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WEYL REEXAMINED: "DAS KONTINUUM" 100 YEARS LATER

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Abstract. Hermann Weyl was one of the greatest mathematicians of the 20th century, with contributions to many branches of mathematics and physics. In 1918, he wrote a famous book, "Das Kontinuum", on the foundations of mathematics. In that book, he described mathematical analysis as a 'house built on sand', and tried to 'replace this shifting foundation with pillars of enduring strength'.

In this paper, we reexamine and explain the philosophical and mathematical ideas that underly Weyl's system in "Das Kontinuum", and show that they are still useful and relevant. We propose a precise formalization of that system, which is the first to be completely faithful to what is written in the book. Finally, we suggest that a certain set-theoretical modern system reflects better Weyl's ideas than previous attempts (most notably by Feferman) of achieving this goal.

§1. Introduction. Hermann Weyl (1885–1955) was one of the greatest mathematicians of the 20th century, with contributions to many branches of mathematics and physics. He was also deeply interested in the philosophy of these disciplines, as his great book [47] shows. The question of the certainty of the propositions of mathematics and the strongly related question of the security of its foundations were of particular importance for him (as well as a source of worry) throughout his scientific career [46]. This worry has caused him to make several contributions to the debate about the foundation of mathematics (like [45], [46], and [47]), in which he changed his mind more than once.[1] In contrast, Weyl practically had just one important contribution to the *research* on the foundations of mathematics: his famous small book "Das Kontinuum" ([43], English translation in [48]). As explained in [19], this book has a great historical significance, since in it the predicativist program [21] for the foundations of mathematics (originally initiated by Poincaré in [33, 35] and partially adopted by Russel in [49]) was seriously developed for the first time, and its viability was demonstrated. Personally [8, 9], I believe that this program is the only one which really succeeds to provide secure foundations and certainty to the most important parts of classical mathematics. However, the aim of this paper is not to describe or defend the

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[1] See [17] for detailed descriptions of Weyl's mathematical career and of its contributions to the study of the foundations of mathematics.

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predicativist program in general; it is to reexamine (and reformalize) Weyl’s specific contribution to it in his seminal book, whose content and goals are described in the following famous words from its first page:[2]

> I shall show that the house of analysis is to a large degree built on sand. I believe that I can replace this shifting foundation with pillars of enduring strength. They will not, however, support everything which today is generally considered to be securely grounded. I give up the rest, since I see no other possibility. [P. 1]

The main goal of this paper is to restudy the philosophical and mathematical ideas that underly Weyl’s system in “Das Kontinuum”, to show that they are still relevant, and to provide a precise and completely faithful formalization of it.[3] (To our best knowledge, this is the first formalization to really be completely faithful to what is written in the book.)

There are two reasons why it might seem strange to devote in 2018 a long paper to a new study and formalization of the content of [43].

1. A lot of good research into predicative mathematics has been made in the 100 years that have passed since [43] was published. Accordingly, it is generally believed that Weyl’s mathematical work on the foundations of analysis is by now “superseded by later developments of arithmetical mathematics” [17]. Therefore, it seems that nowadays the study of “Das Kontinuum” has *only* the historical value noted above, but has no true *mathematical* value anymore.
2. In any case, it is well-known that [43] has already been carefully and thoroughly studied by S. Feferman in [17] and [19]. In [17] Feferman has also fully formalized in modern terms the system which is described and used in [43], and his formalization is usually accepted by almost everyone as very faithful and adequate. (See, e.g., Chapter 9 of [28].) So what reason can there be for examining and formalizing it again?

In what follows, we show that both arguments above against pursuing the goals of this paper are mistaken. First, none of Feferman’s formalizations of Weyl’s system is really faithful to those of Weyl.[4] Second, Weyl’s work actually transcends what Feferman calls above ‘arithmetical mathematics’. Third, I believe that Weyl’s system is *superior* to its modern counterparts for the task of serving as a basis of a predicatively justified, natural development of classical analysis. Fourth, there are some important hidden ideas in “Das

[2]Henceforth, the page references in all quotes from “Das Kontinuum” are according to [48]. In case a quote is from the Appendix of [48] (which is a translation of [44]), the letter ‘A’ appears before the page number. Words are emphasized in quotes if and only if they were emphasized in the original text. The book itself is referred to as [43].

[3]The notion of “formalization” is meant here in the traditional way in mathematical logic, that is: obtaining a corresponding formal system on paper. Note that in computer science (and, in particular, in the field of formalization of mathematics), this term also has the connotation that the formalization is computerized in some proof assistant.

[4]This fact, as well as the one mentioned at the next (second) point, had been noted in [4] well before I independently reached the same conclusions. See Section 3.5.

Kontinuum" that have so far been completely ignored in the literature on it, but will be uncovered and exploited below.

NOTE 1. Unfortunately, this paper contains several critical remarks concerning Feferman's papers [17] and [19]. It should be emphasized that these remarks do not diminish my great admiration of Feferman's foundational work in general, and his research on predicativity in particular.

The structure of this paper is as follows. Section 2 describes in more detail Weyl's motivations for writing [43]. Then in Section 3, we present our formalization WA of Weyl's system in [43], and compare it to previous ones. Unlike the order things are presented in Weyl's book, the ideas of Weyl that have led to WA are explained after that, in Section 4. Section 5 is devoted to the semantics of Weyl's system, and to possible extensions of it that have been considered by him. Section 6 describes how analysis has been developed by Weyl in WA. In Section 7, we discuss some drawbacks of Weyl's approach. Finally, in Section 8, we give a brief description of how most of Weyl's ideas are reflected and implemented, while the drawbacks of his system are avoided, in the predicative set theory PZF of [8].

§2. What has Weyl rejected, and why? There are two (not unrelated) reasons for Weyl's dissatisfaction with the usual accepted foundations of analysis that can immediately be seen in his book. First, and most important, Weyl's totally rejected as "*vague*" the modern notions of an *arbitrary* set, and of a function as an arbitrary set of pairs. Second, Weyl thought that the "currently accepted foundations of analysis" commits the sin of *vicious circularity*. He said about it:

> [B]ecause of its vague concept of set and function and its manner of applying the concepts of existence and identity (particularly to the real numbers), finds itself caught in a *vicious circle*. [P. 44]

As is well known, the diagnosis that the paradoxes in Cantor's set theory are due to the use of definitions which involve vicious circles was first due to Poincaré, and then adopted by Russell. They both rejected the introduction of objects using definitions which involve a collection (or 'totality') to which the defined object belongs. Definitions of this sort were called by them *impredicative*, and the principle that forbids such definitions was called the 'Vicious Circle Principle (VCP)', because they thought that "it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities" [49]. The VCP was given by Russell in [49] a rather vague formulation: "Whatever involves all of a collection must not be one of the collection". Following Poincaré and Russell, in [43] Weyl too allowed only predicative (i.e., not impredicative) definitions of sets, and strongly objected to the use of impredicative ones. Thus, he writes the following concerning *legitimate principles of definitions of properties* of objects (and so of sets of objects):

> [I]t would be meaningless to include among these principles an assertion such as the following: If $\mathcal{A}$ is a property of properties, then

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> one may form that property $P_{\mathcal{A}}$ which belongs to an object x if and only if there is a property constructed by means of these principles which belongs to x and itself possesses the property $\mathcal{A}$. That would be a blatant *circulus vitiosus*; yet our current version of analysis commits this error and I consider this ground for censure. [A:P. 113]

In current notation and terminology, Weyl is forbidding here definitions of elements of the powerset 2^S of a set S of, for example, the following form:

$$(*)\ P_{\mathcal{A}} = \{x \in S \mid \exists X. X \in 2^S \wedge \mathcal{A}(X) \wedge x \in X\}.$$

The reason that such definitions are rejected as involving a vicious circle is again that they include quantification over a collection, 2^S, to which the defined set is supposed to belong. The circularity in this definition becomes evident once we recall that the only meaning of " 2^S" that makes sense for Weyl is as the collection of subsets of S which are *definable* by acceptable principles of definitions. (This is reflected at this very quote!)

Note 2. An obvious objection to the above argument, first made by Hölder in [23], is that it depends on artificially including 2^S in $(*)$. Hölder argued that it would suffice to let $P_{\mathcal{A}} = \{x \in S \mid \exists X. \mathcal{A}(X) \wedge x \in X\}$. If $\mathcal{A}$ has already been defined in a noncircular way, then no circularity is involved in *this* shorter definition either. As far as I was able to check, Weyl has never directly responded to this objection. However, it is clear what his response would have been: In order to know whether an object satisfies $\mathcal{A}$, we need first to identify it as a property of elements of S, that is: as an element of 2^S. So the omission of 2^S from the definition of $P_{\mathcal{A}}$ only hides the problem; it does not eliminate it. This omission is indeed impossible in Weyl's system, since Weyl adopted in it a strict type discipline. (See Section 3.2.) Because of it, the suggested shorter definition is simply not available in his system. (Hölder's objection is accepted in the system PZF, which is presented in Section 8. We return to it also at the end of Section 5.2 and in Section 7.)

A particularly important example of impredicativity in analysis that Weyl explicitly rejected, is given by the least upper bound (LUB) principle. The LUB of a bounded set $\mathcal{A}$ of reals is defined as the minimal element of the set of upper bounds of $\mathcal{A}$. *This is an impredicative definition.* Moreover, if real numbers are taken (as Weyl does) as Dedekind cuts, then the *existence* of the LUB of $\mathcal{A}$ is shown by defining it as:

$$LUB(\mathcal{A}) = \{q \in \mathbb{Q} \mid \exists X. X \in 2^{\mathbb{Q}} \wedge X \in \mathcal{A} \wedge q \in X\}.$$

Again, *this involves an impredicative definition*. (Note that this definition of the LUB is a particular instance of the scheme $(*)$ above.)

Weyl should have been aware that there are impredicative definitions in Russell's sense that are perfectly acceptable, like 'the least prime number'. (This is a legitimate definition according to Weyl's views.) Therefore, he needed a better criterion than the VCP for characterizing impredicative definitions. Such a criterion was already given by Poincaré himself in [34]. As Crosilla explains (with quotes) in [14], "For Poincaré impredicative definitions were problematic as they treat as completed infinite classes which are instead open-ended or incomplete by their very nature." A nonsuperficial reading of *Das Kontinuum* reveals that *this* (and not so much the alleged circularities) is also what Weyl really saw as problematic. This is explicitly said on [P. 234] of [47] (concerning the set-theoretical antinomies):

> The deepest root of the trouble lies elsewhere: a field of possibilities open into infinity has been mistaken for a closed realm of things existing in themselves.

Now what the example above of an harmless 'impredicative definition' does, is just to select an (already given) object from an already given, *closed* collection, that is: a collection of the type called in [14] (following Poincaré) 'stable and invariant'. In contrast, a definition is illegitimate (that is: impredicative) if it tries to select an object from an *open* collection, treating the latter as if it were closed. This happens, for example, if the definition involves quantification over such a collection. In fact, *every definition that involves quantification over a collection which is not 'stable and invariant' should be rejected as impredicative* from this point of view (and not only definitions of elements of such a collection).

As we have already noted, Weyl's views about legitimate definitions are practically the same as those just described; but his terminology is different. What Crosilla calls 'stable and invariant' he calls 'extensionally determinate', and he characterizes a collection of this sort as a "closed aggregate which is intrinsically determined and demarcated" [A:P. 109]. About such collections, he writes:

> Suppose P is a property pertinent to the objects falling under a concept C. And suppose P has a clear and unambiguous sense. ... if the concept C is extensionally determinate, then not only the question "Does a have the property P?" (where a is an arbitrary object falling under C), but also the existential question "Is *there* an object falling under C which has the property P?", possesses a sense which is intrinsically clear. Corresponding remarks apply to relations. [A:P. 109]

Here, Weyl implicitly divides all collections into three sorts:

(I). The *extensionally determinate* collections, that is, those that are stable and invariant, and so quantification over them has a definite truth-value. (Examples for Weyl are finite sets of objects [P. 20], and the collection $\mathbf{N}$ of the natural numbers.)

(II). Collections which may not be extensionally determinate (and so are open), but are definite in the sense that the question whether a given object belongs to them has a definite answer. ($\mathbb{R}$, the set of real numbers, is the most important example. Weyl explicitly says on [P. 67] that it is an object in his universe, while on [P. 111] he says that it is not extensionally determinate. See Section 4.2.)

(III). Collections that are not even definite. (The collection of continuous functions over $\mathbb{R}$ is an example here. See Section 5.1 for discussion and relevant quotes, and Section 6.5.)

Only a collection of sort (I) *or sort* (II)*, that is, one with a definition that "has a clear and unambiguous sense", can be an object in Weyl's universe. In addition, Weyl allowed to use in definitions only quantifications over collections which are extensionally determinate.*

At this point, a natural question arises: given a definition of a collection, how can we decide to which of the above three sorts it belongs? Most of the first chapter (out of two) of Weyl's book is devoted to this question. What is important about his effort is that unlike Poincaré, he is not just providing there a vague negative characterization of 'predicative' definitions. He attaches no less importance to *precise positive* criteria for such definitions,[5] that is: to providing a list of effective rules for producing legitimate definitions, which is sufficiently strong for a safe development of analysis (though it might not be complete).

> The *sense* of cognition directed toward the physical world thoroughly eludes me if I am not able to anchor the concepts "number", "set", and "function" in logical principles of construction in the manner I attempted in my treatise. [A:P. 113]

§3. Exact formalization of Weyl's system.

3.1. Problems in understanding "Das Kontinuum". For current readers, Weyl's book is not easy to follow. (As a result, several conflicting formalizations of his system, to be reviewed in Section 3.5, can be found in the literature.) We now describe the main problems in understanding this book, and how they are overcome in this paper.

The main source of confusion might be the fact that Weyl's system is officially (and fully) presented *only* at Section 8 of the first chapter of [43]. The sections before it are mainly devoted to explaining Weyl's *road* to his final system. Concerning their content, Weyl explicitly writes the following at Section 8: "Let us withdraw all our provisional remarks (i.e., the whole of Sections 4–7). For we are now going to present the definitive formulation of the principles which are to govern the formation of relations." (Actually, Weyl does need in this 'definite formulation' three technical things from Sections 6 and 7: the notion of 'absolute sphere of operation', and the precise formulation of the principles of substitution and iteration.) It seems to me that because of this method of exposition, the content of Sections 4–7 has caused much confusion about Weyl's intentions and system. Our formalization of Weyl's system is mainly based on Section 8, together with the explanation of it that Weyl provided in [44]. Unfortunately, even Section 8 leaves open some technical questions that should be faced when one

[5] Note that Weyl himself has not used the concept of predicativity in his book.

tries to fully formalize the system described there. However, these questions can be answered by a careful examination of what is done in the *second chapter* of [43], in which the system developed in the first chapter is put into use. (The formalizations of Feferman were exclusively based on the first chapter.)

Another significant problem is Weyl's terminology. Some of his notions are vague; some are overloaded; and some are different from those we use today. Though imprecise, the following "dictionary" might be helpful:

- category: type
- blank: (free) variable
- (pertinent) judgement, proposition: sentence, closed formula
- judgement scheme: open formula
- multidimensional set: set of tuples
- intensional: syntactic
- extensional: semantic
- sphere of operation: universe, or more accurately: structure

Finally, a major source of problems in understanding "Das Kontinuum", is that it seems that when he was writing it, Weyl was not yet aware of the need to strictly separate between syntax and semantics. As a result, there are cases in which he is treating the same thing sometimes as a semantic object, and sometimes as a syntactic one. A typical case in which this confusion is clearly seen to modern eyes is in his principle 5 on P.10, which allows to 'fill in a blank in a relation by an *object*' (i.e., substitute an object for a free variable of a formula). By an "object" Weyl almost always means an element of his universe(s), that is: this concept belongs to the semantics of his system. Accordingly, what Weyl really meant here is substituting a (closed) *term* of the language for a variable. (In [17], Feferman corrected this by replacing "object" here by "constant symbol".) In this paper, we disambiguate Weyl's use of notions, strictly separating syntax and semantics.

3.2. WA**: Weyl's formal system for analysis.** *Conventions:* We use σ and τ as metavariables for types, t, s as metavariables for terms, and φ, ψ as metavariables for formulas. We also employ x, y, z, w as general variables for objects, n, k, m as variables for objects of type **N**, f, g as variables for objects of types of functions, X, Y, Z for objects of types of sets. We let $\overline{\sigma} = \sigma_1 \times \cdots \times \sigma_k$, $\overline{\tau} = \tau_1 \times \cdots \times \tau_n$, $\vec{x} = x_1, \ldots, x_n$, $\vec{w} = w_1, \ldots, w_n$, $\vec{y} = y_1, \ldots, y_k$, $\vec{z} = z_1, \ldots, z_k$, $\vec{f} = f_1, \ldots, f_m$, $\vec{X} = X_1, \ldots, X_m$, $\forall x \ldots = \neg\exists x \neg \ldots$, $\exists x : \sigma \ldots = \exists x^{\sigma} \ldots$, $\forall \vec{z} : \overline{\sigma} \ldots = \forall z_1 : \sigma_1 \cdots \forall z_k : \sigma_k \ldots$.

3.2.1. Language.

Types.

(1) **N** is a basic type.
(2) If $\sigma_1, \ldots, \sigma_k$ and $\tau_1, \ldots, \tau_n$ are types, where $k \geq 0$ and $n \geq 1$, then $(\sigma_1 \times \cdots \times \sigma_k) \to S(\tau_1 \times \cdots \times \tau_n)$ is a type.

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Terms and their type(s).

(1) $x^\sigma : \sigma$ whenever x^σ is a variable of type σ.[6] (We assume an infinite supply of variables x^σ for each type σ.)
(2) $f(t_1,\dots,t_k) : S(\overline{\tau})$ in case $f : \overline{\sigma} \to S(\overline{\tau})$ and $t_i : \sigma_i$ for $1 \le i \le k$.
(3) $\{(x_1,\dots,x_n) \mid \psi\} : S(\overline{\tau})$ whenever $n \ge 1$, $x_i : \tau_i$ for $1 \le i \le n$, and ψ is a *delimited* formula.
(4) $\lambda y_1,\dots,y_k.t : \overline{\sigma} \to S(\overline{\tau})$ in case $t : S(\overline{\tau})$ and $y_i : \sigma_i$ for $1 \le i \le k$.
(5) $IT^i_m(f_1,\dots,f_m) : \mathbf{N} \times \overline{\sigma} \times S(\overline{\tau})^m \to S(\overline{\tau})$ if $m > 0$, and for $1 \le i \le m$, either $f_i : \overline{\sigma} \times S(\overline{\tau}) \to S(\overline{\tau})$ or $f_i : \mathbf{N} \times \overline{\sigma} \times S(\overline{\tau}) \to S(\overline{\tau})$.

