to those of [4], we include it for the sake of completeness. Start by recalling that the twofold convolution $\sigma_{d-1} * \sigma_{d-1}$ defines a measure supported on the ball $B_2 \subset \mathbb{R}^d$, which is absolutely continuous with respect to the Lebesgue measure on B_2, and whose Radon-Nikodym derivative equals

$$(\sigma_{d-1} * \sigma_{d-1})(x) = \frac{\omega_{d-2}}{2^{d-3}} \frac{1}{|x|} (4 - |x|^2)_+^{\frac{d-3}{2}}. \tag{4.1}$$

Here, $\omega_{d-2} := \sigma_{d-2}(\mathbb{S}^{d-2}) = 2\pi^{\frac{d-1}{2}} \Gamma(\frac{d-1}{2})^{-1}$ denotes the surface area of $\mathbb{S}^{d-2}$, $y_+ := \max\{0, y\}$ for $y \in \mathbb{R}$, and

$$(4 - |x|^2)_+^{\frac{d-3}{2}} := \big((4 - |x|^2)_+\big)^{\frac{d-3}{2}};$$

see, for instance, [1, Lemma 5].

Let $h_1, h_2 \in \mathrm{Lip}(\mathbb{S}^{d-1})$. From [12, Appendix A.2], we know that the function u_{12} defined by the relation $(h_1\sigma_{d-1} * h_2\sigma_{d-1})(x) = u_{12}(x)(\sigma_{d-1} * \sigma_{d-1})(x)$ for $0 < |x| \leqslant 2$ and $u_{12}(x) = 0$ for $|x| > 2$ can be expressed as

$$u_{12}(x) = ⨍_{\Gamma_x} h_1(\nu) h_2(x - \nu) \, \mathrm{d}\sigma_x(\nu), \tag{4.2}$$

where $\Gamma_x = \mathbb{S}^{d-1} \cap (x + \mathbb{S}^{d-1})$, and $⨍$ denotes the averaged integral on the $(d-2)$-dimensional sphere Γ_x; see also [3] for a careful discussion of the case $d = 3$.

The case $d = 2$ merits some further remarks. In this case, if $0 < |x| < 2$, then Γ_x consists of two points, which we identify with $\mathbb{S}^0$. Let $x^\perp$ be the 90°-counterclockwise rotation of x, so that $x^\perp \cdot x = 0$ and $|x^\perp| = |x|$. Given $x \in B_2 \setminus \{0\} \subset \mathbb{R}^2$, there exist unique-up-to-permutation $x_1, x_2 \in \mathbb{S}^1$, such that $x = x_1 + x_2$. The vectors x_1, x_2 are explicitly given by

$$x_1 = \frac{x}{2} + \left(1 - \frac{|x|^2}{4}\right)^{\frac{1}{2}} \frac{x^\perp}{|x|}, \quad x_2 = \frac{x}{2} - \left(1 - \frac{|x|^2}{4}\right)^{\frac{1}{2}} \frac{x^\perp}{|x|}.$$

Given $h_1, h_2 \in \mathrm{Lip}(\mathbb{S}^1)$, the convolution $h_1\sigma_1 * h_2\sigma_1$ can be written in the following way: If $0 < |x| \leqslant 2$, then

$$(h_1\sigma_1 * h_2\sigma_1)(x) = 2\frac{h_1(x_1)h_2(x_2) + h_1(x_2)h_2(x_1)}{|x|\sqrt{4 - |x|^2}},$$

and for $|x| > 2$ one obviously has that $(h_1\sigma_1 * h_2\sigma_1)(x) = 0$. In this case, identity (4.2) is then seen to reduce to

$$u_{12}(x) = \frac{1}{2}(h_1(x_1)h_2(x_2) + h_1(x_2)h_2(x_1)), \text{ if } 0 < |x| \leqslant 2.$$

Lemma 4.1. *Let $d \geqslant 2$ and $x, x' \in B_2 \setminus \{0\} \subset \mathbb{R}^d$. Then*

$$|u_{12}(x) - u_{12}(x')| \leqslant C \|h_1\|_{\mathrm{Lip}(\mathbb{S}^{d-1})} \|h_2\|_{\mathrm{Lip}(\mathbb{S}^{d-1})} \left(|x - x'|^{1/2} + \left|\frac{x}{|x|} - \frac{x'}{|x'|}\right|\right),$$

for some universal constant $C < \infty$.

Proof. The integral (4.2) defining u_{12} can be equivalently written as

$$u_{12}(x) = \omega_{d-2}^{-1} \int_{\mathbb{S}_x^{d-2}} h_1(\tfrac{x}{2} + \rho(x)\omega) h_2(\tfrac{x}{2} - \rho(x)\omega) \, \mathrm{d}\sigma_{d-2}(\omega),$$

where the function $\rho \geqslant 0$ satisfies $\rho(x)^2 + (|x|/2)^2 = 1$, and the unit sphere $\mathbb{S}_x^{d-2}$ is contained in the $(d-1)$-dimensional subspace of $\mathbb{R}^d$ orthogonal to x and is therefore parallel to the hyperplane containing Γ_x. It is elementary to check that $|\rho(x) - \rho(x')| \leqslant ||x| - |x'||^{1/2}$ for every $x, x' \in B_2 \setminus \{0\}$.

Let us start by considering the case $x' = \lambda x$ for some $\lambda > 0$. We then have that $\mathbb{S}_x^{d-1} = \mathbb{S}_{x'}^{d-1}$, and so

$$\left|\left(\frac{x}{2} + \rho(x)\omega\right) - \left(\frac{x'}{2} + \rho(x')\omega\right)\right| \leqslant \frac{1}{2}|x - x'| + |\rho(x) - \rho(x')| \lesssim |x - x'|^{1/2}.$$

In a similar way,

$$\left|\left(\frac{x}{2} - \rho(x)\omega\right) - \left(\frac{x'}{2} - \rho(x')\omega\right)\right| \lesssim |x - x'|^{1/2}.$$

Denote $x_1 = \frac{x}{2} + \rho(x)\omega, x_2 = \frac{x}{2} - \rho(x)\omega$ and $\tilde{u}(x) = h_1(x_1)h_2(x_2)$. Because h_1, h_2 are Lipschitz functions, we have that

$$\begin{aligned}
|\tilde{u}(x) - \tilde{u}(x')| &= |h_1(x_1)h_2(x_2) - h_1(x_1')h_2(x_2')| \\
&\leqslant |h_2(x_2)||h_1(x_1) - h_1(x_1')| + |h_1(x_1')||h_2(x_2) - h_2(x_2')| \\
&\leqslant \|h_2\|_{L^\infty}\|h_1\|_{\mathrm{Lip}}|x_1 - x_1'| + \|h_1\|_{L^\infty}\|h_2\|_{\mathrm{Lip}}|x_2 - x_2'| \\
&\lesssim \|h_1\|_{\mathrm{Lip}}\|h_2\|_{\mathrm{Lip}}|x - x'|^{1/2}.
\end{aligned}$$

It then follows by integration over $\mathbb{S}_x^{d-2}$ that

$$|u_{12}(x) - u_{12}(x')| \lesssim \|h_1\|_{\mathrm{Lip}}\|h_2\|_{\mathrm{Lip}}|x - x'|^{1/2}.$$

We now consider the case $|x| = |x'| \in (0, 2]$. We then have that $\rho(x) = \rho(x')$. Let $\Theta \in \mathrm{SO}(d)$ denote a rotation that fixes the space $(\mathrm{span}\{x, x'\})^\perp$ and sends $x/|x|$ to $x'/|x'|$. It is not difficult to see that $|\Theta - I| \leqslant |x/|x| - x'/|x'||$. We can then write

$$u_{12}(x') = \omega_{d-2}^{-1}\int_{\mathbb{S}_x^{d-2}} h_1(\tfrac{x'}{2} + \rho(x)\Theta\omega)h_2(\tfrac{x'}{2} - \rho(x')\Theta\omega)\,\mathrm{d}\sigma_{d-2}(\omega),$$

so that, for $\epsilon \in \{-1, 1\}$,

$$\begin{aligned}
\left|\left(\frac{x}{2} + \epsilon\rho(x)\omega\right) - \left(\frac{x'}{2} + \epsilon\rho(x')\Theta\omega\right)\right| &\leqslant \frac{1}{2}|x - x'| + \rho(x)|(\Theta - I)\omega| \\
&\leqslant \frac{1}{2}|x - x'| + \rho(x)\left|\frac{x}{|x|} - \frac{x'}{|x'|}\right| \\
&= \left(\rho(x) + \frac{|x|}{2}\right)\left|\frac{x}{|x|} - \frac{x'}{|x'|}\right| \lesssim \left|\frac{x}{|x|} - \frac{x'}{|x'|}\right|.
\end{aligned}$$

Reasoning as before, we conclude that

$$|u_{12}(x) - u_{12}(x')| \lesssim \|h_1\|_{\mathrm{Lip}}\|h_2\|_{\mathrm{Lip}}\left|\frac{x}{|x|} - \frac{x'}{|x'|}\right|.$$

For general $x, x' \in B_2 \setminus \{0\}$ we proceed as follows. Let $y = |x|x'/|x'|$, so that $|y| = |x|$ and $x' = \lambda y$ for $\lambda = |x'|/|x| > 0$. Then

$$\begin{aligned}
|u_{12}(x) - u_{12}(x')| &\leqslant |u_{12}(x) - u_{12}(y)| + |u_{12}(y) - u_{12}(x')| \\
&\lesssim \|h_1\|_{\mathrm{Lip}}\|h_2\|_{\mathrm{Lip}}\left(|x' - y|^{1/2} + \left|\frac{y}{|y|} - \frac{x}{|x|}\right|\right)
\end{aligned}$$

$$= \|h_1\|_{\mathrm{Lip}}\|h_2\|_{\mathrm{Lip}}\left(\left||x|-|x'|\right|^{1/2} + \left|\frac{x'}{|x'|} - \frac{x}{|x|}\right|\right)$$
$$\leqslant \|h_1\|_{\mathrm{Lip}}\|h_2\|_{\mathrm{Lip}}\left(|x-x'|^{1/2} + \left|\frac{x}{|x|} - \frac{x'}{|x'|}\right|\right).$$

This completes the proof of the lemma. □

The following consequence of Lemma 4.1 will be useful in the forthcoming analysis.

Corollary 4.2. *Let $d \geqslant 3$ and $x, x' \in B_2 \setminus \{0\} \subset \mathbb{R}^d$. Then*

$$\Big||x|(h_1\sigma_{d-1} * h_2\sigma_{d-1})(x) - |x'|(h_1\sigma_{d-1} * h_2\sigma_{d-1})(x')\Big|$$
$$\leqslant C\|h_1\|_{\mathrm{Lip}(\mathbb{S}^{d-1})}\|h_2\|_{\mathrm{Lip}(\mathbb{S}^{d-1})}\left(|x-x'|^{1/2} + \left|\frac{x}{|x|} - \frac{x'}{|x'|}\right|\right),$$

for some universal constant $C < \infty$.

Proof. From (4.1) and (4.2), for $|x| \leqslant 2$ we have that

$$|x|(h_1\sigma_{d-1} * h_2\sigma_{d-1})(x) = 2^{-d+3}\omega_{d-2}(4-|x|^2)^{\frac{d-3}{2}}u_{12}(x).$$

The function $(4-|x|^2)^{\frac{d-3}{2}}\mathbb{1}_{B_2}(x)$ belongs to $\Lambda_{1/2}(\mathbb{R}^d)$ if $d \geqslant 4$ and to $\Lambda_{1/2}(B_2)$ if $d = 3$. The desired conclusion follows easily from this and Lemma 4.1. □

4.2. *The case* $(d, n) = (3, 3)$

In the course of this subsection only, we shall simplify the notation by writing $\mathrm{d}\sigma = \mathrm{d}\sigma_2$. Our goal is to establish a Hölder estimate for the threefold convolution $h_1\sigma * h_2\sigma * h_3\sigma$, where $\{h_j\}_{j=1}^3$ are Lipschitz functions on the unit sphere $\mathbb{S}^2$.

Proposition 4.3. *Given $h_1, h_2, h_3 \in \mathrm{Lip}(\mathbb{S}^2)$, let $H = h_1\sigma * h_2\sigma * h_3\sigma$. Then there exists a universal constant $C < \infty$ such that, for every $x, x' \in \mathbb{R}^3$,*

$$|H(x) - H(x')| \leqslant C\prod_{j=1}^3 \|h_j\|_{\mathrm{Lip}(\mathbb{S}^2)}|x-x'|^{1/3}.$$

Proof. By homogeneity, we may assume $\|h_j\|_{\mathrm{Lip}} = 1$, $1 \leqslant j \leqslant 3$. Because the function H is compactly supported, it is enough to consider $x, x' \in \mathbb{R}^3$ for which[4] $|x-x'| \ll 1$. From (4.1) and (4.2), the function $u_{12}(x) := (2\pi)^{-1}|x|(h_1\sigma * h_2\sigma)(x)$ is given by

$$u_{12}(x) = \fint_{\Gamma_x} h_1(\nu)h_2(x-\nu)\,\mathrm{d}\sigma_x(\nu), \tag{4.3}$$

where $\Gamma_x = \mathbb{S}^2 \cap (x + \mathbb{S}^2)$. We further have that

$$H(x) = \int_{\mathbb{S}^2}(h_1\sigma * h_2\sigma)(x-\omega)h_3(\omega)\,\mathrm{d}\sigma(\omega) = 2\pi\int_{\mathbb{S}^2}\frac{\mathbb{1}_{|x-\omega|<2}(\omega)}{|x-\omega|}u_{12}(x-\omega)h_3(\omega)\,\mathrm{d}\sigma(\omega),$$

[4]We will write $|x-x'| \ll 1$ to mean that the quantity $|x-x'|$ is sufficiently small for the purposes of the corresponding proof. For instance, in the course of the proof of Proposition 4.3, we can and will assume that $|x-x'| \leqslant 100^{-1}$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

and so

$$(2\pi)^{-1}(H(x)-H(x')) = \int_{\mathbb{S}^2} \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\,(u_{12}(x-\omega)-u_{12}(x'-\omega))\,h_3(\omega)\,\mathrm{d}\sigma(\omega) + \int_{\mathbb{S}^2}\left(\frac{\mathbb{1}_{|x-\omega|<2}(\omega)}{|x-\omega|}-\frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\right)u_{12}(x-\omega)h_3(\omega)\,\mathrm{d}\sigma(\omega).$$

We denote the integrals on the right-hand side of the latter identity by I and II, respectively. We start by estimating the first integral.

