A GENERALIZATION OF AN INEQUALITY OF HARDY AND LITTLEWOOD

K. T. SMITH

1. Introduction. A well-known inequality of Hardy-Littlewood reads as follows **(4)**: if $p > 1$ and $f > 0$, then

$$\int_a^b \bar{f}(x)^p dx \leqslant A \int_a^b f(x)^p dx,$$

where $\bar{f}(x)$ is defined as the supremum of the numbers

$$\frac{1}{v+u}\int_{x-u}^{x+v} f(t)\, dt;$$

the constant depends on p only. The statement obtained by putting $p = 1$ is false; its substitute reads:

$$\int_a^b \bar{f}(x)\, dx \leqslant A \int_a^b f(x)\, dx + B\int_a^b f(x) \log^+ f(x)\, dx + \epsilon;$$

the constants depend on ϵ but not on f. The Hardy-Littlewood inequality has had several important applications: to function theory, harmonic functions, Fourier series, and the strong differentiability of multiple integrals—to mention those with which the author is acquainted. The application to harmonic functions is the following **(4)**:

Let $u(r, \phi)$ be a non-negative harmonic function in the unit circle, and for each ϕ define

$$\tilde{u}(\phi) = \sup_{0<r<1} u(r, \phi).$$

Then if $p > 1$,

$$\int_0^{2\pi} \tilde{u}(\phi)^p d\phi \leqslant A \sup_{0<r<1} \int_0^{2\pi} u(r, \phi)^p d\phi = A \int_0^{2\pi} u(\phi)^p d\phi,$$

where $u(\phi) = u(1, \phi)$ is the boundary function for u and where A is a constant depending on p only.

Since the original appearance of the inequality there have been a number of generalizations. It was formulated for n-dimensional space by Wiener **(14)** and used to prove dominated individual ergodic theorems. The n-dimensional case was used also by Calderón and Zygmund **(3)** to prove dominated pointwise convergence of singular integrals. It was formulated for certain types of

Received February 28, 1955. Work performed under contract with Office of Naval Research N58304.

locally compact topological groups with Haar measure by Calderón **(2)** and used again to prove ergodic theorems. It was formulated in a weaker version for metric spaces with an outer measure of the type considered below by Rauch **(8)** and used to prove ergodic theorems and theorems about analytic functions of several complex variables. The latter are described after Theorem 4 below.[1]

The object of this note is to give the inequality a general form valid for certain types of measures on metric spaces and to give applications of the general form of the inequality to harmonic functions, subharmonic functions, and strong differentiability of multiple integrals.

2. The inequality. Most of the arguments are based upon a simple covering theorem which appears implicitly in Banach's proof of the Vitali covering theorem. It can be stated as follows.[2]

LEMMA 1. *Let $\mathfrak{S}$ be a family of spheres in a metric space. If $\mathfrak{S}$ satisfies the conditions* (i) *and* (ii) *below, then it contains a disjoint sequence $\{S(x_n, r_n)\}$ such that*

$$\sum_{S(x,r)\in\mathfrak{S}} S(x, r) \subset \sum_{n=1}^{\infty} S(x_n, 4r_n).$$

The conditions are as follows:

(i) *There is a number R such that for every $S(x, r) \in \mathfrak{S}$, $0 < r \leqslant R$.*

(ii) *If $\{S(x_n, r_n)\}$ is any disjoint sequence in $\mathfrak{S}$, then $r_n \to 0$.*

By using the notation $S(x, r)$ for the sphere with center x and radius r we agree tacitly that a sphere is an object determined by a center and a radius. In the applications of Lemma 1 it is the set of points included in the sphere which is important. In order to apply the Lemma to a family of sets each of which is a sphere with respect to several centers and several radii it will be necessary to demonstrate the possibility of choosing for each set one center and one radius in such a way that the hypotheses of the Lemma are satisfied. This arrangement has been picked in order to avoid an unnecessary hypothesis excluding isolated points in the metric space. When the space consists solely of a finite number of isolated points the inequality becomes an inequality on finite sums of some interest in itself. It is to this special case that the greater part of the proof of Hardy and Littlewood is devoted.

We shall consider a metric space B on which there is a regular outer measure subject to the conditions which follow.

[1]Until told by the referee, the author was not aware of the work of Wiener, Calderón, and Rauch. Recently Rauch has supplemented his note **(8)** with a paper **(9)** which will be found elsewhere in this journal; in the latter he obtains the full Hardy-Littlewood inequality by a method of Wiener, but with less precise constants than those in the theorems below.

[2]Wiener, Calderón, and Rauch use similar covering theorems. A proof is given in **(1)**.

If E is any set, $|E|$ is its measure and $\delta(E)$ is its diameter.

