RATIONAL VALUED AND REAL VALUED PROJECTIVE CHARACTERS OF FINITE GROUPS

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It is well-known [**3**; V.13.7] that each irreducible complex character of a finite group G is rational valued if and only if for each integer m coprime to the order of G and each $g \in G$, g is conjugate to g^m. In particular, for each positive integer n, the symmetric group on n symbols, $S(n)$, has all its irreducible characters rational valued. The situation for projective characters is quite different. In [**5**], Morris gives tables of the spin characters of $S(n)$ for $n \leq 13$ as well as general information about the values of these characters for any symmetric group. It can be seen from these results that in no case are all the spin characters of $S(n)$ rational valued and, indeed, for $n \geq 6$ these characters are not even all real valued. In section 2 of this note, we obtain a necessary and sufficient condition for each irreducible character of a group G associated with a 2-cocycle α to be rational valued. A corresponding result for real valued projective characters is discussed in section 3. Section 1 contains preliminary definitions and notation, including the definition of projective characters given in [**2**].

1. Let G be a finite group and H be a representation group for G so that H has a subgroup A such that

$$\text{(i) } A \leq Z(H) \cap H'; \qquad \text{(ii) } G \cong H/A \quad \text{and} \quad \text{(iii) } A \cong H^2(G, \mathbb{C}^\times),$$

where $\mathbb{C}^\times$ denotes the set of non-zero complex numbers. Let $\theta : G \to H/A$ be an isomorphism fixed throughout and, for each $g \in G$, let $r(g)$ be an element of H such that

$$\theta(g) = r(g)A$$

with the convention that $r(1) = 1$. By Proposition 1.1 of [**2**], we may suppose that the transversal $\{r(g) \mid g \in G\}$ is chosen to be conjugacy preserving in the sense that $r(x)$ is conjugate to $r(y)$ in H whenever x is conjugate to y in G.

Given $x, y \in G$, there exists an element $A(x, y)$ of A such that

$$r(x)r(y) = r(xy)A(x, y),$$

where

$$A(1, g) = 1 = A(g, 1),$$

for all $g \in G$. Now let λ be a homomorphism from A to $\mathbb{C}^\times$ and define $\alpha : G \times G \to \mathbb{C}^\times$ by

$$\alpha(x, y) = \lambda(A(x, y))$$

for all $x, y \in G$. It follows that α is a 2-cocycle of G satisfying

$$\alpha(1, g) = 1 = \alpha(g, 1),$$

for all $g \in G$. We refer to α as the special cocycle associated with λ. Now if P is a projective representation of G with special cocycle α, the map D on H defined by

$$D(ar(g)) = \lambda(a)P(g),$$

for all $a \in A$ and all $g \in G$, is a linear representation of H. We say that D linearizes P. The projective character ξ of P is defined by

$$\xi(g) = \text{trace } P(g),$$

for all $g \in G$, so that if D linearizes P and χ is the character of D,

$$\chi(r(g)) = \xi(g),$$

for all $g \in G$.

An element g of G is said to be α-regular if

$$\alpha(g, x) = \alpha(x, g),$$

for all $x \in C_G(g)$. It is easily checked that an element g of G is α-regular if and only if each conjugate of g is α-regular. The following result is Corollary 4.6 of [**2**].

PROPOSITION 1. *Let G be a finite group and α be a special cocycle. An element g of G is α-regular if and only if there exists an irreducible α-projective character ξ such that $\xi(g) \neq 0$.*

2. In this section we obtain a necessary and sufficient condition for the character of each irreducible α-projective representation of a group G to be rational valued.

THEOREM 2. *Let G be a finite group and α be the special cocycle associated with λ. Each irreducible α-projective character of G is rational valued if and only if for each α-regular $g \in G$ and each integer m greater than 1 and coprime to $|G|$,*

(i) *g is conjugate to g^m, and*
(ii) $\alpha(g, g)\alpha(g, g^2) \ldots \alpha(g, g^{m-1}) = 1$.

Proof. For convenience, we will denote

$$A(g, g)A(g, g^2) \ldots A(g, g^{m-1})$$

by $f_A(g, m)$ and

$$\alpha(g, g)\alpha(g, g^2) \ldots \alpha(g, g^{m-1})$$

by $f_\alpha(g, m)$ so that

$$f_\alpha(g, m) = \lambda(f_A(g, m)).$$

Suppose firstly that conditions (i) and (ii) hold. Let P be an irreducible α-projective representation of G with character ξ and suppose P is linearized by the irreducible representation D of H. By a result of Alperin and Kuo [**3**; V.24.5], H has exponent dividing $|G|$. Thus if ω is a primitive $|G|$-th root of unity, for any $g \in G$, the eigenvalues of $D(r(g))$ $(= P(g))$ are powers of ω. By Proposition 1, $\xi(g) = 0$ if g is not α-regular, so in order to show that ξ is rational valued, we need only consider the case where g is α-regular.