Estimating I. The first step is to restrict the domain of integration to the region where $x-\omega, x'-\omega \in B_2$, plus a remainder, which is $O(|x-x'|)$. With this purpose in mind, decompose $\mathbb{S}^2 = U \cup U' \cup V \cup W$, where

$$U := \{\omega \in \mathbb{S}^2 : |x'-\omega| < 2 \leqslant |x-\omega|\},\ U' := \{\omega \in \mathbb{S}^2 : |x-\omega| < 2 \leqslant |x'-\omega|\},\quad V := \{\omega \in \mathbb{S}^2 : |x-\omega|, |x'-\omega| < 2\},\ W := \{\omega \in \mathbb{S}^2 : 2 \leqslant |x'-\omega|, |x-\omega|\}. \tag{4.4}$$

The integrand of I vanishes on the region $U' \cup W$, and so we are left to analyse the integrals over U and V. We claim that $\sigma(U) = O(|x-x'|)$. Indeed, if $\omega \in U$, then $|x'-\omega| < 2 \leqslant |x-\omega|$, so that as $|x'-\omega| \geqslant |x-\omega| - |x-x'| \geqslant 2 - |x-x'|$ we obtain

$$U \subseteq \{\omega \in \mathbb{S}^2 \colon 2 - |x-x'| \leqslant |x'-\omega| \leqslant 2\}. \tag{4.5}$$

This shows that the region U is contained in the intersection of $\mathbb{S}^2$ with a spherical shell of thickness $|x-x'|$ centred at x'. The claim follows. The contribution of U to the integral I can then be bounded in the following way:

$$\int_U \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}|u_{12}(x'-\omega)h_3(\omega)|\,\mathrm{d}\sigma(\omega) \leqslant \int_U \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\,\mathrm{d}\sigma(\omega).$$

If $\omega \in U$, then $|x'-\omega| \geqslant 2 - |x-x'| > 1$ because $|x-x'| \ll 1$. As a consequence, the latter integral can be crudely bounded as follows:

$$\int_U \frac{\mathrm{d}\sigma(\omega)}{|x'-\omega|} \leqslant \sigma(U) \lesssim |x-x'|. \tag{4.6}$$

To handle the contribution of the region V, note that Lemma 4.1 implies the pointwise estimate

$$|u_{12}(x-\omega)-u_{12}(x'-\omega)| \lesssim |x-x'|^{1/2} + \left|\frac{x-\omega}{|x-\omega|} - \frac{x'-\omega}{|x'-\omega|}\right|. \tag{4.7}$$

The contribution of the region

$$R := \left\{\omega \in V : \left|\frac{x-\omega}{|x-\omega|} - \frac{x'-\omega}{|x'-\omega|}\right| \leqslant |x-x'|^{1/2}\right\} \tag{4.8}$$

to the integral I is easy to estimate. In view of (4.7) and (4.8),

$$\left|\int_R \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\,(u_{12}(x-\omega)-u_{12}(x'-\omega))\,h_3(\omega)\,\mathrm{d}\sigma(\omega)\right| \lesssim \left(\int_R \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\,\mathrm{d}\sigma(\omega)\right)|x-x'|^{1/2} \lesssim |x-x'|^{1/2}.$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

In the second estimate, we used the elementary fact that there exists a universal constant $C < \infty$ such that

$$\int_R \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\, \mathrm{d}\sigma(\omega) \leqslant \int_{\mathbb{S}^2} \frac{\mathrm{d}\sigma(\omega)}{|x'-\omega|} \leqslant C < \infty,$$

for all $x' \in \mathbb{R}^3$. If $\omega \in V \setminus R$, then

$$|x-x'|^{1/2} < \left| \frac{x-\omega}{|x-\omega|} - \frac{x'-\omega}{|x'-\omega|} \right| \leqslant \frac{2|x-\omega||x-x'|}{|x-\omega|\,|x'-\omega|}, \tag{4.9}$$

from which we obtain $|x'-\omega| \leqslant 2|x-x'|^{1/2}$. The contribution of this region can then be estimated as follows:

$$\begin{aligned}
&\left| \int_{V\setminus R} \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|} \left(u_{12}(x-\omega) - u_{12}(x'-\omega)\right) h_3(\omega)\, \mathrm{d}\sigma(\omega) \right| \\
&\qquad \lesssim \int_{V\setminus R} \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|} \left| \frac{x-\omega}{|x-\omega|} - \frac{x'-\omega}{|x'-\omega|} \right| h_3(\omega)\, \mathrm{d}\sigma(\omega) \\
&\qquad \lesssim \left(\int_{\mathbb{S}^2 \cap B(x', 2|x-x'|^{1/2})} \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|}\, \mathrm{d}\sigma(\omega) \right) \|h_3\|_{L^\infty} \\
&\qquad \lesssim |x-x'|^{1/2}.
\end{aligned}$$

From the third line to the fourth line, we used the fact that

$$\phi(x') := \int_{\mathbb{S}^2 \cap B(x',\varepsilon)} \frac{\mathrm{d}\sigma(\omega)}{|x'-\omega|} \tag{4.10}$$

defines a radial function of x' that satisfies

$$\phi(x') \lesssim \sigma(\mathbb{S}^2 \cap B(x',\varepsilon))^{1/2} \lesssim \varepsilon.$$

This concludes the verification of the bound $|I| \lesssim |x-x'|^{1/2}$.

Estimating II. The integral II is bounded by

$$\int_{\mathbb{S}^2} \left| \frac{\mathbb{1}_{|x-\omega|<2}(\omega)}{|x-\omega|} - \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|} \right| \mathrm{d}\sigma(\omega).$$

By symmetry, it is enough to consider

$$\int_T \left(\frac{\mathbb{1}_{|x-\omega|<2}(\omega)}{|x-\omega|} - \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|} \right) \mathrm{d}\sigma(\omega), \tag{4.11}$$

where the integral is taken over the region

$$T := \left\{ \omega \in \mathbb{S}^2 : \frac{\mathbb{1}_{|x-\omega|<2}(\omega)}{|x-\omega|} > \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|} \right\}.$$

Decompose $T = U'' \cup V''$, where

$$\begin{aligned}
U'' &:= \{\omega \in T : |x-\omega| < 2 \leqslant |x'-\omega|\}, \\
V'' &:= \{\omega \in T : |x-\omega| < |x'-\omega| < 2\}.
\end{aligned}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

We have that $U'' = U' \cap T$, and therefore $\sigma(U'') = O(|x - x'|)$. Moreover,

$$\int_{U''} \left(\frac{\mathbb{1}_{|x-\omega|<2}(\omega)}{|x-\omega|} - \frac{\mathbb{1}_{|x'-\omega|<2}(\omega)}{|x'-\omega|} \right) \mathrm{d}\sigma(\omega) = \int_{U''} \frac{\mathrm{d}\sigma(\omega)}{|x-\omega|} \lesssim |x - x'|,$$

where the last inequality follows as in (4.6). The contribution of the region V'' to the integral in (4.11) is slightly more delicate to estimate. We consider two cases as before. Outside the ball $|x'-\omega| \geqslant |x-x'|^{1/3}$, we use the estimate $|x - \omega| \geqslant |x' - \omega| - |x - x'| \gtrsim |x - x'|^{1/3}$, which implies

$$\left| \frac{1}{|x-\omega|} - \frac{1}{|x'-\omega|} \right| = \left| \frac{|x'-\omega| - |x-\omega|}{|x-\omega||x'-\omega|} \right| \leqslant \frac{|x'-x|}{|x-\omega||x'-\omega|}$$
$$\lesssim |x-x'|^{-2/3}|x-x'| = |x-x'|^{1/3}.$$

Inside the ball $|x' - \omega| \leqslant |x - x'|^{1/3}$, we also have $|x - \omega| \leqslant |x - x'|^{1/3}$, as $\omega \in V''$. The contribution of this region to the integral in (4.11) is at most two times the integral

$$\phi(x') = \int_{\mathbb{S}^2 \cap B(x',\delta)} \frac{\mathrm{d}\sigma(\omega)}{|x'-\omega|},$$

where $\delta = |x - x'|^{1/3}$. Proceeding as in (4.10), one is led to the bound $\phi(x') \lesssim \delta$, whence the term in question is $O(|x - x'|^{1/3})$. This establishes the bound $|II| \lesssim |x - x'|^{1/3}$. The proof of the proposition is now complete. □

Remark 4.4. Proposition 4.3 implies that if $n \geqslant 4$, then $G_n := h_1\sigma * \cdots * h_n\sigma \in \Lambda_{1/3}(\mathbb{R}^3)$ whenever $\{h_j\}_{j=1}^3 \subset \mathrm{Lip}(\mathbb{S}^2)$ and $\{h_j\}_{j=4}^n \subset L^1(\mathbb{S}^2)$. This can be improved under the additional assumption $\{h_j\}_{j=1}^n \subset \mathrm{Lip}(\mathbb{S}^2)$, in which case we have, for instance, that $G_6 \in \Lambda_{2/3}(\mathbb{R}^3)$. In dimensions $d \geqslant 4$, a similar argument to that in the proof of Proposition 4.3 shows that, if $\{h_j\}_{j=1}^3 \subset \mathrm{Lip}(\mathbb{S}^{d-1})$, then $h_1\sigma_{d-1} * h_2\sigma_{d-1} * h_3\sigma_{d-1} \in \Lambda_\alpha(\mathbb{R}^d)$ for some $\alpha > 0$. Consequently, if $n \geqslant 3$ and $\{h_j\}_{j=1}^n \subset \mathrm{Lip}(\mathbb{S}^{d-1})$, then $h_1\sigma_{d-1} * \cdots * h_n\sigma_{d-1} \in \Lambda_\alpha(\mathbb{R}^d)$ for some $\alpha > 0$.

4.3. *The case* $(d, n) = (2, 4)$

In the course of this subsection only, we shall simplify the notation by writing $\mathrm{d}\sigma = \mathrm{d}\sigma_1$. Our goal is to establish a Hölder-type estimate for the fourfold convolution $h_1\sigma * h_2\sigma * h_3\sigma * h_4\sigma$, where $\{h_j\}_{j=1}^4$ are Lipschitz functions on the unit circle $\mathbb{S}^1$. We start with some preparatory work. As in Subsection 4.1, let

$$u_{12}(x) = \frac{1}{2}(h_1(x_1)h_2(x_2) + h_1(x_2)h_2(x_1))\mathbb{1}_{B_2}(x), \tag{4.12}$$

$$u_{34}(x) = \frac{1}{2}(h_3(x_1)h_4(x_2) + h_3(x_2)h_4(x_1))\mathbb{1}_{B_2}(x), \tag{4.13}$$

both of which satisfy the conclusion of Lemma 4.1. For brevity, we write

$$F(x) := (\sigma * \sigma)(x) = 4|x|^{-1}(4 - |x|^2)^{-1/2}\mathbb{1}_{B_2}(x), \tag{4.14}$$

as in (4.1), with $d = 2$. We will make repeated use of the upper bound

$$\frac{1}{|x|\sqrt{4-|x|^2}} = \frac{\sqrt{4-|x|^2}}{4|x|} + \frac{|x|}{4\sqrt{4-|x|^2}} \leqslant \frac{1}{|x|} + \frac{1}{\sqrt{2-|x|}}, \text{ for all } |x| \leqslant 2, \tag{4.15}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

together with the estimate

$$\sigma^{*4}(x) \lesssim (1 + |\log|x||)\mathbb{1}_{B_4}(x), \text{ for all } x \in \mathbb{R}^2. \tag{4.16}$$

Inequality (4.16) follows from [24, Eq. (3.21)] and, in particular, implies that

$$|\cdot|^{\beta}\sigma^{*4} \in L^{\infty}(\mathbb{R}^2), \text{ for every } \beta > 0. \tag{4.17}$$

Setting $H_\gamma(x) = |x|^\gamma\big((u_{12}F) * (u_{34}F)\big)(x)$, we then have that $H_\gamma \in L^\infty(\mathbb{R}^2)$, for any $\gamma > 0$ and $\{h_j\}_{j=1}^4 \subset L^\infty(\mathbb{S}^1)$. This will be used in Proposition 4.6. The following preparatory result quantifies the smallness of the function $(\mathbb{1}_E(\sigma * \sigma)) * (\sigma * \sigma)$ for certain sets $E \subset \mathbb{R}^2$ of small Lebesgue measure.

Lemma 4.5. *Set $F = \sigma * \sigma$. Let $x \in B_4 \subset \mathbb{R}^2$. Then, for every $\gamma \in (0, 1]$ and $s \in (0, \frac{\gamma}{2(\gamma+1)})$, there exists a constant $C_{\gamma,s} < \infty$ such that, for all $\varepsilon \in (0, 1)$,*

$$|x|^\gamma \int_{A(x,\varepsilon)} F(y)F(x-y)\,\mathrm{d}y \leqslant C_{\gamma,s}\varepsilon^{\min\{\frac{1}{6}, \frac{\gamma}{2(\gamma+1)} - s\}}, \tag{4.18}$$

$$|x|^\gamma \int_{B_2\cap B(x,\varepsilon)} F(y)F(x-y)\,\mathrm{d}y \leqslant C_{\gamma,s}\varepsilon^{\min\{\frac{1}{2}, \gamma - s\}}, \tag{4.19}$$

where $A(x,\varepsilon) := \{y \in B_2 : 2 - \varepsilon \leqslant |x-y| \leqslant 2\}$.

Before embarking on the proof of Lemma 4.5, we discuss a coordinate system that will prove convenient for the argument. Let $x \in \mathbb{R}^2$, $x \neq 0$ be given. A point $y \in \mathbb{R}^2$ is uniquely determined by the pair $(|y|, |x-y|)$, up to reflection with respect to the line spanned by x. This gives rise to the so-called (two-center) *bipolar coordinates*, defined by $(r, s) = (|y|, |x-y|)$; see [10, §2] and [12, §2.2] for the use of this coordinate system in a related setting. The map $y \mapsto (r, s) = (|y|, |x-y|)$ is a two-to-one map from $\mathbb{R}^2 \setminus \operatorname{span}\{x\}$ to the region determined by the relations $|r-s| < |x| < r+s$, whose Jacobian is given by

$$\mathrm{d}y = \frac{2rs}{(|x|^2 - (r-s)^2)^{\frac{1}{2}}((r+s)^2 - |x|^2)^{\frac{1}{2}}}\,\mathrm{d}r\,\mathrm{d}s. \tag{4.20}$$

After the change of variables $a = r - s$ $b = r + s$, the Jacobian becomes

$$\mathrm{d}y = \frac{(a+b)(b-a)}{4(|x|^2 - a^2)^{\frac{1}{2}}(b^2 - |x|^2)^{\frac{1}{2}}}\,\mathrm{d}a\,\mathrm{d}b. \tag{4.21}$$

Proof of Lemma 4.5. From (4.17), it follows that the left-hand sides of (4.18), (4.19) define bounded functions of x, and therefore $\varepsilon > 0$ can be taken as small as needed in the argument below. We may also assume that $x \neq 0$; otherwise, (4.18) and (4.19) are trivial.

Let us start with (4.18). Note that $|A(x,\varepsilon)| \lesssim \varepsilon$ and that if $y \in A(x,\varepsilon)$, then $|x-y| \geqslant 2 - \varepsilon > 1$. As a consequence, the left-hand side of (4.18) can be bounded as follows:

$$|x|^\gamma \int_{A(x,\varepsilon)} \frac{\mathrm{d}y}{|y|\sqrt{4-|y|^2}|x-y|\sqrt{4-|x-y|^2}} \lesssim |x|^\gamma \int_{A(x,\varepsilon)} \frac{\mathrm{d}y}{|y|\sqrt{4-|y|^2}\sqrt{2-|x-y|}}.$$

We then use the upper bound (4.15),

$$\frac{1}{|y|\sqrt{4-|y|^2}} \leqslant \frac{1}{|y|} + \frac{1}{\sqrt{2-|y|}}, \text{ for } |y| \leqslant 2, \tag{4.22}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

and are left to analyse the following integrals:

$$\phi_1(x,\varepsilon) := |x|^\gamma \int_{A(x,\varepsilon)} \frac{\mathrm{d}y}{|y|\sqrt{2-|x-y|}}, \quad \phi_2(x,\varepsilon) := |x|^\gamma \int_{A(x,\varepsilon)} \frac{\mathrm{d}y}{\sqrt{2-|y|}\sqrt{2-|x-y|}}.$$

Analysis of $\phi_1(x,\varepsilon)$. We perform a dyadic decomposition of $A(x,\varepsilon)$ via

$$A_j = \{y \in B_2 \colon 2-2^{-j}\varepsilon \leqslant |x-y| \leqslant 2-2^{-(j+1)}\varepsilon\}, \quad j \in \mathbb{N}_0, \tag{4.23}$$

so that

$$\phi_1(x,\varepsilon) = |x|^\gamma \int_{A(x,\varepsilon)} \frac{\mathrm{d}y}{|y|\sqrt{2-|x-y|}} \simeq |x|^\gamma \sum_{j=0}^{\infty} (2^{-j}\varepsilon)^{-1/2} \int_{A_j} \frac{\mathrm{d}y}{|y|}.$$

Further, consider $\delta \in (0,\frac{1}{2})$ and decompose $A_j = A_{j,1} \cup A_{j,2}$, where $A_{j,1} = \{y \in A_j : |y| > (2^{-j}\varepsilon)^{1/2-\delta}\}$ and $A_{j,2} = \{y \in A_j : |y| \leqslant (2^{-j}\varepsilon)^{1/2-\delta}\} = A_j \cap B_{(2^{-j}\varepsilon)^{1/2-\delta}}$. Then the contribution of $\{A_{j,1}\}_{j\geqslant 0}$ to $\phi_1(x,\varepsilon)$ can be bounded as follows:

$$|x|^\gamma \sum_{j=0}^{\infty} (2^{-j}\varepsilon)^{-1/2} \int_{A_{j,1}} \frac{\mathrm{d}y}{|y|} \lesssim \sum_{j=0}^{\infty} (2^{-j}\varepsilon)^{-1+\delta} |A_{j,1}| \lesssim \sum_{j=0}^{\infty} (2^{-j}\varepsilon)^{-1+\delta} 2^{-j}\varepsilon \lesssim_\delta \varepsilon^\delta, \tag{4.24}$$

where we used that $|A_{j,1}| \leqslant |A_j| \lesssim 2^{-j}\varepsilon$. We now proceed to bound the contribution of the sets $A_{j,2}$ with the help of bipolar coordinates. We have

$$\begin{aligned}
\int_{A_{j,2}} \frac{\mathrm{d}y}{|y|} &\simeq \int_{\substack{|r-s|<|x|<r+s \\ 0\leqslant r \leqslant (2^{-j}\varepsilon)^{1/2-\delta} \\ 2-2^{-j}\varepsilon \leqslant s \leqslant 2-2^{-(j+1)}\varepsilon}} \frac{s\,\mathrm{d}r\,\mathrm{d}s}{\sqrt{|x|^2-(r-s)^2}\sqrt{(r+s)^2-|x|^2}} \\
&\simeq \int_{\substack{|a|<|x|<b \\ 0\leqslant \frac{a+b}{2} \leqslant (2^{-j}\varepsilon)^{1/2-\delta} \\ 2-2^{-j}\varepsilon \leqslant \frac{b-a}{2} \leqslant 2-2^{-(j+1)}\varepsilon}} \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{|x|^2-a^2}\sqrt{b^2-|x|^2}} \\
&\simeq \int_{\substack{|a|<|x|<b \\ 0\leqslant \frac{a+b}{2} \leqslant (2^{-j}\varepsilon)^{1/2-\delta} \\ 2-2^{-j}\varepsilon \leqslant \frac{b-a}{2} \leqslant 2-2^{-(j+1)}\varepsilon}} \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{|x|-|a|}\sqrt{b-|x|}} \\
&\simeq \int_{\substack{|a|<|x| \\ -2+2^{-(j+1)}\varepsilon \leqslant a \leqslant -2+2^{-j}\varepsilon + (2^{-j}\varepsilon)^{1/2-\delta}}} \int_{\substack{b>|x| \\ -a\leqslant b \leqslant -a+2(2^{-j}\varepsilon)^{1/2-\delta} \\ a+4-2^{-j+1}\varepsilon \leqslant b \leqslant a+4-2^{-j}\varepsilon}} \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{|x|-|a|}\sqrt{b-|x|}} \\
&\lesssim \max\{(2^{-j}\varepsilon)^{1/2}, (2^{-j}\varepsilon)^{1/4-\delta/2}\} \min\{(2^{-j}\varepsilon)^{1/2}, (2^{-j}\varepsilon)^{1/4-\delta/2}\} \\
&= (2^{-j}\varepsilon)^{1/2}(2^{-j}\varepsilon)^{1/4-\delta/2},
\end{aligned}$$

where we used that in the domain of integration $|a| \simeq b \simeq 1$, so that $\sqrt{b+|x|} \simeq 1$ and $\sqrt{|x|+|a|} \simeq 1$. Therefore, the contribution of $\{A_{j,2}\}_{j\geqslant 0}$ to $\phi_1(x,\varepsilon)$ can be bounded as follows:

$$\begin{aligned}
|x|^\gamma \sum_{j=0}^{\infty} (2^{-j}\varepsilon)^{-1/2} \int_{A_{j,2}} \frac{\mathrm{d}y}{|y|} &\lesssim \sum_{j\geqslant 0} (2^{-j}\varepsilon)^{-1/2}(2^{-j}\varepsilon)^{1/2}(2^{-j}\varepsilon)^{1/4-\delta/2} \\
&= \sum_{j\geqslant 0} (2^{-j}\varepsilon)^{1/4-\delta/2} \simeq_\delta \varepsilon^{1/4-\delta/2}.
\end{aligned} \tag{4.25}$$

Taking $\delta = \frac{1}{6}$, we conclude from (4.24) and (4.25) that $\phi_1(x,\varepsilon) \lesssim \varepsilon^{1/6}$, which is an acceptable contribution, in the sense that it is smaller than a multiple of the right-hand side of (4.18).