(a) *Each sphere is measurable and has finite measure.*

(b) *There is a constant K such that $|S(x, 4r)| \leqslant K|S(x, r)|$ for every closed sphere $S(x, r)$.*

(c) *If $\{S_n\}$ is a sequence of closed spheres such that $|S_n| \to 0$, then*[3] $\delta(S_n) \to 0$.

(d) *If $\{S_n\}$ is a sequence of closed spheres such that $\delta(S_n) \to \infty$, then $|S_n| \to \infty$.*

It is known that these conditions are sufficient to ensure that every Borel set in B is measurable.

When f is a non-negative measurable function belonging to some class L^p, $p \geqslant 1$, on B we make use of the following notations:

(i) $\bar{f}(x)$ is the supremum of the averages of f over all the closed spheres centered at x; that is,

$$\bar{f}(x) = \sup \frac{1}{|S|} \int_S f(y)\, dy,$$

the supremum being taken over all closed spheres S centered at x.

(ii) $f^*(t)$, defined for t real and >0, is the non-increasing equimeasurable rearrangement of f. (that is, $|E_t[f^*(t) > a]| = |E_t[f(x) > a]|$ for all $a > 0$.) It is well known that for any measurable set $E \subset B$,

$$\int_E f(x)\, dx \leqslant \int_0^{|E|} f^*(t)\, dt,$$

and equality holds if $E = B$.

(iii)
$$\beta_f(t) = \frac{1}{t} \int_0^t f^*(s)\, ds.$$

β_f is a continuous non-increasing function, strictly decreasing except possibly in an interval beginning with 0 where it can be constant.

(iv) β^f is the (upper semi-continuous) inverse function to β_f; if $s > \sup \beta_f(t)$, then[4] $\beta^f(s) = 0$.

Lemma 2. *If $f \geqslant 0$ belongs to L^p, $p \geqslant 1$, then $\bar{f}$ is lower semi-continuous. Consequently $\bar{f}$ is measurable.*

Proof. Suppose that $x_n \to x$. If S is an arbitrary closed sphere with center x and radius r, let S_n be the closed sphere with center x_n and radius $r + d(x_n, x)$. Since $S = \lim S_n$, it follows both that $|S| = \lim |S_n|$ and that

$$\int_S f(y)\, dy = \lim \int_{S_n} f(y)\, dy$$

[3]The author's original condition was somewhat less general. The change to this condition and a modification in the proof of Theorem 1 required by the change were suggested by N. Aronszajn.

[4]The properties of f^*, β_f, and β^f are described briefly in Calderón and Zygmund **(3)**.

https://doi.org/10.4153/CJM-1956-019-5 Published online by Cambridge University Press

and hence that

$$\frac{1}{|S|}\int_S f(y)\,dy = \lim \frac{1}{|S_n|}\int_{S_n} f(y)\,dy \leqslant \lim\inf \bar{f}(x_n).$$

THEOREM 1. *If $f \geqslant 0$ belongs to L^p, $p > 1$, then $\bar{f}$ also belongs to L^p and*

$$\int \bar{f}(x)^p dx \leqslant K\left(\frac{p}{p-1}\right)^p \int f(x)^p dx$$

where K is the constant of hypothesis (b) *on B.*

Proof. For the first part of the argument we suppose only $p \geqslant 1$. We begin by noting that if E is any measurable set of positive measure over which the average of f is $\geqslant t > 0$, then by Hölder's inequality

$$t \leqslant \frac{1}{|E|}\int_E f(x)\,dx \leqslant \frac{1}{|E|^{1/p}}\left\{\int f(x)^p dx\right\}^{1/p},$$

so that

$$|E| \leqslant \frac{1}{t^p}\int f(x)^p dx.$$

That is, $|E|$ is bounded by a constant independent of E. If $\{E_n\}$ is a disjoint sequence of sets over which the average of f is $\geqslant t$, then

$$E = \sum_{n=1}^{\infty} E_n$$

is also a set over which the average of f is $\geqslant t$. Consequently $\sum |E_n| = |E| < \infty$, so that $|E_n| \to 0$.

Now, if $t > 0$, let B_t denote the set of points x such that $\bar{f}(x) > t$. For each $x \in B_t$, t fixed, let S_x be a closed sphere centered at x over which the average of f is $\geqslant t$. Furthermore, choose S_x with positive measure and so that it admits a non-zero radius. Let $\mathfrak{S}$ be the family of these spheres. It will be shown that it is possible to choose for each $x \in B_t$ a radius $r(x)$ such that $S_x = S(x, r(x))$ and such that with this choice of centers and radii for the spheres in $\mathfrak{S}$, $\mathfrak{S}$ satisfies the conditions of Lemma 1.

