THE ISOMORPHISM BETWEEN GRAPHS AND THEIR ADJOINT GRAPHS

V. V. Menon

(received May 25, 1964)

1. Introduction. A graph G is defined as a set $X = \{x_1, \ldots, x_n\}$ of elements x_i called vertices, and a collection Γ of (not necessarily distinct) unordered pairs of distinct vertices, called edges. An edge (x_i, x_j) is said to be incident to x_i and x_j which are its end-vertices.

DEFINITION 1. The adjoint (or the interchange graph) $I(G)$ of a given graph $G = (X, \Gamma)$ is defined as follows. The edges of G form the vertices of $I(G)$, and two vertices in $I(G)$ are joined by zero, one or two edges according as the corresponding edges in G have zero, one or two common end-vertices respectively.

For example, in Fig. 1 we see the graphs G_1, G_2 and G_3 and their adjoints $I(G_1)$, $I(G_2)$ and $I(G_3)$. The edges have been called e_1, e_2, e_3.

DEFINITION 2. $I^n(G)$ is defined recursively by

$$I^n(G) = I[I^{n-1}(G)], \quad n \geq 2 .$$

DEFINITION 3. Two graphs G and G' are isomorphic if there exists a one-one correspondence between their vertices such that if $x_i, x_j \in G$ correspond to vertices $x_{i'}, x_{j'} \in G'$

Canad. Math. Bull. vol. 8, no. 1, February 1965

https://doi.org/10.4153/CMB-1965-002-7 Published online by Cambridge University Press

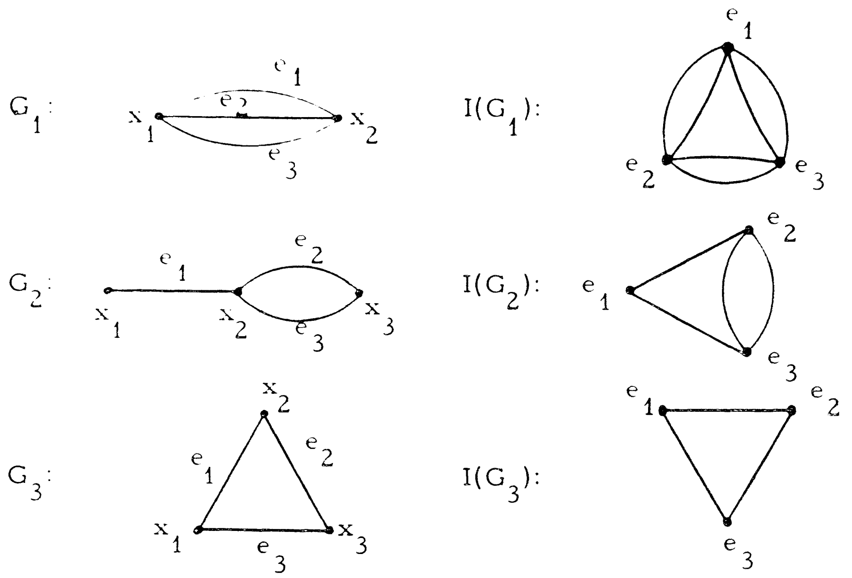

Fig. 1

respectively, then the edge (x_i, x_j) exists in G if and only if the edge $(x_{i'}, x_{j'})$ exists in G'.

DEFINITION 4. The <u>degree</u> of a vertex x_i is the number of edges incident to it.

The problem dealt with in this paper is that of determining graphs which are isomorphic to their adjoints; and in general, of graphs G which are isomorphic to $I^n(G)$ for some n. The latter is the generalisation of a problem suggested in Ore [1].

The solution of this problem occurs as Theorem 2 in section 3. The theorem 1 is a general result applicable to any graph. The proofs of these theorems also appear in section 3. In section 2 are given certain obvious results which are useful in simplifying the proof of the main theorem.

2. <u>Preliminary remarks.</u> First we define the connected components of a graph. A graph is said to be connected if for any pair of vertices x_i, x_j there exists a sequence $u_1, \ldots, u_k$ of edges of the graph such that (1) u_1 is incident to x_i and u_k

https://doi.org/10.4153/CMB-1965-002-7 Published online by Cambridge University Press

to x_j, and (2) u_{i-1} is incident to one end-vertex of u_i, and u_{i+1} to the other, for $2 \leq i \leq k-1$. In other words, between every pair of vertices there exists a chain of edges. Any given graph can be partitioned into components, called the connected components of the graph, such that each component is a connected graph and there are no edges joining vertices belonging to different components.