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

Analysis of $\phi_2(x,\varepsilon)$. The contribution of the region $A' := \{y \in A(x,\varepsilon) : \sqrt{2-|y|} \geqslant \varepsilon^\delta\}$ to $\phi_2(x,\varepsilon)$ can be estimated as follows:

$$|x|^\gamma \int_{A'} \frac{\mathrm{d}y}{\sqrt{2-|y|}\sqrt{2-|x-y|}} \leqslant |x|^\gamma \varepsilon^{-\delta} \int_{A'} \frac{\mathrm{d}y}{\sqrt{2-|x-y|}} \lesssim \varepsilon^{-\delta} \int_{A(x,\varepsilon)} \frac{\mathrm{d}y}{\sqrt{4-|x-y|^2}}$$
$$= \varepsilon^{-\delta} \int_0^{2\pi} \int_{2-\varepsilon}^2 \frac{r}{\sqrt{4-r^2}}\,\mathrm{d}r\,\mathrm{d}\theta \lesssim \varepsilon^{\frac12-\delta}. \tag{4.26}$$

If $y \in A'' := A(x,\varepsilon) \setminus A'$, then $2-\varepsilon^{2\delta} \leqslant |y| \leqslant 2$ and $2-\varepsilon \leqslant |x-y| \leqslant 2$. Therefore, A'' is contained in the intersection of two annuli of small thickness and located at distance comparable to 2 from the origin. We may further assume that $|x| \geqslant \varepsilon^\delta$, because otherwise, given any $s \in (0,\gamma)$,

$$\phi_2(x,\varepsilon) \leqslant (\varepsilon^\delta)^{(\gamma-s)} |x|^s \sigma^{*4}(x) \lesssim_s \varepsilon^{(\gamma-s)\delta}, \tag{4.27}$$

so that $\phi_2(x,\varepsilon) = O_\alpha(\varepsilon^\alpha)$ for every $\alpha \in (0,\gamma\delta)$. We now apply the same dyadic decomposition of $A(x,\varepsilon)$ as in (4.23) together with a similar one on the second annulus,

$$D_k = \{y \in B_2 \colon 2-2^{-k}\varepsilon^{2\delta} \leqslant |y| < 2-2^{-(k+1)}\varepsilon^{2\delta}\},\ k \in \mathbb{N}_0,$$

so that $A'' = \cup_{j,k\geqslant 0} A_j \cap D_k$. This yields

$$|x|^\gamma \int_{A''} \frac{1}{\sqrt{2-|y|}} \frac{1}{\sqrt{2-|x-y|}}\,\mathrm{d}y \lesssim |x|^\gamma \varepsilon^{-1/2-\delta} \sum_{j,k\geqslant 0} 2^{(j+k)/2} |A_j \cap D_k|. \tag{4.28}$$

We now use bipolar coordinates to bound $|A_j \cap D_k|$. First consider the case where, in addition to $|x| \geqslant \varepsilon^\delta$, we have $|x| \leqslant 4-\varepsilon^\delta$, so that A_j and D_k intersect *transversely*; the intersection consists of two connected components that are symmetric with respect to the line spanned by x. Using bipolar coordinates, we have that

$$|A_j \cap D_k| \simeq \int_{\substack{|r-s|<|x|<r+s \\ 2-2^{-k}\varepsilon^{2\delta} \leqslant r \leqslant 2-2^{-(k+1)}\varepsilon^{2\delta} \\ 2-2^{-j}\varepsilon \leqslant s \leqslant 2-2^{-(j+1)}\varepsilon}} \frac{rs\,\mathrm{d}r}{\sqrt{|x|^2-(r-s)^2}\sqrt{(r+s)^2-|x|^2}}$$
$$\simeq \int_{\substack{|a|<|x|<b \\ 2-2^{-k}\varepsilon^{2\delta} \leqslant \frac{a+b}{2} \leqslant 2-2^{-(k+1)}\varepsilon^{2\delta} \\ 2-2^{-j}\varepsilon \leqslant \frac{b-a}{2} \leqslant 2-2^{-(j+1)}\varepsilon}} \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{|x|^2-a^2}\sqrt{b^2-|x|^2}}. \tag{4.29}$$

Given (a,b) in the domain of integration from (4.29), it holds that $0 \leqslant 4-b \leqslant \varepsilon^{2\delta}$ and $|a| \leqslant \varepsilon^{2\delta}$, so that under the working assumption $|x| \geqslant \varepsilon^\delta$, we have $|x|-|a| \gtrsim_\delta |x|$ and $|x|+|a| \simeq |x|$. If, in addition, $|x| \leqslant 4-\varepsilon^\delta$, then $b-|x| \gtrsim_\delta \varepsilon^\delta$, and therefore (4.29) yields

$$|A_j \cap D_k| \lesssim_\delta |x|^{-1}\varepsilon^{-\delta}(2^{-j}\varepsilon)(2^{-k}\varepsilon^{2\delta}) = |x|^{-1}2^{-(j+k)}\varepsilon^{1+\delta},$$

and (4.28) can be bounded as follows:

$$|x|^\gamma \int_{A''} \frac{1}{\sqrt{2-|y|}} \frac{1}{\sqrt{2-|x-y|}}\,\mathrm{d}y \lesssim_\delta |x|^{-(1-\gamma)} \sum_{j,k\geqslant 0} 2^{-(j+k)/2}\varepsilon^{1/2} \lesssim \varepsilon^{\frac12-\delta(1-\gamma)}. \tag{4.30}$$

In the complementary case when $4-\varepsilon^\delta \leqslant |x| \leqslant 4$, we have $|x|-|a| \simeq 1$ and $b+|x| \simeq 1$ in the domain of integration from (4.29), so that

$$\begin{aligned}
|A_j \cap D_k| &\simeq \int_{\substack{|a|<|x|<b \\ 2-2^{-k}\varepsilon^{2\delta}\leqslant \frac{a+b}{2}\leqslant 2-2^{-(k+1)}\varepsilon^{2\delta} \\ 2-2^{-j}\varepsilon\leqslant \frac{b-a}{2}\leqslant 2-2^{-(j+1)}\varepsilon}} \frac{\mathrm{d}a\,\mathrm{d}b}{\sqrt{b-|x|}} \\
&= \int_{\substack{b>|x| \\ b\geqslant 4-2^{-j}\varepsilon-2^{-k}\varepsilon^{2\delta} \\ b\leqslant 4-2^{-(j+1)}\varepsilon-2^{-(k+1)}\varepsilon^{2\delta}}} \int_{\substack{|a|<|x| \\ 2-2^{-k}\varepsilon^{2\delta}\leqslant \frac{a+b}{2}\leqslant 2-2^{-(k+1)}\varepsilon^{2\delta} \\ 2-2^{-j}\varepsilon\leqslant \frac{b-a}{2}\leqslant 2-2^{-(j+1)}\varepsilon}} \mathrm{d}a \frac{1}{\sqrt{b-|x|}}\,\mathrm{d}b \\
&\lesssim \min\{2^{-j}\varepsilon, 2^{-k}\varepsilon^{2\delta}\} \int_{\max\{4-2^{-j}\varepsilon-2^{-k}\varepsilon^{2\delta},|x|\}}^{4-2^{-(j+1)}\varepsilon-2^{-(k+1)}\varepsilon^{2\delta}} \frac{\mathrm{d}b}{\sqrt{b-|x|}} \\
&\lesssim \min\{2^{-j}\varepsilon, 2^{-k}\varepsilon^{2\delta}\}\max\{(2^{-j}\varepsilon)^{1/2}, 2^{-k/2}\varepsilon^{\delta}\} \\
&= 2^{-(j+k)/2}\varepsilon^{1/2+\delta}\min\{2^{-j/2}\varepsilon^{1/2}, 2^{-k/2}\varepsilon^{\delta}\}.
\end{aligned}$$

In this way, (4.28) is bounded as follows:

$$\begin{aligned}
|x|^\gamma \int_{A''} \frac{1}{\sqrt{2-|y|}}\frac{1}{\sqrt{2-|x-y|}}\,\mathrm{d}y &\lesssim |x|^\gamma \sum_{j,k\geqslant 0} \min\{2^{-j/2}\varepsilon^{1/2}, 2^{-k/2}\varepsilon^{\delta}\} \\
&\lesssim \sum_{j\geqslant 0}(2^{-j}\varepsilon)^{1/2} + \sum_{j\geqslant 0}(2^{-j}\varepsilon)^{1/2}|\log(2^{-j}\varepsilon^{1-2\delta})| \\
&\lesssim_s \sum_{j\geqslant 0}(2^{-j}\varepsilon)^{1/2} + \sum_{j\geqslant 0}\varepsilon^{\delta}(2^{-j}\varepsilon^{1-2\delta})^{1/2-s} \\
&\lesssim_s \varepsilon^{\frac{1}{2}-s},
\end{aligned} \tag{4.31}$$

for any $s \in (0, \frac{1}{2})$. In the passage from the first line to the second line above, we split the sum in k according to the partition $\mathbb{N}_0 = \{k \in \mathbb{N}_0 : k \geqslant |\log_2(2^{-j}\varepsilon^{1-2\delta})|\} \cup \{k \in \mathbb{N}_0 : k < |\log_2(2^{-j}\varepsilon^{1-2\delta})|\} =: E_1 \cup E_2$, respectively, where $\log_2(\cdot)$ denotes the base 2 logarithm. On E_1, the sum is a convergent geometric series whose value is proportional to the first term; hence it is $\lesssim (2^{-j}\varepsilon)^{1/2}$. On E_2, the summands are constant, and the contribution to the sum equals $(2^{-j}\varepsilon)^{1/2}|E_2| \simeq (2^{-j}\varepsilon)^{1/2}|\log(2^{-j}\varepsilon^{1-2\delta})|$. As a result of (4.26), (4.27), (4.30) and (4.31), we conclude the upper bound $\phi_2(x,\varepsilon) \lesssim_{\delta,s} \max\{\varepsilon^{\frac{1}{2}-\delta}, \varepsilon^{(\gamma-s)\delta}\}$ for every $\delta \in (0, \frac{1}{2})$ and $s \in (0,\gamma)$. Optimising in δ, we are thus led to the estimate $\phi_2(x,\varepsilon) \lesssim_s \varepsilon^{\frac{\gamma}{2(\gamma+1)}-s}$ for every $s > 0$. This concludes the verification of (4.18).

To handle (4.19), start by noting that

$$|x|^\gamma \int_{B_2\cap B(x,\varepsilon)} F(y)F(x-y)\,\mathrm{d}y = 16|x|^\gamma \int_{B_2\cap B(x,\varepsilon)} \frac{1}{|y|\sqrt{4-|y|^2}}\frac{1}{|x-y|\sqrt{4-|x-y|^2}}\,\mathrm{d}y.$$

Because $\varepsilon < 1$, we may remove the term $\sqrt{4-|x-y|^2}$ from the latter integrand at the expense of a universal constant. After an application of (4.22), we are then left to study the following integrals:

$$\phi_3(x,\varepsilon) := |x|^\gamma \int_{B_2\cap B(x,\varepsilon)} \frac{\mathrm{d}y}{|y||x-y|}, \quad \phi_4(x,\varepsilon) := |x|^\gamma \int_{B_2\cap B(x,\varepsilon)} \frac{\mathrm{d}y}{\sqrt{2-|y|}|x-y|}. \tag{4.32}$$

Analysis of $\phi_3(x,\varepsilon)$. Decompose the region of integration $B_2 \cap B(x,\varepsilon) = A_1 \cup A_2$, where

$$\begin{aligned}
A_1 &:= B(x,\varepsilon) \cap \{y \in B_2 : |y| \geqslant \varepsilon^{1/2}\}, \\
A_2 &:= B(x,\varepsilon) \cap \{y \in B_2 : |y| < \varepsilon^{1/2}\}.
\end{aligned}$$

On the region A_1, we may simply estimate

$$|x|^\gamma \int_{A_1} \frac{\mathrm{d}y}{|y||x-y|} \leqslant |x|^\gamma \varepsilon^{-1/2} \int_{B(x,\varepsilon)} \frac{\mathrm{d}y}{|x-y|} = 2\pi |x|^\gamma \varepsilon^{-1/2}\varepsilon \lesssim \varepsilon^{1/2}.$$

We further split $A_2 = A_2' \cup A_2''$, with

$$A_2' := A_2 \cap \{y : |y| \geqslant |x-y|\}, \text{ and } A_2'' := A_2 \cap \{y : |y| < |x-y|\}.$$

If $y \in A_2'$, then $|y| \geqslant \frac{1}{2}|x|$, and therefore

$$|x|^\gamma \int_{A_2'} \frac{\mathrm{d}y}{|y||x-y|} \lesssim \int_{A_2'} \frac{\mathrm{d}y}{|y|^{1-\gamma}|x-y|}.$$

Now, $|y|^{-(1-\gamma)}\mathbb{1}_{B_2} \in L^p(\mathbb{R}^2)$ for every $1 \leqslant p < \frac{2}{1-\gamma}$, and $|y-x|^{-1}\mathbb{1}_{B_2} \in L^q(\mathbb{R}^2)$ for every $1 \leqslant q < 2$. Taking $2 < p < \frac{2}{1-\gamma}$, its conjugate satisfies $\frac{2}{1+\gamma} < p' < 2$, and so by Hölder's inequality we have that

$$\begin{aligned}\int_{A_2'} \frac{\mathrm{d}y}{|y|^{1-\gamma}|x-y|} &\leqslant \left(\int_{B_{\varepsilon^{1/2}}} \frac{\mathrm{d}y}{|y|^{p(1-\gamma)}}\right)^{\frac{1}{p}} \left(\int_{B(x,\varepsilon)} \frac{\mathrm{d}y}{|x-y|^{p'}}\right)^{\frac{1}{p'}} \\ &= \frac{2\pi}{(2-p(1-\gamma))^{\frac{1}{p}}(2-p')^{\frac{1}{p'}}}\varepsilon^{\frac{p(1+\gamma)-2}{2p}}.\end{aligned}$$

Note that $(p(1+\gamma)-2)/(2p)$ strictly increases to γ as p increases to $2/(1-\gamma)$. In this way, we obtain

$$|x|^\gamma \int_{A_2'} \frac{\mathrm{d}y}{|y||x-y|} \lesssim_s \varepsilon^{\gamma-s},$$

for every $s \in (0,\gamma)$. If $y \in A_2''$, then $|x-y| \geqslant \frac{1}{2}|x|$ and $|y| < |x-y| \leqslant \varepsilon$; in particular, $A_2'' \subset B_\varepsilon$. Therefore, if $2 < p < \frac{2}{1-\gamma}$, then

$$\begin{aligned}|x|^\gamma \int_{A_2''} \frac{\mathrm{d}y}{|y||x-y|} &\lesssim \int_{A_2''} \frac{\mathrm{d}y}{|y||x-y|^{1-\gamma}} \leqslant \left(\int_{B_\varepsilon} \frac{\mathrm{d}y}{|y|^{p'}}\right)^{\frac{1}{p'}} \left(\int_{B(x,\varepsilon)} \frac{\mathrm{d}y}{|x-y|^{p(1-\gamma)}}\right)^{\frac{1}{p}} \\ &\lesssim \varepsilon^{\frac{2-p'}{p'}}\varepsilon^{\frac{2-p(1-\gamma)}{p}} = \varepsilon^\gamma.\end{aligned}$$

We conclude that $\phi_3(x,\varepsilon) \lesssim_s \varepsilon^{\min\{\frac{1}{2},\gamma-s\}}$ for every $s \in (0,\gamma)$.