For each $x \in B_t$ let $r_0(x)$ be the infimum and $r_1(x)$ the supremum of the numbers r such that $S_x = S(x, r)$. If $r_0(x) \geqslant |S_x|$, then take $r(x) = r_0(x)$. If $r_0(x) < |S_x|$, then take

$$r(x) = \min\left[\frac{r_0(x) + r_1(x)}{2}, |S_x|\right].$$

Clearly $r(x) \neq 0$ and $S_x = S(x, r(x))$. From the first paragraph of the proof it follows that the numbers $|S_x|$ for $S_x \in \mathfrak{S}$ are bounded, and then from condition (d) on B it follows that the numbers $\delta(S_x)$ for $S_x \in \mathfrak{S}$ are bounded. Now, either $r(x) = r_0(x) \leqslant \delta(S_x)$ or $r(x) \leqslant |S_x|$, so the numbers $r(x)$ for $S_x \in \mathfrak{S}$ are bounded; and condition (i) of Lemma 1 is verified. If

$$\{S_{x_n}\}$$

is a disjoint sequence in $\mathfrak{S}$, then again from the first paragraph it follows that

$$|S_{x_n}| \to 0,$$

and from condition (c) on B it follows that

$$\delta(S_{x_n}) \to 0.$$

Thus $r(x_n) \to 0$, and condition (ii) of Lemma 1 is verified.

Having verified (i) and (ii), we can apply Lemma 1 to extract from $\mathfrak{S}$ a disjoint sequence $\{S(x_n, r_n)\}$ such that

$$B_t \subset \sum_{S(x,r)\in\mathfrak{S}} S(x, r) \subset \sum_{n=1}^{\infty} S(x_n, 4r_n).$$

Then

$$|B_t| \leqslant \sum_{n=1}^{\infty} |S(x_n, 4r_n)| \leqslant K\sum_{n=1}^{\infty} |S(x_n, r_n)| = K|E|,$$

where K is the constant in hypothesis (b), and $E = \sum S(x_n, r_n)$. As before, the average of f over E is $\geqslant t$. (Therefore the first paragraph provides a bound for $|E|$, but this bound is not sharp enough.) We have however,

$$t \leqslant \frac{1}{|E|}\int_E f(x)\,dx \leqslant \frac{1}{|E|}\int_0^{|E|} f^*(t)\,dt = \beta_f(|E|),$$

and by inverting β_f, $|E| \leqslant \beta^f(t)$. Finally, therefore, $|B_t| \leqslant K\beta^f(t)$.

Now let $p > 1$. In the following chain of inequalities we use the fact that

$$\lim_{s\to\infty} s\beta_f(s)^p = \lim_{s\to 0} s\beta_f(s)^p = 0,$$

and we use the substitution $t = \beta_f(s)$. We have[5]

$$\begin{aligned}\int \bar{f}(x)^p dx &= \int_0^\infty pt^{p-1}|B_t|\,dt \leqslant K\int_0^\infty pt^{p-1}\beta^f(t)\,dt \\ &= -K\int_0^\infty s\,d\beta_f(s)^p = K\int_0^\infty \beta_f(s)^p ds = K\int_0^\infty \frac{1}{s^p}\left\{\int_0^s f^*(t)\,dt\right\}^p ds \\ &\leqslant K\left(\frac{p}{p-1}\right)^p \int_0^\infty f^*(t)^p dt = K\left(\frac{p}{p-1}\right)^p \int_B f(x)^p dx.\end{aligned}$$

THEOREM 2. *If f is non-negative and measurable, and if $f(x)\log^+ f(x)$ is integrable, then for any measurable set E,*

$$\int_E \bar{f}(x)\,dx \leqslant 2K\int f(x)\log^+ f(x)\,dx + \left(\frac{4K}{e} + 2\right)|E|.$$

[5]This concluding calculation can be found in Calderón and Zygmund **(3)**. The last inequality in the chain is a well-known inequality of Hardy.

https://doi.org/10.4153/CJM-1956-019-5 Published online by Cambridge University Press

Proof. It is well known that if $f \log^+ f$ is integrable, then f itself is integrable over every set of finite measure. This is all that is necessary to the formation of the function $\bar{f}$. It does not guarantee the existence of f^*, however, so we write $f = g + h$, where $g(x) = f(x)$ if $f(x) < 2$ and $g(x) = 0$ otherwise.

It is clear that $\int_E \bar{g}(x)\,dx \leqslant 2\,|E|$, so if it can be proved that

$$\int_E \bar{h}(x)\,dx \leqslant 2K \int h(x) \log^+ h(x)\,dx + \frac{4K}{e}|E|, \tag{2.1}$$

then we will have

$$\int_E \bar{f}(x)\,dx \leqslant \int_E \bar{g}(x)\,dx + \int_E \bar{h}(x)\,dx \leqslant 2K \int f(x) \log^+ f(x)\,dx + \left(\frac{4K}{e} + 2\right)|E|.$$

Now, h is in fact an integrable function, so the proof will be complete if we prove (2.1) for integrable functions.

