

A FIXED-POINT THEOREM FOR COMMUTING MONOTONE FUNCTIONS

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Hamilton (1) proved that a hereditarily unicoherent, hereditarily decomposable metric continuum has the fixed-point property for homeomorphisms. In this paper we shall generalize this result by showing that if X is a hereditarily unicoherent, hereditarily decomposable Hausdorff continuum and S is an abelian semigroup of continuous monotone functions from X into X , then S leaves a point of X fixed.

Let X be a Hausdorff continuum. X is unicoherent if, whenever $X = A \cup B$, where A and B are subcontinua of X , $A \cap B$ is a continuum. If each subcontinuum of X is unicoherent, X is hereditarily unicoherent. X is decomposable if X is the union of two of its proper subcontinua. If each subcontinuum of X which contains more than one point is decomposable, X is hereditarily decomposable. If S is a semigroup of functions from X into X , we say that $x \in X$ is fixed under S if x is a fixed point of each element of S . A function $f: X \rightarrow X$ is monotone if $f^{-1}x$ is connected for every $x \in X$.

Before proceeding to the main result of this paper, we prove two lemmas which appear to be of independent interest.

LEMMA 1. *Let X be a hereditarily unicoherent Hausdorff continuum and let \mathcal{C} be a collection of subcontinua of X such that $A, B \in \mathcal{C}$ implies $A \cap B \neq \emptyset$. Then $H = \bigcap \{A: A \in \mathcal{C}\}$ is a non-empty subcontinuum of X .*

Proof. Since X is hereditarily unicoherent, H is a continuum. We must show that $H \neq \emptyset$. It suffices to show that \mathcal{C} has the finite intersection property. Assume that for some fixed integer $n \geq 1$, if $A_1, \dots, A_n \in \mathcal{C}$, then $A_1 \cap \dots \cap A_n \neq \emptyset$. Let $A_1, \dots, A_{n+1} \in \mathcal{C}$. Let

$$K = (A_1 \cap \dots \cap A_{n-1}) \cup A_{n+1}.$$

Then the continuum $K \cap A_n$ is the union of two closed sets $A_n \cap A_{n+1}$ and $A_1 \cap \dots \cap A_{n-1} \cap A_n$, each of which is non-empty by the inductive hypothesis. Hence, $A_1 \cap \dots \cap A_{n+1} \neq \emptyset$, and the proof is complete.

LEMMA 2. *Let X be a hereditarily unicoherent Hausdorff continuum. For each positive integer n , let*

$$X = A_n \cup B_n,$$

where A_n and B_n are non-empty subcontinua of X . Assume that for each pair of

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positive integers n and m , $A_n \cap A_m \neq \emptyset$ if and only if $A_{n+1} \cap A_{m+1} \neq \emptyset$. Suppose that the sequence $\{B_n\}$ has the same property. Then one of

$$\bigcap \{A_n: n = 1, \dots\}, \quad \bigcap \{B_n: n = 1, \dots\}$$

is a non-empty subcontinuum of X .

Proof. Since none of the A_n or B_n are empty, we may assume without loss of generality that $A_1 \cap A_2 \neq \emptyset$. We prove that one of the following must hold:

- (I) $A_1 \cap A_k \neq \emptyset$ for every k ;
- (II) $B_1 \cap B_k \neq \emptyset$ for every k .

Assume that (I) fails to hold. Let $m > 2$ be the least integer such that $A_1 \cap A_m = \emptyset$. We shall prove that

- (III) $A_1 \cap A_k = \emptyset$ for every $k \geq m$.

Assume that (III) is false. Let n be the least integer greater than m for which $A_1 \cap A_n \neq \emptyset$. Set

$$H = A_2 \cup A_m \cup A_{m+1} \cup \dots \cup A_n.$$

Then the continuum $(A_1 \cap A_2) \cup (A_1 \cap A_n) = A_1 \cap H$ is the union of two non-empty closed sets, and hence $A_2 \cap A_n \neq \emptyset$. Therefore $A_1 \cap A_{n-1} \neq \emptyset$, and this contradicts the choice of n . Hence (III) holds.

Our hypothesis shows that $A_p \cap A_{k+p-1} = \emptyset$ for every $k \geq m$ and $p \geq 1$. Therefore we have the two inclusions

$$\bigcup \{A_k: k \geq m\} \subset B_1, \quad \bigcup \{A_k: k \geq m + p - 1\} \subset B_p$$

for every $p > 1$. Hence, $\bigcup \{A_k: k \geq m + p - 1\} \subset B_1 \cap B_p$ for every p , and (II) holds.