Analysis of $\phi_4(x,\varepsilon)$. Proceeding as before, we decompose the region of integration $B_2 \cap B(x,\varepsilon) = D_1 \cup D_2$, where

$$\begin{aligned}D_1 &:= B(x,\varepsilon) \cap \{y \in B_2 \colon \sqrt{2-|y|} \geqslant \varepsilon^{1/2}\}, \\ D_2 &:= B(x,\varepsilon) \cap \{y \in B_2 \colon \sqrt{2-|y|} < \varepsilon^{1/2}\}.\end{aligned}$$

On the region D_1, we may simply estimate

$$|x|^\gamma \int_{D_1} \frac{\mathrm{d}y}{\sqrt{2-|y|}|x-y|} \leqslant |x|^\gamma \varepsilon^{-1/2} \int_{B(x,\varepsilon)} \frac{\mathrm{d}y}{|x-y|} \lesssim \varepsilon^{1/2}.$$

If $y \in D_2$, then $2-\varepsilon < |y| \leqslant 2$, and so $2-2\varepsilon \leqslant |x| \leqslant 2+\varepsilon$. We may apply a dyadic decomposition,

$$V_j = \{y \in D_2 \colon 2^{-(j+1)}\varepsilon \leqslant |x-y| \leqslant 2^{-j}\varepsilon\},\ j \in \mathbb{N}_0,$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

so that, letting $P(V_j)$ denote the image of V_j under the polar coordinate map, and further writing $P(V_j) = \{(r,\theta)\colon \theta \in \Theta, r \in R(\theta)\}$ for some $\Theta \subseteq [0,2\pi)$ and $R(\theta) \subseteq [0,\infty)$, we have that

$$|x|^\gamma \int_{D_2} \frac{\mathrm{d}y}{\sqrt{2-|y|}|x-y|} \lesssim |x|^\gamma \sum_{j=0}^\infty 2^j \varepsilon^{-1} \int_{V_j} \frac{\mathrm{d}y}{\sqrt{4-|y|^2}} = |x|^\gamma \sum_{j=0}^\infty 2^j \varepsilon^{-1} \int_{P(V_j)} \frac{r\,\mathrm{d}r\,\mathrm{d}\theta}{\sqrt{4-r^2}}$$
$$\lesssim |x|^\gamma \sum_{j=0}^\infty 2^j \varepsilon^{-1} \int_\Theta |R(\theta)|^{1/2}\,\mathrm{d}\theta \lesssim |x|^\gamma \sum_{j=0}^\infty (2^{-j}\varepsilon)^{-1}(2^{-j}\varepsilon)^{1/2}(2^{-j}\varepsilon) \lesssim \varepsilon^{1/2}.$$

In the second-to-last inequality, we used the fact that the length of the intersection of any line with the annulus V_j is $O(2^{-j}\varepsilon)$ (so that $|R(\theta)|^{1/2} \lesssim (2^{-j}\varepsilon)^{1/2}$), whereas the angular span Θ has measure $O(2^{-j}\varepsilon)$ given that $|x| \gtrsim 1$ and $V_j \subseteq B(x, 2^{-j}\varepsilon)$. We conclude that $\phi_4(x,\varepsilon) \lesssim \varepsilon^{1/2}$, and therefore (4.19) is verified. This finishes the proof of the lemma. □

Proposition 4.6. *Given $\gamma > 0$ and $\{h_j\}_{j=1}^4 \subset \mathrm{Lip}(\mathbb{S}^1)$, let $H_\gamma = |\cdot|^\gamma(h_1\sigma * h_2\sigma * h_3\sigma * h_4\sigma)$. Then there exist $\tau > 0$ and $C < \infty$ such that, for every $x, x' \in \mathbb{R}^2$,*

$$|H_\gamma(x) - H_\gamma(x')| \leqslant C|x-x'|^\tau, \tag{4.33}$$

where $C \leqslant C_0 \prod_{j=1}^4 \|h_j\|_{\mathrm{Lip}(\mathbb{S}^1)}$ for some constant $C_0 < \infty$ depending only on γ.

The proof of Proposition 4.6 will reveal that one can take any $\tau < \min\{\frac{1}{14}, \frac{\gamma}{2(3\gamma+2)}\}$. To a large extent, the proof follows similar lines to those of Proposition 4.3, and so at times we shall be brief. The main difference is that now the extra singularity of $(\sigma * \sigma)(x)$ along the boundary circle $|x| = 2$ also needs to be accounted for.

Proof of Proposition 4.6. Because the case $\gamma > 1$ follows from that of $\gamma \in (0,1]$, the latter condition will be assumed throughout the proof. By homogeneity, we may assume that $\|h_j\|_{\mathrm{Lip}} = 1$, $1 \leqslant j \leqslant 4$. Because H_γ is compactly supported, it is enough to consider $x, x' \in \mathbb{R}^2$ satisfying $|x-x'| \ll 1$; we further assume $|x| \leqslant \min\{4, |x'|\}$. With the notation introduced above (recall (4.12)–(4.14)), we have that

$$\begin{aligned} |H_\gamma(x) - H_\gamma(x')| &= \Big||x|^\gamma (u_{12}F * u_{34}F)(x) - |x'|^\gamma (u_{12}F * u_{34}F)(x')\Big| \\ &\leqslant |x|^\gamma \Big|(u_{12}F * u_{34}F)(x) - (u_{12}F * u_{34}F)(x')\Big| + \big||x|^\gamma - |x'|^\gamma\big| \Big|(u_{12}F * u_{34}F)(x')\Big|. \end{aligned} \tag{4.34}$$

The second summand in (4.34) satisfies the upper bound

$$\begin{aligned} \big||x|^\gamma - |x'|^\gamma\big| \Big|(u_{12}F * u_{34}F)(x')\Big| &\leqslant \big||x|^\gamma - |x'|^\gamma\big| \sigma^{*4}(x') \lesssim_\gamma \big||x| - |x'|\big|^\gamma \sigma^{*4}(x') \\ &\leqslant |x-x'|^s |x'|^{\gamma - s} \sigma^{*4}(x') \\ &\lesssim_{\gamma,s} |x-x'|^s, \end{aligned}$$

for any $s \in (0,\gamma)$, where in the third inequality we used $|x| \leqslant |x'|$ to obtain $\big||x| - |x'|\big| \leqslant |x'|$ and in the last inequality we invoked (4.17). The first summand in (4.34) can be rewritten as the sum of two integrals,

$$\begin{aligned} &|x|^\gamma \Big((u_{12}F * u_{34}F)(x) - (u_{12}F * u_{34}F)(x')\Big) \\ &\qquad = |x|^\gamma \int_{B_2} u_{12}(y)F(y)F(x'-y)\,(u_{34}(x-y) - u_{34}(x'-y))\,\,\mathrm{d}y \\ &\qquad\qquad + |x|^\gamma \int_{B_2} u_{12}(y)F(y)\,(F(x-y) - F(x'-y))\,u_{34}(x-y)\,\mathrm{d}y. \end{aligned}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

We denote the integrals on the right-hand side of the latter identity by I and II, respectively, and proceed to estimate them separately.

Estimating I. The first step is to restrict the domain of integration to the region where $x-y, x'-y \in B_2$, plus a $O(|x-x'|^{\alpha})$ remainder, for some $\alpha > 0$ to be determined. With this purpose in mind, decompose $B_2 = U \cup U' \cup V \cup W$, where

$$\begin{aligned} U &:= \{y \in B_2 : |x'-y| < 2 \leqslant |x-y|\},\ U' := \{y \in B_2 : |x-y| < 2 \leqslant |x'-y|\}, \\ V &:= \{y \in B_2 : |x-y|, |x'-y| < 2\},\ W := \{y \in B_2 : 2 \leqslant |x'-y|, |x-y|\}. \end{aligned} \tag{4.35}$$

The integrand of I vanishes on $U' \cup W$, and so we are left to analyse the integrals over the regions U and V. As in (4.5), we have that

$$U \subseteq \{y \in B_2 \colon 2 - |x-x'| \leqslant |x'-y| \leqslant 2\} =: A(x', |x-x'|) \tag{4.36}$$

and, therefore,

$$\begin{aligned} |x|^{\gamma} \int_U |u_{12}(y) u_{34}(x'-y)| F(y) F(x'-y)\, \mathrm{d}y &\leqslant |x|^{\gamma} \int_{A(x',|x-x'|)} F(y) F(x'-y)\, \mathrm{d}y \\ &\lesssim_{\gamma,s} |x-x'|^{\min\{\frac{1}{6}, \frac{\gamma}{2(\gamma+1)} - s\}}, \end{aligned}$$

for every $s \in (0, \frac{\gamma}{2(\gamma+1)})$, where the latter inequality follows from estimate (4.18). We now consider the integral over the set V. To begin with, note that Lemma 4.1 implies the pointwise estimate

$$|u_{34}(x-y) - u_{34}(x'-y)| \lesssim |x-x'|^{1/2} + \left| \frac{x-y}{|x-y|} - \frac{x'-y}{|x'-y|} \right|, \tag{4.37}$$

provided that $x-y, x'-y \in B_2$. The contribution of the region

$$R := \left\{ y \in V : \left| \frac{x-y}{|x-y|} - \frac{x'-y}{|x'-y|} \right| \leqslant |x-x'|^{1/2} \right\} \tag{4.38}$$

to the integral I is easy to estimate. In view of (4.37) and (4.38), because $|x| \leqslant |x'|$,

$$\begin{aligned} &|x|^{\gamma} \left| \int_R u_{12}(y) F(y) F(x'-y) \left(u_{34}(x-y) - u_{34}(x'-y)\right)\, \mathrm{d}y \right| \\ &\qquad \lesssim |x'|^{\gamma} \left(\int_R F(y) F(x'-y)\, \mathrm{d}y \right) |x-x'|^{1/2} \\ &\qquad \leqslant |x'|^{\gamma} \sigma_2^{*4}(x') |x-x'|^{1/2} \lesssim_{\gamma} |x-x'|^{1/2}, \end{aligned}$$

where in the latter inequality we invoked (4.17). If $y \in V \setminus R$, then $|x'-y| \leqslant 2|x-x'|^{1/2}$ as in (4.9). The contribution of the region $V \setminus R$ can then be estimated as follows:

$$\begin{aligned} &|x|^{\gamma} \left| \int_{V\setminus R} u_{12}(y) F(y) F(x'-y) \left(u_{34}(x-y) - u_{34}(x'-y)\right)\, \mathrm{d}y \right| \\ &\qquad \lesssim |x|^{\gamma} \int_{V \setminus R} |u_{12}(y)| F(y) F(x'-y) \left| \frac{x-y}{|x-y|} - \frac{x'-y}{|x'-y|} \right|\, \mathrm{d}y \\ &\qquad \leqslant 2|x|^{\gamma} \int_{V \cap B(x', 2|x-x'|^{1/2})} F(y) F(x'-y)\, \mathrm{d}y \\ &\qquad \lesssim_{\gamma,s} |x-x'|^{\min\{\frac{1}{4}, \frac{\gamma}{2} - s\}}, \end{aligned}$$

for every $s \in (0, \frac{\gamma}{2})$. The latter inequality is a consequence of estimate (4.19).

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

Estimating II. The integral II is bounded in absolute value by

$$|x|^\gamma \int_{B_2} F(y)\,|F(x-y)-F(x'-y)|\ \mathrm{d}y.$$

Decompose $B_2 = U \cup U' \cup V \cup W$ as in (4.35) and note that the integrand of II vanishes on W. The contribution of the region $U \cup U'$ can be handled with estimate (4.18) as follows (recall (4.36)):

$$\begin{aligned}|x|^\gamma \int_{U\cup U'} F(y)|F(x-y)-F(x'-y)|\,\mathrm{d}y &\leqslant 2|x|^\gamma \int_{A(x,|x-x'|)} F(y)F(x-y)\,\mathrm{d}y\\ &\lesssim_\gamma |x-x'|^{\min\{\frac{1}{6},\frac{\gamma}{2(\gamma+1)}-s\}},\end{aligned}$$

for every $s \in (0, \frac{\gamma}{2(\gamma+1)})$. The estimate on the region V is more delicate, and we split the analysis into two cases. Inside the ball $|x-y| \leqslant |x-x'|^{1/4}$, we also have that $|x'-y| \leqslant |x-x'|+|x-y| \lesssim |x-x'|^{1/4}$. In order to bound the corresponding piece of II, it suffices to consider the integral

$$\varphi(x,x') := |x'|^\gamma \int_{V\cap B(x',|x-x'|^{1/4})} F(y)F(x'-y)\,\mathrm{d}y,$$

which by (4.19) satisfies $\varphi(x,x') \lesssim_s |x-x'|^{\frac{1}{4}\min\{\frac{1}{2},\gamma-s\}}$ for every $s \in (0,\gamma)$. We proceed with the analysis of the complementary region; that is, where $|x-y| > |x-x'|^{1/4}$. If $y \in B_2$, then

$$F(y) = \frac{4}{|y|\sqrt{4-|y|^2}} = \frac{\sqrt{4-|y|^2}}{|y|} + \frac{|y|}{\sqrt{4-|y|^2}}$$

and, as a consequence,

$$\begin{aligned}&|F(x-y)-F(x'-y)|\\ &\leqslant \left|\frac{\sqrt{4-|x-y|^2}}{|x-y|}-\frac{\sqrt{4-|x'-y|^2}}{|x'-y|}\right| + \left|\frac{|x-y|}{\sqrt{4-|x-y|^2}}-\frac{|x'-y|}{\sqrt{4-|x'-y|^2}}\right|\\ &\leqslant \sqrt{4-|x-y|^2}\left|\frac{1}{|x-y|}-\frac{1}{|x'-y|}\right| + \frac{1}{|x'-y|}\left|\sqrt{4-|x-y|^2}-\sqrt{4-|x'-y|^2}\right|\\ &\quad + |x-y|\left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right| + \frac{1}{\sqrt{4-|x'-y|^2}}\Big||x-y|-|x'-y|\Big|.\end{aligned}$$

Using the triangle inequality and recalling that $F(x'-y) = \frac{4}{|x'-y|\sqrt{4-|x'-y|^2}}$,

$$\begin{aligned}|F(x-y)-F(x'-y)| &\lesssim \frac{|x-x'|}{|x-y||x'-y|} + \left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\\ &\quad + \frac{|x-x'|^{1/2}}{|x'-y|} + \frac{|x-x'|}{\sqrt{4-|x'-y|^2}}\\ &\lesssim \frac{|x-x'|}{|x-y||x'-y|} + \left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\\ &\quad + |x-x'|^{1/2}F(x'-y).\end{aligned}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

If $|x-y|>|x-x'|^{1/4}$, then $|x'-y|\geqslant|x-y|-|x-x'|\gtrsim|x-x'|^{1/4}$. Then for $y\in V\cap B(x,|x-x'|^{1/4})^{\complement}$ we obtain

$$|F(x-y)-F(x'-y)|\lesssim|x-x'|^{1/2}+\left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\\+|x-x'|^{1/2}F(x'-y).$$

It follows that the contribution of this region to the integral II is bounded by

$$\begin{aligned}&|x|^\gamma|x-x'|^{1/2}\int_V F(y)\,\mathrm{d}y+|x|^\gamma|x-x'|^{1/2}\int_V F(y)F(x'-y)\,\mathrm{d}y\\&+|x|^\gamma\int_{V\cap B(x,|x-x'|^{1/4})^{\complement}}F(y)\left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\mathrm{d}y\\&\lesssim|x-x'|^{1/2}+|x|^\gamma\int_{V\cap B(x,|x-x'|^{1/4})^{\complement}}F(y)\left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\mathrm{d}y,\end{aligned}$$

where we used that $|x|\leqslant|x'|$, $|x|^\gamma\int_V F(y)F(x'-y)\,\mathrm{d}y\leqslant|x'|^\gamma\sigma^{*4}(x')\leqslant C_\gamma<\infty$ and $\int_V F(y)\,\mathrm{d}y\leqslant\sigma(\mathbb{S}^1)^2$. The last integral left to analyse is

$$|x|^\gamma\int_{V\cap B(x,|x-x'|^{1/4})^{\complement}}F(y)\left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\mathrm{d}y. \tag{4.39}$$