Let us call the integrable function f, rather than h, so that the notations used in Theorem 1 will be appropriate. Let $B_t' = E \cap B_t$. Then $|B_t'| \leqslant |E|$ and, as was proved in Theorem 1, $|B_t'| \leqslant |B_t| \leqslant K\beta^f(t)$. Hence

$$\int_E \bar{f}(x)\,dx = \int_0^\infty |B_t'|\,dt \leqslant \int_0^{t_0} |E|\,dt + K \int_{t_0}^\infty \beta^f(t)\,dt$$

for any t_0, while also, for $t_0 = \beta_f(|E|)$,

$$\int_{t_0}^\infty \beta^f(t)\,dt = -\int_0^{|E|} s\,d\beta_f(s) = -\,|E|\beta_f(|E|) + \int_0^{|E|} \beta_f(s)\,ds.$$

Furthermore,

$$\int_0^{|E|} \beta_f(s)\,ds = \int_0^{|E|} \frac{ds}{s} \int_0^s f^*(s')\,ds'$$

$$= \int_0^{|E|} f^*(s') \log \frac{|E|}{s'}\,ds' \leqslant 2 \int_0^{|E|} f^*(s') \log^+ f^*(s')\,ds' + \frac{2}{e} \int_0^{|E|} \left(\frac{|E|}{s'}\right)^{\frac{1}{2}} ds'$$

$$\leqslant 2 \int_B f(x) \log^+ f(x)\,dx + \frac{4}{e}\,|E|.$$

Therefore

$$\int_E \bar{f}(x)\,dx \leqslant (1 - K)\,|E|\,\beta_f(|E|) + 2K \int_B f(x) \log^+ f(x)\,dx + \frac{4K}{e}\,|E|.$$

Since necessarily $K \geqslant 1$, (2.1) follows.

Once the estimate for $|B_t|$ is obtained, the evaluation is almost identical with that given in Calderón and Zygmund **(3)** in the proof of Theorem 2. One of the inequalities used in the chain is that of W. H. Young, namely $ab \leqslant a \log a + e^{b-1}$.

THEOREM 3. *If f is non-negative and integrable, and $0 < \epsilon < 1$, then for every measurable set E,*

$$\int_E \bar{f}(x)^{1-\epsilon} dx \leqslant \left(\frac{1}{\epsilon}\right) |E|^\epsilon K^{1-\epsilon} \left\{ \int f(x)\,dx \right\}^{1-\epsilon}.$$

https://doi.org/10.4153/CJM-1956-019-5 Published online by Cambridge University Press

Proof. The proof is the same as the last part of the proof of Theorem 3 of Calderón and Zygmund **(3)**. Use must be made, of course, of our previous estimate for $|B_t|$.

3. Applications—Harmonic functions. We propose to apply the inequality to the case where B is a smooth surface bounding a bounded domain D in Euclidean n-dimensional space R^n, $n \geqslant 3$. The explicit smoothness assumptions are as follows.[6]

(a) *B is a C^1 surface; that is, each point of B has an n-dimensional neighborhood V which can be mapped in* 1–1 *fashion on an n-dimensional cube by a transformation T such that T and T^{-1} are C^1 transformations and such that $T(B \cap V)$ is the intersection of the cube with one of the coordinate hyperplanes.*

(b) *B is of bounded curvature in the large sense, that is, if $\alpha(x, y)$ denotes the angle between the exterior normals at x and y, then*

$$\frac{1}{\rho_0} = \sup_{x \neq y} \frac{\sin \frac{1}{2}\alpha(x, y)}{\frac{1}{2}|x - y|} < \infty.$$

The metric in B is its metric as a subset of Euclidean space. The measure on B is the area measure, definable in the classical manner because of the smoothness conditions.

We shall use capital letters P, Q, etc. to designate points in the interior of D, and small letters x, y, etc., to designate points on the boundary B. Each point P at distance less than ρ_0 from B lies on a unique line segment of length less than ρ_0 and normal to B. We write x_P for the point at which this segment meets B. Proceeding from the opposite direction, we write $P(\rho, x)$ for the point at distance ρ from x measured along the interior normal through x. Finally, we write $\mathfrak{F}_p$ for the class of functions $f(P)$ harmonic in D and such that

$$\sup_{0<\rho<\rho_0} \int_B |f[P(\rho, x)]|^p \, dx < \infty.$$

(Note that $f[P(\rho, x)]$ is the restriction of $f(P)$ to the surface parallel to B at distance ρ.)

It is known that if $f \in \mathfrak{F}_p$ then $f(P)$ has a limit $f(x)$ as $P \to x$ non-tangentially **(13)** for almost every point $x \in B$. The so defined function $f(x)$, which belongs to L^p on B, is called the boundary function of f. When $p > 1$, the functions $f[P(\rho, x)]$ converge in mean of order p to the boundary function as $\rho \to 0$, and f is the Poisson integral of the boundary function.[7] We shall make

[6]A forthcoming note by Aronszajn will contain proofs of all the needed properties of such surfaces. The object of his note is to exhibit the best possible constants in all cases. Here we do not need the best constants, but only the qualitative sense of the properties, and for the most part this is classical information.

