J. Aust. Math. Soc. **78** (2005), 323–338

WEYL QUANTIZATION AND A SYMBOL CALCULUS FOR ABELIAN GROUPS

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(Received 8 February 2003; revised 20 October 2003)

Communicated by A. H. Dooley

Abstract

We develop a notion of a $*$-product on a general abelian group, establish a Weyl calculus for operators on the group and connect these with the representation theory of an associated Heisenberg group. This can all be viewed as a generalization of the familiar theory for $\mathbb{R}$. A symplectic group is introduced and a connection with the classical Cayley transform is established. Our main application is to finite groups, where consideration of the symbol calculus for the cyclic groups provides an interesting alternative to the usual matrix form for linear transformations. This leads to a new basis for $\mathrm{sl}(n)$ and a decomposition of this Lie algebra into a sum of Cartan subalgebras.

2000 *Mathematics subject classification*: primary 44A45 17B05 11F27.

1. Introduction

In this paper we show that a certain circle of ideas related to harmonic analysis on $\mathbb{R}$ has a natural generalization to any locally compact abelian group satisfying a certain parity requirement. The topics with which we are concerned are:

(1) the Weyl calculus for pseudo-differential operators on $\mathbb{R}$ (see for example Hormander [5]);

(2) the Moyal product, or $*$-product, for functions on $T^*\mathbb{R}$ (see Bayen *et al.* [2], Moyal [9]); and

(3) the convolution structure on the Heisenberg group.

The close connection between these subjects has been described in Howe [6, 7], Grossman, Loupias and Stein [4], Folland [3] and Lion and Vergne [8]. We wish here

to point out that these three theories can be built around any locally compact abelian group subject to a certain parity condition. This parity condition is satisfied by $\mathbb{R}$ but not by the circle group, which is an explanation for why there is no good symbol calculus for the latter. More generally, the crucial condition is that multiplication by 2 ought to be an automorphism of the group. For a finite abelian group, this simply requires that the order of the group be odd. The significance of this factor 2 was already mentioned in Howe [6, 7]. For quite another approach using Kac algebras, see Aldrovandi and Saeger [1].

To avoid technicalities, we restrict our attention to the case of G a finite abelian group; since our formulae hold in general however, we have used integrals instead of sums throughout.

We begin by considering the group $\widetilde{G} = G \times \widehat{G}$ (following Weil [10]) and its unique Haar measure. We show there is a canonical algebra structure on the space of functions on $\widetilde{G}$ which is given by an integral formula similar to one which may be used to define the Moyal product on $T^*\mathbb{R}$.

Furthermore, if $|G| = n$ is odd (which we henceforth assume) then there is a canonical identification of such functions as symbols of operators on $C(G)$ so, that this algebra structure corresponds to operator multiplication.

This shows that abstractly $C(\widetilde{G})$ is isomorphic to $M(n, \mathbb{C})$. Subgroups of $\widetilde{G}$ are shown to naturally lead to subalgebras of $M(n, \mathbb{C})$.

The above algebra structure is related to the representation theory of an associated Heisenberg group which we define. We study the associated oscillator representation of the symplectic group which arises as 'quadratic' symmetries of $C(\widetilde{G})$ and show how the familiar Cayley transform arises in a natural way when we derive formulae for this representation. We are here combining well known ideas from the special case $G = \mathbb{R}$ (as found, say, in Howe [6] or Lion and Vergne [8]) with the general approach of Weil [10].

It is perhaps useful to point out the direction of the applications we have in mind. The idea of representing a linear transformation of $\mathbb{R}^n$ by a matrix with respect to an ordered basis permeates much of modern mathematics. This is simply one convenient way of encoding n^2 bits of information and corresponds to viewing the matrix as a function on $\mathbb{Z}_n \times \mathbb{Z}_n$. However in infinite dimensional functional analysis it is occasionally useful to encode an operator (say, on $L^2(\mathbb{R})$) by its symbol. In the finite dimensional case, this corresponds to representing a linear transformation by a function on $\mathbb{Z}_n \times \hat{\mathbb{Z}}_n$. This is an entirely equivalent formulation—strictly speaking no information is either lost or gained. There will be situations in which such a symbol representation of an operator poses distinct advantages to the matrix representation.

As an application therefore we consider the cyclic group and give an associated symbol calculus for n by n matrices, culminating in a new decomposition of the Lie algebra $\mathrm{sl}(n)$ into a sum of Cartan subalgebras.

2. Notation and preliminaries

For any finite set S, denote the space of complex-valued functions on S by $C(S)$. If S has measure ds, introduce the inner product $(\cdot\,,\cdot)$ on $C(S)$

$$(f_1, f_2) = \int_S f_1(s)\overline{f_2(s)}\,ds, \quad \forall f_1, f_2 \in C(G),$$

and let $\delta_s \in C(S)$ be the function such that

$$\int_S \delta_s(s')f(s')\,ds' = f(s), \quad \forall f \in C(S).$$

If S_1, S_2 are sets and $\sigma : S_1 \to S_2$ a map, let $a^\sigma = a \circ \sigma$ for all $a \in C(S_2)$. We let $\mathrm{Perm}(S)$ denote the group of permutations of the set S.

Now let G be a finite abelian group with $|G| = n$ and with typical element x, operation addition and identity 0. Let $\widehat{G}$ be the dual group with typical element Y and denote the pairing between $x \in G$ and $Y \in \widehat{G}$ by $\langle Y, x\rangle$. Let dx denote a Haar measure on G (unique up to a constant).

The Fourier transform $\hat{}\,: C(G) \to C(\widehat{G})$ is given by

$$f\hat{}\,(Y) = \int_G \overline{\langle Y, x\rangle} f(x)\,dx, \quad \forall f \in C(G).$$

Normalize Haar measure dY on $\widehat{G}$ so that the inverse $\check{}$ of the Fourier transform is given by

$$h\check{}\,(x) = \int_{\widehat{G}} \langle Y, x\rangle h(Y)\,dY, \quad \forall h \in C(\widehat{G}).$$

Then both the Fourier transform and its inverse are unitary operators. For $x \in G$, let $\hat{\hat{x}}$ denote the element of $C(\widehat{G})$ given by $\hat{\hat{x}}(Y) = \langle Y, x\rangle$. The map $x \to \hat{\hat{x}}$ provides us with a natural identification of G and $\widehat{\widehat{G}}$. For a general locally compact abelian group, there is of course no isomorphism between G and $\widehat{G}$ but when G is finite there is, although it is not unique. This follows from the basic structure theorem that states that any finite abelian group is isomorphic to a direct product of cyclic groups along with the obvious fact that the dual of the cyclic group $\mathbb{Z}_n$ is itself isomorphic to $\mathbb{Z}_n$.