An easy induction argument shows that for any positive integer $m>1$,

$$r(g)^m = f_A(g, m)r(g^m).$$

Thus

$$\begin{aligned} P(g)^m &= D(r(g))^m \\ &= D(r(g)^m) \\ &= D(f_A(g, m)r(g^m)) \\ &= f_\alpha(g, m)D(r(g^m)) \\ &= f_\alpha(g, m)P(g^m). \end{aligned}$$

Hence if g is α-regular and m is coprime to $|G|$, condition (ii) implies that $P(g)^m = P(g^m)$. It then follows that if $\omega^{a_1}, \dots, \omega^{a_d}$ are the eigenvalues of $P(g)$, then $P(g^m)$ has eigenvalues $\omega^{ma_1}, \dots, \omega^{ma_d}$. Therefore

$$\xi(g^m) = \sum_{i=1}^{d} \omega^{ma_i} = \left(\sum_{i=1}^{d} \omega^{a_i}\right)\theta_m = (\xi(g))\theta_m$$

where θ_m is the automorphism of $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ defined by

$$\omega\theta_m = \omega^m.$$

Condition (i) together with the fact that ξ is a class function implies that $\xi(g)$ is fixed by each element of the Galois group of $\mathbb{Q}[\omega]$ over $\mathbb{Q}$ and so $\xi(g)$ is rational.

Conversely, suppose that each α-projective character of G is rational valued. Let g be an α-regular element and $m>1$ be an integer coprime to $|G|$. Let ξ be the character of an α-projective representation P of G and suppose P is linearized by D. Then

$$\begin{aligned} P(g^m) &= D(r(g^m)) \\ &= D(f_A^{-1}(g, m)r(g)^m) \\ &= f_\alpha^{-1}(g, m)D(r(g)^m) \\ &= f_\alpha^{-1}(g, m)P(g)^m. \end{aligned}$$

Thus

$$\xi(g^m) = f_\alpha^{-1}(g, m)(\xi(g))\theta_m \tag{1}$$

By Proposition 1, there is an irreducible α-projective character ξ_0 such that $\xi_0(g) \neq 0$. Since $\xi_0(g^m)$ and $\xi_0(g)$ are rational and $f_\alpha^{-1}(g, m)$ is a root of unity, equation (1) implies that $f_\alpha^{-1}(g, m)$ is either 1 or -1.

Suppose $f_\alpha(g, m) = -1$. Equation (1) would then imply that $\xi(g^m) = -\xi(g)$ for all irreducible α-projective characters ξ. Since g is α-regular, the second orthogonality relation [**2**; Lemma 4.4] gives

$$\sum \xi(g)\overline{\xi(g)} = |C_G(g)|,$$

where the summation is over all irreducible α-projective characters of G. Also

$$\sum \xi(g)\overline{\xi(g^m)} = -\sum \xi(g)\overline{\xi(g)}$$
$$= -|C_G(g)|,$$

which is impossible. Therefore $f_\alpha(g, m) = 1$ establishing (ii). Equation (1) then implies that $\xi(g^m) = \xi(g)$ for all irreducible α-projective characters of G so that g is conjugate to g^m as required.

Examples. (i) Let G be a group with a special cocycle α such that G has precisely one irreducible α-projective character. Thus G has precisely one α-regular conjugacy class which therefore consists of the identity element of G. Proposition 1 now implies that the irreducible α-projective character of G is rational valued.

In [**1**], De Meyer and Janusz construct examples of such a group G. In their terminology, a group H is of central type if H has an irreducible linear character χ with

$$\chi(1)^2 = |H : Z(H)|.$$

Theorem 1 of [**1**] asserts that if H is of central type, then there is a cocycle β on $G = H/Z(H)$ such that G has precisely one irreducible β-projective representation. Since any cocycle β is cohomologous to a special cocycle α and the number of irreducible β-projective representations is equal to the number of irreducible α-projective representations, the groups constructed in section 4 of [**1**] provide examples of groups all of whose irreducible α-projective characters are rational valued.

(ii) All known examples of groups of central type are soluble. In [**6**] Morris gives projective character tables for various exceptional Weyl groups. Among these, E_8 provides an example of a non-soluble group with a cocycle α for which each irreducible projective character associated with α is rational valued.