[7]A proof of the mean convergence can be found in **(1)**. A related result concerning the constant surfaces of the Green's function (for fixed pole) rather than the parallel surfaces is proved by Privaloff and Kouznetzoff **(7)**.

https://doi.org/10.4153/CJM-1956-019-5 Published online by Cambridge University Press

use of an inequality between the values of f in D and the mean values of the boundary function over spheres in B.

Mean value inequality. *If the harmonic function $f(P)$ is the Poisson integral of its boundary function $f(x)$, and if $\bar{f}(x)$ denotes the supremum of the averages of $|f(y)|$ over the closed spheres in B centered at x, then for every P within distance ρ_0 of B we have $|f(P)| \leqslant A\,\bar{f}(x_P)$. The constant A depends only on ρ_0 and on the dimension*[8] *n.*

THEOREM 4. *For each f in $\mathfrak{F}_p$ let*

$$\tilde{f}(x) = \sup_{0<\rho<\rho_0} |f[P(\rho, x)]|.$$

For $p > 1$, the following assertion holds: if f belongs to $\mathfrak{F}_p$, then $\tilde{f}$ belongs to L^p on B, and

$$\int_B \tilde{f}(x)^p dx \leqslant K\left(\frac{p}{p-1}\right)^p A^p \int_B |f(x)|^p dx = \lim_{\rho\to 0} K\left(\frac{p}{p-1}\right)^p A^p \int_B |f[P(\rho, x)]|^p dx.$$

Proof. The metric space B and its measure are obviously of the type considered in the second section, so Theorem 4 follows directly from Theorem 1 and the mean-value inequality.

In the case of the circle in the plane this is the theorem of Hardy and Littlewood quoted in the introduction. The related theorem of Rauch **(8; 9)** on analytic functions is as follows:

If D is the sphere, if f is complex valued, and if n is even and the variables can be paired so that f is an analytic function of $n/2$ complex variables, then the assertion of Theorem 4 holds for any exponent $p > 0$.

Rauch's theorem is obtained from the special case of exponent 2 in Theorem 6 below by putting $s(P) = |f(P)|^{\frac{1}{2}p}$.

THEOREM 5. *If f belongs to $\mathfrak{F}_1$, then for each ϵ, $0 < \epsilon < 1$,*

$$\int_B \tilde{f}(x)^{1-\epsilon} dx < \infty.$$

Proof. The function $f \in \mathfrak{F}_1$ has a boundary measure ν in terms of which it can be represented as a Poisson-Stieltjes integral. The mean value inequality is valid here in a suitably modified form; namely, $|f(P)|$ is less than or equal to a constant times the upper bound of the quotients $\nu(C)/|C|$ taken over the closed spheres in B centered at x_P. We do not give more of the proof for it is essentially the same as the proof of Theorem 7 below on subharmonic functions.

Remark. The proof of Theorem 5 is not based on Theorem 3, for f does not necessarily have a boundary function of which it is the Poisson integral. It is plain that if f does have such a boundary function, then certain conclusions can be drawn from Theorems 2 and 3. It does not seem necessary to state the conclusions.

[8]This is a special case of an inequality which is proved in Aronszajn and Smith **(1)**. This special case was obtained for the circle in the plane by Hardy and Littlewood **(4)**.

Subharmonic functions.

THEOREM 6. *Let $s(P)$ be a non-negative subharmonic function in D, and let*

$$\tilde{s}(x) = \sup_{0<\rho<\rho_0} s[P(\rho, x)].$$

For $p > 1$, we have[9]

$$\int_B \tilde{s}(x)^p dx \leqslant K\left(\frac{p}{p-1}\right)^p A^p \sup_{0<\rho<\rho_0} \int_B s[P(\rho, x)]^p dx.$$

Proof. We suppose that the right side is finite. For each ρ, $0 < \rho < \rho_0$, we write B_ρ for the set of points in D at distance ρ from B, and D_ρ for the sub-domain of D bounded by B_ρ. We write f_ρ for the harmonic function in D_ρ with the same boundary values[10] as s. It is well known that the harmonic function f_ρ converge increasingly as $\rho \to 0$ to a harmonic function $f \in \mathfrak{F}_p$ which dominates s **(10)**. In addition the functions $s[P(\rho, x)]$ converge weakly in L^p on B as $\rho \to 0$ to the boundary function $f(x)$ for f. Theorem 6 follows from Theorem 4 and the lower semi-continuity of the norm in L^p with respect to weak convergence.

By using the results of F. Riesz on the representation of subharmonic functions by potentials we can prove similar theorems for subharmonic functions which are not necessarily non-negative. For the sake of simplicity we confine the discussion to the sphere, though the results are equally valid for the more general domains of the last paragraph, as the proofs will show.