Given $\delta\in(0,\frac{1}{2})$, we further decompose the domain of integration, $V\cap B(x,|x-x'|^{1/4})^{\complement}$, into the subregion where $4-|x-y|^2\geqslant|x-x'|^\delta$ and its complement. If $y\in V$ satisfies $4-|x-y|^2\geqslant|x-x'|^\delta$, then $4-|x'-y|^2\gtrsim|x-x'|^\delta$, and so

$$\left|\frac{1}{\sqrt{4-|x-y|^2}}-\frac{1}{\sqrt{4-|x'-y|^2}}\right|\lesssim\frac{|x-x'|^{1/2}}{\sqrt{4-|x-y|^2}\sqrt{4-|x'-y|^2}}\lesssim|x-x'|^{\frac{1}{2}-\delta}.$$

Therefore, the contribution of this region to the integral (4.39) is bounded by

$$|x|^\gamma|x-x'|^{\frac{1}{2}-\delta}\int_{B_2}F(y)\,\mathrm{d}y\lesssim|x-x'|^{\frac{1}{2}-\delta}.$$

Finally, if $4-|x-y|^2<|x-x'|^\delta$, then $2-|x-y|\leqslant\frac{1}{2}|x-x'|^\delta$, so that this region is contained in the annular domain

$$A(x,\varepsilon):=\{y\in B_2\colon 2-\varepsilon\leqslant|x-y|\leqslant 2\},$$

for $\varepsilon=\frac{1}{2}|x-x'|^\delta$. Because we also have $2-|x'-y|\leqslant|x-x'|^\delta$ if $|x-x'|\ll 1$, the region is also contained in $A(x',2\varepsilon)$. The triangle inequality implies that the integral over the latter region is bounded by (two times) the quantity

$$\tilde{\varphi}(x,x'):=|x|^\gamma\int_{A(x,|x-x'|^\delta)}F(y)F(x-y)\,\mathrm{d}y.$$

One last application of estimate (4.18) reveals that $\tilde{\varphi}(x,x')\lesssim_{\gamma,s}|x-x'|^{\delta\min\{\frac{1}{6},\frac{\gamma}{2(\gamma+1)}-s\}}$ for every $s\in(0,\frac{\gamma}{2(\gamma+1)})$. This concludes the proof of the proposition. □

Remark 4.7. More generally, all higher convolutions $G_n:=h_1\sigma*\cdots*h_n\sigma$, $n\geqslant 5$, are Hölder continuous functions whenever $\{h_j\}_{j=1}^n\subset\mathrm{Lip}(\mathbb{S}^1)$. Indeed, this can be verified for the fifth convolution $G_5=h_1\sigma*\cdots*h_5\sigma$ by writing $G_5=(|\cdot|^{-\gamma}H_\gamma)*h_5\sigma$ for any $\gamma\in(0,1)$, studying the differences

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

$|G_5(x) - G_5(x')|$ and using Proposition 4.6 together with the methods employed in its proof. Once it is known that $G_5 \in \Lambda_\alpha(\mathbb{R}^2)$ for some $\alpha > 0$, it is immediate that $G_n \in \Lambda_\alpha(\mathbb{R}^2)$ for every $n \geqslant 5$. This can be improved, for example, by noting that $G_{10} \in \Lambda_{2\alpha}(\mathbb{R}^2)$.

5. $\mathcal{H}^s$-bound for a restricted convolution operator

Consider a function $H : \mathbb{R}^d \to \mathbb{C}$ supported on the ball $B_R \subset \mathbb{R}^d$, for some $R > 0$, satisfying, for some $\alpha \in (0, 1)$ and $C < \infty$,

$$|H(x) - H(x')| \leqslant C|x - x'|^\alpha + C\left|\frac{x}{|x|} - \frac{x'}{|x'|}\right|, \quad \text{for every } x, x' \in B_R \setminus \{0\}. \tag{5.1}$$

Then $H \in L^\infty(\mathbb{R}^d)$ and is continuous in $B_R \setminus \{0\}$. Given $\gamma \in [0, 1]$, let $K_\gamma = |\cdot|^{-\gamma} H$ and define the corresponding linear operator $\mathcal{K}_\gamma : C^0(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ via

$$(\mathcal{K}_\gamma f)(\omega) = \int_{\mathbb{S}^{d-1}} f(\nu) K_\gamma(\omega - \nu)\, \mathrm{d}\sigma_{d-1}(\nu). \tag{5.2}$$

Lemma 5.1. *Let $d \geqslant 3$ and $\gamma \in [0, 1]$ or $d = 2$ and $\gamma \in [0, 1)$. Let $R > 0$ and $\mathcal{K}_\gamma$ be the linear operator defined in (5.2) above. Then there exists $\delta = \delta(d, \gamma, R) > 0$ such that $\mathcal{K}_\gamma$ extends to a bounded operator from $L^2(\mathbb{S}^{d-1})$ to $\mathcal{H}^\delta(\mathbb{S}^{d-1})$.*

Proof. Let us start by considering the case $\gamma = 1$ in dimensions $d \geqslant 3$. Henceforth, $K_1, \mathcal{K}_1$ will be denoted by $K, \mathcal{K}$, respectively. Implicit constants may depend on d, R, as well as on the constant C from (5.1). Consider the function $\delta(x)$ as in the proof of Lemma 3.1. Introduce a radial partition of unity on B_R, $\{\phi_j\}_{j\geqslant 0}$, where $\phi_j = \delta(2^j R^{-1}\cdot)$ is supported where $2^{-j-1}R \leqslant |x| \leqslant 2^{-j+1}R$, and $\sum_{j\geqslant 0} \phi_j(x) = 1$ for every $x \in B_R \setminus \{0\}$. Let $K_j = K\phi_j$, so that $\|K_j\|_{L^\infty} \leqslant 2^{j+1}R^{-1}\|H\|_{L^\infty}$, and K_j is supported in the spherical shell

$$A_j(R) := \{x \in \mathbb{R}^d : 2^{-j-1}R \leqslant |x| \leqslant 2^{-j+1}R\}.$$

For $x, x' \in A_j(R)$, we have that

$$\begin{aligned}
|K_j(x) - K_j(x')| &= ||x|^{-1}H(x)\phi_j(x) - |x'|^{-1}H(x')\phi_j(x')| \\
&\leqslant ||x|^{-1} - |x'|^{-1}||H(x)|\phi_j(x) + |x'|^{-1}|H(x) - H(x')|\phi_j(x) \\
&\quad + |x'|^{-1}|H(x')||\phi_j(x) - \phi_j(x')| \\
&\lesssim \left|\frac{1}{|x|} - \frac{1}{|x'|}\right| + 2^j|x - x'|^\alpha + 2^j\left|\frac{x}{|x|} - \frac{x'}{|x'|}\right| + 2^{2j}|x - x'| \\
&\lesssim 2^{2j}|x - x'| + 2^j|x - x'|^\alpha + 2^j\left(\frac{1}{|x|} + \frac{1}{|x'|}\right)|x - x'| \\
&\lesssim 2^{2j}|x - x'|^\alpha.
\end{aligned} \tag{5.3}$$

If $x, x' \in B_R$, $x \in \operatorname{supp}(K_j)$ but $x' \notin \operatorname{supp}(K_j)$, then $|K_j(x) - K_j(x')| = |K_j(x)| \lesssim 2^j$.

To each K_j there is a corresponding operator $\mathcal{K}_j$, so that $\mathcal{K} = \sum_{j\geqslant 0} \mathcal{K}_j$. The claimed boundedness of $\mathcal{K}$ is ensured if the operator norms of the $\mathcal{K}_j$ are summable in j. In turn, the operator $\mathcal{K}_j$ is bounded on $L^2(\mathbb{S}^{d-1})$, with operator norm $\|\mathcal{K}_j\|_{L^2\to L^2} = O(2^{-(d-2)j})$. Indeed, by Schur's test, we have that

$$\begin{aligned}
\sup_{\nu\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |K_j(\omega - \nu)|\, \mathrm{d}\sigma_{d-1}(\omega) &= \sup_{\omega\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} |K_j(\omega - \nu)|\, \mathrm{d}\sigma_{d-1}(\nu) \\
&\lesssim 2^j \sup_{\omega\in\mathbb{S}^{d-1}} \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\{2^{-j-1}R\leqslant|\omega-\nu|\leqslant 2^{-j+1}R\}}(\nu)\, \mathrm{d}\sigma_{d-1}(\nu)
\end{aligned}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

$$\lesssim 2^{-(d-2)j}.$$

Moreover, $\mathcal{K}_j$ maps $L^2(\mathbb{S}^{d-1})$ to $\Lambda_\alpha(\mathbb{S}^{d-1})$. To see why this is the case, given $\omega, \omega' \in \mathbb{S}^{d-1}$, define the sets

$$\begin{aligned} U(\omega,\omega') &:= \{\nu \in \mathbb{S}^{d-1} \colon \omega - \nu \in \operatorname{supp}(K_j), \omega' - \nu \notin \operatorname{supp}(K_j)\},\\ U(\omega',\omega) &:= \{\nu \in \mathbb{S}^{d-1} \colon \omega' - \nu \in \operatorname{supp}(K_j), \omega - \nu \notin \operatorname{supp}(K_j)\},\\ V &:= \{\nu \in \mathbb{S}^{d-1} \colon \omega - \nu, \omega' - \nu \in \operatorname{supp}(K_j)\}. \end{aligned}$$

Observe that

$$\sigma_{d-1}(V) \leqslant \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\{|\omega-\nu|\leqslant 2^{-j+1}R\}}(\nu)\, \mathrm{d}\sigma_{d-1}(\nu) \lesssim 2^{-(d-1)j}.$$

On the other hand, and similar to (4.5), the following inclusion holds:

$$\begin{aligned} U(\omega,\omega') \subseteq \{\nu \in \mathbb{S}^{d-1} \colon 2^{-j-1}R - |\omega-\omega'| \leqslant |\omega'-\nu| \leqslant 2^{-j-1}R\} \\ \cup \{\nu \in \mathbb{S}^{d-1} \colon 2^{-j+1}R - |\omega-\omega'| \leqslant |\omega-\nu| \leqslant 2^{-j+1}R\}. \end{aligned}$$

In particular, $\sigma_{d-1}(U(\omega,\omega')) \lesssim 2^{-(d-2)j}|\omega-\omega'|$. By the same argument, we also have that $\sigma_{d-1}(U(\omega',\omega)) \lesssim 2^{-(d-2)j}|\omega-\omega'|$. Then we may use (5.3) and estimate

$$\begin{aligned} |(\mathcal{K}_j f)(\omega) - (\mathcal{K}_j f)(\omega')| &\leqslant \int_{U(\omega,\omega')\cup U(\omega',\omega)\cup V} |K_j(\omega-\nu) - K_j(\omega'-\nu)||f(\nu)|\, \mathrm{d}\sigma_{d-1}(\nu)\\ &\lesssim 2^{2j}|\omega-\omega'|^\alpha \int_V |f(\nu)|\, \mathrm{d}\sigma_{d-1}(\nu) + 2^j \int_{U(\omega,\omega')} |f(\nu)|\, \mathrm{d}\sigma_{d-1}(\nu)\\ &\lesssim 2^{2j}|\omega-\omega'|^\alpha 2^{-\frac{d-1}{2}j}\|f\|_{L^2} + 2^{-\frac{d-4}{2}j}|\omega-\omega'|^{1/2}\|f\|_{L^2}\\ &\lesssim 2^{-\frac{d-5}{2}j}|\omega-\omega'|^{\min\{\frac12,\alpha\}}\|f\|_{L^2}. \end{aligned} \tag{5.4}$$

No generality is lost in assuming that $\alpha \leqslant \frac12$. Inequality (5.4) implies that $\mathcal{K}_j$ maps L^2 to $\mathcal{H}^\alpha$ boundedly and, moreover,

$$\|\mathcal{K}_j f\|_{\mathcal{H}^\alpha} \lesssim \|\mathcal{K}_j f\|_{L^2} + 2^{-\frac{d-5}{2}j}\|f\|_{L^2} \lesssim 2^{-\frac{d-5}{2}j}\|f\|_{L^2}. \tag{5.5}$$

From the definition of the $\mathcal{H}^s$-spaces, one directly checks the following interpolation bounds:

$$\|f\|_{\mathcal{H}^{\theta s+(1-\theta)t}} \leqslant C\|f\|_{\mathcal{H}^s}^{\theta}\|f\|_{\mathcal{H}^t}^{1-\theta}, \text{ for all } \theta \in [0,1],\ 0 \leqslant s,t < 1.$$

Using this to interpolate (5.5) with the $\mathcal{H}^0$-bound $\|\mathcal{K}_j f\|_{L^2} \lesssim 2^{-(d-2)j}\|f\|_{L^2}$ reveals that if $\delta > 0$ is chosen sufficiently small depending on $d \in \{3,4,5\}$ and $\delta = \alpha$ if $d \geqslant 6$, then $\mathcal{K}_j$ maps L^2 to $\mathcal{H}^\delta$ boundedly, with operator norm $O(2^{-cj})$ for some $c > 0$ that does not depend on j. This implies that $\|\mathcal{K}\|_{L^2\to\mathcal{H}^\delta} < \infty$.

We now discuss the case $\gamma \in [0,1)$. If $d = 2$, then the argument above works for the kernel $K_\gamma = |\cdot|^{-\gamma}H$ for any $\gamma \in (0,1)$, because the $L^2 \to L^2$ operator norm of the corresponding $\mathcal{K}_{\gamma,j}$ is then $O(2^{-(1-\gamma)j})$. If $d \geqslant 3$, then we write $K_\gamma = |\cdot|^{1-\gamma}K_1$ and see that the Hölder estimate for K_1 easily yields a corresponding statement for K_γ for every $\gamma \in (0,1)$; in particular, the above argument also transfers. The argument for $\mathcal{K}_0$ is similar but simpler (details omitted). The proof of the lemma is now complete. □

We are finally ready to establish a suitable replacement of Lemma 3.4 that handles the cases when $(d,m) \in \partial\mathfrak{U}$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

Lemma 5.2. *Given $(d,m)\in\partial\mathfrak{U}$, there exists $\alpha>0$ with the following property. If $\{h_j\}_{j=1}^m\subset \operatorname{Lip}(\mathbb{S}^{d-1})$ and $g\in L^2(\mathbb{S}^{d-1})$, then $\mathrm{M}(h_1,\dots,h_m,g)\in\mathcal{H}^\alpha$. Moreover, the following estimate holds:*

$$\|\mathrm{M}(h_1,\dots,h_m,g)\|_{\mathcal{H}^\alpha}\lesssim\prod_{j=1}^m\|h_j\|_{\operatorname{Lip}(\mathbb{S}^{d-1})}\|g\|_{L^2(\mathbb{S}^{d-1})}.$$

Proof. We consider three distinct cases:

Case $d\geqslant 3$, $m=2$. From Corollary 4.2, the function $G=|\cdot|\,(h_1\sigma_{d-1}*h_2\sigma_{d-1})$ satisfies

$$|G(x)-G(x')|\lesssim\|h_1\|_{\operatorname{Lip}}\|h_2\|_{\operatorname{Lip}}\left(|x-x'|^{1/2}+\left|\frac{x}{|x|}-\frac{x'}{|x'|}\right|\right).$$

The conclusion then follows from Lemma 5.1 with $\gamma=1$.

Case $(d,m)=(3,3)$. In view of Proposition 4.3, the function $h_1\sigma_2*h_2\sigma_2*h_3\sigma_2$ belongs to $\Lambda_{1/3}(\mathbb{R}^3)$. The conclusion then follows from Lemma 5.1 with $\gamma=0$.

Case $(d,m)=(2,4)$. In view of Proposition 4.6, given $\gamma>0$, there exists $\tau\in(0,1)$ such that the function $|\cdot|^\gamma(h_1\sigma_1*h_2\sigma_1*h_3\sigma_1*h_4\sigma_1)$ belongs to $\Lambda_\tau(\mathbb{R}^2)$. The conclusion then follows from Lemma 5.1 applied to any $\gamma\in(0,1)$. □

6. Smoothness of critical points

This section is devoted to the proof of Theorem 1.1. Before starting the proof in earnest, we present two further results that will simplify the forthcoming analysis.

Given $(d,m)\in\mathfrak{U}$ and smooth functions $\{\varphi_j\}_{j=1}^m\subset C^\infty(\mathbb{S}^{d-1})$, we define the linear operator $\mathrm{L}=\mathrm{L}[\varphi_1,\dots,\varphi_m]\colon L^2(\mathbb{S}^{d-1})\to L^2(\mathbb{S}^{d-1})$ via

$$\mathrm{L}[\varphi_1,\dots,\varphi_m](g)=\mathrm{M}(\varphi_1,\dots,\varphi_m,g).$$

Lemmas 3.4 and 5.2 together imply the bound $\|\mathrm{L}(g)\|_{\mathcal{H}^\alpha}\leqslant C\|g\|_{L^2}$, for some constant C that depends on d,m, and on the functions $\{\varphi_j\}$. For our purposes, the precise dependence of the constant C on $\{\varphi_j\}$ is not important; however, it is essential that L defines a bounded operator from $L^2(\mathbb{S}^{d-1})$ to $\mathcal{H}^\alpha$ for some exponent $\alpha>0$ that is independent of the functions $\{\varphi_j\}$. Lemmas 3.4 and 5.2 can be recast in terms of the operator L as follows.