3. In this section we prove an analogous result to Theorem 2 for real valued projective representations.

Theorem 3. *Let G be a finite group and α be a special cocycle. Each irreducible α-projective character of G is real valued if and only if for each α-regular element g of G*

(i) *g is conjugate to g^{-1}, and*

(ii) *$\alpha(g, g^{-1}) = 1$.*

Proof. Suppose conditions (i) and (ii) hold. Let P be an irreducible α-projective representation of G with character ξ and suppose P is linearized by D. By Proposition 1 we only need consider α-regular elements of G. However, for any $g \in G$,

$$P(g^{-1}) = \alpha(g, g^{-1})P(g)^{-1}.$$

Thus condition (ii) implies that if g is α-regular $P(g^{-1}) = P(g)^{-1}$ and so

$$D(r(g^{-1})) = P(g^{-1}) = P(g)^{-1} = D(r(g)^{-1}).$$

Hence, if χ is the character of D,

$$\chi(r(g^{-1})) = \overline{\chi(r(g))}$$

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and so $\xi(g^{-1}) = \overline{\xi(g)}$. Since g is conjugate to g^{-1} and ξ is a class function, we deduce that $\xi(g)$ is real.

Conversely, suppose every irreducible α-projective character of G is real valued. Then for any $g \in G$,

$$\begin{aligned}\xi(g^{-1}) &= \alpha(g, g^{-1})\overline{\xi(g)} \\ &= \alpha(g, g^{-1})\xi(g).\end{aligned} \tag{2}$$

Now let g be α-regular and ξ_0 be an irreducible α-projective character of G such that $\xi_0(g) \neq 0$. Equation (2) implies that $\alpha(g, g^{-1})$ is a real valued root of unity and so is either 1 or -1. However, if $\alpha(g, g^{-1}) = -1$, as in the proof of Theorem 2, we would obtain a contradiction to the second orthogonality relation. Therefore $\alpha(g, g^{-1}) = 1$ and equation (2) then implies that g is conjugate to g^{-1}.

REMARKS. (i) Given an irreducible linear character χ of a group G, a well-known result of Frobenius-Schur [**4**; 4.5] gives a necessary and sufficient condition for χ to be real valued. Thus, defining

$$\nu_2(\chi) = \sum_{g \in G} \chi(g^2),$$

$\nu_2(\chi)$ is zero if and only if χ is not real valued. This result fails for projective characters. For example, let G be the alternating group on four symbols and ω be a primitive cube root of unity. Let

$$P(123) = \begin{bmatrix} \omega & 0 \\ 0 & \omega^2 \end{bmatrix} \text{ and } P((12)(34)) = \begin{bmatrix} i/\sqrt{3} & -\sqrt{(2/3)} \\ \sqrt{(2/3)} & -i/\sqrt{3} \end{bmatrix}.$$

Then $\langle P(123), P((12)(34))\rangle$ is a representation group for G. Thus P can be extended to a projective representation of G in such a way that the cocycle α associated with P is special and so the character ξ of P is a class function. Then ξ has value -1 on elements of order 3 and value 0 on elements of order 2, so that ξ is real valued but $\nu_2(\xi)$ is zero. On the other hand, the character ξ_1 of the projective representation P_1 obtained by taking the tensor product of P with a non-identity one-dimensional linear character of G is not real valued but $\nu_2(\xi_1)$ is non-zero.

(ii) An element g of a group G is real if g is conjugate to g^{-1}. It is well-known that the number of real conjugacy classes of G is equal to the number of irreducible complex linear characters of G which are real valued. The example of the symmetric groups make it clear that the number of real α-regular conjugacy classes is not the number of irreducible α-projective characters of G which are real valued.

In view of Theorem 3, it might be conjectured that the number of irreducible α-projective characters of a group G which are real valued is equal to the number of α-regular real conjugacy classes of G in which $\alpha(g, g^{-1}) = 1$. However this is also false. Take G to be $S(4)$ and

$$P(1234) = \begin{bmatrix} -\omega\sqrt{(2/3)} & -i\omega/\sqrt{3} \\ i\omega^2/\sqrt{3} & \omega^2\sqrt{(2/3)} \end{bmatrix}.$$

Then the projective representation P in (i) extends to a projective representation of $S(4)$

whose character is not real. The character of the representation P_1 obtained by taking the tensor product of P with the non-identity one-dimensional linear representation of G is also non-real. In fact G has one irreducible projective representation whose character is real valued. However, for each element of order 3, $\alpha(g, g^{-1}) = 1$ so G has two α-regular real conjugacy classes in which $\alpha(g, g^{-1}) = 1$.

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