Let $\widetilde{G} = G \times \widehat{G}$ with typical element $z = (x, Y)$ and Haar measure $dz = dx\,dY$. Note that this measure is canonical; it is independent of the original choice of dx. In fact, $\int_{\widetilde{G}} dz = |G| = n$. Furthermore we find that with these normalizations $\delta_{z_0}(z)$ is the function of z that is zero unless $z = z_0$, in which case it is $|G| = n$. From this also follows the useful formula

$$\int_{\widetilde{G}} \langle z, z_1\rangle\,dz = \delta_0(z_1)$$

https://doi.org/10.1017/S1446788700008569 Published online by Cambridge University Press

which we will use throughout. $\widetilde{G}$ also has the canonical function $e(x, Y) = \langle Y, x\rangle$. Now $(\widetilde{G})\widehat{\ } \simeq \widehat{G} \times G$ with typical element $Z = (Y, x)$ and Haar measure $dYdx$. Define $\tau : (\widetilde{G})\widehat{\ } \to \widetilde{G}$ by $\tau(Y, x) = (x, -Y)$. Then τ is a canonical isomorphism from $(\widetilde{G})\widehat{\ }$ to $\widetilde{G}$.

For $z_i = (x_i, Y_i) \in \widetilde{G}$, $i = 1, 2$, define $\langle z_1, z_2\rangle = \langle Y_1, x_2\rangle\overline{\langle Y_2, x_1\rangle}$. For $k > 2$ and $z_i = (x_i, Y_i) \in \widetilde{G}$, $i = 1, \ldots, k$, define $\langle z_1, \ldots, z_k\rangle = \langle z_1, z_2\rangle \cdots \langle z_{k-1}, z_k\rangle\langle z_k, z_1\rangle$. It is evident that for all $k \geq 2$, $\langle z_1, z_2, \ldots, z_k\rangle = \overline{\langle z_k, \ldots, z_2, z_1\rangle}$ and for $k > 2$ the expression $\langle z_1, \ldots, z_k\rangle$ is invariant under cyclic permutations of the z_i's.

For $z \in \widetilde{G}$, let $\hat{z} \in C(\widetilde{G})$ be defined by $\hat{z}(z') = \langle z, z'\rangle$ for all $z' \in \widetilde{G}$.

3. The $*$-product

We will be interested in a particular algebra structure canonically defined on $C(\widetilde{G})$. For $a_1, a_2 \in C(\widetilde{G})$, define $a_1 * a_2 \in C(\widetilde{G})$ by

$$a_1 * a_2(z_3) = \int_{\widetilde{G}}\int_{\widetilde{G}} \langle z_1, z_2, z_3\rangle a_1(z_1)a_2(z_2)\, dz_1\, dz_2, \quad \forall z_3 \in \widetilde{G}.$$

THEOREM 3.1. *$(C(\widetilde{G}), *)$ is an associative algebra with identity the constant function* 1.

PROOF. Let $a_1, a_2, a_3 \in C(\widetilde{G})$. Then for all $z_4 \in \widetilde{G}$,

$$(a_1 * a_2) * a_3(z_4) = \int_{\widetilde{G}} \cdots \int_{\widetilde{G}} \langle z_1, z_2, z\rangle\langle z, z_3, z_4\rangle a_1(z_1)a_2(z_2)a_3(z_3)\, dz_1\, dz_2\, dz_3\, dz,$$

while

$$a_1 * (a_2 * a_3)(z_4) = \int_{\widetilde{G}} \cdots \int_{\widetilde{G}} \langle z_1, z, z_4\rangle\langle z_2, z_3, z\rangle a_1(z_1)a_2(z_2)a_3(z_3)\, dz_1\, dz_2\, dz_3\, dz.$$

Thus associativity of $*$ is equivalent to the validity for all $z_1, z_2, z_3, z_4 \in \widetilde{G}$ of the equation

$$\int_{\widetilde{G}} \langle z_1, z_2, z\rangle\langle z, z_3, z_4\rangle\, dz = \int_{\widetilde{G}} \langle z_1, z, z_4\rangle\langle z_2, z_3, z\rangle\, dz.$$

Now

$$\begin{aligned}\int_{\widetilde{G}} \langle z_1, z_2, z\rangle\langle z, z_3, z_4\rangle\, dz &= \int_{\widetilde{G}} \langle z_1, z_2\rangle\langle z_3, z_4\rangle\langle z, z_1 - z_2 + z_3 - z_4\rangle\, dz \\ &= \delta_{z_4}(z_1 - z_2 + z_3)\langle z_1, z_2, z_3\rangle \\ &= \int_{\widetilde{G}} \langle z_4, z_1\rangle\langle z_2, z_3\rangle\langle z, z_1 - z_2 + z_3 - z_4\rangle\, dz \\ &= \int_{\widetilde{G}} \langle z_1, z, z_4\rangle\langle z_2, z_3, z\rangle\, dz\end{aligned}$$

https://doi.org/10.1017/S1446788700008569 Published online by Cambridge University Press

which proves associativity of $*$.

For $a \in C(\widetilde{G})$ and $z_3 \in \widetilde{G}$,

$$\begin{aligned} a * 1(z_3) &= \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_1, z_2, z_3 \rangle a(z_1)\, dz_1\, dz_2 \\ &= \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_2, z_3 - z_1 \rangle \langle z_3, z_1 \rangle a(z_1)\, dz_1\, dz_2 \\ &= \int_{\widetilde{G}} \delta_0(z_3 - z_1) \langle z_3, z_1 \rangle a(z_1)\, dz_1 = a(z_3). \end{aligned}$$

Similarly $1 * a = a$ for all $a \in C(\widetilde{G})$. □

The computation in the above theorem also shows the following.