Delimited formulas (d.f.).

(1) If $t : \mathbf{N}$ and $s : \mathbf{N}$, then $Succ(t,s)$ is a d.f.
(2) If $t : \mathbf{N}$ and $s : \mathbf{N}$, then $t = s$ is a d.f.
(3) If $t_1,\dots,t_n$ are terms of types $\tau_1,\dots,\tau_n$, respectively, and $s : S(\overline{\tau})$, then $(t_1,\dots,t_n) \in s$ is a d.f.
(4) If φ and ψ are d.f., then so are $\neg\varphi$, $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$.
(5) If x is a variable *of type* $\mathbf{N}$, and φ is a d.f., then so is $\exists x \varphi$.

Formulas.

(1) If $t : \mathbf{N}$ and $s : \mathbf{N}$, then $Succ(t,s)$ is a formula.
(2) If t and s are terms *of the same type,* then $t = s$ is a formula.
(3) If $t_1,\dots,t_n$ are terms of types $\tau_1,\dots,\tau_n$, respectively, and $s : S(\overline{\tau})$, then $(t_1,\dots,t_n) \in s$ is a formula.
(4) If φ and ψ are formulas, then so are $\neg\varphi$, $(\varphi \wedge \psi)$ and $(\varphi \vee \psi)$.
(5) If x is a variable and φ is a formula, then $\exists x \varphi$ is a formula.

3.2.2. Logic and axioms.

Logic. Classical many-sorted first-order logic with variable-binding terms operators [22], and with equality in *all* sorts (i.e., types).
Axioms:

Comprehension Schemas.

- $\forall \vec{w}.(\vec{w}) \in \{(\vec{x}) \mid \psi\} \leftrightarrow \psi[\vec{w}/\vec{x}]$,
- $\forall \vec{z}.(\lambda \vec{y}.t)(\vec{z}) = t[\vec{z}/\vec{y}]$.

Extensionality Schemas.

- $\forall X : S(\overline{\tau}) \forall Y : S(\overline{\tau}).X = Y \leftrightarrow \forall \vec{w} : \overline{\tau}.\vec{w} \in X \leftrightarrow \vec{w} \in Y$,
- $\forall f : \overline{\sigma} \to S(\overline{\tau}) \forall g : \overline{\sigma} \to S(\overline{\tau}).f = g \leftrightarrow \forall \vec{z} : \overline{\sigma}.f(\vec{z}) = g(\vec{z})$.

The standard axioms for *Succ*.

- $\exists! n \forall k.\neg Succ(k,n)$,
- $\forall k \exists! n.Succ(k,n)$,
- $\forall k \forall m \forall n.Succ(k,n) \wedge Succ(m,n) \to k = m$.

[6] We shall usually omit the superscript, writing just $x : \sigma$.

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Induction Schema.

$\psi\{0/n\}\wedge(\forall n\forall k.Succ(n,k)\wedge\psi\rightarrow\psi\{k/n\})\rightarrow\forall n\psi$,

Axiom Schemas for iteration.

For each $1\leq i\leq m$:

- $\forall\vec{z}\forall\vec{f}\forall\vec{X}.IT^i_m(\vec{f})(1,\vec{z},\vec{X})=f_i([1,]\vec{z},\vec{X})$,
- $\forall n\forall k\forall\vec{z}\forall\vec{f}\forall\vec{X}.Succ(n,k)\rightarrow IT^i_m(\vec{f})(k,\vec{z},\vec{X})=$
$IT^i_m(\vec{f})(n,\vec{z},f_1([k,]\vec{z},\vec{X}),\dots,f_m([k,]\vec{z},\vec{X}))$.

Where depending on the type of f_i, $f_i([k,]\vec{z},\vec{X})$ means either $f_i(\vec{z},\vec{X})$ or $f_i(k,\vec{z},\vec{X})$, and similarly with $f_i([1,]\vec{z},\vec{X})$.

3.3. Some explanations.

Note 3. It should be emphasized that $\tau_1\times\cdots\times\tau_n$ is *not* a type of WA in case $n>1$. It is only an expression which may occur in notation for types. (For possible motivations, see the discussion in Section 6.3.3.)

Note 4. Weyl did not use of course the λ-notation, which was not known at the time he wrote his book. Therefore, the notation he did use and the accompanying explanations are sometimes obscure. Thus in Section 7 of Chapter 1, one can find expressions like ‘ $R(n;xx'\mid X)$’, where a semicolon ‘;’ is suddenly used without any explanation of its role!

Note 5. Weyl took substitution of arbitrary terms for free variables as one of the construction principles of his system (denoted by “Pr. 7”). In our formalization, we have from the start closed all atomic formulas under this rule. It is straightforward to show that our whole language is closed under this rule too. More precisely: it is easy to prove by structural induction that if $x:\tau$ is a variable, $s:\sigma$ is a term, φ is a (delimited) formula, and $t:\tau$ is a term which is free for x in s and φ, then $s\{t/x\}$ is a valid term of type σ, and $\varphi\{t/x\}$ is a (delimited) formula.

Note 6. In the definition of terms, no conditions are imposed on the variables which are mentioned in clauses 3 and 4 (except for their types). This might seem strange, since a direct reading of Weyl's text might give the impression that we should have demanded in clause 3 that each x_i is free in ψ, and in clause 4 that each y_i is free in t. However, Weyl's remark on P. 38 of [48] that “In each category of set there is an empty and a universal set.” shows that Weyl either from the start did not intend to impose these conditions, or that he saw that the following theorem (which can easily be shown to be equivalent to the content of Weyl's remark) obtains.

Theorem. *Let* WA_w *be the system which is obtained from* WA *by imposing the above-mentioned conditions. Then* WA *and* WA_w *are equivalent.*

Proof. We first show that in the language of WA_w, there is a closed term t_τ in every type $\tau\neq\mathbf{N}$. The proof is by induction on the complexity of τ. For every type σ which has smaller complexity than that of τ, choose some variable x^σ of type σ. Let $s_\sigma=x^\sigma$ if $\sigma=\mathbf{N}$, and some closed term of type σ (which exists by the induction hypothesis)

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otherwise. Define for each σ, the formula φ_σ to be $x^\sigma = x^\sigma$ in case $\sigma = \mathbf{N}$, and $\exists x^{\mathbf{N}}.(s_{\tau_1'},\ldots,s_{\tau_l'}) \in x^\sigma(s_{\sigma_1'},\ldots,s_{\sigma_m'})$ in case $\sigma = \sigma_1' \times \cdots \times \sigma_m' \to S(\tau_1' \times \cdots \times \tau_l')$. Then for each σ of complexity smaller than τ, φ_σ is a delimited formula whose sole free variable is x^σ. Suppose now that $\tau = (\sigma_1 \times \cdots \times \sigma_k) \to S(\tau_1 \times \cdots \times \tau_n)$. Let $\psi = \varphi_{\sigma_1} \wedge \cdots \wedge \varphi_{\sigma_k} \wedge \varphi_{\tau_1} \wedge \cdots \wedge \varphi_{\tau_n}$. Then $\lambda x^{\sigma_1},\ldots,x^{\sigma_k}.\{(x^{\tau_1},\ldots,x^{\tau_n}) \mid \psi\}$ is a closed term of type τ.

Next, let $U_{S(\overline{\tau})} = \{\overline{x} \mid \overline{x} \in t \vee \overline{x} \notin t\}$, where t is some closed term of type $S(\overline{\tau})$. Then $U_{S(\overline{\tau})}$ denotes the universal set of $S(\overline{\tau})$. Similarly, $\emptyset_{S(\overline{\tau})} = \{\overline{x} \mid \overline{x} \in t \wedge \overline{x} \notin t\}$ is a closed term that denotes its empty set.

Finally, suppose ψ is a delimited formula, and $x_1,\ldots,x_n$ variables of type $\tau_1,\ldots,\tau_n$ (respectively). Then $\psi \wedge x_1 \in U_{S(\tau_1)} \wedge \cdots \wedge x_n \in U_{S(\tau_n)}$ is equivalent in WA_w to ψ. This fact easily implies the theorem. ⊣

NOTE 7. In the two last clauses in the definitions above of formulas and delimited formulas, we are following Weyl in the most faithful way, since in his six logical principles for constructing new formulas from old ones, Weyl indeed chose to take negation, disjunction, and conjunction as the basic connectives, and the existential quantifier as the sole basic quantifier. Weyl partially explained the second of these choices by the fact that $\forall$ can be defined in terms of $\exists$ and $\neg$. (This is explicitly noted in the second paragraph of P. 12.) However, Weyl noted also [P. 11] that conjunction can be defined in terms of $\vee$ and $\neg$. Hence, there is some mystery here: why in the case of $\vee$ and $\wedge$ Weyl chose to include both as primitives, while in the case of the two quantifiers he chose to take only one of them as primitive? And why $\exists$ rather than $\forall$? Was this just an arbitrary choice, or was there a deeper reason for it? No answer can be found to any of these questions in the works of Weyl. Nevertheless, Weyl's choices here might be due to some strong intuitions of a great mathematician, since in Section 8, we shall see that Weyl's choices are precisely the right ones!

3.4. Example: cardinalities. As an example of how our formalization works, we now formalize in it the definition of the notion of cardinality that is given in Chapter 1 of [43].

In what follows we use, like Weyl, k,m,n as variables of type $\mathbf{N}$. For $X : S(\mathbf{N})$, let $X - \{m\}$ denote $\{k : \mathbf{N} \mid k \in X \wedge k \neq m\}$. Define:

$$\mathcal{N} = \{n \mid n = n\}, \qquad \mathcal{P} = \lambda X : S(\mathbf{N})\{Y : S(\mathbf{N}) \mid \forall n.n \in Y \to n \in X\},$$

$$\mathsf{d} = \lambda Y : S(S(\mathbf{N})).\{X : S(\mathbf{N}) \mid \exists m.m \in X \wedge X - \{m\} \in Y\},$$

$$Card = \lambda X : S(\mathbf{N}).\{n : \mathbf{N} \mid X \in IT_1^1(\mathsf{d})(n, \mathcal{P}(\mathcal{N}))\}.$$

Obviously, $\mathcal{N}$ denotes the set of natural numbers (as an *object* rather than as a type) and $\mathcal{P}(\mathcal{N})$ denotes its powerset. Therefore, it is easy to show (by induction on n in the metalanguage) that $IT_1^1(\mathsf{d})(n, \mathcal{P}(\mathcal{N}))$ denotes the collection of sets of natural numbers that have at least n elements. In turn, this implies that if X is a set of natural numbers that has exactly n elements, then $Card(X)$ is the set $\{1,2,\ldots,n\}$, while if X is infinite then $Card(X) = \mathcal{N}$. Note that $\mathsf{d} : S(S(\mathbf{N})) \to S(S(\mathbf{N}))$!

It is important to note that in Chapter 2 of [43], Weyl proved in his system some basic properties that $Card(X)$, as defined above, should be expected to have. The first of them is that if $n' \in Card(X)$ (where n' is the successor of n), then $n \in Card(X)$. (From this it easily follows that $Card(X)$ is an initial segment $\mathcal{N}$.) Using the comprehension axioms and extensionality axioms of WA, this is equivalent to:

$$\forall n \forall k \forall X : S(\mathbf{N}). Succ(n,k) \wedge X \in IT_1^1(\mathsf{d})(k, \mathcal{P}(\mathcal{N})) \rightarrow X \in IT_1^1(\mathsf{d})(n, \mathcal{P}(\mathcal{N})).$$

This proposition is proved in Chapter 2 by induction on n. Note that what is proved by this induction is *not* a delimited formula. Hence it needs the full power of the induction *schema*. (This fact was first observed in [3].)

3.5. Previous formalizations of "Das Kontinuum". The above formalization of Weyl's system in [43, 48] is not the first one that has been suggested. This section provides short descriptions of those previous ones that I am aware about. In the sequel, it will become clear (I hope) that none of them is fully satisfactory, while the one given here does reflect *exactly* the system that Weyl was describing and using in his book.

As noted in the introduction, the most well-known and generally accepted work on the subject is that of Feferman in [17] and [19]. In these papers, Feferman has explicitly associated with "Das Kontinuum" four different modern formalizations. What he took to be the most adequate formalization of what he believed that Weyl had had in mind is the *second-order* system $\mathbf{K}^{(\alpha)}$ which he presented in detail in [17], and described in [20] as "the one customarily ascribed to Weyl". $\mathbf{K}^{(\alpha)}$ (so he wrote) "formalizes that part of Weyl's system that meets his aim of purely arithmetical interpretation". (We shall see in Section 4.4.2 that Weyl had no such aim, at least not in the sense of 'arithmetical' that Feferman had in mind.) Then he noted that $\mathbf{K}^{(\alpha)}$ is practically equivalent to the system called $\mathbf{ACA}_0$ in the area of reverse mathematics [40]. Since $\mathbf{ACA}_0$ is simpler (and better known) than $\mathbf{K}^{(\alpha)}$, it is the one which is presented in [19] as "a modern formulation of Weyl's system". However, we have seen in Section 3.4 an important example of an induction made in [43] that cannot be reduced to the induction *axiom* of $\mathbf{ACA}_0$. The only possible alternative to $\mathbf{ACA}_0$ as a modern counterpart of Weyl's system which is (briefly) mentioned in [17] and [19], is indeed the much stronger version $\mathbf{ACA}$ of $\mathbf{ACA}_0$, which is obtained from the latter by replacing its single induction *axiom* by the full induction *schema* that is adopted also in WA. Finally, a forth candidate which is briefly mentioned in [17] is a *third*-order system called $\mathbf{K}^{(\beta)}$. The need to introduce it arose because Feferman had realized that regardless of the strength of induction, Weyl's system as a whole is necessarily stronger than $\mathbf{K}^{(\alpha)}$—a fact that he (wrongly) attributed to an incoherence in Weyl's principles. Feferman claimed that in $\mathbf{K}^{(\beta)}$ Weyl's system *can* fully be captured.[7] However, since Weyl allowed sets of arbitrary order, neither $\mathbf{ACA}$ nor $\mathbf{K}^{(\beta)}$ can possibly capture his system.

[7] Surprisingly, no official presentation of $\mathbf{K}^{(\beta)}$ can be found in [17] (or elsewhere). However, from Feferman's hints, one might guess what system he had in mind.

Though it is not impossible that $\mathbf{K}^{(\beta)}$ (or **ACA**) suffices for every piece of mathematics that Weyl actually developed in his book, the potential strength of his system is much higher than that of $\mathbf{K}^{(\beta)}$ or **ACA**. (Other important differences between Feferman's formalizations and ours will be noted in the sequel.)

NOTE 8. The second part of [17] is devoted to Feferman's system **W** (named after Weyl). That system is not presented as a possible exact formalization of Weyl's system in [43]. It is only described there as very close to it in its spirit and intentions. (This is debatable, since Feferman uses in **W** the machinery of his explicit mathematics. However, the operational approach, on which the latter is based, is not really compatible with Weyl's views, since Weyl gave priority to relations over functions.) Like WA, and unlike $\mathbf{K}^{(\alpha)}$ and $\mathbf{K}^{(\beta)}$, **W** is based on a system of *types*, which includes types of any finite order. However, that system is much more complicated and extensive than that used by Weyl, since it is a system of what Feferman called *flexible* types. Thus unlike WA, **W** has *variables for types*, and related equality formulas.

A different formalization of Weyl's system in [43] is outlined (more or less) in S. Pollard's introduction to [48]. Unlike Feferman's official formalizations, Pollard's one is not limited to any particular order, but allows sets of any finite order. The corresponding universe is created in an accumulative way, so there are still no type distinctions. (As a result, Pollard found it necessary to introduce the extensionality axiom in a strange way, which is not even hinted in Weyl's book.) Another significant difference between Pollard's formalization and all the other ones mentioned here is that according to it, Weyl's functions are not objects in Weyl's universe. (Hence there are no variables for functions in that formalization.)

An almost fully accurate and faithful previous formalization of Weyl's system in [43] is the one described by R. Adams and Z. Luo in [2–4]. Except for one significant difference (explained below), the formalization that is provided in the present paper is very similar to theirs (though it was independently done). Unfortunately, the emphasis in their papers was on how their formalization of [43] is implemented in the authors' general framework of Logic-enriched type theory. Therefore, no attempt was made in them to give support to their interpretation of [43] from the text. They also gave very little attention to describing and analyzing the philosophical assumptions and views that underlie [43], or to explain its intended semantics (as we do here). As a result, Adams and Luo's work has passed practically unknown among people interested in the foundations of mathematics, and has not got the attention that it deserved.

Adams and Luo's formalization differs from ours in the following points:

- The most significant difference between our formalization and that of [4] is with respect to the treatment of equality. In [4] only equality between terms of the basic type **N** (or, more generally, the same *basic*

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type in versions where other basic types exist) is taken as primitive, while equality in other types is taken as *defined*. However, Weyl is very clear about this issue in his official description of his system:

> [T]hese basic relations are to be augmented by the identity relation (whose blanks can now be affiliated with *any* category of object of the expanded sphere, both blanks with the same category of course). [P. 44]

Their misinterpretation of Weyl's use of equality causes serious complications to Adams and Luo in translating Weyl's theorems into their system. Thus they had to introduce a notion of "extensional function" that was never used by Weyl, and write "Let $f : X \to Y$ be an *extensional* function" instead of "Let $f : X \to Y$ be a function".

- In order to simplify their system and its implementation, Adams and Luo explicitly chose to make the extensions below to Weyl's system (correctly noting in what ways these are extensions). Their system is most probably conservative over Weyl's system, but their extensions are in conflict with some of his basic principles and ideas.
  - They introduce $\times$ as an operation on types, with corresponding new operations on objects (the pairing operation, and its two converses). This approach surely has its advantages; but it also has the disadvantage that $(\tau \times \sigma) \times \theta$ and $\tau \times (\sigma \times \theta)$ are different types. It seems that Weyl has considered this distinction to be artificial. (See Section 6.3.3 for a detailed discussion.)
  - Instead of the single basic *relation Succ* on type $\mathbf{N}$, they use a *constant* 0 and a unary *operation* s. This change (and the introduction of the operations connected with $\times$) is not compatible with the priority that Weyl gave to relations over functions.
  - They allowed function types of the form $\sigma \to \tau$ for *arbitrary* τ, and together with it function terms of the form $\lambda y_1, \ldots, y_k.t$, where t may be a term of *any* type (not only a type of sets). This is undoubtly a very natural extension from the point of view of modern type theories. However, it is not coherent with Weyl's quite original and deep ideas concerning functions, and the importance that he has explicitly attached to his special notion of a function ([48], p. 34). We shall elaborate on these notion and ideas in Section 4.3.