The theorem of F. Riesz states that if $s(P)$ is a subharmonic function in the domain D, then a necessary and sufficient condition that $s(P)$ be the sum of a harmonic function and the Green's potential of a negative Borel measure on D is that $s(P)$ be bounded above in D by a harmonic function; the harmonic function which figures in the representation is the smallest harmonic function which bounds $s(P)$ above **(10)**. This function is called the smallest harmonic majorant of $s(P)$. If the positive part of $s(P)$, which we call $s^+(P)$, satisfies the condition

$$\sup_{0<\rho<1} \int_B s^+(\rho\, x)\, dx < \infty,$$

then the smallest harmonic majorant $h(P)$ exists and satisfies

$$\sup_{0<\rho<1} \int_B |h(\rho\, x)|\, dx < \infty.$$

Therefore, as $s(P) = -\int_D G(P, Q)\, d\mu(Q) + h(P)$, where μ is a positive Borel measure on D and $G(P, Q)$ is the Green's function of D; and as $h(P)$ satisfies the hypotheses of Theorem 5, the analogue of Theorem 5 for subharmonic

[9]For the circle in the plane this is a result of Hardy and Littlewood **(4)**.

[10]The surfaces B_ρ are also C^1 and of bounded curvature. The curvature constant ρ_0' for $B_{\rho'}$ is $\rho_0 - \rho'$. f_ρ is defined by the Poisson integral over B_ρ.

functions will result from an analysis of the first term, the Green's potential, alone. Before stating the theorem we observe that the upper bound of $|s(P)|$ along the various radii will be identically infinite whenever the Green's potential is infinite at the origin. Therefore a small sphere with center at 0 must be removed from D before taking the upper bounds.

THEOREM 7. *Let ρ_0 be a fixed number between* 0 *and* 1, *and put*

$$\tilde{s}(x) = \sup_{\rho_0 < \rho < 1} |s(\rho x)|$$

for each $x \in B$. If $s(P)$ is subharmonic in D and if

$$\sup_{0<\rho<1} \int_B s^+(\rho x)\, dx < \infty,$$

then for each ϵ, $0 < \epsilon < 1$,

$$\int_B \tilde{s}(x)^{1-\epsilon} dx < \infty.$$

Proof. As we have mentioned, it results from Theorem 5 and the theorem of F. Riesz that we need only prove the theorem for functions of the type $s(P) = -\int_D G(P, Q)\, d\mu(Q) = -u(P)$, where μ is a positive Borel measure on D and $G(P, Q)$ is the Green's function for D. The explicit expression for the Green's function is well known.

$$G(P, Q) = \frac{1}{\omega_n(n-2)}\left[\frac{1}{|P-Q|^{n-2}} - \frac{r^{n-2}}{|Q|^{n-2}} \frac{1}{|P-Q'|^{n-2}}\right], \tag{3.1}$$

where

$$Q' = \frac{r^2}{|Q|^2} Q,$$

and ω_n is the area of the surface of the unit sphere.

It is known that the Green's potential $u(P) = \int_D G(P, Q)\, d\mu(Q)$ either is identically $+\infty$ or is finite except at a set of points of outer capacity 0. For the latter to be the case it is necessary and sufficient that [11] $\int_D (r - |Q|)\, d\mu(Q) < \infty$. For each $x \in B$ and each real ξ, $0 \leqslant \xi \leqslant 2r$, let $C(x, \xi)$ be the sphere in B with center x and radius ξ; and let $S(x, \xi)$ be the conical sector in D generated by joining each point of $C(x, \xi)$ to the origin. Let

$$I(x, \xi) = \int_{S(x,\xi)} (r - |Q|)\, d\mu(Q),$$

and let

$$m(x) = \sup_{\xi} \frac{I(x, \xi)}{|C(x, \xi)|}.$$

[11] This is clear for the sphere. In the case of more general domains the integrand $r - |Q|$ is replaced by $|x_Q - Q|$, the distance from Q to the boundary. In this form the fact was observed by Privaloff and Kousnetzoff **(7)**.

For the present we assume the following Lemma.

LEMMA 3. *There is a constant A such that $\tilde{u}(x) \leqslant Am(x)$, where $\tilde{u}$ is defined like $\tilde{s}$.*