PROPOSITION 3.2. *For $z_1, z_2, z_3 \in \widetilde{G}$, $\delta_{z_1} * \delta_{z_2} * \delta_{z_3} = \langle z_1, z_2, z_3 \rangle \delta_{z_4}$, where $z_4 = z_1 - z_2 + z_3$.*

LEMMA 3.3. *For $a_1, a_2 \in C(\widetilde{G})$, $a_2 * a_1 = \overline{\bar{a}_1 * \bar{a}_2}$.*

PROOF. For $z_3 \in \widetilde{G}$,

$$\begin{aligned} a_2 * a_1(z_3) &= \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_1, z_2, z_3 \rangle a_2(z_1) a_1(z_2)\, dz_1\, dz_2 \\ &= \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_2, z_1, z_3 \rangle a_1(z_1) a_2(z_2)\, dz_1\, dz_2 \\ &= \int_{\widetilde{G}} \int_{\widetilde{G}} \overline{\langle z_1, z_2, z_3 \rangle} a_1(z_1) a_2(z_2)\, dz_1\, dz_2 = \overline{\bar{a}_1 * \bar{a}_2}(z_3). \end{aligned}$$ □

LEMMA 3.4. *If $a_i \in C(\widetilde{G})$ and $a_i(x, Y) = a_i(x)$, $i = 1, 2$ then $a_1 * a_2 = a_1 a_2$. The same holds if $a_i(x, Y) = a_i(Y)$, $i = 1, 2$.*

PROOF. Suppose that $a_i(x, Y) = a_i(x)$. Then for $z_3 = (x_3, Y_3) \in \widetilde{G}$,

$$\begin{aligned} a_1 * a_2(z_3) &= \int_G \int_{\widehat{G}} \int_G \int_{\widehat{G}} \langle Y_1, x_2 - x_3 \rangle \langle Y_2, x_3 - x_1 \rangle \langle Y_3, x_1 - x_2 \rangle \\ &\quad \times a_1(x_1) a_2(x_2)\, dY_1\, dx_1\, dY_2\, dx_2 \\ &= \int_G \int_G \delta_0(x_2 - x_3) \delta_0(x_3 - x_1) \langle Y_3, x_1 - x_2 \rangle a_1(x_1) a_2(x_2) dx_1\, dx_2 \\ &= a_1(x_3) a_2(x_3). \end{aligned}$$ □

PROPOSITION 3.5. *For $z_1, z_2 \in \widetilde{G}$, $\hat{z}_1 * \hat{z}_2 = \langle z_2, z_1 \rangle (z_1 + z_2)\hat{}$.*

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PROOF.

$$\begin{aligned}\hat{z}_1 * \hat{z}_2(z_3) &= \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_1', z_2', z_3\rangle \langle z_1, z_1'\rangle \langle z_2, z_2'\rangle \, dz_1' \, dz_2' \\ &= \int_{\widetilde{G}} \delta_0(z_2' - z_3 - z_1) \langle z_2', z_3\rangle \langle z_2, z_2'\rangle \, dz_2' \\ &= \langle z_1, z_3\rangle \langle z_2, z_1 + z_3\rangle = \langle z_2, z_1\rangle (z_1 + z_2)\hat{}(z_3). \end{aligned}$$ □

COROLLARY 3.6. *For $z \in \widetilde{G}$ and $a \in C(\widetilde{G})$,*

(1) $\hat{z} * a(z') = \hat{z}(z') a(z' + z)$;
(2) $a * \hat{z}(z') = \hat{z}(z') a(z' - z)$ *for all* $z' \in \widetilde{G}$.

PROOF. Proposition 3.5 shows the formulae hold when $a = \hat{z}_0$ for $z_0 \in \widetilde{G}$ and thus generally since $\{\hat{z}_0 | z_0 \in \widetilde{G}\}$ spans $C(\widetilde{G})$. □

Proposition 3.5 provides us with a ready supply of subalgebras of $(C(\widetilde{G}), *)$, for if H is any subgroup of $\widetilde{G}$ it follows that the span of $\{\hat{z} \mid z \in H\}$ is a subalgebra of $C(\widetilde{G})$.

PROPOSITION 3.7. *For $a_1, a_2 \in C(\widetilde{G})$, $(a_1, a_2) = \int_{\widetilde{G}} a_1 * \bar{a}_2(z) \, dz$.*

PROOF.

$$\begin{aligned}\int_{\widetilde{G}} a_1 * \bar{a}_2(z) \, dz &= \int_{\widetilde{G}} \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_1, z_2\rangle \langle z_2, z\rangle \langle z, z_1\rangle a_1(z_1) \overline{a_2(z_2)} \, dz_1 \, dz_2 \, dz \\ &= \int_{\widetilde{G}} \int_{\widetilde{G}} \delta_0(z_1 - z_2) \langle z_1, z_2\rangle a_1(z_1) \overline{a_2(z_2)} \, dz_1 \, dz_2 \\ &= \int_{\widetilde{G}} a_1(z_1) \overline{a_2(z_1)} \, dz_1 = (a_1, a_2). \end{aligned}$$ □

From Proposition 3.7 and Lemma 3.3 one easily obtains the following.

PROPOSITION 3.8. *For $a_1, a_2, a_3 \in C(\widetilde{G})$,*

(1) $\int_{\widetilde{G}} a_1 * a_2(z) \, dz = \int_{\widetilde{G}} a_2 * a_1(z) \, dz$;
(2) $(a_1, a_2 * a_3) = (a_1 * \bar{a}_3, a_2)$;
(3) $(a_1 * a_2, a_3) = (a_2, \bar{a}_1 * a_3)$.

There are a number of other algebra structures on $C(\widetilde{G})$ that can be derived from the $*$-product. Suppose for example that σ is a permutation of $\widetilde{G}$. For $a_1, a_2 \in C(\widetilde{G})$ define $a_1 *_\sigma a_2 \in C(\widetilde{G})$ by $a_1 *_\sigma a_2 = (a_1^\sigma * a_2^\sigma)^{\sigma^{-1}}$. Then for all $z_3 \in \widetilde{G}$,

$$a_1 *_\sigma a_2(z_3) = \int_{\widetilde{G}} \int_{\widetilde{G}} \langle \sigma(z_1), \sigma(z_2), \sigma(z_3)\rangle a_1(z_1) a_2(z_2) \, dz_1 \, dz_2,$$

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and the algebra structure $*_\sigma$ is isomorphic to $*$. In particular if $k \in \mathbb{Z}$ and k and $|G| = n$ are relatively prime, then the map $\sigma\big((x, Y)\big) = (kx, Y)$ is a permutation of $\widetilde{G}$ and we have

$$a_1 *_\sigma a_2(z_3) = \int_{\widetilde{G}} \int_{\widetilde{G}} \langle z_1, z_2, z_3\rangle^k a_1(z_1) a_2(z_2)\, dz_1\, dz_2.$$

In this case we will write $*_\sigma$ as $*_k$. Then k plays the role that Planck's constant $\hbar$ plays in the theory of the $*$-product for $\mathbb{R}$. It is easily seen that the constant function 1 is the identity for all $*_k$ and that for $z_1, z_2 \in \widetilde{G}$, Proposition 3.5 becomes

$$\hat{z}_1 *_k \hat{z}_2 = \langle z_2, z_1/k\rangle (z_1 + z_2)\hat{} = \langle z_2/k, z_1\rangle (z_1 + z_2)\hat{},$$

where $z \to z/k$ is the inverse map to $z \to kz$. The case $k = 2$ will be of special importance for what follows.