§4. The ideas and principles behind Weyl's system.

4.1. Objects, categories, sets, relations, and formulas. Like Russell and Whitehead in [49], Weyl tries to ensure the predicativity of his system by using a combination of two *independent* means, and (again like in [49]), one of these means is the use of *types*. The corresponding notion which is used by Weyl is that of a *category*. In this paper, we reserve (as far as possible) this notion of Weyl for the *semantic* level (in which it corresponds to a certain collection of objects), and use 'type' for the syntactic one. Similarly, we follow Weyl in referring to the elements of the categories in his universe (i.e., the intended semantics of his system) as 'objects', but using, as usual, the name 'terms' for the syntactical creatures that denote such objects. Here are the basic principles that underlie Weyl's system of categories, objects, and relations between objects:

P1. Every object, as well as each place in a relation, is affiliated with a definite category.
P2. A few (but at least one) of the categories are taken as *basic*. These categories should be extensionally determinate. Associated with each of them are few, particularly simple and well-understood,[8] *primitive relations*.[9] One of these relations should be the identity relation.
P3. The concept of a natural number is a basic, well-understood, mathematical concept, and there should always be a basic category which corresponds to it.
P4. On top of the basic categories, there is an infinite hierarchy of 'ideal' categories. The objects of the ideal categories are sets and functions.
P5. Existence can be attributed only to a set (or a function) which has a (legitimate) *definition*.
P6. Every set is the extension of some relation.
P7. Relations (and so sets) are introduced genetically: they are derived by adequate, "logical" means from some small collection of relations that are taken as basic. As a result, the relations in Weyl's system are those that are defined by open formulas in an appropriate formal language which is an extension of the (possibly many-sorted) first-order language with equality that corresponds to the basic categories (taken as the sorts).
P8. The use of quantification over a collection of sets (or functions) and the use of equality between sets (or functions) should be forbidden *in definitions of objects*.

In the next notes, we provide some explanations of these principles and the related concepts.

NOTE 9. In the description of his system, Weyl distinguishes between three different sorts of entities: judgement schemes, relations, and sets. (In the multidimensional case, the latter are also called "functional connections".) Judgement schemes are purely syntactic entities, and are exactly what we call today open formulas (i.e., formulas with free variables). Sets are mathematical objects of the intended semantics. Finally, relations are intermediate *intensional collections* of objects. They are induced (or defined) by formulas, and in turn induce sets by what Weyl calls "the mathematical process". The central difference between a relation and a set is given by principle **P6**: while two relations of the same type are considered identical

[8] Weyl's terminology is "immediately exhibited".

[9] Sometimes Weyl uses 'basic relation' or 'fundamental relation' as synonyms for 'primitive relation'. We shall do the same in what follows.

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iff their underlying defining formulas are (logically) equivalent, two sets (of the same category) are identical iff they have exactly the same elements.[10]

NOTE 10. Footnote 12 to Chapter 1 of [43] says the following about closed judgment schemes (i.e., *sentences*) concerning the basic categories:

> That they all have one meaning, i.e., express a judgment, is a precise formulation of the hypothesis …regarding the "complete system of self-existent objects". [P. 119]

From this footnote, as well as from the content of the book and its system as a whole, it is clear that concerning the basic categories, Weyl's intention is, essentially, that the (many-sorted, in case there is more than one basic category) first-order language with equality which is induced by the collection of the basic categories (and the primitive relations connected with them) has a definite (and intuitive) intended semantics. With respect to this intended semantics *of the basic categories and the associated primitive relations*, Weyl seems to have a platonist view: those categories and their objects are "self-existent", that is: they have an independent existence; and relative to that semantics every proposition of that language (including those which contain quantifiers) has an *absolute* meaning and an *absolute truth value*. I believe that in addition, by describing basic relations as "immediately exhibited" (our footnote 8), Weyl means that they should be sufficiently simple to be taken as *decidable*. (This should be understood here in the intuitive sense described in P. 18 of [48]: having "a definite, methodical proof procedure for reaching (in finitely many steps) a decision concerning the truth or falsity of every pertinent judgment").

NOTE 11. The justification Weyl gives to **P3** is that "The intuition of iteration assures us that *the concept 'natural number' is extensionally determinate*." [A:P. 110]. Not only this, but following Poincaré, Weyl maintains that "[*T*]*he idea of iteration, i.e., of the sequence of the natural numbers, is an ultimate foundation of mathematical thought.*" [P. 48]. In particular: "Our grasp of the basic concepts of set theory depends on a prior intuition of iteration and of the sequence of natural numbers" [P. 24]. Accordingly, he took this category to be *absolute*, and so always one of the basic categories. In his system for analysis (and therefore in WA), it is the *sole* basic category.

Weyl does not explicitly explain why he thinks that the sequence of the natural numbers is an ultimate foundation of mathematical thought.

[10]Unfortunately, Weyl is not always consistent in his use of the notions of relation and judgement scheme, and often confuses them. For example: the formulation of his first six closure principles Pr.1–Pr.6 explicitly refers to judgement schemes, and relations are not mentioned in them. In contrast, his remaining two closure principles, Pr.7 and Pr.8, explicitly apply to relations, and judgement schemes are not mentioned in them. Nevertheless, as the numbering shows, all these principles belong to one list, which can be viewed either as a list of principles for deriving new formulas from previous ones, or as a list of principles for deriving new relations from previous ones. Weyl does not seem to always distinguish between these two interpretations.

However, a clear hint is provided in his discussion (and rejection) of the claim (of Hilbert, though his name is not mentioned) that arithmetical axioms are mere stipulations, and that asserting a proposition means asserting that it is a consequence of the axioms. He writes:[11]

> The interpretation under consideration proves to be feasible only when one knows that the axioms are *consistent* and *complete* ... But we do not *know* this (although we may believe it). And if this belief is one day to be transformed into insight, then, clearly, since logical inference consists of iterating certain elementary logical inferences, we will attain this insight only through our intuition of iteration, i.e., of the infinite repetition of a procedure. But from this intuition we also directly obtain the fundamental arithmetical insights into the natural numbers on the basis of which the whole *mathesis pura* is logically constructed. [P. 19]

What I think Weyl had in mind here, is that our very notions of mathematical propositions and mathematical proofs are based on the idea of iteration. Thus as explained below, in the case of propositions, Weyl provided in his book six principles for obtaining more and more “complex judgements schemes” from simple ones, using *iteration*. Proofs, in turn, are obtained by iterating the use of certain rules of inference. Therefore, the abilities to formulate propositions and to prove them depend on prior understanding of the general idea of iteration/recursion/induction. But it makes no sense to claim understanding of iteration in this case, but still deny understanding of the simplest case of iteration: that which produces the natural numbers from a single basic object 1 by iterating a single unary operation: the successor. (In fact, in several places in [43] Weyl *identifies* iteration with the use of the sequence of natural numbers.)

Note 12. Recall that a basic category is an extensionally determinate collection of objects, equipped with a few simple basic relations. The above characterization of the natural numbers implies that

> A single basic relation, whose meaning is immediately exhibited, underlies this category — namely, the relation $S(x, y)$ which holds between two natural numbers x, y when y is the immediate successor of x. [P. 25]

In particular, the operations of addition and multiplication are *not* primitive in Weyl’s system. This reflects the fact that Weyl viewed *relations* as primary, not functions. This view is also reflected by the fact that there are no primitive function symbols, or even primitive constants, in Weyl’s system in [43], and functions are introduced in those systems only as a sort of a generalization of relations. (We shall elaborate on this fact in Section 4.3.)

[11] Note that Weyl is conjecturing here (in 1918!) the incompleteness of arithmetics. (This interesting fact has been noted already in [26].)

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NOTE 13. **P4** is a crucial point in which our understanding of Weyl strongly differs from that of Feferman in [17, 19]. However, examples like that given in Section 3.4, where Weyl employs a function $\mathsf{d} : S(S(\mathbf{N})) \to S(S(\mathbf{N}))$, as well as plenty of others in Section 6 below, can leave no doubt about it. The same is true for quotes like the following one:

> [R]elations between objects are themselves objects — between which new relations can obtain. [P. 40, footnote 35]

It should be noted that this quote is actually *imprecise*, since what are objects in Weyl's universe are the *sets* that correspond to the relations, not the relations themselves. This is clearly stated in the *definite*, final presentation of his system (in Section 8 of Chapter 1):

> Joining the objects of the basic categories are objects of new ideal categories, i.e., the sets and functions; [P. 43].

With that understood, it follows from the first quote that Weyl's universe includes basic objects, sets of basic objects, sets of sets of basic objects, etc. Moreover, for every category τ and $n \geq 1$, the category of the n-dimensional sets[12] of objects of category τ is always different from τ (and for $k \neq n$ also from the k-dimensional sets of such objects). Thus Weyl's universe includes objects of any finite order (unlike in $\mathrm{K}^{(\alpha)}$ or $\mathrm{K}^{(\beta)}$).

NOTE 14. **P5** is a positive formulation of Weyl's complete rejection of the notion of "arbitrary set" (sometimes called "quasi-combinatorial set"), which underlies Cantor's set theory, and on which the current standard foundations of mathematics are based. In Weyl's own words:[13]

> No one can describe an infinite set other than by indicating properties which are characteristic of the elements of the set. And no one can establish a correspondence among infinitely many things without indicating a rule, i.e., a relation, which connects the corresponding objects with one another. The notion that an infinite set is a "gathering" brought together by infinitely many individual arbitrary acts of selection, assembled and then surveyed as a whole by consciousness, is nonsensical; "inexhaustibility" is essential to the infinite. [P. 23]

NOTE 15. Practically, the important point concerning relations and sets is that what is directly given a definition are the *relations*, not the corresponding sets which are obtained from them by the mathematical process. So the first crucial question concerning what sets are available to us is: *What constitutes*

[12] Recall that Weyl does not use Cartesian products. So in places we would usually talk on subsets of the Cartesian product of τ and σ, Weyl talks about two-dimensional sets whose first component ('blank') comes from τ and the second—from σ.

[13] The difference between property and relation is that the former is one-dimensional, while the latter may be multidimensional. Like Weyl, henceforth, we take properties to be a special kind of relations, and frequently use the name 'relation' for both.

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a legitimate definition of a relation? The answer is given by Principle **P7**. Thus Weyl writes:

> [O]ne is to restrict oneself to those properties and relations which can be defined in a purely logical way on the basis of the few properties and relations which are given immediately in intuition along with the relevant categories of object. [A:P. 112]

Now, the concept of “logical” in “defined in a purely logical way” is ambiguous in Weyl’s book. At first it refers to the iteration of six principles for defining new relations from old ones, that are listed on pp. 9–11 of [48] (and referred to later as Pr. 1–Pr. 6). In modern terms, these six principles allow the use (in defining new relations from old ones) of negation, conjunction, disjunction, the existential quantifier, and substitutions of variables and constants for free variables of the same type. Once terms for sets and functions are introduced into his language, Weyl allows also substitutions of arbitrary terms for variables of the same type. (This is Pr. 7, in his list of principles of definitions.) As long as “purely logical definability” is limited to the use of these seven principles, it can easily be seen to be equivalent to the classical (many-sorted) definability in the first-order language with equality that is based on the given basic relations. However, later Weyl introduced an eighth, very important, method of deriving new relations from old ones: iteration (Pr. 8 in his list of principles of definitions). Therefore, it seems that by “purely logical definability”, Weyl has in mind something stronger than first-order definability. (We shall return to this question in Section 4.5.2, after a thorough discussion of Pr. 8 in Section 4.4.) These observations are reflected in Principle **P7**.

NOTE 16. Another crucial question is: *does every relation induce a set?* Unfortunately, many sentences of [43] give the impression that this is the case. However, again Weyl is more cautious about what he writes in the definite presentation of his system (Section 8 of Chapter 1) and in the definite explanation of that system in [44]. He makes it clear there that only certain *special* relations, which he calls “*delimited*”,[14] actually induce sets.[15] This constraint of using only delimited relations is the second means (besides the classification of objects/terms into categories/types) that Weyl

[14]Unfortunately, the notion of ‘relation’ is overloaded in [43] and [48], and is not always used consistently. The fact that frequently Weyl writes just ‘relation’ where he has in mind ‘delimited relation’ is a case in point. Another one is when a relation is treated as a syntactic entity. There are also cases in which the notion of relation is used in the meta-meta-language, like in: “Now the claim we are making here about the two judgment schemes U and V does not explicitly mention any intrinsic relation at all which holds between them” [P. 21]. Finally, when Weyl talks about ‘basic relation’ (or ‘primitive relation’), he usually means what we now call ‘atomic predicate’.

[15]There are places (e.g., Chapter 2, footnote 3) where the sets induced by the delimited relations are called “delimited sets”. This seems to imply that there are nondelimited sets. (Those which are induced by relations which are not delimited, that is: by arbitrary open formulas?) However, no use is made by Weyl of such sets, and only the sets which are induced by delimited relations are objects in his universe.

uses in order to ensure predicativity and to prevent vicious circles. Weyl's fundamental observation here is that arguments closely related to those used in Richard's paradox show the following:

> [T]he universal concept "object" is not extensionally determinate, — nor is the concept "property", nor even just "property of natural numbers". [A:P. 110]

Since the natural numbers are the most basic objects in Weyl's universes, the last quote implies that *Weyl saw any (infinite) collection of sets (or functions) as open*. Therefore, by the discussion in Section 2, the demand of predicativity of definitions of objects implies Principle **P8**:

> *The essential thing is that in defining the relations no use is made of the concepts of the equality and existence of sets*; thereby, but also only thereby, do we avoid the meaninglessness of circular definition. [P. 41]

Since the universal quantifier is definable in Weyl's system in terms of the existential one, the above quote forbids to use *in definitions of sets* any quantification over sets. In addition, it allows the use of equality in definitions of objects only if it is between objects of the same *basic* category. The reason for this second constraint is that equality between two sets of objects of some category is equivalent (by extensionality) to a certain universal quantification over that category. Hence the second constraint follows from the first. (Actually, this argument shows that the use of equality in definitions should also be allowed in case the equality is between *sets* of objects of the same basic category. However, because of the noted equivalence, this would add no extra definability power.)

It should be emphasized that from the description above, it follows that *formulas have two different roles in Weyl's system*. One is the usual one, of expressing proposition schemes (or "judgment schemes", in Weyl's terminology) about the objects in the intended universe(s). Their other role, for which only delimited formulas can be used, is as the main tool for defining (and by this creating) those objects that Weyl calls "ideal", that is: sets and functions. This fact might be responsible for the confusion one can find in the literature about the use of quantifiers in [43]. Thus in [17] Feferman wrote: "It is not clear what position Weyl would take on quantification over relations and functions as part of the language, since they are certainly to be excluded in defining conditions of relations and functions." This is surprising, for two reasons. First, Weyl is actually very clear on this point in his definitive description of his systems:[16]

> The current meaning of "pertinent judgment" emerges from Section 2: they are those judgments (in the proper sense, i.e., without blanks) which arise, through *unrestricted* application of principles 1 through

[16]Principle 6 mentioned in this quote allows quantification over variables. Note that the emphasis on 'unrestricted' in this quote was done by Weyl himself.

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> 6 of Section 2, from the above basic relations of the expanded sphere of operation. [PP. 43–4]

Second, and more important, formulas in which quantifications is made on real numbers (which are sets in Weyl's system—see Section 4.2), and even on functions from real numbers to real numbers, are repeatedly made in Chapter 2 of [43]—and Weyl explicitly noted this! (We shall encounter some examples in the sequel.)

Actually, it is not peculiar to Weyl's system that formulas are used for two different purposes, and that for one of them, not every formula can be used. The same observation applies to every set theory in which an attempt is made to avoid the usual set-theoretical paradoxes. Thus in ZF, both $x \neq x$ and $x \notin x$ are formulas that serve as proposition schemes. However, while $x \neq x$ can be used to define a set in the universe of ZF, $x \notin x$ cannot be used for this purpose. What has been special about Weyl's system is that like its collection of all formulas, the collection of formulas that can be used for defining sets (the delimited formulas) is also *syntactically* defined using specific recursive *principles of definitions* (which are very similar to the rules for defining arbitrary formulas).

It should be clear now how the principles **P1**–**P8** are reflected in WA. Thus, in addition to the usual two main syntactic categories which we have in first-order languages (terms, denoting objects, and formulas), WA includes a syntactic category of *types*, which reflect Weyl's categories. As usual in simple type theories, each term is associated with a unique type.[17] Among the types, some types are *basic*; they are the syntactic counterpart of Weyl's basic categories. (In WA, the only basic type is the type $\mathbf{N}$ of the natural numbers.) All other types are types of terms for sets or functions, and are derived from the basic types by repeated use of two operations: one for introducing types of terms for sets, the other for introducing types of terms for functions. The latter will be explained in Section 4.3. The former is obtained as follows: If $\tau_1, \ldots, \tau_n$ are n arbitrary types, then $S(\tau_1 \times \cdots \times \tau_n)$ is the type of terms for n-dimensional subsets of (intuitively) $\tau_1 \times \cdots \times \tau_n$. As usual, complex formulas are obtained from atomic ones using the standard first-order connectives and quantifiers, while the atomic formulas are obtained by applying basic predicates to terms of the appropriate types. These basic predicates always include (and in WA also only include):[18]

[17]In principle, Weyl does not make this uniqueness demand. On the contrary: on pp. 61–62 of [48] he notes that the constructed set of fractions (i.e., positive rational numbers) can be taken as a new basic category. He further notes that in such a case, this new basic category will be a subcategory of the nonbasic category we denote by $S(\mathbf{N}^2)$. However, he continues that "except for an expanded vocabulary, this approach offers us no more than does the direct continuation of our construction begun on the basis of the single basic category 'natural number'." This seems to indicate that Weyl did not like the possibility of an object belonging to two distinct categories, even though he does not reject it. Anyway, in his system for analysis, every object is indeed affiliated with a unique category. This is in sharp contrast to Feferman's **W**, in which types may have infinitely many subtypes.