Corollary 6.1. *Let $(d,m)\in\mathfrak{U}$. There exists $\alpha>0$ such that $\mathrm{L}[\varphi_1,\dots,\varphi_m](g)\in\mathcal{H}^\alpha$ for any $\{\varphi_j\}_{j=1}^m\subset C^\infty(\mathbb{S}^{d-1})$ and $g\in L^2(\mathbb{S}^{d-1})$. Moreover, the following estimate holds:*

$$\|\mathrm{L}[\varphi_1,\dots,\varphi_m](g)\|_{\mathcal{H}^\alpha}\leqslant C\|g\|_{L^2(\mathbb{S}^{d-1})},\tag{6.1}$$

where $C<\infty$ depends only on d,m, and on the functions $\{\varphi_j\}_{j=1}^m$.

We shall find it necessary to expand the expressions $(\Theta-I)\mathrm{M}(f_1,\dots,f_{m+1})$ and $(\Theta-I)^2\mathrm{M}(f_1,\dots,f_{m+1})$, after a suitable decomposition $f_j=\varphi_{j,0}+\varphi_{j,1}$, $1\leqslant j\leqslant m+1$, has been performed. A model case for this situation is summarised in the following result. The list of $\{\varphi_j\}$ with the ith term removed will be denoted by $[\varphi_1,\dots,\mathring{\varphi}_i,\dots,\varphi_{m+1}]:=[\varphi_1,\dots,\varphi_{i-1},\varphi_{i+1},\dots,\varphi_{m+1}]$.

Lemma 6.2. *Let $(d,m)\in\mathfrak{U}$, let $\varepsilon\in(0,1)$ and let $\{f_j\}_{j=1}^{m+1}\subset L^2(\mathbb{S}^{d-1})$. For each j, decompose $f_j=\varphi_{j,0}+\varphi_{j,1}$, with $\|\varphi_{j,0}\|_{L^2(\mathbb{S}^{d-1})}<\varepsilon\|f_j\|_{L^2(\mathbb{S}^{d-1})}$ and $\varphi_{j,1}\in C^\infty(\mathbb{S}^{d-1})$. Then, for any $\Theta\in SO(d)$,*

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

the following estimates hold:

$$
\begin{aligned}
&\|(\Theta - I)\mathrm{M}(f_1,\dots,f_{m+1})\|_{L^2(\mathbb{S}^{d-1})}\\
&\qquad\lesssim \sum_{i=1}^{m+1}\|(\Theta - I)\mathrm{L}[\varphi_{1,1},\dots,\mathring{\varphi}_{i,1},\dots,\varphi_{m+1,1}](\varphi_{i,0})\|_{L^2(\mathbb{S}^{d-1})}\\
&\qquad\quad+\sum_{i=1}^{m+1}\varepsilon\|(\Theta - I)\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j=1,j\neq i}^{m+1}\|f_j\|_{L^2(\mathbb{S}^{d-1})}\\
&\qquad\quad+\sum_{i=1}^{m+1}\|(\Theta - I)\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j=1,j\neq i}^{m+1}\|f_j\|_{L^2(\mathbb{S}^{d-1})},
\end{aligned}\tag{6.2}
$$

and

$$
\begin{aligned}
&\|(\Theta - I)^2\mathrm{M}(f_1,\dots,f_{m+1})\|_{L^2(\mathbb{S}^{d-1})}\\
&\lesssim \sum_{i=1}^{m+1}\|(\Theta - I)\mathrm{L}[\varphi_{1,1},\dots,\mathring{\varphi}_{i,1},\dots,\varphi_{m+1,1}]((\Theta - I)\varphi_{i,0})\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\sum_{i=1}^{m+1}\varepsilon\|(\Theta - I)^2\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j=1,j\neq i}^{m+1}\|f_j\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\sum_{i=1}^{m+1}\|(\Theta - I)^2\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j=1,j\neq i}^{m+1}\|f_j\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\sum_{\substack{1\leqslant i<j\leqslant m+1\\(\varepsilon_i,\varepsilon_j)\in\{0,1\}^2}}\|(\Theta - I)\varphi_{i,\varepsilon_i}\|_{L^2(\mathbb{S}^{d-1})}\|(\Theta - I)\varphi_{j,\varepsilon_j}\|_{L^2(\mathbb{S}^{d-1})}\prod_{k=1,k\notin\{i,j\}}^{m+1}\|f_k\|_{L^2(\mathbb{S}^{d-1})}.
\end{aligned}\tag{6.3}
$$

Estimates (6.2) and (6.3) exhibit a certain degree of asymmetry with respect to the role played by the functions $\varphi_{i,0}$ and $\varphi_{i,1}$. This is in order to ensure that the less smooth terms $\|(\Theta - I)\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})}$ and $\|(\Theta - I)^2\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})}$ always carry a mitigating factor of ε.

Proof of Lemma 6.2. Decompose each $f_j = \varphi_{j,0} + \varphi_{j,1}$ as in the statement of the lemma. Substituting this into $g := \mathrm{M}(f_1,\dots,f_{m+1})$ and using the multilinearity of M together with the permutation symmetry (3.3), we have that

$$
\begin{aligned}
g &= \sum_{(\varepsilon_1,\dots,\varepsilon_{m+1})\in\{0,1\}^{m+1}}\mathrm{M}(\varphi_{1,\varepsilon_1},\dots,\varphi_{m+1,\varepsilon_{m+1}})\\
&= \mathrm{M}(\varphi_{1,1},\dots,\varphi_{m+1,1}) + \sum_{i=1}^{m+1}\mathrm{L}[\varphi_{1,1},\dots,\mathring{\varphi}_{i,1},\dots,\varphi_{m+1,1}](\varphi_{i,0})\\
&\quad+\sum_{\substack{(\varepsilon_1,\dots,\varepsilon_{m+1})\in\{0,1\}^{m+1}\\ \varepsilon_1+\cdots+\varepsilon_{m+1}\leqslant m-1}}\mathrm{M}(\varphi_{1,\varepsilon_1},\dots,\varphi_{m+1,\varepsilon_{m+1}}).
\end{aligned}
$$

The first, second and third summands in the latter expression correspond to those cases in which exactly none, one or at least two of the ε_is are equal to 0, respectively. Therefore,

$$
(\Theta - I)g = (\Theta - I)\mathrm{M}(\varphi_{1,1},\dots,\varphi_{m+1,1}) + \sum_{i=1}^{m+1}(\Theta - I)\mathrm{L}[\varphi_{1,1},\dots,\mathring{\varphi}_{i,1},\dots,\varphi_{m+1,1}](\varphi_{i,0})
$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

$$+ \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_{m+1})\in\{0,1\}^{m+1} \\ \varepsilon_1+\cdots+\varepsilon_{m+1}\leqslant m-1}} (\Theta - I)\mathrm{M}(\varphi_{1,\varepsilon_1},\ldots,\varphi_{m+1,\varepsilon_{m+1}}). \tag{6.4}$$

In order to L^2-bound the terms coming from the latter sum in (6.4), we appeal to identity (3.5) for each summand and obtain a further sum of terms of the form

$$\mathrm{M}(\varphi_{1,\varepsilon_1},\ldots,\varphi_{i-1,\varepsilon_{i-1}},(\Theta - I)\varphi_{i,\varepsilon_i},\Theta\varphi_{i+1,\varepsilon_{i+1}},\ldots,\Theta\varphi_{m+1,\varepsilon_{m+1}}).$$

The corresponding L^2-norms can be bounded via the basic estimate (3.9), yielding

$$\begin{aligned}\|\mathrm{M}(\varphi_{1,\varepsilon_1},\ldots,\varphi_{i-1,\varepsilon_{i-1}},(\Theta - I)\varphi_{i,\varepsilon_i},\Theta\varphi_{i+1,\varepsilon_{i+1}},\ldots,\Theta\varphi_{m+1,\varepsilon_{m+1}})\|_{L^2(\mathbb{S}^{d-1})} \\ \lesssim \|(\Theta - I)\varphi_{i,\varepsilon_i}\|_{L^2(\mathbb{S}^{d-1})} \prod_{j=1,\, j\neq i}^{m+1} \|\varphi_{j,\varepsilon_j}\|_{L^2(\mathbb{S}^{d-1})}.\end{aligned} \tag{6.5}$$

As noted before, the condition $\varepsilon_1 + \cdots + \varepsilon_{m+1} \leqslant m - 1$ implies the existence of at least two distinct indices $i' \neq j'$ such that $\varepsilon_{i'} = \varepsilon_{j'} = 0$. In this way, (6.5) is bounded by

$$\varepsilon\|(\Theta - I)\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})} \prod_{j=1, j\neq i}^{m+1} \|f_j\|_{L^2(\mathbb{S}^{d-1})}$$

if $\varepsilon_i = 0$ or even better by

$$\varepsilon^2\|(\Theta - I)\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})} \prod_{j=1, j\neq i}^{m+1} \|f_j\|_{L^2(\mathbb{S}^{d-1})}$$

if $\varepsilon_i = 1$. Finally, observe that

$$\|(\Theta - I)\mathrm{M}(\varphi_{1,1},\ldots,\varphi_{m+1,1})\|_{L^2(\mathbb{S}^{d-1})} \leqslant \sum_{i=1}^{m+1} \|(\Theta - I)\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})} \prod_{j=1, j\neq i}^{m+1} \|f_j\|_{L^2(\mathbb{S}^{d-1})}.$$

Adding up all of the contributions, we obtain (6.2). Considering now (6.3), we start from (6.4), apply $\Theta - I$ to both sides and obtain

$$\begin{aligned}(\Theta - I)^2 g =& (\Theta - I)^2\mathrm{M}(\varphi_{1,1},\ldots,\varphi_{m+1,1}) \\ &+ \sum_{i=1}^{m+1} (\Theta - I)^2\mathrm{M}(\varphi_{1,1},\ldots,\mathring{\varphi}_{i,1},\ldots,\varphi_{m+1,1},\varphi_{i,0}) \\ &+ \sum_{\substack{(\varepsilon_1,\ldots,\varepsilon_{m+1})\in\{0,1\}^{m+1} \\ \varepsilon_1+\cdots+\varepsilon_{m+1}\leqslant m-1}} (\Theta - I)^2\mathrm{M}(\varphi_{1,\varepsilon_1},\ldots,\varphi_{m+1,\varepsilon_{m+1}}).\end{aligned} \tag{6.6}$$

Using (3.5) twice together with the basic estimate (3.9), the first term on the latter right-hand side can be bounded as follows:

$$\begin{aligned}&\|(\Theta - I)^2\mathrm{M}(\varphi_{1,1},\ldots,\varphi_{m+1,1})\|_{L^2(\mathbb{S}^{d-1})} \\ &\lesssim \sum_{1\leqslant i<j\leqslant m+1} \|(\Theta - I)\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})}\|(\Theta - I)\varphi_{j,1}\|_{L^2(\mathbb{S}^{d-1})} \prod_{k:k\notin\{i,j\}} \|f_k\|_{L^2(\mathbb{S}^{d-1})}\end{aligned}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

$$+\sum_{i=1}^{m+1}\|(\Theta-I)^2\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j:j\neq i}\|f_j\|_{L^2(\mathbb{S}^{d-1})}.$$

An upper bound similar to the preceding one also applies to each term from the third sum in (6.6), but this can be refined as follows:

$$\begin{aligned}
&\|(\Theta-I)^2\mathrm{M}(\varphi_{1,\varepsilon_1},\dots,\varphi_{m+1,\varepsilon_{m+1}})\|_{L^2(\mathbb{S}^{d-1})}\\
&\lesssim\sum_{1\leqslant i<j\leqslant m+1}\|(\Theta-I)\varphi_{i,\varepsilon_i}\|_{L^2(\mathbb{S}^{d-1})}\|(\Theta-I)\varphi_{j,\varepsilon_j}\|_{L^2(\mathbb{S}^{d-1})}\prod_{k:k\notin\{i,j\}}\|f_k\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\sum_{i=1,\varepsilon_i=1}^{m+1}\varepsilon^2\|(\Theta-I)^2\varphi_{i,1}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j:j\neq i}\|f_j\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\sum_{i=1,\varepsilon_i=0}^{m+1}\varepsilon\|(\Theta-I)^2\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})}\prod_{j:j\neq i}\|f_j\|_{L^2(\mathbb{S}^{d-1})}.
\end{aligned}$$

Lastly, each of the terms coming from the second sum in (6.6) can be bounded as follows:

$$\begin{aligned}
&\|(\Theta-I)^2\mathrm{M}(\varphi_{1,1},\dots,\mathring{\varphi}_{i,1},\dots,\varphi_{m+1,1},\varphi_{i,0})\|_{L^2(\mathbb{S}^{d-1})}\\
&\lesssim\sum_{\substack{1\leqslant j<k\leqslant m+1\\ j\neq i,k\neq i}}\varepsilon\|(\Theta-I)\varphi_{j,1}\|_{L^2(\mathbb{S}^{d-1})}\|(\Theta-I)\varphi_{k,1}\|_{L^2(\mathbb{S}^{d-1})}\prod_{\ell\notin\{j,k\}}\|f_\ell\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\sum_{j=1,j\neq i}^{m+1}\|(\Theta-I)\varphi_{i,0}\|_{L^2(\mathbb{S}^{d-1})}\|(\Theta-I)\varphi_{j,1}\|_{L^2(\mathbb{S}^{d-1})}\prod_{k\notin\{i,j\}}\|f_k\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad+\|(\Theta-I)\mathrm{L}[\varphi_{1,1},\dots,\mathring{\varphi}_{i,1},\dots,\varphi_{m+1,1}]((\Theta-I)\varphi_{i,0})\|_{L^2(\mathbb{S}^{d-1})}.
\end{aligned}$$

Adding up all of the contributions yields (6.3). This completes the proof of the lemma. □

6.1. *Proof of Theorem 1.1*

We are now ready to start with the proof of Theorem 1.1 in earnest. As a first step, we establish an initial regularity kick. Henceforth we assume the parameter λ in equation (1.12) to be nonzero, in which case λ can be absorbed into the function a; see the final remark in Subsection 6.3. We are thus interested in solutions of the equation

$$a\cdot\mathrm{M}(R^{k_1}(f),\dots,R^{k_{m+1}}(f))=f,\quad\sigma_{d-1}\text{-a.e. on }\mathbb{S}^{d-1}.\tag{6.7}$$

Proposition 6.3. *Let $(d,m)\in\mathfrak{U}$ and $(k_1,\dots,k_{m+1})\in\{0,1\}^{m+1}$. Assume that $a\in\Lambda_\kappa(\mathbb{S}^{d-1})$ for some $\kappa\in(0,1)$. Then, given any complex-valued solution $f\in L^2(\mathbb{S}^{d-1})$ of equation (6.7), there exists $s>0$ such that $f\in\mathcal{H}^s$.*

Proof. Let $f\in L^2(\mathbb{S}^{d-1})$ be a complex-valued solution of (6.7), and let $\varepsilon\in(0,1)$ be a small constant, to be chosen in the course of the argument. We may decompose $f=g_\varepsilon+\varphi_\varepsilon$, where $\|g_\varepsilon\|_{L^2}<\varepsilon\|f\|_{L^2}$, and $\varphi_\varepsilon\in C^\infty$. In this way, we have that $\|\varphi_\varepsilon\|_{L^2}\leqslant(1+\varepsilon)\|f\|_{L^2}\leqslant 2\|f\|_{L^2}$; it is important that the latter bound is independent of ε. By multilinearity of M, no generality is lost in assuming that f is L^2-normalised, $\|f\|_{L^2}=1$. In (6.7), we further suppose that $k_i=0$ for every $1\leqslant i\leqslant m+1$. This assumption is made for notational purposes only, because the exact same argument applies in general.[5] Substituting

[5]Note that the operator R is a linear isometry and that $\|(\Theta-I)f\|_{L^2}=\|(\Theta-I)f_\star\|_{L^2}$ for every $\Theta\in\mathrm{SO}(d)$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

$f = g_\varepsilon + \varphi_\varepsilon$ into the right-hand side of (6.7), we then see that the function g_ε satisfies the equation

$$g_\varepsilon = a \cdot \mathrm{M}(f, \ldots, f) - \varphi_\varepsilon.$$

Given $\Theta \in \mathrm{SO}(d)$, apply $\Theta - I$ to both sides of the latter identity, yielding

$$(\Theta - I)g_\varepsilon = (\Theta - I)a \cdot \Theta\mathrm{M}(f, \ldots, f) + a \cdot (\Theta - I)\mathrm{M}(f, \ldots, f) - (\Theta - I)\varphi_\varepsilon.$$

Consequently,

$$\begin{aligned}\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \leqslant\ & \|(\Theta - I)a\|_{L^\infty(\mathbb{S}^{d-1})}\|\mathrm{M}(f, \ldots, f)\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &+ \|a\|_{L^\infty(\mathbb{S}^{d-1})}\|(\Theta - I)\mathrm{M}(f, \ldots, f)\|_{L^2(\mathbb{S}^{d-1})}.\end{aligned}$$