4. The symbol calculus

Define a locally compact abelian group G to be *odd* if the map $x \to 2x$ is an automorphism of G. Then for example $\mathbb{R}$ is odd while the unit circle S^1 is not. A finite abelian group G is odd if and only if $|G| = n$ is odd. Throughout this section we will assume that the finite abelian group G is odd, so that $x/2$ is well-defined for all $x \in G$.

For $a \in C(\widetilde{G})$, define $W^a : C(G) \to C(G)$ by

$$W^a f(x) = \int_{\widehat{G}} \int_G \langle Y', x' - x\rangle a\big((x + x')/2, Y'\big) f(x')\, dx'\, dY' \tag{4.1}$$

for all $f \in C(G)$ and $x \in G$. This is essentially the same formula as used in $\mathbb{R}$ to define an operator from a 'Weyl' symbol. The following facts are immediate.

LEMMA 4.1. (1) *If $a(x, Y) = a(x)$, then W^a is multiplication by $a(x)$.*
(2) *If $a(x, Y) = a(Y)$ then W^a is multiplication by $a(Y)$ on the Fourier transform side.*
(3) *If $a = 1$ then W^a is the identity.*

PROPOSITION 4.2. *For $a \in C(\widetilde{G})$ the adjoint of W^a is $W^{\bar{a}}$.*

PROOF. Let $a \in C(\widetilde{G})$ and $f_1, f_2 \in C(G)$. Then

$$\begin{aligned}(W^a f_1, f_2) &= \int_G \int_{\widehat{G}} \int_G \langle Y', x' - x\rangle a\big((x + x')/2, Y'\big) f_1(x') \overline{f_2(x)}\, dx'\, dY'\, dx \\ &= \int_G \int_{\widehat{G}} \int_G f_1(x') \overline{\langle Y', x - x'\rangle \bar{a}\big((x + x')/2, Y'\big) f_2(x)}\, dx\, dY'\, dx' \\ &= (f_1, W^{\bar{a}} f_2).\end{aligned}$$

□

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If $a \in C(\widetilde{G})$, then $a^{\tau} \in C(\widehat{G} \times G)$ so that reversing the roles of G and $\widehat{G}$ in (4.1) we may regard $W^{a^{\tau}}$ as an operator from $C(\widehat{G})$ to $C(G)$.

PROPOSITION 4.3. *For $a \in C(\widetilde{G})$ let $b = a^{\tau^{-1}} \in C((\widetilde{G})\widehat{\ })$. Then for all $f \in C(G)$, $(W^a f)\widehat{\ } = W^b f\widehat{\ }$.*

PROOF.

$$(W^a f)\widehat{\ }(Y) = \int_G \int_{\widehat{G}} \int_G \overline{\langle Y, x\rangle} \langle Y', x' - x\rangle a\big((x + x')/2, Y'\big) f(x')\, dx'\, dY'\, dx$$
$$= \int_G \int_{\widehat{G}} \overline{\langle Y, x\rangle}\ \overline{\langle Y', x\rangle} a(x/2, Y') f\widehat{\ }(-2Y' - Y)\, dY'\, dx.$$

Replace Y' with $(Y' - Y)/2$ to get

$$(W^a f)\widehat{\ }(Y) = \int_G \int_{\widehat{G}} \overline{\langle (Y' + Y)/2, x\rangle} a(x/2, (Y' - Y)/2) f\widehat{\ }(Y')\, dY'\, dx.$$

Finally replace x with $2x$ and Y' with $-Y'$ to get

$$(W^a f)\widehat{\ }(Y) = \int_G \int_{\widehat{G}} \langle Y' - Y, x\rangle a\big(x, -(Y' + Y)/2\big) f\widehat{\ }(Y')\, dY'\, dx$$
$$= W^b(f\widehat{\ })(Y). \qquad \square$$

PROPOSITION 4.4. *For $z = (x_0, Y_0) \in \widetilde{G}$ and $f \in C(G)$,*

$$W^{\hat{z}} f(x) = \langle Y_0, x_0/2\rangle \langle Y_0, x\rangle f(x + x_0) \quad \forall x \in G.$$

PROOF.

$$W^{\hat{z}} f(x) = \int_{\widehat{G}} \int_G \langle Y', x' - x\rangle \langle Y_0, (x + x')/2\rangle \overline{\langle Y', x_0\rangle} f(x')\, dx'\, dY'$$
$$= \int_G \delta_0(x' - x - x_0) \langle Y_0, (x + x')/2\rangle f(x')\, dx'$$
$$= \langle Y_0, x\rangle \langle Y_0, x_0/2\rangle f(x + x_0). \qquad \square$$

We will be interested in recovering $a \in C(\widetilde{G})$ from a knowledge of W^a. For $Y \in \widehat{G}$ and $x \in G$,

$$W^a Y(x) = \int_{\widehat{G}} \int_G \langle Y', x' - x\rangle a\big((x + x')/2, Y'\big) \langle Y, x'\rangle\, dx'\, dY'$$
$$= \int_{\widehat{G}} \int_G \langle Y', x' - 2x\rangle a\big(x'/2, Y'\big) \langle Y, x' - x\rangle\, dx'\, dY'$$
$$= \overline{\langle Y, x\rangle} \int_{\widehat{G}} \int_G \langle Y', 2x' - 2x\rangle \langle Y, 2x'\rangle a(x', Y')\, dx'\, dY'.$$

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Then if we define $c_a \in C(\widetilde{G})$ by $c_a(z) = c_a(x, Y) = W^a Y(x)$ then

$$c_a(z) = \overline{e(z)} \int_{\widetilde{G}} \langle z', -2z\rangle (e^2 a)(z')\, dz'.$$

Taking the Fourier transform gives

$$e^2 a(z') = \int_{\widetilde{G}} \langle 2z', z\rangle (ec_a)(z)\, dz.$$

We have proven the following.