[18]In principle, '$=$' is not just one predicate, but a family of predicates, one for each type. Similarly '$\in$' should in fact be taken as a family of predicate, one for every tuple $(\tau_1, \ldots, \tau_n)$ of types. However, this is not a significant issue.

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- The binary predicate *Succ* on **N**. (Weyl uses S instead of *Succ*.)
- The equality predicate $=$ between terms of the same type.
- The relation $\in$ between tuples of objects and sets of such tuples.

As for the syntactic category of *terms* (which is needed for determining the syntactic category of atomic formulas, and so of formulas in general), we note that there are no terms in Weyl's system of type **N**, except for the variables of this type. In contrast, it has a rich system of terms for sets. Those terms are formal counterparts of the delimited relations discussed above. This is not surprising, since in formal languages objects are denoted by terms, and sets are objects in Weyl's universe. Therefore, instead of Weyl's trichotomy: judgement schemes—delimited relations—sets (where the first is syntactic, the third semantic, and the middle intermediate) we have the trichotomy: formulas—set terms—sets (where the first two are syntactic, the third semantic). This means that set terms of type $S(\tau_1 \times \cdots \times \tau_n)$ are of the form $\{(x_1,\dots,x_n) \mid \psi\}$, where $x_i : \tau_i$ for $1 \leq i \leq n$, and ψ is a *delimited* formula. In turn, according to **P8**, a delimited formula of WA is a formula in which all quantifications are over variables of type **N**, and all equalities are between two terms of type **N**.

4.2. The real numbers. The real numbers are introduced in [43] as Dedekind cuts in $\mathbb{Q}$. Since elements of the latter are essentially represented in [43] by quadruples of natural numbers (where $\langle k,l,m,n\rangle$ represents $k/l-m/n$), a real number is in WA an object of the category/type $S(\mathbf{N}^4)$. (See Section 6.3 for details.) Now the only quantifications used in the definition of what is a Dedekind cut are on the rational numbers, and so can be reduced to quantifications over **N**. However, since real numbers are sets, $\mathbb{R}$ (the set of all real numbers) is a set of sets.[19] Therefore, in contrast to the set of natural numbers, $\mathbb{R}$ is *open*:

> [*T*]*he concept "real number" is not extensionally determinate.*
> [A:P. 111]

It follows, first of all, that quantifications over $\mathbb{R}$ is not allowed in *definitions of sets and functions*. Other important implications of the fact that $\mathbb{R}$ is open will be discussed in Section 6.

4.3. Functions.

4.3.1. Weyl's notion of a function. We turn, at last, to the nature of the third sort of objects in Weyl's universes: functions. As was emphasized above, Weyl totally rejected the modern view of a function as any set of pairs that satisfies the functionality condition. He insisted that (like sets) functions can be determined only by *rules*. Since he was mainly interested in functions with range $\mathbb{R}$ (the set of real numbers), and every element of $\mathbb{R}$ is of the category $S(\mathbf{N}^4)$, Weyl limited his notion of a function to functions whose range is of the form $S(\tau_1 \times \cdots \times \tau_n)$, that is, a category of sets. Now recall that up to now, every term of such a category which is not a variable has

[19]Note that Weyl explicitly says on P. 67 that $\mathbb{R}$ is an object in his universe.

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the form $\{(x_1,\dots,x_n) \mid \psi\}$, where $x_i : \tau_i$ $(i = 1,\dots,n)$. Suppose now that $y_1 : \sigma_1,\dots,y_k : \sigma_k$ are free variables of ψ such that $\{x_1,\dots,x_n\} \cap \{y_1,\dots,y_k\} = \emptyset$. Then it is natural to define a 'function of several variables' by assigning to each tuple $(a_1,\dots,a_k)$ such that a_i belongs to the category (denoted by) σ_i $(i = 1,\dots,k)$ the set of all tuples $(b_1,\dots,b_n)$ such that $(a_1,\dots,a_k,b_1,\dots,b_n)$ satisfies ψ. This is exactly the sort of functions that Weyl accepted and employed. Accordingly, in WA, there are types for categories of functions of this sort, and terms for denoting such functions. These types have the form $(\sigma_1 \times \cdots \times \sigma_k) \to S(\tau_1 \times \cdots \times \tau_n)$, and the *main* form of terms of such a type have the form $\lambda y_1,\dots,y_k.\{(x_1,\dots,x_n) \mid \psi\}$.

Function terms can of course be applied. This obvious fact implies that once they are introduced, each type of the form $S(\tau_1 \times \cdots \times \tau_n)$ has new terms, in addition to those that had been available before the introduction of function terms. Thus $f(x) : S(\tau)$ whenever f and x are variables such that $f : \sigma \to S(\tau)$ and $x : \sigma$. Accordingly, Weyl's system has other function terms in addition to those called above the 'main' ones. In general, given any term $t : S(\tau_1 \times \cdots \times \tau_n)$, and variables $y_1 : \sigma_1,\dots,y_k : \sigma_k$ that are free in t, we have a corresponding function term of the form $\lambda y_1,\dots,y_k.t : (\sigma_1 \times \cdots \times \sigma_k) \to S(\tau_1 \times \cdots \times \tau_n)$. Such a function term, in which t is not of the main form, and even contains parameters, is used, for example, at the proof (in the second chapter of [43]) that any function which is continuous over the unit interval is uniformly continuous there. The proof starts with the words: "if f is the given real-valued function …". In these words, f is a *variable* of type $S(\mathbf{N}^4) \to S(\mathbf{N}^4)$, and the words express the *assumption* that $f(x)$ is a real number whenever x is a real number. Later in the proof, Weyl uses the expression f^*, which denotes the function term $\lambda \mathsf{l} : \mathbb{Q}.f(\mathsf{l}^*)$, where l^* is the Dedekind's cut determined by l, that is, $f^* = \lambda \mathsf{l} : \mathbb{Q}.f(\{x : \mathbb{Q} \mid x < \mathsf{l}\})$.[20] Note that f^* is a function term which *contains a free variable* (f), and it is not of the canonical form $\lambda y_1,\dots,y_k.\{(x_1,\dots,x_n) \mid \psi\}$.

The following should be noted concerning the syntax of function terms:

1. As the example discussed in the previous paragraph shows, a function term $\lambda y_1,\dots,y_k.t$ *may have free variables* (i.e., t may have other free variables beside $y_1,\dots,y_k$). This fact is not clear at all from the description of the system in the first chapter of [43]. On the contrary, what is written there gives the opposite impression. However, like in the above example, the content of the second chapter clearly shows that this is the case. Moreover, Weyl writes this there explicitly:[21]

   > All this carries over *mutatis mutandis* to *function sequences*, i.e., to cases where the relation $R(\mathsf{l}|n)$, *which defines the sequence, contains blanks in addition to those indicated.* [P. 76]

[20] Since $\mathbb{Q}$ is not officially a type/category in Weyl's system, the actual term which is denoted by f^* is more complicated. Thus, the real term should start with something like $\lambda k : \mathbf{N}, l : \mathbf{N}, m : \mathbf{N}, n : \mathbf{N}$ rather than with $\lambda \mathsf{l} : \mathbb{Q}$. On P. 82, Weyl notes this explicitly.

[21] By 'the relation $R(l \mid n)$', Weyl means what we have denoted by $\lambda n.\{l \mid \psi\}$, where ψ is the formula which defines R.

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2. Function terms may denote functions whose range is *any* category of the form $S(\tau_1 \times \cdots \times \tau_n)$, not only $\mathbb{R}$ (or rather $S(\mathbf{N}^4)$). Thus one of Weyl's first examples (at the end of Chapter 1, Section 6) is $\lambda X : S(\tau).\{x : \tau \mid x \notin X\}$, the complement function on sets of objects from a category τ. (Recall that another, particularly illuminating, example was given in Section 3.4.) This is another demonstration how far is the type system in [43] from being second-order.
3. Since **N** is a basic category, *there are no function terms or function types in Weyl's system that have* **N** *as their range*. Therefore, all functions from **N** to **N** are taken to be *relations*, and are treated as such. In particular, addition and multiplication on **N** are for Weyl ternary relations on **N** [P. 26]. On the other hand, Weyl noted that

   > [T]he cardinal number $p(n)$ of the prime natural numbers up to $n, \ldots$ is a *function* of n in our precise sense. The same holds for all other "number-theoretic functions".[22] [P. 58]

So far about the syntax of function terms and function types. Their intuitive semantics is what one would expect: The objects of the category which corresponds to the type $(\sigma_1 \times \cdots \times \sigma_k) \to \tau$ are intended to be functions that assign to *each* tuple of objects of the categories (which correspond to) $\sigma_1, \ldots, \sigma_k$ an object of the category (which corresponds to) τ. What function each function term induces is obvious. One should only note that like in the case of sets, functions are extensional objects. In other words, the functions that correspond to two function terms of the same type are equal iff to each tuple of possible arguments they assign the same value.

Another very important property of functions in Weyl's universe(s) is that *they are always defined:* If f is an object of $(\sigma_1 \times \cdots \times \sigma_k) \to \tau$, and $a_1, \ldots, a_k$ are, respectively, objects of $\sigma_1, \ldots, \sigma_k$, then $f(a_1, \ldots, a_k)$ exists, and is an object of the category τ.[23] This implies that whenever we want to define some real function from $\mathbb{R}$ to $\mathbb{R}$ with some properties, then we have to define it using a function term of the type $S(\mathbf{N}^4) \to S(\mathbf{N}^4)$ (Section 4.2), and make sure that the value that the corresponding function F assigns to an argument from $\mathbb{R}$ is also in $\mathbb{R}$, and that the reduction of F to $\mathbb{R}$ has the required properties. (See the discussion on pp. 67–8 of [48], where the notion of a real-valued function is defined.) For example, any function term which we use for introducing the inverse function on $\mathbb{R} - \{0\}$ will induce a function that is defined on 0 as well. [24]

[22]This is due to the fact that according to Weyl's definition, the cardinal of a finite set is not a number, but a finite initial segment of $\mathcal{N}$.

[23]This is in very sharp contrast with what happens in Feferman's system **W** from [17], that is supposed to be based on Weyl's ideas. Not only **W** allows functions to be partial—it employs a special logic (called LPT, for 'Logic of Partial Terms') for dealing with terms that might not denote anything.

[24]It is interesting to note that for the function I which is induced by the *natural* definition of the inverse, we have $I(0) = \emptyset_{S(\mathbf{N}^4)}$ (where $\emptyset_{S(\mathbf{N}^4)}$ is the empty set of the category $S(\mathbf{N}^4)$). Note that this set in $S(\mathbf{N}^4)$ is *not* a real number.

4.3.2. Discussion. Weyl's explanation of his motivation to introduce functions in the way he did is rather short and ambiguous. Since this is a very important subject, we would like to devote to it a deeper and more detailed discussion.

As was emphasized above, Weyl totally rejected the modern concept of a function. On the other hand, Weyl did consider ([48], [P. 33]) as an alternative reasonable choice (to the one he finally chose) at least one other notion of a function, a notion that at first sight might seem to be more general than the one described above. It is functions as a special kind of delimited relations: those which satisfy the functionality condition. Weyl admitted that "This is one possible formulation of the concept of function". However, he thought that the one chosen by him "seems more natural". He did not give further explanations why he thought so, except for showing that with the more general notion, even a proposition like 'The sum of two real-valued functions is itself a real-valued function' would fail to be correct. However, this is a subtle matter, so let us try to clarify it.

To begin with, Weyl's description of this possible notion of a function is rather vague. Focusing on "single-variable functions", but generalizing a bit what is written on pp. 33–34 about this issue (and translating it into modern language), what Weyl considers here is the principle that if $\forall x \exists! y.(x, y) \in s$, where $x : \sigma$, $y : \tau$, and $s : S(\sigma \times \tau)$, then s defines a function from σ to τ. But how should this condition be understood? It cannot be understood semantically (i.e., as being true in the intended semantics), since then the property of being a function *would not be absolute*, that is: its truth value might not be stable. (We return to this in Section 5.1.) Because of Weyl's principles, this would make any formalization of analysis in his system extremely cumbersome and unnatural. Hence the condition $\forall x \exists! y.(x, y) \in s$ can only be understood syntactically (i.e., as being provable in Weyl's system). In other words: Weyl is considering here the use of a limited form of the extension by definitions procedure [39], in which the formula involved is delimited. However, doing this has the great drawback of destroying a very important feature of this system: So far, the introduction of new terms and relations to the language of the system has been a purely linguistic matter, not requiring to prove something before. *This remains the case when Weyl's notion of a function is used.* But things change radically once the more general notion is adopted, together with the provability interpretation. In such a case, the task of introducing the language cannot be separated from the task of introducing the proof system. An even more serious problem with treating functions using the extension by definitions procedure is that according to it, a formula of the form $\psi\{s(t)/y\}$ is just an abbreviation for $\exists y.(t, y) \in s \wedge \psi$, and so *it is not delimited* (unless the type of y is $\mathbf{N}$). In other words: delimited formulas are not closed under substitution of terms for variables. One result of this unfortunate failure is Weyl's observation that under this interpretation, even the sum of the real-valued functions f_1 and f_2 is in general not available as a function, since the obvious definition of this sum, $\{(x, y) \mid \forall q.q \in y \leftrightarrow \exists q_1 \exists q_2.q_1 \in f_1(x) \wedge q_2 \in f_2(x) \wedge q = q_1 + q_2\}$, would not be a valid term.

Another very interesting remark which Weyl made about his concept of a function is that "[O]nce we become aware of it, we also immediately grasp its significance" [P. 34]. Again he gave no further explanations about this significance. However, well before I read Weyl's book, I had independently used in [7] the same notion of a function in order to provide a unified theory of constructions and operations as they are used in different branches of mathematics and computer science, including set theory, computability theory, and database theory. As explained in [11], my notion has been based on the following two basic principles:

- From an abstract logical point of view, the focus of a general theory of computation should be on functions of the form:

  $$\lambda y_1,\dots,y_k.\{\langle x_1,\dots,x_n\rangle \in S^n \mid S \models \varphi(x_1,\dots,x_n,y_1,\dots,y_k)\},$$

  where S is a structure for some first-order signature σ, φ is some formula of σ, and $\{\{x_1,\dots,x_n\},\{y_1,\dots,y_k\}\}$ is a partition of the set of the free variables of φ. Here, φ is used to define a *query* with parameters $y_1,\dots,y_k$. Accordingly, the tuple $\langle y_1,\dots,y_k\rangle$ provides here the input, while the output is the set of answers to the resulting specific query.

  Note that usual functions to S^n can be identified with functions of the above form in which the output is a singleton.
- An allowable query should be *safe* in the sense that the answer to it does not depend on the exact domain of S, but only on the values of the parameters $\{y_1,\dots,y_k\}$ and the part of S which is relevant to them and to the query, under certain conditions concerning the language and the structures that are taken as relevant to the query.

Obviously, the notion of a function described in the first principle is exactly Weyl's notion. It is also rather clear that the notion of safety described in the second principle is closely related to Poincaré's notion of invariability, and so there is a close connection between 'safe formulas' and 'delimited formulas'. We refer the reader to [7] and [11] for further explanations of these two principles, including their use for characterizing Church Thesis. Here we shall be content with some examples of the usefulness and universality in mathematics of Weyl's notion of a function.

1. Construction problems in Euclidean geometry. Here what one is usually required to do is to find a procedure that given some finite list of points, produces *all* the points that have a certain relation with the given points. Thus the procedure for "computing" the function $\lambda O\lambda A.\{X \mid XO = AO\}$ is to construct the circle with center O that passes through A, while that for "computing" the function $\lambda B\lambda A.\{X \mid XB = XA\}$ is the construction of the perpendicular bisector of AB.
2. Procedures for solving equations and inequalities. Examples here are the procedures learned in high school for solving inequalities of the form $ax^2 + bx + c > 0$, or linear trigonometric equations of the form $Asin(nx) + Bcos(nx) = C$. Another example is the procedure given

in Linear Algebra for solving a system of linear equations with an arbitrary number of unknowns and arbitrary number of equations.

3. Queries in logic programming and queries in relational databases.
4. Practical computations with natural numbers. In every computerized system, what is taken as the type of natural numbers is actually only some finite initial segment of the full set of natural numbers. Therefore, a safe query should be one that has the same answer in all implementations in which this initial segment includes the inputs to the query and the natural numbers mentioned in it. Note that for this, addition, multiplication, etc., should practically be taken as *relations*—exactly as they are treated in Weyl's book.

4.4. Iteration and recursion.

4.4.1. The principle and its justification. As noted above, in Weyl's system for analysis, the natural numbers form the sole basic category. Hence the only quantification which is allowed in it within definitions of sets and functions, is over the natural numbers. This constraint, together with the fact that the successor relation is the only primitive relation which is available for the natural numbers, impose very severe limitations on the power of his system. In fact, it is easy to see that the principles of Weyl that we have described so far are not even sufficient for defining addition or multiplication of natural numbers. To be able to develop analysis in a reasonable way nevertheless, Weyl added to his logical principles of construction a *mathematical* principle which he called 'the principle of iteration' and labelled it as Pr. 8. (The classification of this principle as mathematical rather than logical is due to Weyl himself. See, however, the discussion in Note 15.) Weyl has repeatedly emphasized in his papers the crucial role that this principle has in his system. Thus in [46] he wrote:

> In this system iteration plays the role which in set-theory was played by the uninhibited application of quantifiers. ... we have adhered to the belief that "there is" and "all" make sense when applied to natural numbers: in addition to logic we rely on this existential creed and the idea of iteration.

So what is the principle of iteration? The basic idea behind it is the following. Suppose σ is a type/category and $F : \sigma \to \sigma$. (In Weyl's original system such σ should be of the form $S(\tau_1 \times \cdots \times \tau_n)$.) One can obtain from F a sequence of functions F^1, F^2, F^3, ..., where for $x : \sigma$, $F^1(x) = F(x)$, $F^2(x) = F(F(x))$, etc. In more precise terms, the sequence is recursively defined (in our metalanguage) by the equations:

$$F^1 = F, \qquad F^{n+1} = \lambda x : \sigma . F^n(F(x)).$$

Intuitively, this sequence can be turned into a function $IT(F) : \mathbf{N} \times \sigma \to \sigma$ by letting $IT(F)(n,x) = F^n(x)$ for $n : \mathbf{N}$ and $x : \sigma$. However, the passage from F to $IT(F)$ cannot be done by using the principles we have so far. Its introduction is the simplest (and the most useful, at least in Weyl's work) form that Weyl gives for his principle of iteration. Then he extended it in three directions. First: allowing the use of parameters. Second: allowing simultaneous mutual recursive definitions of functions. Third: allowing the application of IT to F also in case $F: \mathbf{N} \times \sigma \rightarrow \sigma$. In WA, all these forms of the iteration principles are subsumed under one single form.