The covering theorem is used as in the proof of the general Hardy-Littlewood inequality. Let B_t, $t > 0$, denote the set of points x such that $\tilde{u}(x) > t$. If $x \in B_t$, then there is a ξ_x such that

$$\frac{I(x, \xi_x)}{|C(x, \xi_x)|} \geqslant \frac{t}{A}.$$

Choosing such a ξ_x for each $x \in B_t$ we have $B_t \subset \sum C(x, \xi_x)$, so by the covering theorem there is a disjoint sequence

$$C(x_n, \xi_n) \ (\xi_n = \xi_{x_n})$$

such that

$$B_t \subset \sum_{n=1}^{\infty} C(x_n, 4\xi_n).$$

If K is chosen so that for all C, $|C(x, 4\xi) \leqslant K|C(x, \xi)|$, then [12]

$$|B_t| \leqslant \sum_{n=1}^{\infty} |C(x_n, 4\xi_n)| \leqslant K \sum_{n=1}^{\infty} |C(x_n, \xi_n)| \leqslant \frac{KA}{t} \sum_{n=1}^{\infty} I(x_n, \xi_n)$$
$$= \frac{KA}{t} \int_E (r - |Q|)\, d\mu(Q),$$

where E is the sum of the disjoint sets $S(x_n, \xi_n)$. Hence $|B_t| \leqslant k'/t$ for $k' = KA \int_D (r - |Q|)\, d\mu(Q)$. Now

$$\int_B \tilde{u}(x)^{1-\epsilon} dx = \int_0^{\infty} (1 - \epsilon)\, t^{-\epsilon} |B_t|\, dt \leqslant k'' \int_0^1 t^{-\epsilon} dt + k'(1 - \epsilon) \int_1^{\infty} \frac{dt}{t^{1+\epsilon}}$$

where k'' is larger than $(1 - \epsilon)$ times the area of the surface of the sphere.

Proof of the Lemma. We shall not give the entire proof. The calculations, which are routine, are achieved by majorizing the Green's function ((3.2) below) and considering separately the integrals over three different parts of the sphere. The majoration for the Green's function is obtained by inspection of the explicit formula (3.1).[13]

(3.2) *There is a constant k such that*

$$0 \leqslant G(P, Q) \leqslant k(r - |P|)(r - |Q|)/\, |P - Q|^n;$$

also
$$G(P, Q) \leqslant \frac{1}{\omega_n(n - 2)} \frac{1}{|P - Q|^{n-2}}.$$

[12] $|E|$ is used for subsets of D to refer to Lebesgue measure and for subsets of B to refer to the area measure on B.

[13] Essentially the same majoration and division of the sphere are used by Littlewood **(6)** to prove that in the case of the circle in the plane a Green's potential has radial limit 0 at almost every boundary point. The majoration is valid for any domain bounded by a C^1-surface of bounded curvature **(11)** and for even more general domains **(5)**.

The division of the sphere is as follows. Let P be fixed with $|P| \geqslant \rho_0 r$, let $x = rP/|P|$, and let $\xi_0 = r - |P|$. One part of the sphere is the exterior of the conical sector $S(x, \xi_0)$; another is that part of $S(x, \xi_0)$ whose points Q satisfy $r - |Q| > \frac{1}{2}(r - |P|)$; the third is that part of $S(x, \xi_0)$ whose points Q satisfy $r - |Q| \leqslant \frac{1}{2}(r - |P|)$. Finally, it is necessary to use an evaluation of $|P - Q|$ in terms of the variable

$$\xi = \left| \frac{r}{|Q|} Q - \frac{r}{|P|} P \right| .$$

(3.3) *There is a constant k such that if $|P| \geqslant \rho_0 r$, then $|P - Q| \geqslant k\xi$.*

The Lemma results from simple calculation with these estimates and the remark that the quotient $|C(x, \xi)|/\xi^{n-1}$ is bounded above and from 0.

Theorem 7 can be improved if it is known that the measure is the indefinite integral of a density subject to certain conditions.

THEOREM 8. *If $u(P) = \int_D G(P, Q) f(Q)\, dQ$ where $f(Q)$ is such that*

$$\int_D (r - |Q|)^p f(Q)^p dQ < \infty, \qquad p > 1,$$

then $\tilde{u}(x)$ belongs to L^p on B, and there is a constant M such that

$$\int_B \tilde{u}(x)^p dx \leqslant M^p \int_D (r - |Q|)^p f(Q)^p dQ.$$

Proof. The proof is similar to the proof of the last theorem, but it is possible to make use of the non-increasing rearrangements as in the proof of Theorem 1 in order to obtain better evaluations. With the notations of the last theorem we have, as we had there,

$$|B_t| \leqslant K \sum_{n=1}^{\infty} |C(x_n, \xi_n)| = K|C|, \quad C = \sum_{n=1}^{\infty} C(x_n, \xi_n).$$

Because of the disjointness it happens in Theorem 1 that

$$t \leqslant \frac{A}{|C|} \int_E (r - |Q|) f(Q)\, dQ$$

(where again $E = \sum S(x_n, \xi_n)$. From the fact that $|E| = r/n|C|$, it follows that

$$t \leqslant \frac{A}{|C|} \int_0^{|E|} g^*(s)\, ds = \frac{rA}{n|E|} \int_0^{|E|} g^*(s)\, ds = \frac{rA}{n} \beta_g(|E|)$$

for $g(Q) = (r - |Q|) f(Q)$. Hence

$$|B_t| \leqslant K|C| = \frac{Kn}{r} |E| \leqslant \frac{Kn}{r} \beta^g\left(\frac{n}{Ar} t\right).$$

The proof is finished in the same manner as the proof of Theorem 1.

https://doi.org/10.4153/CJM-1956-019-5 Published online by Cambridge University Press

Strong differentiability of double integrals. The general Hardy-Littlewood inequality yields a generalization of the theorem of Jessen, Marcinkiewicz, and Zygmund on the strong differentiability of multiple integrals **(12)**. However, we need the inequality in a slightly stronger form.