PROPOSITION 4.5. *Let $T : C(G) \to C(G)$ be any linear operator. Define $d_T \in C(\widetilde{G})$ by $d_T(x, Y) = TY(x)$. Then $T = W^a$, where*

$$a(z) = \tilde{e}^2(z) \int_{\widetilde{G}} \langle 2z, z'\rangle e(z')\, d_T(z')\, dz'.$$

COROLLARY 4.6. *The map $a \to W^a$ is a linear isomorphism from $C(\widetilde{G})$ to* End$(C(G))$.

We now relate the $*$-product to the algebra structure on $C(\widetilde{G})$ arising from its identification with End $(C(G))$.

THEOREM 4.7. *For $a, b \in C(\widetilde{G})$, suppose $W^a W^b = W^c$, $c \in C(\widetilde{G})$. Then $c = a *_2 b$.*

PROOF. For $(x, Y) \in \widetilde{G}$,

$$W^a W^b Y(x) = \int_{\widehat{G}} \int_G \int_{\widehat{G}} \int_G \langle Y', x' - x\rangle a\big((x + x')/2, Y'\big) \langle Y'', x'' - x'\rangle \times b\big((x' + x'')/2, Y''\big) \langle Y, x''\rangle\, dx''\, dY''\, dx'\, dY'.$$

Then from Proposition 4.5, for $(x_1, Y_1) \in \widetilde{G}$,

$$\langle Y_1, x_1\rangle^2 c(x_1, Y_1) = \int_{\widehat{G}} \cdots \int_G \langle 2Y_1, x\rangle \overline{\langle 2Y, x_1\rangle} \langle Y, x\rangle \langle Y', x' - x\rangle \langle Y'', x'' - x'\rangle \langle Y, x''\rangle \times a\big((x + x')/2, Y'\big) b\big((x' + x'')/2, Y''\big)\, dx''\, dY''\, dx'\, dY'\, dx\, dY.$$

Replace x'' with $x'' - x'$ to get

$$\langle Y_1, x_1\rangle^2 c(x_1, Y_1) = \int_{\widehat{G}} \cdots \int_G \langle 2Y_1, x\rangle \overline{\langle 2Y, x_1\rangle} \langle Y, x\rangle \langle Y', x' - x\rangle \langle Y'', x'' - 2x'\rangle \langle Y, x'' - x'\rangle \times a\big((x + x')/2, Y'\big) b\big(x''/2, Y''\big)\, dx''\, dY''\, dx'\, dY'\, dx\, dY.$$

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Replace x' with $x' - x$ to get

$$\begin{aligned}
&\langle Y_1, x_1\rangle^2 c(x_1, Y_1) \\
&\quad = \int_{\widehat{G}} \cdots \int_G \langle 2Y_1, x\rangle \overline{\langle 2Y, x_1\rangle} \langle Y, x\rangle \langle Y', x' - 2x\rangle \langle Y'', x'' - 2x' + 2x\rangle \\
&\qquad \times \langle Y, x'' - x' + x\rangle a\big(x'/2, Y'\big) b\big(x''/2, Y''\big)\, dx''\, dY''\, dx'\, dY'\, dx\, dY \\
&\quad = \int_{\widehat{G}} \cdots \int_G \langle 2Y_1, x\rangle \overline{\langle 2Y, x_1\rangle} \langle Y, x\rangle \langle Y', 2x' - 2x\rangle \langle Y'', 2x'' - 4x' + 2x\rangle \\
&\qquad \times \langle Y, 2x'' - 2x' + x\rangle a(x', Y') b(x'', Y'')\, dx\, dY\, dx'\, dY'\, dx''\, dY''.
\end{aligned}$$

After integrating with respect to Y and x and cancelling terms, we obtain

$$\begin{aligned}
c(x_1, Y_1) = \int_{\widehat{G}} \int_G \int_{\widehat{G}} \int_G &\langle 2Y_1, x' - x''\rangle \langle 2Y', x'' - x_1\rangle \langle 2Y'', x_1 - x'\rangle \\
&\times a(x', Y') b(x'', Y'')\, dx'\, dY'\, dx''\, dY'' = a *_2 b(x_1, Y_1). \qquad \square
\end{aligned}$$

COROLLARY 4.8. *If G is an odd finite abelian group, then $(C(\widetilde{G}), *)$ is isomorphic to the algebra of $n \times n$ matrices.*

PROOF. This follows from Theorem 4.7 and the observation in Section 3 that if G is odd, then $(C(\widetilde{G}), *)$ and $(C(\widetilde{G}), *_2)$ are isomorphic. $\square$

For a general abelian group G, $(C(\widetilde{G}), *)$ is isomorphic to a direct sum of matrix algebras (this actually follows from the results of Section 3). However if G is not odd there will generally be more than one component in this sum. For example if $G = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ then one may check directly that $(C(\widetilde{G}), *)$ is a direct sum of 1-dimensional algebras, and therefore abelian.

5. The Heisenberg group

Throughout this section G will be an odd finite abelian group. Let S_G be the subset of the unit circle $S^1 \subseteq \mathbb{C}$ given by $S_G = \{\langle Y, x\rangle \mid Y \in \widehat{G}, x \in G\}$.

LEMMA 5.1. *S_G is a subgroup of S^1.*

PROOF. Suppose that $G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$. Then S_G consists of all complex numbers of the form $\gamma_1 \cdots \gamma_l$ with γ_i an n_ith root of unity. This is clearly a subgroup of S^1. $\square$

Any finite subgroup of S^1 is cyclic. Let $|S_G| = m$, so that S_G consists of all mth roots of unity. If $G \simeq \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_l}$ then m is the least common multiple of the n_i's, $i = 1, \ldots, l$. It follows that for any $k \in \mathbb{Z}$, $(k, m) = 1$ if and only if $(k, n) = 1$.

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Let $H_G = \widetilde{G} \times S_G$ and introduce a multiplication in H_G by

$$(z_1, \gamma_1) \cdot (z_2, \gamma_2) = (z_1 + z_2, \gamma_1\gamma_2\langle z_2/2, z_1\rangle).$$

Then H_G is a finite nilpotent group with identity $(0, 1)$ and centre

$$Z_G = \{(0, \gamma) \mid \gamma \in S_G\} \simeq S_G.$$

We call H_G the Heisenberg group of G.