What justifies in Weyl's view his principle of iteration? Exactly what justified taking the natural numbers as a basic category, and principle **P3**. As we have emphasized, for Weyl, the natural numbers and the *intuition* of iteration cannot be separated, and together they constitute the most basic intuition of mathematics. This intuition also stands behind Weyl's *principle* of iteration. This is clearly explained, for example, in the following quote from [45]:[25]

> The sequence of natural numbers, and the intuition of iteration underlying it, are ultimate foundations of mathematical thought. The crucial importance of these foundations in the construction of all of mathematics is reflected in our iteration principle.

4.4.2. The coherence of the principle of iteration. As explained in [17], Feferman (and independently G. Longo too) believed that he had found a "prima facie incoherence of Weyl's principles" that is caused by Weyl's principle of iteration. Unfortunately, because of Feferman's authority, this wrong belief has become a generally accepted wisdom. Thus, both Mancuso ([28], Chapter 9) and Bernard [12] rely on [17] in saying that Weyl's iteration principle is problematic. Similarly, in [31] Parsons says that what Weyl developed in [43] is what we would call arithmetic analysis, and that Feferman shows in [17] that "Weyl's recursion principle creates a difficulty for this". As we will now show, both claims are simply wrong.

The starting point of this alleged incoherence is one of the most important ideas of the work presented in [43]: to abandon the ramified hierarchy used in [49], according to which subsets of **N** are divided into infinitely many distinct levels. As is well-known, the use of that hierarchy forced Russell to adopt the axiom of reducibility. Concerning this, Weyl wrote the following in [47] (P. 50):

> Russell, in order to extricate himself from the affair, causes reason to commit hara-kiri, by postulating the above assertion in spite of its lack of support by any evidence. In a little book *Das Kontinuum*, published in 1918, I have tried to draw the honest consequence and constructed a field of real numbers of the first level, within which the most important operations of analysis can be carried out.

Weyl's achievement here is correctly described in [17] as follows:

> Weyl saw that he could remain faithful to the tenets of predicativity by confining himself entirely to **N** and subsets of **N** of lowest level

[25]This is a paper of Weyl in which he explained his theory in [43], and explicitly emphasized its merit, but then declared to give it up in favor of Brouwer's intuitionism.

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> (and thereby not requiring any level distinctions), while at the same time accomplishing substantial portions of classical analysis.

However, Feferman then made the following wrong observation:[26]

> This development … may be considered to be a form of arithmetical mathematics, since the sets of level 0 are just those which are arithmetically definable.

From this wrong observation, Feferman derived the following wrong conclusion concerning Weyl's original forms of his principle of iteration:

> [E]ven accepting the first (also the simplest) of these principles conflicts with Weyl's decision to stick to sets of lowest level in the ramified hierarchy.

This was done in [17] by constructing "a relation definable by the iteration principle (8) which is not arithmetical, hence not definable at level 0". Hence Feferman concludes that

> This shows that Weyl's principle (8) is in conflict with his plan to work only with sets definable at the lowest level. Clearly, Weyl was not aware of this problem with his principles.

It should be obvious from all these quotes that the "problem" with Weyl's principles which Feferman believed to have found completely depends on his identification of Weyl's first-level sets of natural numbers with what is now known as the arithmetical sets. This is at best an anachronism. The modern technical notion of an arithmetical set was not known at the time Weyl wrote [43], so Weyl could not possibly decide to stick to those sets. But even had Weyl been acquainted with this notion, he almost certainly would *not* identify the collection of these sets with the collection of first-level sets. First, the principle of iteration is so fundamental for him that there is no way he would have given it up or weaken it in order to stick to the arithmetical sets. Second, unlike in $\mathbf{ACA}_0$ and in Feferman's related formal systems, primitive recursion in general and the operations of addition and multiplication in particular, are *not* taken by Weyl as primitive. As we noted in Section 4.1, Weyl emphasized in [43] that the only primitive relation or function that is (and should be) associated with $\mathbf{N}$ is the relation of successor. Therefore, the *only* way to define arithmetical sets and relations in his system is via his principle of iteration. Without it, the resulting collection of first-level sets would have been extremely weak. So *for Weyl,* the first-level sets definitely included every set of natural numbers which is definable in his system. Actually, this is clearly implied by the content of the following quote, which completely refutes Feferman's claim about the incoherence in Weyl's principles, since Weyl explicitly said in it that the introduction of the principle of iteration effectively destroys the distinction between sensible definability in general and first-level definability:

[26] What Feferman calls here 'level 0' is what Weyl calls 'first level'.

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> We see now, given that the principles of substitution and iteration are to be added, that we can no longer adhere to the notion of a production of relations and corresponding sets in separate *levels*.
> [P. 40]

A similar indication of the rather broad sense that Weyl has given to the notion of first-level definability can be found in the description he gave in [46] of his system in [43]:

> The temptation to pass beyond the first level of construction must be resisted; instead, one should try to make the range of constructible relations as wide as possible by enlarging the stock of basic operations.

This quote also shows that *Weyl has sought to make the collection of first-level sets as extensive as possible, rather than to restrict it.* This fact is in sharp contrast with the belief that he would have decided to restrict his first-level sets to the arithmetical ones had he known the latter.

Finally, in Section 5.1, we will show that not only Weyl did not decide to stick to the arithmetical sets—but also, he explicitly refused to stick to *any* definite collection of sets of natural numbers!

To sum up: the 'incoherence' and conflict that Feferman has spotted were between *his* principles and those of Weyl; not in Weyl's principles themselves.

4.5. The proof system. The rich language of Weyl's system already provides a lot of information about his universe(s). Still, a language does not suffice for making a mathematical system. To prove theorems we also need axioms, together with a logic for deriving theorems from them.

4.5.1. Axioms. The most obvious group of nonlogical axioms in Weyl's system are those that characterize the single basic relation which is assumed by him: the successor relation *Succ* on the natural numbers. Here we simply have the usual Peano's axioms for *Succ*, including *induction*:

> [T]he elementary truths about numbers can, by copious use of induction, be derived from the two "axioms": 'every number has a unique immediate successor' and 'every number other than 1 has a unique immediate predecessor.' [P. 58]

There is one important problem with the content of above quote: it mentions induction, but it does not explain what form of it might be employed. This might be the reason for Feferman's claim in [17] and [19] that there is an ambiguity in Weyl's system concerning this issue, because it is not clear whether Weyl would have admitted the *full scheme of induction*, or just the limited one which is used in $\mathbf{ACA}_0$ and **W** (namely: that a set of natural numbers which includes 1, and is closed under the successor operation, is equal to **N**). However, Feferman is wrong here, because it *is* clear that Weyl does mean here the full scheme. First: there has been no possible reason for Weyl to reject it. On the contrary: accepting it is almost dictated by his ideas. Second (and more important): Weyl's principle of definition by iteration is not limited to the second-order level, but can be used for defining higher-order functions. However, it would be impossible to prove anything nontrivial about functions defined in this way without having sufficiently strong corresponding axioms of induction. A case in which we have already seen such a use of strong induction is Weyl's treatment of cardinalities of sets of numbers that was reviewed in Section 3.4. This example can leave no doubt about the strength of induction that Weyl accepts in [43].

The answer to the question what else is taken as *nonlogical* axioms in Weyl's system depends of course on the question what is taken to be the logic employed there. The only answer to the first question that can be found in Weyl's book is given in the following quote:

> Our principles for the formation of derived relations can be formulated as axioms concerning sets and functions; and, in fact, mathematics will proceed in such a way that it draws the logical consequences of these axioms. [P. 44]

The principles referred to in this quote can be divided (as usual in set theory, second-order logic, and typed or untyped λ-calculi) into two sorts:

Comprehension axioms. which characterize sets and functions (by specifying conditions for elementhood in the case of sets, and the results of applications to arguments in the case of functions).

Extensionality axioms. which specify when two sets or two functions are taken to be equal.

The axioms of both sorts that underly Weyl's system are the standard ones, and exactly what one would expect.

NOTE 17. Note that in the first quote from Weyl in this subsection, Weyl refers to the two basic axioms concerning the natural numbers as "axioms", implying (so it seems) that they are not axioms in the usual sense of his time. Moreover, in [46], Weyl wrote that in *Das Kontinuum,* he "was able to build up in a purely constructive way, and without axioms, a fair part of classical analysis". This seems to me to contradict the two quotes from *Das Kontinuum* brought in this subsection, unless Weyl views all the axioms mentioned here as *logical* axioms. If so, this would mean that Weyl took even the basic axioms of the successor relation as "logical", which is surprising. In any case, it again shows that Weyl did not like to call "axioms" the propositions that function as axioms in his system. Maybe (and this admittedly would need further support) this was due to his objection to Hilbert's views about the role of axioms in mathematics. Still, from the modern point of view, at least the two principles concerning the natural numbers which are mentioned above are definitely *axioms*, without quotation marks.

4.5.2. Logic. It is well-known that Weyl turned to intuitionism not long after the publication of "Das Kontinuum". Therefore, it is important to emphasize that in "Das Kontinuum" itself Weyl used classical logic freely, without expressing any intuitionist tendencies. Thus he wrote:

> "Fermat's last theorem", for example, is intrinsically meaningful and either true or false. But I cannot rule on its truth or falsity

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> by employing a systematic procedure for sequentially inserting all numbers in both sides of Fermat's equation. [P. 49]

What is more, Weyl explicitly accepted and used in [43] the main classical principles that are rejected by the intuitionists: the law of excluded middle, and the equivalence of $\forall x\varphi$ with $\neg\exists x\neg\varphi$:

> Taking U to be any judgment at all, $U \vee \neg U$ is self-evident [P. 17]

> "Every object has such and such a property" means "There is no object which lacks the relevant property." [P. 12]

The last quote is actually the place in the book where Weyl *introduces* the universal quantifier, taking it to be a *defined* quantifier.

According to Feferman, Weyl accepted classical logic in general, and applications of the law of excluded middle to quantified formulas in particular, only with respect to domains which he took as extensionally determinate, but not necessarily for 'open' domains. Thus he writes in [17]:

> In "Das Kontinuum", only one closed infinite totality is assumed, namely, that of the set $\mathbf{N} = \{0, 1, 2, \ldots\}$ of natural numbers. The definiteness of this concept is reflected in the assumption of classical logic with its consequence, for each number theoretic property $P(n)$, that either $P(n)$ is true for all n in $\mathbf{N}$ or $P(n)$ is false for some n in $\mathbf{N}$, that is, in logical symbolism, $\forall nP(n) \vee \exists n\neg P(n)$.... Though Weyl may have been equivocal on this point of underlying logic, either way he clung to the definiteness of the natural number concept, ...

This understanding of Feferman seems indeed to be supported by Weyl's distinction between extensionally determinate collections (like $\mathbf{N}$), in which an existential statement "possesses a sense which is intrinsically clear", and those which are not extensionally determinate (like $\mathbb{R}$), in which this is not the case. On the other hand, the definition given in the first chapter of the universal quantifier in terms of the existential one and negation remains in force throughout the book; nowhere there did Weyl describe a new, independent sense of it when it comes to propositions about real numbers and real-valued functions. This means that classical logic is the official logic of all parts of Weyl's system. (And indeed, the law of excluded middle is applied to the reals several times in Weyl's development of analysis. See Section 6.5.) However, in contrast to what Feferman has thought, there is no conflict between this fact and between Weyl's view of $\mathbb{R}$ as an open collection. We shall explain and clarify this point in Section 5.1.

The next question is: what about equality? The answer to that has already been given in Section 3.5. The same answer is also provided by the way analysis is developed in the first five sections of the second chapter of [43]. It is clear that Weyl took there for granted the axioms of equality that ensure that it is a *congruence* relation. Since Weyl's system is a typed system which employs abstraction terms, it follows that the logic which underlies it *includes* *the whole of classical many-sorted first-order logic with variable binding term operators and equality (where the latter is available in all sorts/types).* This is reflected in WA.

Is the logic used in [43] *identical* with first-order logic with equality? The answer to this question depends on what one takes as 'logic'. As we said in Note 15, Weyl himself seems to be confused concerning this point, and the word 'logical' is ambiguous in his book. Thus he talked there several times about 'general logical principles'. (See, e.g., P. 46 and P. 70.) But what he means by that seems to change from one place to another. On one hand, there are places in which by 'logic' Weyl indeed means just what is called now 'first-order logic'. For example:

> [T]he list of the six principles of definition is, just by itself (unless we are wrong to think it *complete*), of considerable importance for logic. [P. 14]

Weyl's six principles are indeed complete for defining first-order languages without function symbols. So Weyl's belief in their completeness makes sense only if 'logic' is identified with first-order logic, and does not include the principle of definition by iteration, that is introduced later in the book. On the other hand, there are places in the book in which all of Weyl's eight principles of definition, *including iteration* (which is the last one), are taken as logical. As a consequence, induction (the inference rule which is connected with the principle of iteration) is taken as a logical rule too:

> [I]t is natural that an expansion of the forms of inference accompany the extension of the table of our principles of definition. Thus, in particular, the principle of iteration carries with it the Bernoullian "inference from n to $n+1$" (or "inference by complete induction"). [P. 39]

Finally, as been observed in Note 17, there are also places in which (under the influence of Russell's approach?) Weyl seems to take his *system as a whole* to be purely logical:

> I may or may not have managed to fully uncover the requisite general logical principles of construction—which are based, on the one hand, on the concepts "and", "or", "not", and "there is", on the other, on the specifically mathematical concepts of set, function, and natural number (or iteration). (In any case, assembling these principles is not a matter of convention, but of logical discernment.) [P. 46–47]

Whatever the views of Weyl about what is 'logical' and what is not really were—determining them is not so relevant for adequately formalizing his system. For the accuracy of such a formalization, it does not really matter, for example, whether the comprehension and extensionality axioms belong to the underlying logic (making it an infinite-order logic) or not. Neither does it make a difference whether induction is taken as a part of the underlying logic, or as an additional axiom schema. Actually, because of the central place that the iteration idea and the natural numbers have in [43], we believe

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that the logic that best suites the *spirit* of this book is ancestral logic $\mathcal{AL}$, the logic obtained by adding to first-order logic a transitive closure operator. (See, e.g., [5, 13, 29, 30, 38].) However, the underlying logic we have chosen for WA reflects better than $\mathcal{AL}$ what is actually written in [43], and unlike $\mathcal{AL}$, it has corresponding sound and complete proof systems. (This is something that Weyl did not know, of course, when he wrote [43].)

§5. The meaning and use of Weyl's system.

5.1. Intended semantics. Once we have turned Weyl's system into a modern formal system WA, we should ask: what is the semantics of WA from a modern point of view?

One possible answer is that in addition to the natural numbers, the universe of Weyl's system for analysis consists of the sets and functions that are defined in his system, that is: those that are defined by closed terms of WA. Following the theory of knowledge representation in general, and database theory in particular, we may call this the 'closed world assumption' (CWA). However, CWA is in conflict with Weyl's fundamental views of the ideal categories as open collections. As we are now going to show, it also contradicts several places in [43], from which it follows in a decisive way that the number of *intended* models that Weyl envisaged for his system was greater than one. We call this answer to the question about the intended semantics of Weyl's system the 'open world assumption' (OWA).

A first place in [43] in which Weyl clearly adopts the OWA, and this is crucial for his conclusions, is his discussion of the subject of cardinalities (described in Section 3.4 above). This subject is important for Weyl, because basing the natural numbers on the idea of iteration (as Weyl does) explains only their role as *ordinals*, but not their roles as *cardinals*. So Weyl found it necessary to justify that role too. Accordingly, he devotes to this issue pp. 38–39 of the first chapter (in which the cardinality of a set is defined), and pp. 55–58 of the second (where the basic properties of finite cardinals are proved in his system). Unlike in Cantor's set theories, according to Weyl's definition all infinite sets *have the same cardinality*, which he denotes by ∞. The reason Weyl gives for that is *the nonabsoluteness* of ordinals and cardinals in the infinite case.[27]

> [C]ontrary to Cantor's proposal, no universal *scale of infinite ordinal and cardinal numbers* applicable to every sphere of operation can exist. (This does not, however, rule out a universal set theory.) [P. 24]

On the other hand, with finite sets things are completely different:

> The numbers can (in any sphere of operation) be used to determine the cardinality of sets of objects of any basic category. [P. 55]

[27]In a famous wager, Weyl made with Pólya in 1918, Weyl rejected even the statement that every infinite set of numbers contains a denumerable subset. (See [36] or [17] for more details on that wager.) This rejection appears also on [P. 79] of [48]: "[T]he presence of an infinite set of real numbers does not in itself guarantee the existence of a sequence f(n) consisting exclusively of numbers of this set".

In both quotes, it is essential for Weyl to speak about *every* (or *any*) sphere of operation, that is, possible universe. The claims would have made no sense had the CWA been adopted!

Another, particularly significant, place in which the OWA is expressed and used is where Weyl discusses the nonabsoluteness of the most important property that functions over $\mathbb{R}$ may have: continuity. He notes:

> We can see that being continuous at a value *a* is *not* a delimited property of a function (and therefore is dependent on a precise demarcation of the scope of the concept "real number"). In the next section, we plan to give an account of the great significance of this fact for analysis, both pure and applied. [P. 81]

The account promised at the end of this quote is given a few pages later:

> If we regard the principles of definition as an "*open*" system, i.e., if we reserve the right to extend them when necessary by making additions, then in general the question of whether a given function is continuous must also remain *open* (though we may attempt to resolve any *delimited* question). For a function which, within our current system, is continuous can lose this property if our principles of definition are expanded and, accordingly, the real numbers "presently" available are joined by others. [P. 87]

Then Weyl added (in footnote 7 to the last paragraph) that

> Of course, in the case of every function one encounters in analysis, this question does *not* remain open, since the negative judgment which asserts their continuity is a logical consequence of the "axioms" into which the principles of definition change when formulated as positive existential judgments concerning sets. [P. 122]

To begin with, these quotes refute again, in a decisive way, two claims of Feferman that have been mentioned above. First, not only Weyl never said that he had decided to stick to the arithmetical sets—but also here he explicitly talks about the right of extending his current methods of definition, and about the implications of this right. Second, in contrast to what Feferman wrote in [17], they make very clear what position Weyl takes on quantification over relations and functions as part of the language. However, what is really important about the content of these very important quotes (which Feferman seems to completely ignore) is the light which they shed on three important issues concerning Weyl's system and approach in [43].