We estimate the third summand on the right-hand side of the latter inequality with the help of Lemma 6.2, yielding

$$\begin{aligned}\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \lesssim\ & \|(\Theta - I)a\|_{L^\infty(\mathbb{S}^{d-1})} + (1 + \|a\|_{L^\infty(\mathbb{S}^{d-1})})\|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &+ \|a\|_{L^\infty(\mathbb{S}^{d-1})}\Big(\|(\Theta - I)\mathrm{L}[\varphi_\varepsilon, \ldots, \varphi_\varepsilon](g_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})} + \varepsilon\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\Big).\end{aligned}$$

We may now choose $\varepsilon \in (0, 1)$ small enough, depending on d, m and $\|a\|_{L^\infty}$, so that the last term on the right-hand side can be absorbed into the left-hand side, yielding

$$\begin{aligned}\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \lesssim\ & \|(\Theta - I)a\|_{L^\infty(\mathbb{S}^{d-1})} + (1 + \|a\|_{L^\infty(\mathbb{S}^{d-1})})\|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &+ \|a\|_{L^\infty(\mathbb{S}^{d-1})}\|(\Theta - I)\mathrm{L}[\varphi_\varepsilon, \ldots, \varphi_\varepsilon](g_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})}.\end{aligned}$$

Choose $s \in (0, 1)$ in such a way that $s \leqslant \kappa$ and $\mathrm{L}[\varphi_\varepsilon, \ldots, \varphi_\varepsilon]$ is bounded from L^2 to $\mathcal{H}^s$, as promised by Corollary 6.1. Such an s can be chosen independent of the function φ_ε and therefore does not depend on ε either (but the implicit constant may depend on ε, which we now take as fixed). Setting $\Theta = e^{tX_{i,j}}$ for some $1 \leqslant i < j \leqslant d$, multiplying by $|t|^{-s}$ and taking the supremum over $|t| \in [0, 1]$ yields

$$\begin{aligned}&\sup_{|t|\leqslant 1} |t|^{-s}\|(e^{tX_{i,j}} - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\ &\qquad \lesssim \|a\|_{\Lambda_s(\mathbb{S}^{d-1})} + (1 + \|a\|_{L^\infty(\mathbb{S}^{d-1})})\|\varphi_\varepsilon\|_{\mathcal{H}^s} + C_\varepsilon\|a\|_{L^\infty(\mathbb{S}^{d-1})}\|g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} < \infty. \qquad (6.8)\end{aligned}$$

Here we are using that the Λ_s-norm can be controlled by the Λ_κ-norm because $s \leqslant \kappa$. Estimate (6.8) implies that $g_\varepsilon \in \mathcal{H}^s$ and therefore $f \in \mathcal{H}^s$ as well. The proof of the proposition is now complete. □

Remark 6.4. If $(d, m) \in \mathfrak{U} \setminus \partial\mathfrak{U}$, then there is an automatic gain in the initial regularity of any complex-valued $f \in L^2(\mathbb{S}^{d-1})$ solution of equation (6.7). Indeed, we claim that in that case f necessarily coincides with a continuous function on $\mathbb{S}^{d-1}$. To see why this must be so, start by considering the case $d, m \geqslant 3$. Writing $m + 1 = (m - 1) + 2$, where $m - 1 \geqslant 2$, we see that the convolution product on the left-hand side of (1.12) can be written as

$$(R^{k_1}(f)\sigma_{d-1} * \cdots * R^{k_{m-1}}(f)\sigma_{d-1}) * (R^{k_m}(f)\sigma_{d-1} * R^{k_{m+1}}(f)\sigma_{d-1}).$$

Because each of the two functions in the preceding convolution belongs to $L^2(\mathbb{R}^d)$, their convolution defines a continuous function of bounded support on $\mathbb{R}^d$. It follows that its restriction to the unit sphere also defines a continuous function on $\mathbb{S}^{d-1}$, as claimed. An analogous argument works for the case $d = 2$ and $m \geqslant 5$.

The second main step is a bootstrapping procedure that will complete the proof of Theorem 1.1. Indeed, in light of Remark 2.2, Propositions 6.3 and 6.5 together imply that a solution f of equation (6.7) (and therefore of equation (1.12) if $\lambda \neq 0$) satisfies $f \in H^r$ for every $r \geqslant 0$. From Sobolev embedding – see, for example, [18, Theorem 2.7] – it then follows that $f \in C^\infty(\mathbb{S}^{d-1})$.

Proposition 6.5. *Let $(d, m) \in \mathfrak{U}$. Let $(k_1, \dots, k_{m+1}) \in \{0,1\}^{m+1}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $a \in C^\infty(\mathbb{S}^{d-1})$. Then there exists $\alpha > 0$ with the following property. Let f be a solution of equation (6.7) satisfying $f \in \mathcal{H}^s$ for some $s > 0$. Then $f \in \mathcal{H}^t$ for every $t \in [0, s + \min\{s - \lfloor s \rfloor, \alpha\}] \setminus \mathbb{Z}$.*

Proof. We make a few initial simplifications. Firstly, we consider the special case $a \equiv 1$ only, because the general case $a \in C^\infty(\mathbb{S}^{d-1})$ brings no additional complications, as shown by the proof of Proposition 6.3. Secondly, we further assume that $k_i = 0$, for every $1 \leqslant i \leqslant m+1$; this considerably simplifies the forthcoming notation but changes nothing fundamental in the analysis. Thirdly, we start by supposing that $s \in (0,1)$. The case $s \geqslant 1$ will be dealt with at a later stage in the proof.

Assume $\|f\|_{L^2} = 1$, and let $\varepsilon \in (0,1)$, to be chosen in the course of the argument. Decompose $f = g_\varepsilon + \varphi_\varepsilon$, with $\varphi_\varepsilon \in C^\infty(\mathbb{S}^{d-1})$ and $\|g_\varepsilon\|_{L^2} < \varepsilon$. In particular, $\|\varphi_\varepsilon\|_{L^2} \leqslant 2$. Because $f \in \mathcal{H}^s$, it follows that $g_\varepsilon \in \mathcal{H}^s$ as well. The equation satisfied by g_ε is

$$g_\varepsilon = \mathrm{M}(f, \dots, f) - \varphi_\varepsilon.$$

Given $\Theta \in \mathrm{SO}(d)$, we have that

$$(\Theta - I)^2 g_\varepsilon = (\Theta - I)^2 \mathrm{M}(f, \dots, f) - (\Theta - I)^2 \varphi_\varepsilon$$

and, therefore,

$$\|(\Theta - I)^2 g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \leqslant \|(\Theta - I)^2 \mathrm{M}(f, \dots, f)\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I)^2 \varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}.$$

Using Lemma 6.2 to estimate the first term on the right-hand side of the preceding inequality, we obtain

$$\begin{aligned}\|(\Theta - I)^2 g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} &\lesssim \|(\Theta - I)\varphi_\varepsilon\|^2_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I)^2 \varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \\ &\quad + \|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \|(\Theta - I) g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I) g_\varepsilon\|^2_{L^2(\mathbb{S}^{d-1})} \\ &\quad + \|(\Theta - I)\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]((\Theta - I) g_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})} + \varepsilon \|(\Theta - I)^2 g_\varepsilon\|^2_{L^2(\mathbb{S}^{d-1})}.\end{aligned}$$

Now choose $\varepsilon \in (0,1)$ small enough, depending on d, m, in such a way that the last term on the latter left-hand side can be absorbed into the right-hand side. With such a choice of ε, the following inequality holds:

$$\begin{aligned}\|(\Theta - I)^2 g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} &\lesssim \|(\Theta - I)\varphi_\varepsilon\|^2_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I)^2 \varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \\ &\quad + \|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} \|(\Theta - I) g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I) g_\varepsilon\|^2_{L^2(\mathbb{S}^{d-1})} \\ &\quad + \|(\Theta - I)\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]((\Theta - I) g_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})}. \qquad (6.9)\end{aligned}$$

Now that ε has been fixed, Corollary 6.1 implies that the operator $\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]$ is bounded from L^2 to $\mathcal{H}^\alpha$, for some $\alpha \in (0,1)$ independent of ε.

Set $\delta = \min\{s, \alpha\}$, where α is as in the previous paragraph. In particular, $\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]$ is bounded from L^2 to $\mathcal{H}^\delta$, with operator norm that may depend on ε. Henceforth we consider $\Theta = \Theta(t) = e^{tX_{k,\ell}}$, $1 \leqslant k < \ell \leqslant d$ and $|t| \leqslant 1$. The following estimate holds:

$$\begin{aligned}&\|(\Theta - I)\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]((\Theta - I) g_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})} \\ &\qquad \leqslant |t|^\delta \sup_{|\tau| \leqslant 1} |\tau|^{-\delta} \|(\Theta(\tau) - I)\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]((\Theta(t) - I) g_\varepsilon)\|_{L^2(\mathbb{S}^{d-1})} \\ &\qquad \leqslant |t|^\delta \|\mathrm{L}[\varphi_\varepsilon, \dots, \varphi_\varepsilon]((\Theta(t) - I) g_\varepsilon)\|_{\mathcal{H}^\delta} \\ &\qquad \leqslant C_\varepsilon |t|^\delta \|(\Theta - I) g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\end{aligned}$$

$$\leqslant C_\varepsilon |t|^{\delta+s}\|g_\varepsilon\|_{\mathcal{H}^s}.$$

Multiplying (6.9) by $|t|^{-(s+\delta)}$ yields

$$\begin{aligned}
&|t|^{-(s+\delta)}\|(\Theta - I)^2 g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\
&\lesssim |t|^{-\delta}\|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}|t|^{-s}\|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} + |t|^{-(s+\delta)}\|(\Theta - I)^2\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad + |t|^{-\delta}\|(\Theta - I)\varphi_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}|t|^{-s}\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}\\
&\quad + |t|^{-\delta}\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})}|t|^{-s}\|(\Theta - I)g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} + C_\varepsilon\|g_\varepsilon\|_{\mathcal{H}^s}.
\end{aligned}$$

Now take the supremum over $|t| \leqslant 1$ and use the facts that $\varphi_\varepsilon \in \mathcal{H}^r$ for all $0 \leqslant r \notin \mathbb{Z}$ and $g_\varepsilon \in \mathcal{H}^s \cap \mathcal{H}^\delta$ (recall that $\delta \leqslant s$). Invoking the characterisation of the $\mathcal{H}^{s+\delta}$-norm by means of second differences as detailed in Subsection 6.2, which applies because $s + \delta \in (0, 2)$, we obtain that

$$\begin{aligned}
\sup_{|t|\leqslant 1} |t|^{-(s+\delta)}\|(\Theta - I)^2 g_\varepsilon\|_{L^2(\mathbb{S}^{d-1})} &\lesssim \|\varphi_\varepsilon\|_{\mathcal{H}^\delta}\|\varphi_\varepsilon\|_{\mathcal{H}^s} + \|\varphi_\varepsilon\|_{\mathcal{H}^{s+\delta}} + \|\varphi_\varepsilon\|_{\mathcal{H}^\delta}\|g_\varepsilon\|_{\mathcal{H}^s}\\
&\quad + \|g_\varepsilon\|_{\mathcal{H}^\delta}\|g_\varepsilon\|_{\mathcal{H}^s} + C_\varepsilon\|g\|_{\mathcal{H}^s} < \infty.
\end{aligned}$$

In this way, again via second differences, we see that $g_\varepsilon \in \mathcal{H}^{s+\delta}$ and therefore $f \in \mathcal{H}^{s+\delta}$ as well.[6] This concludes the proof of the proposition in the special case when $s \in (0, 1)$.

Repeated applications of the previous step reveal that if $f \in \mathcal{H}^s$ for some $s \in (0, 1)$, then $f \in \mathcal{H}^{1+\gamma}$ for some $\gamma \in (0, 1)$. We complete the proof of the proposition by induction. In order to treat exponents $s = k+\gamma$, with $k \in \mathbb{N}$ and $\gamma \in (0, 1)$, we use the product rule (3.7) and differentiate k times identity (6.7) with respect to $X \in \{X_{i,j} : 1 \leqslant i < j \leqslant d\}$, thus obtaining an equation for $X^k f \in \mathcal{H}^\gamma$. Decomposing $X^k f = g_\varepsilon + \varphi_\varepsilon$, with $\varphi_\varepsilon \in C^\infty(\mathbb{S}^{d-1})$ and $\|g_\varepsilon\|_{L^2} \leqslant \varepsilon\|X^k f\|_{L^2}$, we can use the same method as before to show that $g_\varepsilon \in \mathcal{H}^t$ for any $t \in [s, s + \min\{\gamma, \alpha\}] \setminus \mathbb{Z}$. In a similar way, we may analyse the mixed derivatives $Yf := Y_1 \dots Y_k f$, where $Y_\ell \in \{X_{i,j} : 1 \leqslant i < j \leqslant d\}$, $1 \leqslant \ell \leqslant k$. In what follows, we provide the details.

For simplicity, we only consider powers of the same vector field X but note that the exact same method would apply to a more general vector field Y as in the previous paragraph. The equation satisfied by $X^k f$ is of the form

$$X^k f = \sum_{\substack{\vec{k}:=(k_1,\dots,k_{m+1})\in\mathbb{N}_0^{m+1}\\ k_1+\cdots+k_{m+1}=k}} c_{\vec{k}}\, \mathrm{M}(X^{k_1} f, \dots, X^{k_{m+1}} f), \tag{6.10}$$

for some constants $c_{\vec{k}} > 0$. Note that $X^{k_j} f \in \mathcal{H}^{1+\gamma}$ if $k_j < k$. Thus, we are led to splitting the sum in (6.10) into two parts, one of them containing precisely those summands that carry the term $X^k f$. There are $m + 1$ of them, so

$$X^k f = \sum_{\vec{k}\in K} c_{\vec{k}}\, \mathrm{M}(X^{k_1} f, \dots, X^{k_{m+1}} f) + (m + 1)\mathrm{M}(f, \dots, f, X^k f), \tag{6.11}$$

where $(k_1, \dots, k_{m+1}) \in K$ if and only if $k_j < k$, for every $1 \leqslant j \leqslant m + 1$, and $k_1 + \cdots + k_{m+1} = k$. The first term on the right-hand side of (6.11) can be easily bounded in $\mathcal{H}^{1+\gamma}$ with (3.8), yielding

$$\sum_{\vec{k}\in K} c_{\vec{k}}\|\mathrm{M}(X^{k_1} f, \dots, X^{k_{m+1}} f)\|_{\mathcal{H}^{1+\gamma}} \lesssim \|f\|_{\mathcal{H}^s}^{m+1}.$$

[6] If $s + \delta = 1$, then $\mathcal{H}^{s+\delta}$ is not defined; however, by using any $\delta' < \delta$ in the reasoning above, the conclusion is that $g_\varepsilon \in \mathcal{H}^t$ for every $t < 1$ and therefore $f \in \mathcal{H}^t$ for every $t < 1$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

To handle the second term, let $\varepsilon \in (0,1)$ and decompose $f = \varphi_0 + \varphi_1$, $X^k f = \psi_0 + \psi_1$, with $\varphi_1, \psi_1 \in C^\infty(\mathbb{S}^{d-1})$ and $\|\varphi_0\|_{L^2} < \varepsilon \|f\|_{L^2}, \|\psi_0\|_{L^2} < \varepsilon \|X^k f\|_{L^2}$. Because $f \in \mathcal{H}^s$, we have that $\varphi_0 \in \mathcal{H}^s$ and $\psi_0 \in \mathcal{H}^\gamma$. Now take $\delta \in (0,1)$ satisfying $\delta \leqslant \min\{\gamma, \alpha\}$; recall that $\gamma = s - \lfloor s \rfloor$ and that α was chosen immediately following (6.9). The equation satisfied by ψ_0 may be derived from (6.11). Applying $(\Theta - I)^2$ to both sides of that equation and invoking Lemma 6.2, we find that, if $\varepsilon > 0$ is small enough, then

$$\begin{aligned}
\|(\Theta - I)^2 \psi_0\|_{L^2(\mathbb{S}^{d-1})} \lesssim{}& \sum_{\vec{k} \in K} c_{\vec{k}} \|(\Theta - I)^2 \mathrm{M}(X^{k_1} f, \ldots, X^{k_{m+1}} f)\|_{L^2(\mathbb{S}^{d-1})} \\
&+ (\|(\Theta - I)^2 \varphi_0\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I)^2 \varphi_1\|_{L^2(\mathbb{S}^{d-1})}) \|X^k f\|_{L^2(\mathbb{S}^{d-1})} \\
&+ (\|(\Theta - I) \varphi_0\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I) \varphi_1\|_{L^2(\mathbb{S}^{d-1})})^2 \|X^k f\|_{L^2(\mathbb{S}^{d-1})} \\
&+ (\|(\Theta - I) \varphi_0\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I) \varphi_1\|_{L^2(\mathbb{S}^{d-1})}) \|(\Theta - I) \psi_0\|_{L^2(\mathbb{S}^{d-1})} \\
&+ (\|(\Theta - I) \varphi_0\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I) \varphi_1\|_{L^2(\mathbb{S}^{d-1})}) \|(\Theta - I) \psi_1\|_{L^2(\mathbb{S}^{d-1})} \\
&+ C_\varepsilon |t|^\delta \|(\Theta - I) \psi_0\|_{L^2(\mathbb{S}^{d-1})} + \|(\Theta - I)^2 \psi_1\|_{L^2(\mathbb{S}^{d-1})}.
\end{aligned}$$

Consequently, by means of second differences, we obtain

$$\sup_{0 < |t| \leqslant 1} |t|^{-(\delta + \gamma)} \|(\Theta - I)^2 \psi_0\|_{L^2(\mathbb{S}^{d-1})} < \infty$$

and, as a result, $\psi_0 \in \mathcal{H}^{\gamma + \delta}$. It follows that $X^k f \in \mathcal{H}^{\gamma + \delta}$ and, because $X \in \{X_{i,j} : 1 \leqslant i < j \leqslant d\}$ was arbitrary,[7] $f \in \mathcal{H}^{s + \delta}$. The proof of the proposition is now complete. □

6.2. Second differences

Given $s \in (0,2)$, we define the space $\mathscr{H}^s = \mathscr{H}^s(\mathbb{S}^{d-1})$ of all functions $f \in L^2(\mathbb{S}^{d-1})$, for which the norm

$$\|f\|_{\mathscr{H}^s} = \|f\|_{L^2(\mathbb{S}^{d-1})} + \sum_{1 \leqslant i < j \leqslant d} \sup_{|t| \leqslant 1} |t|^{-s} \|(e^{tX_{i,j}} - I)^2 f\|_{L^2(\mathbb{S}^{d-1})} \tag{6.12}$$

is finite. We see that

$$(e^{tX_{i,j}} - I)^2 f = f \circ e^{2tX_{i,j}} - 2f \circ e^{tX_{i,j}} + f$$

resembles a second difference of f. From the definition, it is immediate that $\|f\|_{\mathscr{H}^s} \leqslant 2\|f\|_{\mathcal{H}^s}$ provided that $s \in (0,1)$, so $\mathcal{H}^s \subseteq \mathscr{H}^s$. The reverse inclusion also holds. Moreover, if $s \in (0,2) \setminus \{1\}$, then $\mathcal{H}^s = \mathscr{H}^s$, and the two norms given by (2.7) and (6.12) are equivalent. These assertions have all appeared in the literature; in what follows, we provide precise references.