Suppose that ρ is any irreducible unitary representation of H_G. Then Z_G acts as scalars so there is a character χ of Z_G such that $\rho(\gamma) = \chi(\gamma)I$ for all $\gamma \in Z_G$. Since any character χ of $Z_G \simeq S_G$ is given by $\chi(\gamma) = \gamma^k$ for some $k \in \mathbb{Z}$, $0 \leq k < m$, we may separate the unitary dual $(H_G)\hat{}$ of H_G into m components

$$(H_G)\hat{} = \bigcup_{k=0}^{m-1} (H_G)\hat{}_k$$

with $\rho \in (H_G)\hat{}_k$ if and only if the central character of ρ is $\gamma \to \gamma^k$.

For $0 \leq k < m$, let

$$C(H_G)_k = \{\phi \in C(H_G) \mid \phi(\gamma h) = \gamma^k\phi(h)\ \forall \gamma \in Z_G, h \in H_G\}.$$

Then we have the orthogonal decomposition $C(H_G) = \bigoplus_{k=0}^{m-1} C(H_G)_k$ and $\rho \in (H_G)\hat{}_k$ if and only if all the matrix coefficients of ρ belong to $C(H_G)_k$ (recall that a matrix coefficient of ρ is a function on H_G of the form $h \to (\rho(h)v, w)$, where v, w are vectors in the space of ρ).

PROPOSITION 5.2. *There is (up to equivalence) exactly one representation in* $(H_G)\hat{}_1$. *It acts on* $C(G)$ *and is given by* $\rho_1(z, \gamma) = \gamma W^{\hat{z}}$.

PROOF. It is easy to check that ρ_1 is a unitary representation of H_G. Since $\{\hat{z} \mid z \in \widetilde{G}\}$ spans $C(\widetilde{G})$ the representation is irreducible and clearly belongs to $(H_G)\hat{}_1$. Now since the space of matrix coefficients of ρ_1 is n^2 dimensional it exhausts $C(H_G)_1$. Thus there are no other representations in $(H_G)_1$. □

6. The symplectic group and the oscillator representation

$\widetilde{G}$ is a finite abelian group with additional structure determined by the 'form' $\langle \cdot, \cdot \rangle$. Let $\mathrm{Hom}(\widetilde{G})$ denote the ring of homomorphisms $\sigma : \widetilde{G} \to \widetilde{G}$, and $\mathrm{Aut}(\widetilde{G})$ the subgroup of automorphisms. For any $\sigma \in \mathrm{Aut}(\widetilde{G})$, there is a unique $\sigma^* \in \mathrm{Aut}(G)$ such that $\langle \sigma^* z_1, z_2\rangle = \langle z_1, \sigma z_2\rangle$ for all $z_1, z_2 \in \widetilde{G}$. In analogy with the case $G = \mathbb{R}$ we define

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the symplectic group $\mathrm{Sp}(\widetilde{G})$ to be the set of automorphisms σ of $\widetilde{G}$ which preserve $\langle \cdot, \cdot \rangle$, that is the set of $\sigma \in \mathrm{Aut}(\widetilde{G})$ such that $\sigma^* = \sigma^{-1}$. It follows immediately that for $\sigma \in Sp(\widetilde{G})$,

$$a^\sigma * b^\sigma = (a * b)^\sigma, \quad \forall a, b \in C(\widetilde{G}), \tag{6.1}$$

so that $*_\sigma = *$.

Let $\mathrm{Perm}(\widetilde{G}, *)$ denote the group of permutations σ of $\widetilde{G}$ which satisfy (6.1). Clearly $\mathrm{Sp}(\widetilde{G})$ is a subgroup of $\mathrm{Perm}(\widetilde{G}, *)$ and we may consider $\widetilde{G}$ as contained in $\mathrm{Perm}(\widetilde{G}, *)$ if we identify $z \in \widetilde{G}$ with the permutation σ_z of $\widetilde{G}$ given by

$$\sigma_z(z') = z' + z, \quad \forall z' \in \widetilde{G}. \tag{6.2}$$

THEOREM 6.1. $\mathrm{Perm}(\widetilde{G}, *) = \widetilde{G} \ltimes \mathrm{Sp}(\widetilde{G})$.

PROOF. $\widetilde{G} \cap Sp(\widetilde{G})$ contains just the identity and $\mathrm{Sp}(\widetilde{G})$ normalizes $\widetilde{G}$. It thus suffices to show that if $\sigma \in \mathrm{Perm}(\widetilde{G}, *)$ and $\sigma(0) = 0$, then $\sigma \in \mathrm{Sp}(\widetilde{G})$.

First note that (6.1) is equivalent to the condition

$$\langle \sigma z_1, \sigma z_2, \sigma z_3 \rangle = \langle z_1, z_2, z_3 \rangle, \quad \forall z_1, z_2, z_3 \in \widetilde{G}.$$

Setting $z_3 = 0$ and using the condition $\sigma(0) = 0$ shows that

$$\langle \sigma z_1, \sigma z_2 \rangle = \langle z_1, z_2 \rangle, \quad \forall z_1, z_2 \in \widetilde{G}.$$

We thus need only show that σ is a homomorphism. Recall that every character of $\widetilde{G}$ is of the form $\hat{z}$ for some $z \in \widetilde{G}$. Then the computation

$$\begin{aligned} (\hat{z})^\sigma(z') &= \hat{z}(\sigma z') = \langle z, \sigma z' \rangle \\ &= \langle \sigma^{-1} z, z' \rangle = (\sigma^{-1} z)\hat{}(z'), \quad \forall z' \in \widetilde{G} \end{aligned}$$

shows that the map $\chi \to \chi^\sigma$ permutes the characters of $\widetilde{G}$.