The above proof shows that either (I) or (II) must hold. We now assume without loss of generality that (I) holds. If n and m are arbitrary positive integers, then $A_n \cap A_m \neq \emptyset$. By Lemma 1, the proof is complete.

THEOREM. *Let X be a hereditarily unicoherent and hereditarily decomposable Hausdorff continuum and let S be an abelian semigroup of continuous monotone functions from X into X . Then S has a fixed point.*

Proof. In the proof we shall make use of the fact that if $f: X \rightarrow X$ is continuous monotone, then $f^{-1}C$ is a continuum for each subcontinuum C of X . Further, since X is hereditarily unicoherent, the intersection over an arbitrary collection of subcontinua of X is a subcontinuum of X . From the above observations, if C is a subcontinuum of X , $f|C$ is continuous monotone.

Let $f \in S$. We shall first show that f leaves some point of X fixed. Let \mathcal{C} be the class of all non-empty subcontinua of X which satisfies $fA \subset A$ for each $A \in \mathcal{C}$. By the minimal principle (2, p. 33), \mathcal{C} contains a minimal element K . Then $fK = K$, and to prove that f has a fixed point we may assume (in this paragraph only) that $K = X$. If X is not a point, there are proper

subcontinua A and B of X such that $X = A \cup B$. Then $X = f^{-n}A \cup f^{-n}B$ for each positive integer n . Hence, the two sequences

$$A, f^{-1}A, f^{-2}A, \dots \quad \text{and} \quad B, f^{-1}B, f^{-2}B, \dots$$

satisfy the hypothesis of Lemma 2. Hence, one of

$$H_1 = \bigcap \{f^{-n}A; n \geq 1\} \cap A, \quad H_2 = \bigcap \{f^{-n}B; n \geq 1\} \cap B$$

is a proper non-empty subcontinuum of X . Since $fH_1 \subset H_1$ and $fH_2 \subset H_2$, this is a contradiction, and X is a point. Thus f has a fixed point.

Now let X be an arbitrary space which satisfies the hypothesis of the theorem. We shall show that S leaves some point of X fixed. For this purpose we observe that by the minimal principle, there is a non-empty subcontinuum K of X such that $fK \subset K$ for every $f \in S$, and K is minimal with respect to this property. It suffices to assume that $K = X$. If X is not a point, then one of the following must hold: if $X = A \cup B$, where A and B are proper non-empty subcontinua of X , then

(A) There is an $f \in S$ such that the fixed point set F of f is a subset of either A or B ;

(B) If $f \in S$, both $A - B$ and $B - A$ contain a fixed point of f .

Assume that (A) holds. We may assume that $F \subset A$. Since S is abelian, if $g, h \in S$, $gF \subset F \subset A$ and $hF \subset F \subset A$, so that $g^{-1}A \cap h^{-1}A \neq \emptyset$ and $g^{-1}A \cap A \neq \emptyset$. By Lemma 1, if

$$D = \bigcap \{g^{-1}A; g \in S\} \cap A,$$

then D is a proper non-empty subcontinuum of X such that $gD \subset D$ for every $g \in S$. This is a contradiction, and case (B) must hold.

Assume that (B) holds. Let $f, g \in S$. Let

$$H = (A \cap f^{-1}B) \cup (B \cap f^{-1}A) \cup (B \cap f^{-1}B).$$

Since B contains a fixed point of f , $B \cap f^{-1}B \neq \emptyset$. Since

$$(A \cap f^{-1}B) \cup (B \cap f^{-1}B) \quad \text{and} \quad (B \cap f^{-1}A) \cup (B \cap f^{-1}B)$$

are continua, so is H . If $y_1 \in B - A$ and $y_2 \in A - B$ are fixed points of f , then $y_1 \notin A \cap f^{-1}A$ and $y_2 \notin H$. Hence, $X = (A \cap f^{-1}A) \cup H$ is a decomposition of X into the union of two proper non-empty subcontinua of X , and therefore $A \cap f^{-1}A$ contains a fixed x of g ; then $x \in f^{-1}A \cap g^{-1}A$. Clearly, $A \cap f^{-1}A \neq \emptyset$ for every $f \in S$. As before, this is a contradiction. The proof is complete.

REFERENCES

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