- Since the intended semantics of Weyl's system consists of an *open* collection of "spheres of operation", its main use cannot be for providing semantics for analysis. It can only be as an *axiomatic theory*, which should be valid for every potential universe for analysis. Despite the fact that Weyl was reluctant to use the word 'axiom' (Note 17), the content of footnote 7 we have just quoted shows that he was very much

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aware of this nature of his system. So does his remark that the fact that there cannot be a universal scale of infinite cardinals does not rule out a universal set *theory*. Actually, Weyl has explicitly said that the goal of his principles is to get a *mathematical theory*:

> If we are to use our principles to erect a mathematical theory, we need a foundation [P. 48]

- The above quotes emphasize again the central role that *absoluteness* plays in Weyl's system. Thus his theory is intended to be able to prove only absolutely true propositions. (Exactly like in set theory, Weyl calls a proposition 'absolute', if its truth value remains "stable and invariant" in the passage from a universe to an expansion of it.)
- One of the examples that Weyl gave of questions that may be asked about a given function of a real variable, is whether it is continuous [P. 54]. Now we see that his answer to this question depends on the model of the theory in which we are working when it is asked. A function that is continuous at our current model might lose this property in an expansion of it.[28] However, in each model, the question does have a definite answer. The same applies to any other proposition φ. It follows that $\neg\varphi \vee \varphi$ always has the truth value true, and so is absolute, even in case φ itself is not. The same applies of course to every other classical tautology. This explains and *fully justifies* the use of *classical logic* in Weyl's system!

NOTE 18. It is very common to read in the literature that the law of excluded middle applies only to a proposition with a definite truth-value (implying, according to [42], that classical logic is valid for a closed universe, while intuitionistic logic should be applied in an open one).[29] This claim is based on a logical confusion between ' φ has an absolute truth value' and $\neg\varphi \vee \varphi$. For the validity of the latter, it suffices that in every relevant universe/structure, φ has a definite truth value. The former means that this definite truth value should be the same in all of them.

5.2. The possibility of iterating the "Mathematical Process". As we saw, Weyl retained the option of adding new principles of definition to his system. In this section, we discuss the only possible candidates that Weyl actually considered in [43] (and decided not to use). Both of these candidates are connected with iterating what Weyl called 'the mathematical process' (Section 4.1). The latter, recall, is the process by which we turn intensional

[28] By a 'given function', Weyl obviously means here *an interpretation of a given function term*. I do not see any other possible way to understand his claim. Note also that this particular proposition is *downward absolute*, that is: when a model is expanded to a bigger one it can change its truth value from true to false, but not vice versa.

[29] Thus Feferman says in [17] that Weyl regarded statements formulated in his terms as "having a definite truth value, true or false, and thus accepted classical logic (first-order predicate calculus) as basic". Similarly, Parsonas writes in [31] that for Weyl, "the question whether …has a definite sense, and the law of excluded middle applies".

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relations into extensional sets and functions. Concerning the use of this process Weyl wrote:

> In the early stages of my presentation, this transition also depended on the concepts of (one- and multi-dimensional) *set* and *function*; and the conception which drove us forward was that of the *iteration* of the construction process ...But in the systematic construction at which I finally arrive (Chapter 1, Section 8) ... *the conception of the iteration* [of the construction process] *is again dropped entirely*, and the concept of set and function must be deferred much longer than was done originally. [A:P. 114]

In order to understand the second part of the last quote, note that the introduction of all the terms of WA (which correspond to the intensional relations accepted by Weyl) is completely independent of applying the ‘mathematical process’. In other words: all the relations, of all categories, are available to us already at the *syntactic* stage of setting the *language*. The ‘mathematical process’, which in practice is imposed by the introduction of the extensionality *axioms*, is then applied to the *whole* system of terms/relations, and it simultaneously provides extensional *semantics* for all of them. This order of development is described by Weyl himself as follows:

> If we imagine, as is appropriate for an intuitive understanding, that the relations and corresponding sets are “produced” genetically, then this production will ... occur in merely parallel individual acts (so to speak). [P. 40]
>
> Only now, at the end, should the *concept of set and function* be introduced. [A:P. 117]

However, Weyl is telling us that at the beginning of the research that has led to [43], he did intend to follow Russel in iterating the construction process and using ramification. By doing this, one gets sets (and functions) that belong to the same category, but have different *levels*. The most important case in which new objects might be obtained in this way is when the corresponding category is $S(\mathbf{N}^n)$ for some n. (Recall that every real numbers is an object of this sort, where $n = 4$.) The first iteration of the mathematical process consists in this case of two steps, of which the second is optional.

1. Adding to $S(S(\mathbf{N}^n))$ a new close term (perhaps in the form of a constant), whose intended semantics is the collection of all the sets that are induced by close terms of type $S(\mathbf{N}^n)$ in WA’s language.
2. Treating the new set as a *new basic category*, by allowing to quantify over its elements in legal definitions of objects.

Allowing only the first step is called by Weyl ‘the narrower procedure’. This process obviously provides a new method for producing new sets of type $S(S(\mathbf{N}^n))$. However, Weyl claims (without a proof) that by adopting it we will not get new sets of type $S(\mathbf{N}^n)$. (In particular: no new real numbers.) In contrast, Weyl proved that the combination of the two steps does provide new real numbers. These numbers are then real numbers of level 2 (where the real numbers of level 1 are those that are definable in WA). By iterating this process (i.e., both steps), one gets real numbers of arbitrary finite level.

There is no place in [43] (or in later works of Weyl) in which the iteration of the process just described is rejected as impredicative.[30] Still, Weyl has decided to avoid any iteration of even the narrower procedure. The reasons he gave for this decision are mainly practical: the resulting system would be too messy, and it seems that nothing important would be gained.

> A "hierarchical" version of analysis is artificial and useless. It loses sight of its proper object, i.e., number. [P. 32]

> As far as I can see, analysis provides no occasion for iterating this expanded mathematical process which includes the principles of construction 7 and 8. [A:P. 117]

Note 19. The last quote explicitly refers to iterating the full process that led to WA, including the substitution principle and the iteration principle (which should not be confused with the iteration of the mathematical process). However, the possibility to iterate the mathematical process was first discussed and rejected already before the introduction of these two principles. In fact, we have seen in Section 4.4.2 that the introduction of the iteration principle is Weyl's *alternative* to the iteration of the 'mathematical process'. Weyl even seems to suggest (in quotes brought there) that there is a conflict between these two methods of construction!

Let us return to the justification of the avoidance of the iteration of the construction process. Though Weyl's practical arguments are convincing, and he is also right in calling such an iteration 'artificial', in my opinion there is also a good philosophical reason (from Weyl's point of view) to reject the idea of even one iteration of the process. The essence of this idea is to treat, for example, the collection $D(\mathbf{N})$ of the sets of natural numbers that are definable by terms of WA as if it is a new *basic* category. The justification of this move seems to be that this collection is (in Weyl's terminology) an extensionally determinate collection. However, according to principle **P2**, this is not enough. A basic category should be equipped with "immediately exhibited" primitive relations between its objects. As I wrote in Note 10, I believe that this means that those relations should be decidable. But even the identity relation on $D(\mathbf{N})$ is obviously undecidable (in the sense that it is undecidable whether two closed terms of WA induce the same set), and it is difficult to see in what sense this relation can be taken as "immediately exhibited". Accordingly, $D(\mathbf{N})$ should not be treated as if it were basic.

Another question, that naturally arises when quantification over $D(\mathbf{N})$ is allowed just because this collection is extensionally determinate, is: why not allowing such quantification over any *present* element of $S(S(\mathbf{N}^n))$ which we

[30] And indeed, a transfinite iteration of this sort is the basis of the famous Feferman–Schütte analysis of the limit of predicativity. (See [16, 21, 37].)

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recognize as extensionally determinate? This was exactly the idea of Hölder in [23] (Note 2 above), and accepting $D(\mathbf{N})$ as legitimate strongly supports Hölder's position. (Recall that Weyl himself considered doing so in the case of the collection $\mathbb{Q}^+$ of fractions. See footnote 17.)

§6. Developing analysis in WA.

In this section, we outline the way Weyl developed the fundamentals of classical analysis in the first five sections of the second chapter of [43]. (Sections 6–8 are less relevant for this, and are left for another occasion.)

6.1. Natural numbers. Recall that the only basic structural elements that Weyl assumes on $\mathbf{N}$ are the binary equality and successor *relations*. So Weyl starts Chapter 2 by showing how to use iteration in order to define addition $(+)$ and multiplication $(\times)$ of natural numbers as *ternary relations* on $\mathbf{N}$. (Thus, e.g., the intended meaning of $+(n,k,m)$ is that m is the sum of n and k.)

Notations From now on, i,j,k,l,m,n will be used as variables of type $\mathbf{N}$. We denote $\{(i,j) \mid i=j\}$ by $Id_{\mathbf{N}}$, and write just IT instead of IT_1^1.

Here is the definition of $+$:

$$F_{Succ} = \lambda X : S(\mathbf{N}^2).\{(m,k) \mid \exists l.Succ(k,l) \wedge (m,l) \in X\},$$

$$R_+ = \{(n,k,m) \mid ((m,k) \in IT(F_{Succ})(n, Id_{\mathbf{N}})\}.$$

Using induction on *n in the metalanguage*, it is easy to see that for every set X of natural numbers, $IT(F_{Succ})(n,X)$ is the set of all pairs (m,k) such that the pair $(m,k+n)$ is in X. This implies that for every n,k,m, (n,k,m) is in R_+ iff $n+k=m$. Hence the above definition of R_+ is indeed adequate.

Next one uses induction on *n within* WA to show that R_+ defines a binary operation on $\mathbf{N}$, that is: $\vdash_{\mathsf{WA}} \forall n \forall k.\exists! m.R_+(k,n,m)$

Handling multiplication of natural numbers in WA using addition is rather similar to the way addition was introduced above using *Succ*:

$$F_+ = \lambda X : S(\mathbf{N}^2).\{(m,k) \mid \exists l.(m,l) \in X \wedge (m,l,k) \in R_+\},$$

$$R_\times = \{(n,m,k) \mid (n=1 \wedge k=m) \vee \exists l.Succ(l,n) \wedge (m,k) \in IT(F_+)(l, Id_{\mathbf{N}})\}.$$

Conventions. In the definition above of $R_\times$, we wrote $n=1$ instead of $\forall l \neg Succ(l,n)$. It is indeed rather inconvenient to continue to use the purely relational official language of WA. Therefore, we shall henceforth use (like Weyl himself did) an extension of the language which has the constant 1, denoting the first natural number, and the usual *function symbols* for the unary successor function and the binary addition and multiplication functions. Thus we shall write $\varphi(1)$ instead of $\exists n.\varphi(n) \wedge \forall k \neg Succ(k,n)$; $\varphi(t')$ instead of $\exists n.\varphi(n) \wedge Succ(t,n)$; and $\varphi(t+s)$ instead of $\exists n.\varphi(n) \wedge (t,s,n) \in R_+$. Since '$\forall k \neg Succ(k,n)$', '$Succ(n,m)$', '$(n,k,m) \in R_+$', and '$(n,k,m) \in R_\times$' are all delimited formulas, the axioms for *Succ* and the above theorem of WA about R_+ can be used for justifying these conventions in a way which is similar (though a little bit more complicated) to the way the usual extension by definition procedure (for ordinary first-order logic) is justified.[31] (See, e.g., [39].) We shall also denote $1'$ by 2, $1''$ by 3, etc.

It is now straightforward to derive in WA all the axioms of PA (first-order Peano's Arithmetics). It follows that WA is at least as strong as PA. Note that in [17] it is shown that the definability power of WA is actually strictly stronger than that of PA, while from results in [3] it follows that the proof-theoretical strength of the former is much bigger than the latter.

6.2. Fractions and rational numbers. After his treatment of the natural numbers, Weyl first turns to the introduction of the fractions (i.e., the positive rational numbers), and then to the introduction of the rational numbers themselves. His treatment of the former is rather standard and straightforward, and is similar to the way it is done, for example, in the classical book [25]. In particular: fractions are essentially equivalence classes of the obvious corresponding equivalence relation on pairs of natural numbers. Here is the formalization of the main definitions:[32]

- $/ = \lambda m,n.\{(k,l) \mid ml = nk\}$ $\quad [/ : \mathbf{N}^2 \to S(\mathbf{N}^2)]$.
- $m/n = /(m,n)$; an element of $S(\mathbf{N}^2)$ of this form is called a *fraction*.
- $\mathbb{Q}^+ = \{q : S(\mathbf{N}^2) \mid \exists m \exists n. q = m/n\}$ $\quad [\mathbb{Q}^+ : S(S(\mathbf{N}^2))]$.
- $+ = \lambda q_1 : S(\mathbf{N}^2), q_2 : S(\mathbf{N}^2).\{(k,l) \mid \exists m_1 \exists n_1 \exists m_2 \exists n_2. q_1 = m_1/n_1 \wedge q_2 = m_2/n_2 \wedge k/l = (m_1 n_2 + m_2 n_1)/(n_1 n_2)\}$ $\quad [+ : (S(\mathbf{N}^2)^2 \to S(\mathbf{N}^2)]$.
- $\times = \lambda q_1 : S(\mathbf{N}^2), q_2 : S(\mathbf{N}^2).\{(k,l) \mid \exists m_1 \exists n_1 \exists m_2 \exists n_2. q_1 = m_1/n_1 \wedge q_2 = m_2/n_2 \wedge k/l = (m_1 m_2)/(n_1 n_2)\}$ $\quad [\times : (S(\mathbf{N}^2)^2 \to S(\mathbf{N}^2)]$.
- $< = \{(q_1,q_2) : (S(\mathbf{N}^2))^2 \mid \exists m \exists n. q_2 = q_1 + m/n\}$ $\quad [<: S((S(\mathbf{N}^2))^2)]$.

Note that in all the definitions above except the first, the equality symbol is put between two terms of type $S(\mathbf{N}^2)$, and so the resulting formulas are equivalent (by extensionality) to delimited formulas.

It is a straightforward matter now to prove in WA that $\mathbb{Q}^+$ is closed under $+$ and $\times$, that these operations and $<$ have in $\mathbb{Q}^+$ all the basic properties we expect (like distributivity, associativity, invertibility under $\times$, etc.), and that $\lambda n.n/1$ is an embedding of $\mathbf{N}$ into $\mathbb{Q}^+$.

The next obvious step, the construction of the rational numbers from the fractions, involves a small complication. The standard and natural way of doing so is to imitate the construction of the fractions from the natural numbers, using the addition of fractions instead of multiplication of natural numbers. This would have been straightforward had the fractions been taken as a new basic category. The definitions would have then been as follows:

- $- = \lambda q_1 : \mathbb{Q}^+, q_2 : \mathbb{Q}^+.\{(r_1,r_2) : (\mathbb{Q}^+)^2 \mid q_1 + r_2 = q_2 + r_1\}$,
- $\mathbb{Q} = \{q : S((\mathbb{Q}^+)^2) \mid \exists q_1 : \mathbb{Q}^+ \exists q_2 : \mathbb{Q}^+. q = q_1 - q_2\}$.

[31] Weyl used these conventions without trying to justify them, taking for granted that his readers would see how to eliminate them, and that their use is conservative.

[32] Like in all texts in mathematics, the symbols $+$, $\times$, and $<$ are overloaded here.

However, $\mathbb{Q}^+$ is *not* a basic category in WA, but a subset of the ideal category $S(\mathbf{N}^2)$. Therefore, we should replace $\mathbb{Q}^+$ in the above two "definitions" by $S(\mathbf{N}^2)$. We get then that $-: (S(\mathbf{N}^2))^2 \to S(S(\mathbf{N}^2))$ (which is a third-order operation), while the definition of $\mathbb{Q}$ becomes illegal.

Weyl's solution to this problem is rather simple and obvious. Since each fraction is representable by a pair of natural numbers, a pair of fractions can be represented by a quadruple of natural numbers. The above two "definitions" are therefore changed into the following ones:

- $- = \lambda q_1 : S(\mathbf{N}^2), q_2 : S(\mathbf{N}^2).\{(m_1, n_1, m_2, n_2) \mid q_1 + m_2/n_2 = q_2 + m_1/n_1\}$.
- $\mathbb{Q} = \{q : S(\mathbf{N}^4) \mid \exists m_1 : \mathbf{N}\ \exists n_1 : \mathbf{N}\ \exists m_2 : \mathbf{N}\ \exists n_2 : \mathbf{N}. q = m_1/n_1 - m_2/n_2\}$.

Note that $-: S(\mathbf{N}^2)^2 \to S(\mathbf{N}^4)$, $\mathbb{Q} : S(S(\mathbf{N}^4))$, and that the relations $=$ and $\subseteq$ between elements of $\mathbb{Q}$ are delimited relations (since those relations between elements of $S(\mathbf{N}^4)$ are delimited). It is now a routine exercise to define on $\mathbb{Q}$ (actually: on the whole of $S(\mathbf{N}^4)$) the operations $+$ of addition and $\times$ of multiplication, as well as the order relation $<$. (Note that the latter is a delimited relation on $S(\mathbf{N}^4)$.) It is also easy to prove then in WA all the standard basic properties of $+, \times, =$, and $<$.

Although $\mathbb{Q}$ is not officially a basic category, it can practically be treated as such, since we can use in delimited formulas equality between its elements, as well as quantification over it. The latter can be done as follows:

- $\forall r : \mathbb{Q}.\varphi(r)$ is an abbreviation of $\forall m, n, k, l.\varphi(m/n - k/l)$.
- $\exists r : \mathbb{Q}.\varphi(r)$ is an abbreviation of $\exists m, n, k, l.\varphi(m/n - k/l)$.
- $\{q : \mathbb{Q} \mid \varphi(q)\}$ is an abbreviation of $\{x \in \mathbf{N}^4 \mid \exists r : \mathbb{Q}.\varphi(r) \wedge x \in r\}$.