Now let $z_1, z_2, z_3 \in \widetilde{G}$. Then since characters separate points of an abelian group,

$$\begin{aligned} z_1 z_2 = z_3 &\iff \chi(z_1)\chi(z_2) = \chi(z_3), && \forall \chi \in (\widetilde{G})\hat{}, \\ &\iff \chi^\sigma(z_1)\chi^\sigma(z_2) = \chi^\sigma(z_3), && \forall \chi \in (\widetilde{G})\hat{}, \\ &\iff \chi(\sigma z_1)\chi(\sigma z_2) = \chi(\sigma z_3), && \forall \chi \in (\widehat{G})\hat{}, \\ &\iff (\sigma z_1)(\sigma z_2) = \sigma z_3. \end{aligned}$$

□

If $\sigma \in \mathrm{Perm}(\widetilde{G}, *)$ then Proposition 3.7 shows that σ preserves the inner product $(\cdot, \cdot)$ on $C(\widetilde{G})$. Now suppose that G is odd. Then $\mathrm{Perm}(\widetilde{G}, *)$ acts as a group of automorphisms of the Hilbert algebra $(C(\widetilde{G}), *_2)$ which we have seen is isomorphic

to $M(n, \mathbb{C})$. Now any automorphism of $M(n, \mathbb{C})$ which preserves the Hilbert space structure must be conjugation by a unitary matrix. Thus for each $\mu \in \mathrm{Perm}(\widetilde{G}, *)$ we can find $c_\mu \in C(\widetilde{G})$ such that for all $b \in C(\widetilde{G})$

$$c_\mu *_2 b *_2 \overline{c}_\mu(z) = b(\mu(z)), \quad \forall z \in \widetilde{G}.$$

Furthermore c_μ is unique up to a multiplicative constant of modulus one. We may restate this by saying that $\mu \to W^{c_\mu}$ is a projective unitary representation of $\mathrm{Perm}(\widetilde{G}, *)$ which we call the oscillator representation. For $\mu \in \mathrm{Sp}(\widetilde{G})$, the correspondence $\mu \to c_\mu$ involves a 'Cayley transform'.

THEOREM 6.2. *Suppose G is odd. Let $\mu \in \mathrm{Sp}(\widetilde{G})$ and suppose that $1 + \mu \in \mathrm{Hom}(\widetilde{G})$ is invertible. Define $C(\mu) = (1 - \mu)/(1 + \mu)$. Then $c_\mu(z) = \langle z, C(\mu)z\rangle$.*

PROOF. For $\sigma \in \mathrm{Hom}(\widetilde{G})$, set $a_\sigma(z) = \langle z, \sigma z\rangle$ for all $z \in \widetilde{G}$. Then for $b \in C(\widetilde{G})$,

$$\begin{aligned} a_\sigma *_2 b *_2 \overline{a}_\sigma(z) &= \int_{\widetilde{G}}\int_{\widetilde{G}}\int_{\widetilde{G}}\int_{\widetilde{G}} \langle z_1, z_2\rangle^2\langle z_2, z_3\rangle^2\langle z_3, z_1\rangle^2\langle z_1, \sigma z_1\rangle b(z_2) \\ &\quad \times \langle z_3, z_4\rangle^2\langle z_4, z\rangle^2\langle z, z_3\rangle^2\langle \sigma z_4, z_4\rangle \, dz_1\, dz_2\, dz_3\, dz_4 \\ &= \int_{\widetilde{G}}\int_{\widetilde{G}}\int_{\widetilde{G}} \langle z_1, z_2\rangle^2 \delta_0(z_1 - z_2 + z_4 - z)\langle z_1, \sigma z_1\rangle b(z_2) \\ &\quad \times \langle z_4, z\rangle^2\langle \sigma z_4, z_4\rangle \, dz_1\, dz_2\, dz_4 \\ &= \int_{\widetilde{G}}\int_{\widetilde{G}} \langle z_1, z_4 - z\rangle^2\langle z_1, \sigma z_1\rangle b(z_1 + z_4 - z)\langle z_4, z\rangle^2 \\ &\quad \times \langle \sigma z_4, z_4\rangle \, dz_1\, dz_4 \end{aligned}$$

Now make the change of variables $t = z_1 + z_4$ and $s = z_1 - z_4$ so that

$$z_1 = (t + s)/2, \quad z_4 = (t - s)/2.$$

The above integral then becomes

$$\begin{aligned} &\int_{\widetilde{G}}\int_{\widetilde{G}} \langle z, s\rangle^2 \left\langle \frac{t+s}{2}, \frac{t-s}{2}\right\rangle^2 \left\langle \frac{t+s}{2}, \sigma\left(\frac{t+s}{2}\right)\right\rangle \\ &\qquad \times \left\langle \sigma\left(\frac{t-s}{2}\right), \frac{t-s}{2}\right\rangle b(t - z)\, dt\, ds \\ &= \int_{\widetilde{G}}\int_{\widetilde{G}} \langle z, s\rangle^2\langle s, t\rangle \left\langle t, \frac{\sigma s}{2}\right\rangle \left\langle s, \frac{\sigma t}{2}\right\rangle b(t - z)\, dt\, ds \\ &= \int_{\widetilde{G}} \delta_0\left(t - 2z - \frac{\sigma^* t}{2} + \frac{\sigma t}{2}\right) b(t - z)\, dt. \end{aligned}$$

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Assume for the moment that $2 - \sigma^* + \sigma : \widetilde{G} \to \widetilde{G}$ is invertible. Then we may solve

$$t = (2 - \sigma^* + \sigma)^{-1}(4z)$$

so that the above integral becomes simply

$$b((2 - \sigma^* + \sigma)^{-1}(4z) - z) = b(\tau(z)),$$

where

$$\alpha(z) = \frac{2 + \sigma^* - \sigma}{2 - \sigma^* + \sigma}(z) = \frac{1 - (\sigma - \sigma^*)/2}{1 + (\sigma - \sigma^*)/2}(z).$$

We have established that if $\sigma \in \mathrm{Hom}(\widetilde{G})$ with $2 - \sigma^* + \sigma$ invertible, then for all $b \in C(\widetilde{G})$, $a_\sigma *_2 b *_2 \overline{a}_\sigma(z) = b(\alpha(z))$, where α is defined in terms of σ above.

Now suppose that $\mu \in Sp(\widetilde{G})$ and that $1 + \mu$ is invertible. Let $\eta = (1 - \mu)/(1 + \mu)$. Then $\eta^* = -\eta$. To see this, let $z_1, z_2 \in \widetilde{G}$ with $z_i = (1 + \mu)w_i$. Then

$$\langle \eta z_1, z_2 \rangle = \langle (1 - \mu)w_1, (1 + \mu)w_2 \rangle = \langle w_1, \mu w_2 \rangle \overline{\langle \mu w_1, w_2 \rangle}$$

and similarly $\langle z_1, \eta z_2 \rangle = \langle \mu w_1, w_2 \rangle \overline{\langle w_1, \mu w_2 \rangle}$ so the claim follows.