Theorems 1, 2, *and* 3 *remain true and their proofs remain correct when* $\bar{f}(x)$ *is redefined to be the supremum of the averages of* $f(y)$ *over all spheres containing* x.

THEOREM 9. *Let* B_1 *and* B_2 *be metric spaces with measures of the kind considered in* §2, *and let* $f(x, y)$ *be a measurable function on* $B_1 \times B_2$. *If*

$$\int_{B_1} \int_{B_2} |f| \log^+ |f| \, dxdy < \infty,$$

then the indefinite integral of f *is almost everywhere derivable in the strong sense. That is, for almost every choice of* (x, y),

$$\lim_n \frac{1}{|S_{1,n}|} \frac{1}{|S_{2,n}|} \int_{S_{1,n}} \int_{S_{2,n}} f(s, t) \, dsdt$$

exists for all sequences $\{S_{1,n}\}$ *and* $\{S_{2,n}\}$ *of closed spheres such that* $x \in S_{1,n}$, $y \in S_{2,n}$, $\delta(S_{1,n}) \to 0$, *and* $\delta(S_{2,n}) \to 0$.

Proof (cf. **12**, pp. 147–149). Several earlier theorems are required (notably, the Vitali covering theorem, the strong density theorem, and the theorem on the strong differentiability of the indefinite integral of a bounded function); these theorems are true in the present case, and the proofs given by Saks are valid after simple modifications.

Remark. It was noticed by Hardy and Littlewood and by Calderón and Zygmund that the Hardy-Littlewood inequality leads to certain results on integral operators. The results are of such a kind as to establish dominated convergence of sequences of transforms. Thus, for example, Hardy and Littlewood show dominated convergence of the Fejer polynomials formed from the Fourier series of a function f; and Calderón and Zygmund show dominated convergence of singular integrals. Our general case of the inequality leads to similar results, which can be used, for example, to give another proof of Theorem 4. However, since we do not have applications which would lead to new results, we shall omit the statement of this theorem on integral operators. In any case it is a re-phrasing in the abstract terms of the theorems of the authors cited.

REFERENCES

1. N. Aronszajn and K. T. Smith, *Functional spaces and functional completion.* To appear shortly in Ann. Inst. Fourier, Grenoble.
2. A. P. Calderón, *A general ergodic theorem.* Ann. Math., *8* (1953), 182–191.
3. A. P. Calderón and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., *88* (1952), 85–139.

4. G. H. Hardy and J. E. Littlewood, *A maximal theorem with function-theoretic applications*, Acta Math., *54* (1930), 81–116.
5. M. Keldych and M. Lavrentieff, *Sur une évaluation pour la fonction de Green*, C.R. Ac. Sci. U.S.S.R., *24* (1939), 22–24.
6. J. E. Littlewood, *On functions subharmonic in a circle*, Lond. Math. Soc., *2* (1927), 192–196.
7. I. I. Privaloff and P. Kouznetzoff, *Sur les problèmes limites et les classes différentes de fonctions harmoniques et subharmoniques définies dans un domaine arbitraire*, Rec. Math. Moscou, *6* (1939), 345–376.
8. H. E. Rauch, *Généralisation d'une proposition de Hardy et de Littlewood et de théorèmes ergodiques qui s'y rattachent*, C.R. Ac. Sci. Paris, *22* (1948), 887–889.
9. ———, *Harmonic and analytic functions of several variables and the maximal theorem of Hardy and Littlewood*, Can. J. Math., *8* (1956), 171–183.
10. F. Riesz, *Sur les fonctions subharmoniques et leur rapport à la théorie du potentiel*, Acta Math., *54* (1930), 321–360.
11. A. Rosenblatt, *Sur la fonction de Green d'un domaine borné de l'espace à trois dimensions*, C.R. Ac. Sci. Paris, *201* (1935), 22–24.
12. S. Saks, *Theory of the Integral* (New York, 1937).
13. C. de la Vallée Poussin, *Propriétés des fonctions harmoniques dans un domaine ouvert limité par des surfaces à courbure bornée*, Ann. Scuola Norm. Sup. Pisa, (2) (1933), 167–197.
14. N. Wiener, *The ergodic theorem*, Duke Math. J., *5* (1939), 1–18.

University of Kansas