6.3. Real numbers.

6.3.1. Introducing the real numbers. As we said in Section 4.2, the real numbers are essentially taken in [43] to be Dedekind cuts. Again there is a problem. Since the rationals are elements of $S(\mathbf{N}^4)$, sets of rationals (as Dedekind cuts are usually taken to be) are elements of $S(S(\mathbf{N}^4))$, and so third-order objects—and as such it would be too complicated to handle the real numbers in WA. Weyl's solution to this problem is to take a real number not as a certain set of rational numbers, but as the *union* of such a set. Formally, let $\mathbf{RN} = S(\mathbf{N}^4)$. (This is just a convenient shorter name that Weyl introduced; $\mathbf{RN}$ is not a new basic category.) Weyl called $M : \mathbf{RN}$ a real number if M is neither empty nor the whole of $\mathbf{N}^4$, and $\forall m, n, k, l.(m, n, k, l) \in M \to m/n - k/l \subseteq M$. (It follows that both the rational numbers and the real numbers are elements of $\mathbf{RN}$ according to Weyl's definitions.) In addition, Weyl uses a terminology that connects the treatment of the reals in his system with the standard one. He says that a rational number 'belongs' to a real number M if it is a subset of M. Following this terminology, we too use sometimes below $\tilde{\in}$ as a synonym of $\subseteq$ (that is: write $r \tilde{\in} M$ instead of $r \subseteq M$). Thus, if M is a real numbers, then $(m_1/n_1 - m_2/n_2)\ \tilde{\in}\ M \leftrightarrow (m_1, n_1, m_2, n_2) \in M$. Using this terminology, Weyl defines $\mathbb{R}$ and basic related notions as follows:

DEFINITION.

1. The formulas $Real(x)$, where $x:$ **RN**, is the conjunction of the following delimited formulas:
   - $\forall m,n,k,l.(m,n,k,l) \in x \to m/n - k/l \subseteq x$;
   - $\exists r:\mathbb{Q}(r\tilde{\in}x) \wedge \exists r:\mathbb{Q}\neg(r\tilde{\in}x)$;
   - $\forall r_1:\mathbb{Q}\forall r_2:\mathbb{Q}.r_1\tilde{\in}x \wedge r_2 < r_1 \to r_2\tilde{\in}x$;
   - $\forall r_2:\mathbb{Q}(r_2\tilde{\in}x \to \exists r_1:\mathbb{Q}(r_1\tilde{\in}x \wedge r_2 < r_1))$.
2. $\mathbb{R} = \{x:\mathbf{RN} \mid Real(x)\}$ $\quad [\mathbb{R}:S(\mathbf{RN})]$.
3. Let $f:\sigma \to$ **RN**. The *domain* $dom(f)$ of f is $\{x:\sigma \mid f(x) \in \mathbb{R}\}$.
4. $f:$ $\mathbf{N} \to \mathbf{RN}$ is a *sequence of real numbers* if $\forall n:$ $\mathbf{N}.f(n) \in \mathbb{R}$.
5. Let $T:S(\mathbf{RN}^n)$. $f:$ $\mathbf{RN}^n \to \mathbf{RN}$ is a *real function on* T (of n variables) if $\forall x:$ $\mathbf{RN}^n.x \in T \to x \in dom(f)$.

Note that being a real number and being a sequence of real numbers are delimited properties. In contrast, being a real function of n variables is in general not *(even for* $n = 1$*).*

Defining the standard order relation on $\mathbb{R}$, and the binary operations $+,\times$ of addition and multiplication (respectively) of real numbers, is now done in the usual way. (E.g., $+ = \lambda x_1:$ $\mathbf{RN}, x_2:$ $\mathbf{RN}.\{(m,n,k,l) \mid \exists r_1:\mathbb{Q}\exists r_2:\mathbb{Q}.r_1\tilde{\in}x_1 \wedge r_2\tilde{\in}x_2 \wedge (m,n,k,l) \in r_1 + r_2\}$.) It is also not difficult to prove in WA the following properties of $\mathbb{R}$:

- $+$ and $\times$ are real functions on $\mathbb{R}^2$.
- For $x:$ **RN** let $x^* = \{(m,n,k,l) \mid \exists r:\mathbb{Q}.r < x \wedge (m,n,k,l) \in r\}$. Then the reduction of $\lambda x \in$ $\mathbf{RN}.x^*$ to $\mathbb{Q}$ is an embedding of $\mathbb{Q}$ in $\mathbb{R}$. Moreover, $\{y:$ $\mathbf{RN} \mid \exists r:\mathbb{Q}.y = r^*\}$ is a dense subset of $\mathbb{R}$.
- $\mathbb{R}$ equipped with $+$, $\times$, and $<$ is an ordered field of characteristics 0 that satisfies Archimedes' axiom. Moreover: there are corresponding *definable* inverse functions $-:$ $\mathbf{RN}^2 \to$ **RN** and $\div:$ $\mathbf{RN}^2 \to$ **RN**. (It is provable in WA that if $x,y \in \mathbb{R}$ then $x \div y \in \mathbb{R}$ iff $y \neq 0^*$. However $x \div y$ is defined of course also in case $y = 0^*$. See footnote 24.)

6.3.2. Algebraic numbers. Weyl's next goal is to define the notion of an *algebraic number*. This is not a straightforward task, though. There is no problem, of course, to define separately for each natural number n the set of real numbers which are roots of some algebraic equation of order n with rational coefficients. (Thus in case $n = 2$, it is defined by $\{x:$ $\mathbf{RN} \mid x \in \mathbb{R} \wedge \exists r_1:\mathbb{Q}\exists r_0:\mathbb{Q}.x \times x + r_1^* \times x + r_2^* = 0^*\}$.) However, in order to define the general notion of algebraic number, we should be able to define the *union* of all these sets, and it is not obvious at first sight how to do it. Nevertheless, Weyl solved this problem rather easily by applying an iteration to the following *third-order function* Δ:

$$\Delta = \lambda L \in S(\mathbf{RN}^2).\{(x,b):\ \mathbf{RN}^2 \mid \exists r:\mathbb{Q}.(x, x \times b + r^*) \in L\}.$$

It is easy to see that for every $n:$ **N** and $L:S(\mathbf{RN}^2)$, $IT(\Delta)(n,L)$ is:

$$\{(x,b):\mathbf{RN}^2 \mid \exists r_0:\mathbb{Q},\ldots,\exists r_{n-1}:\mathbb{Q}.(x, x^n \times b + r_{n-1} \times x^{n-1} + \cdots + r_0) \in L\}.$$

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Let $L_0 = \{(a,b) : \mathbf{RN}^2 \mid a \in \mathbb{R} \wedge b = 0^*\}$. Then the set of algebraic numbers is defined by the term: $\{x : \mathbf{RN} \mid x \in \mathbb{R} \wedge \exists n.(x, 1^*) \in IT(\Delta)(n, L_0)\}$.

6.3.3. Complex numbers and pairs. Section 3 (on real numbers) of Chapter 2 of [43] ends with the introduction of the complex numbers. There is nothing surprising or original in the way *this* is done: a complex number is defined as a pair of real numbers. However, Weyl's approach to the notion of a *pair* is certainly unusual.

Recall that although the formal expression $(x_1, \ldots, x_n)$ is used in our formal terms, this is *not* a term itself. (See footnote 12 and Note 3.) Actually, it seems that Weyl was very reluctant to use the notion of a pair: it is *never* mentioned in Chapter 1 of [43] (where Weyl's system is introduced). In Chapter 2, on the other hand, the notion *is* used, but in different ways. Thus it is used in an unofficial way on [P. 51] (of [48]), where the notion of 'set of pair of natural numbers' is explicitly identified with that of 'two-dimensional set of natural numbers', without giving 'a pair' an independent meaning. In contrast, on [P. 66] 'a set of pairs of fractions' is identified with 'a four-dimensional set of natural numbers'. The first time a precise *technical* notion of a pair is used is on [P. 74], where the complex numbers are introduced. It is defined there only for any two *nonempty sets* of some type. The definition goes as follows. Let $A : S(\sigma_1 \times \cdots \times \sigma_n)$ and $B : S(\tau_1 \times \cdots \times \tau_k)$. If $n = k = 1$ then '$\langle A, B\rangle$' is just the Cartesian product of A and B. In the general case, it is the result of *flattening* this product. (The condition that A and B should be nonempty is in order to ensure that $\langle A_1, B_1\rangle = \langle A_2, B_2\rangle$ iff $A_1 = A_2$ and $B_1 = B_2$.) Note that in all cases $\langle A, B\rangle : S(\sigma_1 \times \cdots \times \sigma_n \times \tau_1 \times \cdots \times \tau_k)$. Weyl notes that for every $\overline{\sigma}$ and $\overline{\tau}$, there is a corresponding function $Pair_{\overline{\sigma},\overline{\tau}}$. It is important to add to this the observation that in contrast, the corresponding projection functions are available only if $\overline{\sigma}$ and $\overline{\tau}$ consist of *basic* types.

Note also that according to Weyl's definition, $\langle\langle A, B\rangle, C\rangle = \langle A, \langle B, C\rangle\rangle$. This crucial fact might explain Weyl's reluctance (not explained in his book) to use the ordinary notion of a pair: He might have wanted to avoid the complications which would have been involved by iterating this operation, or by introducing Cartesian product as an independent operation on types. Thus by introducing the product operator, we would get infinitely many practically "basic" types (e.g., $\mathbf{N} \times (\mathbf{N} \times \mathbf{N})$), and various distinctions that it seems that Weyl found useless.

Finally, what are the complex numbers according to Weyl's notion of a pair? Since a real number is for Weyl a nonempty element of $\mathbf{RN} = S(\mathbf{N}^4)$, a pair of real numbers is an element of $S(\mathbf{N}^8)$. Hence Weyl's definitions imply that complex numbers are equivalence classes of 8-tuples of natural numbers under a certain equivalence relation. Therefore, each complex number is characterized by some 8-tuple of natural numbers.

6.4. Sequences and convergence principles.

NOTATION. *For convenience, from now on, we shall not distinguish between a rational number r and the corresponding real number r^*, and usually write just the former where in principle it should be the latter.*

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A sequence of real numbers is just a function from the natural numbers to the reals. Therefore, the type of such a sequence is $\mathbf{N} \to \mathbf{RN}$. The first main theorem about the real numbers that Weyl proves in [43] is that any Cauchy sequence of reals converges to some real number. The definitions of the notions involved in this theorem are practically identical to the usual ones. Thus $\mathcal{CAUCHY}$, the collection of Cauchy sequences, is the following element of $S(\mathbf{N} \to \mathbf{RN})$ (where $|a|$ is defined as usual):

$$\{f : \mathbf{N} \to \mathbf{RN} \mid \forall \varepsilon : \mathbb{Q}.\varepsilon > 0 \to \exists k \forall m \forall n.m > k \wedge n > k \to |f(n) - f(m)| < \varepsilon\}.$$

Similarly, given $a : \mathbf{RN}$ and $f : \mathbf{N} \to \mathbf{RN}$, we say that f *converges* to a (or that a is the *limit* of f) if the following condition obtains:

$$a \in \mathbb{R} \wedge \forall \varepsilon : \mathbb{Q}.\varepsilon > 0 \to \exists k \forall n.n > k \to |f(n) - a| < \varepsilon.$$

NOTE 20. The only (rather insignificant) difference between these definitions and the usual ones is the restriction of ε to $\mathbb{Q}$. It ensures that the collection of Cauchy sequences of reals (and so, by the next theorem, also the collection of convergent sequences) is a set, and the convergence of a sequence of reals to a specific real number is a delimited relation. (Note that the usual definition of the collection of convergent sequences is not delimited, since it involves quantification over $\mathbb{R}$.)

CAUCHY'S CONVERGENCE PRINCIPLE. A sequence of real numbers converges to some real number iff it is a Cauchy sequence.

For proving the difficult part of this theorem (i.e., that every Cauchy sequence converges), Weyl defines:

$$\underline{lim} = \lambda f : \mathbf{N} \to \mathbf{RN}.\{r : \mathbb{Q} \mid \exists q : \mathbb{Q} \exists n.q > r \wedge \forall m > n.q \in f(m)\}.$$

Then $\underline{lim} : (\mathbf{N} \to \mathbf{RN}) \to \mathbf{RN}$, and it is easy to see that for any $f : \mathbf{N} \to \mathbf{RN}$, $\underline{lim}(f)$ is either the whole of $\mathbb{R}$ (which is denoted in this context by ∞), or the empty set (which is denoted in this context by $-\infty$), or a Dedekind cut, that is, some real number. One then shows that if f is a Cauchy sequence then the last case obtains, and f converges to $\underline{lim}(f)$.

The next topic that Weyl discusses in this section is sequences of *functions*. His main observation here is that what is usually written, for example as a sequence of real functions $F_1(x), F_2(x), \dots$ where $F_i : \mathbf{RN} \to \mathbf{RN}$, is really a function $F : \mathbf{N} \times \mathbf{RN} \to \mathbf{RN}$. Therefore, the limit of the sequence can be defined as $\lambda x : \mathbf{RN}.\underline{lim}(\lambda n.F(n,x))$. (Obviously, this limit is a function from $\mathbb{R}$ to $\mathbb{R}$ in case $\forall x.x \in \mathbb{R} \to \lambda n.F(n,x) \in \mathcal{CAUCHY}$.)

Cauchy's convergence principle is one of several principles that are "allegedly equivalent" [P. 76] in the standard development of analysis. Weyl's next determines which of them is provable in his system.

The provable ones are:

1. The intersection of a nested sequence of closed intervals whose length converges to 0 is a singleton.
2. Every monotone and bounded sequence of real numbers converges.

The unprovable ones are:

(i) Dedekind's cut principle;
(ii) The LUB principle;
(iii) Bolzano–Weierstrass theorem for sets;
(iv) Heine–Borel theorem.

NOTE 21. Weyl actually stated that Principles (i)–(iv) are *false*. However, he did *not prove* this, or even that none of them is provable in WA.

Although Principles (i)–(iv) cannot be maintained in WA, most of them have satisfactory weaker versions that *are* provable there, like:

(ii)' The *sequential* LUB principle: Every bounded *sequence* of real numbers has an LUB and a GLB.
(iii)' Bolzano–Weierstrass theorem for *sequences*: Every bounded *sequence* of real numbers has a convergent subsequence.
(iv)' The *sequential* Heine–Borel theorem: If $[0,1] \subseteq \cup_{i=1}^{\infty}\Delta_i$, where Δ_i is an open interval for each i, then $[0,1] \subseteq \cup_{i=1}^{n}\Delta_i$ for some n.

NOTE 22. By a "sequence of intervals" Weyl officially meant (P. 77) two sequences f and g of real numbers such that for all $n, f(n) < g(n)$. Therefore the formalization of (iv)' according to what he had in mind is:

$$\begin{aligned}&\forall f: \mathbf{N} \to \mathbf{RN} \forall g: \mathbf{N} \to \mathbf{RN}.\\&(\forall x: \mathbf{RN}.x \in \mathbb{R} \wedge 0 \leq x \wedge x \leq 1 \to \exists i.f(i) < x \wedge x < g(i)) \to\\&\to \exists n \forall x: \mathbf{RN}.x \in \mathbb{R} \wedge 0 \leq x \wedge x \leq 1 \to \exists i.i \leq n \wedge f(i) < x \wedge x < g(i).\end{aligned}$$

To this version of the theorem, Weyl provided a proof by contradiction that works directly with cuts. The key construction is to show that if the theorem fails for some f and g, then the following term

$$\{r : \mathbb{Q} \mid \exists n \forall q : \mathbb{Q}.q \leq r \to (q < 0 \vee \exists i \leq n.f(i) < q \wedge q < g(i))\}$$

defines a real number in the unit interval that leads to a contradiction. However, using the strong means of construction by iteration and proof by induction that WA provides, it is possible to prove a more general (and closer to the usual formulation of Heine–Borel theorem) version of (iv)':

$$\begin{aligned}&\forall \Delta: \mathbf{N} \to S(\mathbf{RN})\\&((\forall i: \mathbf{N}.Open(\Delta(i))) \wedge (\forall x: \mathbf{RN}.x \in \mathbb{R} \wedge 0 < x < 1 \to \exists i: \mathbf{N}.x \in \Delta(i))\\&\to \exists n: \mathbf{N} \forall x: \mathbf{RN}.x \in \mathbb{R} \wedge 0 < x \wedge x < 1 \to \exists i: \mathbf{N}.i \leq n \wedge x \in \Delta(i)),\end{aligned}$$

where $Open(\Delta)$ abbreviates:

$$\forall x.x \in \Delta \to \exists r : \mathbb{Q}.r > 0 \wedge \forall y.(y \in \mathbb{R} \wedge x - r < y \wedge y < x + r) \to y \in \Delta.$$

The proof of the last claim can be done by one of the usual methods (e.g., by using a repeated bisection of the unit interval).

Section 4 of Chapter 2 (with which the present section of this paper deals) ends with a very brief outline of how the general principle of iteration enables us to derive the theory of *infinite series* (of real numbers and of real-valued functions) from the theory of sequences just described. This, in turn, can be used to develop the theory of power series (and the theory of infinite products). Finally, the *elementary functions* "can be defined by using any of the infinite processes usually employed for this purpose" [P. 80].

6.5. Continuous functions. One great advantage of Weyl's development of analysis in his system over the way this is done, for example, in $\mathbf{ACA}_0$, is that *Weyl is using no coding*. Thus the formalization of his definition of the continuity of a function $f: \mathbf{RN} \rightarrow \mathbf{RN}$ at an element $a: \mathbf{RN}$ such that $a \in \mathbb{R} \wedge f(a) \in \mathbb{R}$ is:

$$\forall \varepsilon : \mathbb{Q}(\varepsilon > 0 \rightarrow \exists \delta : \mathbb{Q}(\delta > 0 \wedge \forall x : \mathbf{RN}(|x-a| < \delta \rightarrow |f(x)-f(a)| < \varepsilon))).$$

Note that the condition $|x-a| < \delta$ implies that $x \in \mathbb{R}$, and the condition that $|f(x)-f(a)| < \varepsilon$ implies that $f(x) \in \mathbb{R}$. Hence the only difference between this definition and the usual one is that in Weyl's definition ε and δ should be rational. However, this is an insignificant difference, since Weyl's definition and the standard one are equivalent both classically and in WA. Most probably, Weyl has chosen the above definition in order to show the exact reason why the property of being continuous at some point is not a delimited property of a function. (It is due to the universal quantification over $x: \mathbf{RN}$ in the above definition.) We have already discussed the great significance of this fact in Section 5.1: It means that being continuous is not an absolute matter; a continuous function might cease to be so when new elements are added to $\mathbb{R}$. Thus Weyl's treatment of continuity is in a sharp contrast with its treatment in $\mathbf{ACA}_0$ [40] or in Feferman's systems [17], where continuous functions are *encoded* by certain functions from $\mathbb{Q}$ to $\mathbb{R}$, and continuity is an absolute property.