Consequently $2 - \eta^* + \eta = 2 + 2\eta = 4/(1 + \mu)$ is invertible, so replacing σ in ( ) with η we get that for all $b \in C(\widetilde{G})$, $c_\mu *_2 b *_2 \overline{c}_\mu(z) = b(\alpha(z))$, where $\alpha(z) = (1 - \eta)/(1 + \eta) = \mu(z)$. □

7. Systems of Cartan subalgebras for sl(n)

We now give some applications of our approach of regarding matrices from a symbol point of view. To be very concrete, we work out some of the ramifications of the preceding approach in the simplest possible case, that when $G = \mathbb{Z}_3 = \{0, 1, 2\}$. The dual group $\widehat{G}$ can also be regarded as $\mathbb{Z}_3$, where the pairing between $k \in G$ and $l \in \widehat{G}$ is $\langle k, l \rangle = \xi^{kl}$ where $\xi = e^{2\pi i/3}$. Writing

$$\widetilde{G} = G \times \widehat{G} = \{z_{kl} = (k, l) \mid 0 \le k, l \le 2\}$$

we get $\langle z_{ij}, z_{kl} \rangle = \xi^{jk - il}$. The space $V = C(G)$ has basis $\{e_0, e_1, e_2\}$ where $e_i(j) = \delta_{ij}$. We get from Proposition 4.4 the formula

$$W^{\widehat{z}_{ij}} e_l(k) = \langle j, i/2 \rangle \langle j, k \rangle \delta_{l,k+i} = \xi^{2ij} \xi^{jk(l-i)} \delta_{l-i,k} = \xi^{j(i+l)} e_{l-i}(k).$$

Thus $W^{\widehat{z}_{ij}} e_l = \xi^{j(i+l)} e_{l-i}$, where of course the subscripts are computed mod 3. We will write $W^{\widehat{z}_{ij}} = W_{ij}$, an operator on V and let M_{ij} denote the matrix of the operator

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in the basis $\{e_0, e_1, e_2\}$. Here are these nine matrices:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad M_{00} \qquad \begin{bmatrix}1&0&0\\0&\xi&0\\0&0&\xi^2\end{bmatrix} \quad M_{01} \qquad \begin{bmatrix}1&0&0\\0&\xi^2&0\\0&0&\xi\end{bmatrix} \quad M_{02}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad M_{10} \qquad \begin{bmatrix}0&\xi^2&0\\0&0&1\\\xi&0&0\end{bmatrix} \quad M_{11} \qquad \begin{bmatrix}0&\xi&0\\0&0&1\\\xi^2&0&0\end{bmatrix} \quad M_{12}$$

$$\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad M_{20} \qquad \begin{bmatrix}0&0&\xi\\\xi^2&0&0\\0&1&1\end{bmatrix} \quad M_{21} \qquad \begin{bmatrix}0&0&\xi^2\\\xi&0&0\\0&1&0\end{bmatrix} \quad M_{22}$$

We now record various properties of these matrices which follow from Section 6.

THEOREM 7.1. (1) $B = \{M_{ij} \mid 0 \leq i, j \leq 2\}$ *forms a basis of* $M(3, \mathbb{C})$.
(2) $B_0 = B \setminus M_{00}$ *forms a basis of* $\mathrm{sl}(3, \mathbb{C})$, *the trace zero* 3×3 *matrices.*
(3) *Each* M_{ij} *is invertible, in fact unitary. Furthermore* $M_{ij}^{-1} = M_{-i\,-j}$.
(4) $M_{ij} M_{kl} = \xi^{2(il-jk)} M_{i+k,j+l}$.
(5) $M_{ij} M_{kl} M_{ij}^{-1} = \xi^{il-jk} M_{kl}$.
(6) $M_{ij}^T = M_{-i,j}$.
(7) $\overline{M_{ij}} = M_{i,-j}$.
(8) *If we define* $M_{ij}^\circ = M_{ji}$ *and extend to the entire algebra linearly, then* $(AB)^\circ = B^\circ A^\circ$.

Some remarks are in order. We will shortly see that Property (2) is potentially useful in Lie theory, where the fact that there is no 'canonical' basis of $\mathrm{sl}(n, \mathbb{C})$ constructed from the usual basis E_{ij} (the matrix with a single 1 in the ij th position) is always a source of some awkwardness. Property (4) show that pairwise the M_{ij} 'almost' commute. Property (5) shows that conjugation by M_{ij} is diagonalized by the subspaces $\langle M_{kl} \rangle$. Properties (6) and (7) reveal a symmetry between the operations of transposition and conjugation not so apparent in the usual formulation. Property (8) reveals another transposition-like symmetry which is canonical in this basis.

We now point out a consequence of our discussions of subalgebras of $C(\widetilde{G})$. There are 4 subgroups of $\widetilde{G}$ of size 3, namely $G_1 = \{(00), (01), (02)\}$, $G_2 = \{(00), (11), (22)\}$, $G_3 = \{(00), (21), (12)\}$ and $G_4 = \{(00), (10), (20)\}$. To each corresponds a 3-dimensional commutative subalgebra of $C(\widetilde{G}) \cong M(3, \mathbb{C})$, spanned by the corresponding matrices M_{ij}. When we pass to $\mathfrak{g} = \mathrm{sl}(3, \mathbb{C})$ regarded as a Lie

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algebra, we get the decomposition

$$(*) \qquad \mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \mathfrak{h}_4,$$

where $\mathfrak{h}_1 = \langle M_{01}, M_{02} \rangle$, $\mathfrak{h}_2 = \langle M_{11}, M_{22} \rangle$, $\mathfrak{h}_3 = \langle M_{21}, M_{12} \rangle$ and $\mathfrak{h}_4 = \langle M_{10}, M_{20} \rangle$.

This implies an interesting decomposition on the group level as well. If H_i is the Cartan subgroup in $G = \mathrm{SL}(3, \mathbb{C})$ corresponding to $\mathfrak{h}_i$, then it is natural to ask to what extent does $(*)$ extend to a decomposition of the form $G = H_1 H_2 H_3 H_4$? It will certainly hold in a neighbourhood of the identity.

The discussion here easily generalizes to $\mathrm{sl}(n)$. To what extent can we decompose other simple Lie algebras as direct sums of Cartan subalgebras?

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