Let us discuss the Euclidean case first. Given $s \in (0,1)$, we defined the Hölder space $\Lambda_s(\mathbb{R}^d)$ to contain precisely those functions $f : \mathbb{R}^d \to \mathbb{C}$ for which the norm

$$\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{|t| > 0} |t|^{-s} \|f(x + t) - f(x)\|_{L^\infty_x(\mathbb{R}^d)}$$

is finite, whereas for $s = k + \delta$, $1 \leqslant k \in \mathbb{N}$, $\delta \in (0,1)$, we have that $f \in \Lambda_s(\mathbb{R}^d)$ if $f \in C^k(\mathbb{R}^d)$ and $\partial^\alpha f \in \Lambda_\delta(\mathbb{R}^d)$, for all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = k$. Given $s \in (0,2)$, consider the norm (defined in terms of second differences)

$$\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{|t| > 0} |t|^{-s} \|f(x + 2t) - 2f(x + t) + f(x)\|_{L^\infty_x(\mathbb{R}^d)}$$

[7] Again, if $s + \delta \in \mathbb{Z}$, then the conclusion is that $f \in \mathcal{H}^t$ for every $t \in [0, s + \delta] \setminus \mathbb{Z}$.

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

and the corresponding space of functions for which the latter norm is finite. These two spaces coincide if $s \in (0,2) \setminus \{1\}$, as dictated by the classical equivalence between Hölder and Zygmund spaces, the latter being defined through higher differences; precise references include [27, Ch. V, Prop. 8] and [30, Ch. 2, §2.6]. More generally, one may consider an L^p-norm in x, $1 \leqslant p \leqslant \infty$, and possibly an additional L^q-norm in t, $1 \leqslant q \leqslant \infty$; see [27, Ch. V, Prop. 8'] and [30, Ch. 2, §2.6].

For the case of the unit sphere $\mathbb{S}^{d-1}$, the equivalence between the $\mathcal{H}^s$- and the $\mathscr{H}^s$-norms, and therefore the equality of the two corresponding spaces, can be found in [15, 16]. These works rely on harmonic extensions, in a similar spirit to the aforementioned chapter in [27]. Of particular relevance are Propositions 4.1 and 4.3 in [15] and Proposition 1.8 in [16]. In the former article [15], the function space $\Lambda(\alpha; p, q)$ is defined for $\alpha > 0$, $1 \leqslant p, q \leqslant \infty$, and shown to be equivalent to a variant thereof using first- and second-order differences; the special case $(p,q) = (2,\infty)$ and $\alpha = s \in (0,1)$ of this equivalence is used to establish that the spaces $\mathcal{H}^s$ and $\mathscr{H}^s$ coincide whenever $s \in (0,1)$. In the latter article [16], spaces of index $\alpha = k+\gamma$, $k \in \mathbb{N}$, are related to those of index γ in a precise way; in turn, this is used to establish the equivalence between the spaces $\mathcal{H}^s$ and $\mathscr{H}^s$ whenever $s \in (1,2)$. It should be pointed out that the norms in terms of first and second differences considered in [15] are slightly different from the ones that we are using to define $\mathcal{H}^s$ and $\mathscr{H}^s$. However, the norms are seen to be equivalent; see [7, Cor. 3.11]. See also [5] and [23, Theorems 3.1 and 3.3].

We proceed to describe an alternative approach to the equivalence discussed in the previous paragraph that perhaps requires less effort from the unfamiliar reader.

Firstly, the equivalence between $\mathcal{H}^s$ and $\mathscr{H}^s$ when $s \in (0,1)$ follows directly from the combinatorial proof of [19, Lemma 1.1], stated in [19] for the case of $\mathbb{R}^d$. For the convenience of the reader, we provide a brief sketch of the argument. As mentioned already, the estimate $\|f\|_{\mathscr{H}^s} \leqslant 2\|f\|_{\mathcal{H}^s}$ follows easily from the definitions. For the reverse inequality, consider the following identity, which is valid for every $t \in \mathbb{R}$, $X \in \{X_{i,j} : 1 \leqslant i < j \leqslant d\}$ and $m \in \mathbb{N}$:

$$2^m(e^{tX} - I) = (e^{2^m tX} - I) - \sum_{i=0}^{m-1} 2^{m-1-i}(e^{2^i tX} - I)^2.$$

Applying this operator to a function $f \in \mathscr{H}^s$, taking the $L^2(\mathbb{S}^{d-1})$-norm on both sides and invoking the triangle inequality yields

$$2^m\|(e^{tX} - I)f\|_{L^2(\mathbb{S}^{d-1})} \leqslant \|(e^{2^m tX} - I)f\|_{L^2(\mathbb{S}^{d-1})} + \sum_{i=0}^{m-1} 2^{m-1-i}\|(e^{2^i tX} - I)^2 f\|_{L^2(\mathbb{S}^{d-1})}.$$

Dividing by 2^m, using that $\|(e^{2^m tX} - I)f\|_{L^2(\mathbb{S}^{d-1})} \leqslant 2\|f\|_{L^2(\mathbb{S}^{d-1})} \leqslant 2\|f\|_{\mathscr{H}^2}$ and letting $m \to \infty$, we then obtain

$$\|(e^{tX} - I)f\|_{L^2(\mathbb{S}^{d-1})} \leqslant \frac{1}{2}\sum_{i=0}^{\infty} 2^{-i}\|(e^{2^i tX} - I)^2 f\|_{L^2(\mathbb{S}^{d-1})}.$$

Multiplying by $|t|^{-s}$ and taking the supremum over $t \in \mathbb{R}$ shows that, when $s < 1$, the following holds:

$$\begin{aligned}\sup_{t\in\mathbb{R}} |t|^{-s}\|(e^{tX} - I)f\|_{L^2(\mathbb{S}^{d-1})} &\leqslant \frac{1}{2}\left(\sum_{i=0}^{\infty} 2^{-i(1-s)}\right)\sup_{t\in\mathbb{R}} |t|^{-s}\|(e^{tX} - I)^2 f\|_{L^2(\mathbb{S}^{d-1})} \\ &= \frac{1}{2(1-2^{-(1-s)})}\sup_{t\in\mathbb{R}} |t|^{-s}\|(e^{tX} - I)^2 f\|_{L^2(\mathbb{S}^{d-1})}. \qquad (6.13)\end{aligned}$$

https://doi.org/10.1017/fms.2021.7 Published online by Cambridge University Press

On the other hand, by 2π-periodicity of e^{tX} and $SO(d)$-invariance of σ_{d-1},

$$\sup_{t\in\mathbb{R}} |t|^{-s}\|(e^{tX}-I)^k f\|_{L^2(\mathbb{S}^{d-1})} \simeq \sup_{|t|<1} |t|^{-s}\|(e^{tX}-I)^k f\|_{L^2(\mathbb{S}^{d-1})}$$

for $k \in \{1,2\}$. This together with (6.13) yields $\|f\|_{\mathcal{H}^s} \leqslant C\|f\|_{\mathscr{H}^s}$ for some $C < \infty$.

Secondly, the equivalence between $\mathcal{H}^s$ and $\mathscr{H}^s$ when $s \in (1,2)$ can be obtained via the techniques in [8, §3] (especially Theorem 3.6) and [7, §2.3], which rely on the modulus of smoothness and Marchaud-type inequalities. Indeed, the equivalence of the norms $\|\cdot\|_{\mathcal{W}_p^{r,\alpha}}$ and $\|\cdot\|_{H_p^{r+\alpha}}$ given by [8, Theorem 3.6] provides the answer after specialising to $(\ell, r\, p, \alpha) = (1, 1, 2, s-1)$, because in this case $\|f\|_{\mathcal{H}^s} \simeq \|f\|_{\mathcal{W}_p^{r,\alpha}} + \|f\|_{\mathcal{H}^{s-1}}$, $\|f\|_{H_p^{r+\alpha}} \simeq \|f\|_{\mathscr{H}^s}$ and, as already remarked, $\|f\|_{\mathcal{H}^{s-1}} \simeq \|f\|_{\mathscr{H}^{s-1}} \lesssim \|f\|_{\mathscr{H}^s}$. The argument is straightforward but lengthy; thus, the reader is directed to the aforementioned references.

6.3. *One final remark*

Our proof of Theorem 1.1 does not in general handle the case when $\lambda = 0$ in (1.12). An exception corresponds to the case when $m = 2k$ is an even integer, $\vec{k} \in \{0,1\}^{m+1}$ satisfies $k_1 + \cdots + k_{m+1} = k-1$ and $a > 0$ on $\mathbb{S}^{d-1}$ (or, more generally, $a = 0$ on a set of σ_{d-1}-measure zero), which corresponds to the Euler-Lagrange equation (1.10) with $\lambda = 0$. In this case, by multiplying both sides of (1.12) by $\overline{f}$ and integrating over $\mathbb{S}^{d-1}$, one concludes that $\|\widehat{f\sigma}_{d-1}\|_{L^{m+2}(\mathbb{R}^d)} = 0$, which clearly forces $f = \mathbf{0}$. It remains unclear whether one should expect general solutions of (1.12) to be smooth when $\lambda = 0$.

Acknowledgments. D.O.S. is supported by the EPSRC New Investigator Award 'Sharp Fourier Restriction Theory', Grant No. EP/T001364/1, and is grateful to Michael Christ, Giuseppe Negro and Po-Lam Yung for valuable discussions during the preparation of this work. The authors thank the anonymous referees for a careful reading of the article and valuable suggestions.

Conflict of Interest: None.

References

[1] E. Carneiro and D. Oliveira e Silva, 'Some sharp restriction inequalities on the sphere', *Int. Math. Res. Not. IMRN* **2015**(17) (2015), 8233–8267.

[2] M. Christ and R. Quilodrán, 'Gaussians rarely extremize adjoint Fourier restriction inequalities for paraboloids', *Proc. Amer. Math. Soc.* **142**(3) (2014), 887–896.

[3] M. Christ and S. Shao, 'Existence of extremals for a Fourier restriction inequality', *Anal. PDE* **5**(2) (2012), 261–312.

[4] M. Christ and S. Shao, 'On the extremizers of an adjoint Fourier restriction inequality', *Adv. Math.* **230**(3) (2012), 957–977.

[5] L. Colzani, 'Lipschitz spaces on compact rank one symmetric spaces', *in Harmonic Analysis (Cortona,* 1982*)*, *Lecture Notes in Mathematics*, Vol. **992** (Springer, Berlin, 1983), 139–160.

[6] F. Dai and Y. Xu, 'Moduli of smoothness and approximation on the unit sphere and the unit ball', *Adv. Math.* **224**(4) (2010), 1233–1310.

[7] F. Dai and Y. Xu, 'Polynomial approximation in Sobolev spaces on the unit sphere and the unit ball', *J. Approx. Theory* **163**(10) (2011), 1400–1418.

[8] F. Dai and Y. Xu, *Approximation Theory and Harmonic Analysis on Spheres and Balls*, *Springer Monographs in Mathematics* (Springer, New York, 2013).

[9] L. Fanelli, L. Vega and N. Visciglia, 'On the existence of maximizers for a family of restriction theorems', *Bull. Lond. Math. Soc.* **43**(4) (2011), 811–817.

[10] D. Foschi, *On the Regularity of Multilinear Forms Associated to the Wave Equation* (PhD Dissertation, Princeton University, 2000).

[11] D. Foschi, 'Global maximizers for the sphere adjoint Fourier restriction inequality', *J. Funct. Anal.* **268** (2015), 690–702.

[12] D. Foschi and D. Oliveira e Silva, 'Some recent progress on sharp Fourier restriction theory', *Anal. Math.* **43**(2) (2017), 241–265.

[13] R. Frank, E. H. Lieb and J. Sabin, 'Maximizers for the Stein-Tomas inequality', *Geom. Funct. Anal.* **26** (2016), no. 4, 1095–1134.

[14] L. Grafakos, *Modern Fourier Analysis*, third ed., *Graduate Texts in Mathematics*, **Vol. 250** (Springer, New York, 2014).

[15] H. C. Greenwald, 'Lipschitz spaces on the surface of the unit sphere in Euclidean n-space', *Pacific J. Math.* **50** (1974), 63–80.

[16] H. C. Greenwald, 'Lipschitz spaces of distributions on the surface of unit sphere in Euclidean n-space', *Pacific J. Math.* **70**(1) (1977), 163–176.

[17] B. Hall, *Lie Groups, Lie Algebras, and Representations. An Elementary Introduction*, second ed., *Graduate Texts in Mathematics*, **Vol. 222** (Springer-Verlag, New York, 2015).
[18] E. Hebey, *Nonlinear Analysis on Manifolds: Sobolev Spaces and Inequalities*, *Courant Lecture Notes in Mathematics*, **Vol. 5** (New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1999).
[19] C. S. Herz, 'Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms', *J. Math. Mech.* **18** (1968–69), 283–323.
[20] L. Hörmander, 'Hypoelliptic second order differential equations', *Acta Math.* **119** (1967), 147–171.
[21] V. Jurdjevic and H. J. Sussmann, 'Control systems on Lie groups', *J. Differ. Equ.* **12** (1972), 313–329.
[22] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications: Vol. I. Die Grundlehren der mathematischen Wissenschaften [Comprehensive Studies in Mathematics]*, Band 181 (Springer-Verlag, New York, 1972). Translated from the French by P. Kenneth.
[23] P.I. Lizorkin and KH. P. Rustamov, 'Nikol'skiĭ-Besov spaces on the sphere in connection with approximation theory', *Proc. Steklov Inst. Math.* **204** (1994), 149–172.
[24] D. Oliveira e Silva and R. Quilodrán, 'Global maximizers for adjoint Fourier restriction inequalities on low dimensional spheres', *J. Funct. Anal.* **280**(7) (2021) 108825, https://doi.org/10.1016/j.jfa.2020.108825.
[25] S. Shao, 'On existence of extremizers for the Tomas-Stein inequality for $\mathbb{S}^1$', *J. Funct. Anal.* **270** (2016), 3996–4038.
[26] S. Shao, 'On smoothness of extremizers of the Tomas-Stein inequality for $\mathbb{S}^1$', Preprint (2016), arXiv:1601.07119v3.
[27] E. M. Stein, *Singular Integrals and Differentiability of Functions* (Princeton University Press, Princeton, NJ, 1970).
[28] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals* (Princeton University Press, Princeton, NJ, 1993).
[29] P. Tomas, 'A restriction theorem for the Fourier transform', *Bull. Amer. Math. Soc.* **81**(2) (1975), 477–478.
[30] H. Triebel, *Theory of Function Spaces II*, *Monographs in Mathematics*, **Vol. 84** (Birkhäuser Verlag , Basel, Switzerland, 1992).