Next, Weyl proves three basic properties of continuous functions:

A: A continuous function assumes all intermediate values.
B: A continuous function on $[0,1]$ has a maximum there.
C: A continuous function on $[0,1]$ is uniformly continuous there.

Again, using the strong means of construction by iteration and proof by induction that WA provides, it is possible to prove these theorems by one of the usual methods (e.g., by using a repeated bisection of the relevant interval), relying for that on the results proved in the previous section (e.g., that the intersection of a nested sequence of closed intervals whose length converges to 0 is a singleton). However, again Weyl prefers not to do this (or even to note this possibility). Instead, he presents direct proofs that use the definition of real numbers in terms of Dedekind cuts. For example, in order to prove Theorem **A**, Weyl assumes that f is a continuous function on the unit interval such that $f(0) < 0$ and $f(1) > 0$, and define $c = \{r : \mathbb{Q} \mid \exists q : \mathbb{Q}.0 < q < 1 \wedge q > r \wedge f(q) < 0\}$. He then notes (leaving proofs to the readers) that c is a real number (that is: a Dedekind cut) in the unit interval, and that $f(c)$ is neither negative nor positive. From the latter fact, he infers that $f(c) = 0$.

NOTE 23. In the last step of the above proof, Weyl is applying the law of excluded middle in the form of the trichotomy $f(c) < 0 \vee f(c) = 0 \vee f(c) > 0$. Then in the proofs of **B** and **C**, this law is applied to more complicated formulas. It should be noted, though, that (as far as I have checked) in "Das Kontinuum" the law of excluded middle is applied only to *delimited* formulas. Therefore, it is plausible that in order to develop analysis in WA, it might suffice to use intuitionistic logic augmented with excluded middle for delimited formulas (rather than full classical logic).

After proving **A**, **B**, and **C**, Weyl notes (without proofs) that **A** can be extended to continuous functions of several real arguments, and that the fundamental theorem of algebra also holds in his version of analysis. Then he proves that a monotone continuous function on $[0,1]$ has an inverse on $[f(0), f(1)]$. He noted that in contrast, an arbitrary function $f: \mathbf{RN} \to \mathbf{RN}$ may not have an inverse on a set T of reals even if for every $y \in T$ there exists a single $x \in \mathbb{R}$ such that $f(x) = y$. Finally, the section on continuous functions ends with the following paragraph, whose content speaks for itself:

> In the realm of continuous functions, differentiation and integration serve as function-generating processes just as they do in contemporary analysis: no change in the foundations is required. Of course, things are not so simple in the case of the more far-reaching integration- and measure-theories of Riemann, Darboux, Cantor, Jordan, Lebesgue and Caratheodory. [P. 86]

The remaining three sections of Chapter 2 (and the book) also have a lot of interest, but are less important for the subject of formalizing Weyl's system in [43], and developing basic analysis in it. Section 6 is devoted to long and deep discussion of the differences between the intuitive continuum and the mathematical one (as developed, e.g., in the previous sections). Section 7 is about magnitudes in general and their measures, and Section 8 is about curves and surfaces.

§7. Limitations and drawbacks of Weyl's system. Although WA has (in our opinion) great advantages over systems like $\mathbf{ACA}_0$ or even Feferman's **W**, it has several serious drawbacks as well.

1. There are terms in the language of WA for all three sorts of collections (I–III) that were described in Section 2, even though collections of sort III (like the universal element of $S(S(S(\mathbf{N})))$ intuitively is) are not really objects according to predicative views in general, and those of Weyl in particular. Even worse is the fact that beyond the types of the form $S(\mathbf{N}^n)$, there is no effective criterion for distinguishing between, for example, terms that denote extensionally determinate collections, and those which do not. Thus no method is provided by Weyl that can allow us to distinguish between the very different nature of the terms $\mathbb{Q}$ (denoting the set of rational numbers) and $\mathbb{R}$ (denoting the set of real numbers). Both terms are of type $S(S(\mathbf{N}^4))$, but according to

Weyl himself, the first intuitively defines an extensionally determinate collection, while the second does not. The same is (respectively) true for the following two terms of type $S(S(\mathbf{N}))$:

- $\{\{n\} \mid n \in \mathbf{N}\} = \{X : S(\mathbf{N}) \mid \exists n.\forall k.k \in X \leftrightarrow k = n\}$.
- $\mathcal{P}(\mathcal{N}) - \{\emptyset\} = \{X : S(\mathbf{N}) \mid \exists n.n \in X\}$.

2. While $\{X : S(N) \mid \forall n.n \in X\}$ is a term of type $S(S(\mathbf{N}))$ that denotes the singleton $\{\mathcal{N}\}$, there is no term of type $S(S(S(\mathbf{N})))$ that denotes the singleton of this set (i.e., $\{\{\mathcal{N}\}\}$). It follows that a finite collection of extensionally determinate objects of some category σ is not always an object of category $S(\sigma)$. This seems to be incoherent with Weyl's most basic principles, since such collections are obviously extensionally determinate. What is more, this fact actually contradicts what Weyl explicitly says (P. 20) about the possibility of characterizing a finite set by simply listing its elements.
3. Following Hölder in [23], and the discussions above at the end of Section 5.2 and in Note 2, it seems strange (according to Weyl's own ideas) to allow quantification only over $\mathcal{N}$, but not over any other extensionally determinate set. An example of an anomaly caused by this unjustified constraint is the fact that $f^{-1}[Y] = \{x : \mathbf{RN} \mid f(x) \in Y\}$ is a set for every subset Y of $\mathbb{R}$ and every f : **RN to RN**, while $f[X]$ is not a set even in case X is extensionally determinate, because $\{y : \mathbf{RN} \mid \exists x : \mathbf{RN}.x \in \mathbf{X} \wedge y = f(x)\}$ is not a legal term.
4. The language of WA makes many duplications and artificial distinctions. For example, let $\mathcal{N}$ and $\mathcal{P}$ be the terms defined in Section 3.4. While $\mathbf{N}$ is the *type* of the natural numbers, $\mathcal{N}$ is a *term* of type $S(\mathbf{N})$, denoting the *set* of these numbers. Hence $\mathbf{N}$ and $\mathcal{N}$ are two completely different things, although their intuitive interpretations are the same. Similarly, while $S(\mathbf{N})$ is the *type* of sets of natural numbers, $\mathcal{P}(\mathcal{N})$ is a term of type $S(S(\mathbf{N}))$, denoting the *set* of sets of such numbers. Hence again $S(\mathbf{N})$ and $\mathcal{P}(\mathcal{N})$ are completely different things, even though practically they denote the same collection.
5. Cardinality is defined by Weyl in WA only for sets of natural numbers. This is certainly not enough. Note that Weyl himself talks in [43] about e.g. *two* functions. In other words: he refers to the cardinality of sets of objects of an ideal type.

§8. The predicative set theory PZF**.** The main goal of this section is to demonstrate the relevance that most of Weyl's ideas in [43] still have for the problems of the foundations of mathematics and the mechanization of mathematical proof checking. For this, we briefly describe our own predicative set theory PZF [8, 10]. This system adheres to almost all the ideas on which Weyl's system in "Das Kontinuum" is based, but does not suffer the problematic aspects of the latter (Section 7). There is only one major idea of Weyl that is rejected in PZF: the use of types. However, types are mainly used by Weyl in order to secure that the terms of his theory define "extensionally determinate" objects. It turned out that the same goal can be achieved (and in a better way) without using types. On the other hand, the set-theoretical framework has the great practical advantage of being the one which the great majority of the mathematicians in the world know and prefer. Thus the basic notions of set theory are nowadays used in any branch and textbook of modern mathematics. Moreover, set theory is almost universally accepted as the foundational theory in which mathematics should be developed.

The following principles of Weyl underlie PZF as well.

1. The natural numbers sequence is a well understood mathematical concept, and as a totality it constitutes a set.
2. The idea of iterating an operation or a relation a finite number of times is accepted as fundamental.
3. Induction on the natural numbers is accepted as a method of proof in its full generality, that is: as an (open) scheme.
4. Higher-order objects, such as sets or functions, are acceptable only when introduced through legitimate definitions.
5. A definition of an object should determine it in a unique, absolute way. The same applies to definitions of relations between objects.
6. Objects should be introduced genetically, and be derived by adequate, *logical* means from few basic objects and relations.
7. The relations of elementhood ($\in$) and equality ($=$) are basic.
8. The use of quantification over a collection of objects should be allowed *in definitions of objects* only if that collection forms an object, and is (in Weyl's terminology) extensionally determinate, that is: it is introduced by a "stable and invariant" definition.
9. Sets are extensional: sets that have the same elements are identical.
10. Using ramification in definitions, and classifying sets of natural numbers according to "levels", are artificial, and should be avoided.
11. The use of classical logic is justified.
12. The possibility of introducing new methods of defining sets is taken into account. Accordingly, the OWA (Section 5.1) is adopted: the 'universe' of sets is seen as *open*. More precisely: our theory has no single 'intended universe'.
13. Set terms of the form $\{(x_1,\dots,x_n) \mid \psi\}$, and operations of the form $\lambda y_1,\dots,y_k.\{(x_1,\dots,x_n) \mid \psi\}$, have a particularly central role.
14. $\neg$, $\wedge$, $\vee$, and $\exists$ should be taken as the basic first-order connectives and quantifiers. (This choice of Weyl is completely justified and inevitable in the case of PZF. See Note 7 and footnote 34.)

There are of course also issues in which Weyl's type-theoretic system and our purely set-theoretic one differ. The main points of difference are due to our following four other principles, which were not shared by Weyl:

- Like in ZF, there is just one category/type of objects: sets.
- Like in ZF, we assume just one basic object ($\emptyset$), and only two basic relations: those that were mentioned in Principle 7 above ($\in$ and $=$).

- *Every* object (i.e., every set) should be 'extensionally determinate'.
- Quantification over a collection of objects is allowed in definitions *whenever* that collection is an extensionally determinate object. (This is Hölder's principle—see Note 2 and the end of Section 5.2.) By the previous items, this means that it is allowed over *any set*.

Unsurprisingly, coping with these principles demands that some of the common principles above are implemented differently in PZF and in WA.

- The *ZF*-like framework of PZF causes apparent problems with implementing the first three principles in the list above. (Those that are connected with the natural numbers.) This is solved by following a suggestion that was made in Section 4.5.2 above: using ancestral logic $\mathcal{AL}$ [5, 13, 29, 30, 38] as our underlying logic rather than first-order logic. In fact, in [5] it was argued that the ability to form the transitive closure of a given relation (like forming the notion of an ancestor from the notion of a parent) should be taken as a major ingredient of our logical abilities (even prior to our understanding of the natural numbers), and that this concept is the key for understanding iteration, as well as inductive reasoning. Here it can be added that the use of $\mathcal{AL}$ eliminates the ambiguity in [43] about the meaning of the word "logical" (discussed in Note 15 and Section 4.5.2), and allows us to fully adhere to Principle 6.[33]
- The most important difference is with respect to the fifth principle in the above list. Recall that Weyl tried to implement this principle using two means: imposing type restrictions on variables, and allowing to use in definitions of objects only delimited formulas. The first of these means is not available in PZF, and even with its help the second one would not be sufficient. (See Section 7.) Therefore, in PZF, the use of these two independent constraints, one connected with a *property* of variables and the other with a *property* of formulas, is replaced by a constraint which is connected with a single *relation* $\succ$ between formulas and set of variables. Following the terminology of database theory [41], we call $\succ$ 'the safety relation of PZF'. The intuitive meaning of '$\varphi(x_1,\dots,x_n,y_1,\dots,y_k) \succ \{x_1,\dots,x_n\}$' is that the formula φ is "extensionally determinate", or 'stable and invariant" with respect to the set of variables $\{x_1,\dots,x_n\}$ for all values of the parameters $y_1,\dots,y_k$. Here we identify these notions with *universe independence*.

Definition. Let T be a set theory, and let φ be a formula in the language of T such that $Fv(\varphi) = \{x_1,\dots,x_n,y_1,\dots,y_k\}$. φ is T-*universe-independent* with respect to $\{x_1,\dots,x_n\}$ if every transitive model of T is closed under the operation $\lambda y_1,\dots,y_k.\{(x_1,\dots,x_n) \mid \varphi\}$, and for every tuple $(a_1,\dots,a_n)$ of

[33]The set of valid formulas of $\mathcal{AL}$ is not r.e. (or even arithmetical). Hence no sound and complete *formal* system for it exists. In PZF, we use the standard sound formal system for $\mathcal{AL}$, as it is presented in [13] (following [29, 30]). One of the crucial rules of that system is a general rule of *mathematical induction*.

sets, the value of this operation on $(a_1,\ldots,a_n)$ is the same in every transitive model of T that contains $a_1,\ldots,a_n$.

Note 24. For a set-theoretical platonist, the above definition is absolutely precise. For someone who is not (like the author of this paper), this is an imprecise definition that nevertheless provides a good intuition about the meaning of the informal notion of “universe independence”—and this intuitive understanding is all we need here.

Note 25. If we identify $\{\}$ with $\emptyset$, and let $\{() \mid \varphi\}$ denote 0 (i.e., $\emptyset$) in case φ is false, and 1 (i.e., $\{\emptyset\}$) in case φ is true, we see that φ is T-universe-independent with respect to $\emptyset$ iff φ is T-*absolute* according to the usual set-theoretical notion of absoluteness. (See, e.g., [24].) Hence our universe-independence relation between formulas and sets of variables is a generalization of Gödel's absoluteness (which is a property of formulas).

Universe-independence is a semantic notion. In order to base a proof system on it, we need (like Weyl) to impose it syntactically (and genetically) by adequate logical means. (See the sixth principle above.) The solution to this problem which is used in PZF has come from the observation that this is an instance of a more general task, not peculiar to set theory. In fact, in [6, 7, 11] an appropriate purely logical framework, that can be used for this task, was introduced. This framework unifies different notions of “safety”, or “domain-independence”, of formulas, coming from different areas of mathematics and computer science, like: domain independence in database theory [1, 41], and absoluteness in set theory.

The system PZF from [8] has been designed according to the principles described above. In the Appendix below, we present an improved version of it. An (incomplete) investigation of its power can be found in [10].[34]

Appendix: The formal system PZF

Language. The language $\mathcal{L}_{PZF}$ is defined by a simultaneous recursion.

Predicates and operations:

- $=$ and $\in$ are binary predicates.
- If φ is a formula such that $\varphi \succ_{PZF} \emptyset$, and $Fv(\varphi) = \{x_1,\ldots,x_n\}$ where $n > 0$, then $[(x_1,\ldots,x_n) \mid \varphi]$ is an n-ary predicate.
- If t is a term such that $Fv(t) = \{y_1,\ldots,y_k\}$, then $\lambda y_1,\ldots,y_k.t$ is a k-ary operation.

Terms:

- Every variable is a term.

[34]Note that in PZF, it was necessary to take $\exists$ and both of $\wedge$ and $\vee$ as primitives of the language, because it is impossible to derive any of the clauses in the definition of $\succ_{PZF}$ from the rest of the clauses. (In sharp contrast, no independent conditions for $\forall$ or $\rightarrow$ are known.) Recall that this is exactly what Weyl did! (See Note 7.)

- If $\varphi \succ_{PZF} \{x\}$, then $\{x \mid \varphi\}$ is a term.
- If F is a k-ary operation, $t_1,\dots,t_k$ are terms, then $F(t_1,\dots,t_k)$ is a term.

Formulas:

- If P is an n-ary predicate, then $P(t_1,\dots,t_n)$ is an *atomic* formula whenever $t_1,\dots,t_n$ are terms.
- If φ and ψ are formulas, and x is a variable, then $\neg\varphi$, $(\varphi\wedge\psi)$, $(\varphi \vee \psi)$, and $\exists x\varphi$ are formulas. ($\forall x\varphi$ and $\varphi \to \psi$ are taken as abbreviations for $\neg\exists x\neg\varphi$ and $\neg(\varphi\wedge\neg\psi)$, respectively.)
- If φ is a formula, t and s are terms, and x and y are distinct variables, then $(TC_{x,y}\varphi)(t,s)$ is a formula, and

$$Fv((TC_{x,y}\varphi)(t,s)) = (Fv(\varphi) - \{x,y\}) \cup Fv(t) \cup Fv(s).$$

The Safety Relation $\succ_{PZF}$:

($\in$): $x \in t \succ_{PZF} \{x\}$ if $x \notin Fv(t)$.
(**At**): $\varphi \succ_{PZF} \emptyset$ if φ is atomic.
(**=**): $\varphi \succ_{PZF} \{x\}$ if $\varphi \in \{x \neq x, x = t, t = x\}$, and $x \notin Fv(t)$.
($\neg$): $\neg\varphi \succ_{PZF} \emptyset$ if $\varphi \succ_{PZF} \emptyset$.
($\vee$): $\varphi \vee \psi \succ_{PZF} X$ if $\varphi \succ_{PZF} X$ and $\psi \succ_{PZF} X$.
($\wedge$): $\varphi \wedge \psi \succ_{PZF} X \cup Y$ if $\varphi \succ_{PZF} X$, $\psi \succ_{PZF} Y$, and either $Y \cap Fv(\varphi) = \emptyset$ or $X \cap Fv(\psi) = \emptyset$.
($\exists$): $\exists y\varphi \succ_{PZF} X - \{y\}$ if $y \in X$ and $\varphi \succ_{PZF} X$.
(TC): $(TC_{x,y}\varphi)(x,y) \succ_{PZF} X$ if either $\varphi \succ_{PZF} X \cup \{x\}$, or $\varphi \succ_{PZF} X \cup \{y\}$.

Logic and axioms.

Logic: Classical $\mathcal{AL}$ with variable-binding terms operators.
Axioms:

Extensionality. $\forall z(z \in x \leftrightarrow z \in y) \to x = y$
Comprehension. The universal closures of formulas of the forms:

- $x \in \{x \mid \varphi\} \leftrightarrow \varphi$
- $[(x_1,\dots,x_n) \mid \varphi](t_1,\dots,t_n) \leftrightarrow \varphi\{t_1/x_1,\dots,t_n/x_n\}$
- $(\lambda y_1,\dots,y_k.t)(s_1,\dots,s_k) = t\{s_1/y_1,\dots,s_k/y_k\}$

$\in$***-induction***. $(\forall x(\forall y(y \in x \to \varphi\{y/x\}) \to \varphi)) \to \forall x\varphi$

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