Considering a graph G, we see that the edges in a connected component of G form the vertices of a connected component of I(G), and vice-versa.

From the definition of an adjoint graph, we can easily verify the following lemmas.

LEMMA 1. Let the graph G consist of n edges in a chain $(n \geq 1)$, as shown in Fig. 2(a), then the adjoint I(G) consists of n-1 edges in a chain, as in Fig. 2(b). Conversely, if I(G) consists of n-1 edges in a chain, then the relevant connected component of G consists of n edges in a chain.

Fig. 2

LEMMA 2. In the graph G let there be a vertex x_1 of degree 1 (called a pendant vertex) such that starting from x_1 there is a chain of n edges $(n \geq 1)$ before the first vertex of a degree exceeding 2 is encountered, as in Fig. 3(a). Then the corresponding portion in I(G) has a similar configuration with n-1 edges, as in Fig. 3(b). Conversely, if I(G) has the form shown in Fig. 3(b), then the relevant connected component of G has the form shown in Fig. 3(a).

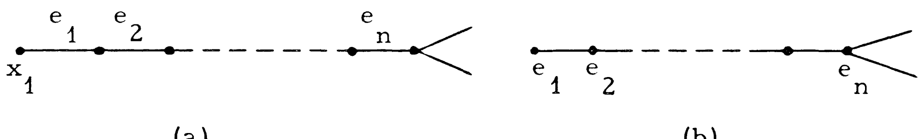

Fig. 3

3. The Main Results.

THEOREM 1. Suppose G is a finite graph without loops (there may be multiple edges); let $x_1, \ldots, x_n$ be its vertices and let d_i be the degree of the vertex x_i, $1 \le i \le n$. Then the number of edges in the adjoint $I(G)$ is

$$\sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2} .$$

Proof: From the construction of adjoints, we see that if there are d_i edges at the vertex x_i of G, then each of the vertices of $I(G)$ corresponding to these edges will be joined by edges to each of the others if $d_i \ge 2$, and there will be no edges in virtue of edges at x_i if $d_i \le 1$. In other words, the number of edges in $I(G)$ contributed by edges (of G) at x_i is $\frac{d_i(d_i - 1)}{2}$ if $d_i \ge 2$, and 0 if $d_i \le 1$. The total number of edges in $I(G)$ is, therefore,

$$\sum \frac{d_i(d_i - 1)}{2} ,$$

where the summation is over all i such that $d_i \ge 2$,

$$= \sum_{i=1}^{n} \frac{d_i(d_i - 1)}{2} .$$

Also if (x_i, x_j) is an edge in G then the vertex (x_i, x_j) of $I(G)$ will be joined by edges to $(d_i - 1)$ vertices in virtue of the edges at x_i in G, and to $(d_j - 1)$ vertices in virtue of the edges at x_j in G. Thus the degree of this vertex in $I(G)$ is $(d_i - 1) + (d_j - 1)$ $= d_i + d_j - 2$.

THEOREM 2. For a finite graph G without loops, the following statements are equivalent.

a) the degree of each vertex of G is 2,

b) G is isomorphic to $I^k(G)$ for all $k \geq 1$,

c) G is isomorphic to $I^k(G)$ for some k, $(k \geq 1)$.

As a corollary it follows that G is isomorphic to $I(G)$ if and only if the degree of each vertex of G is 2.

Proof: We shall prove the following implications

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a) ,$$

Let x_i, $1 \leq i \leq n$ be the vertices of G and d_i, $1 \leq i \leq n$, their corresponding degrees. Since (b) ⇒ (c) obviously, we shall only prove

$$(a) \Rightarrow (b) \text{ and } (c) \Rightarrow (a) .$$

1. (a) ⇒ (b). For if each $d_i = 2$, then every connected component must be of the following form, as in Fig. 4, (called an elementary cycle), where the vertices are $x_1, x_2, \ldots, x_{\ell-1}, x_\ell$ (For different components, the value of ℓ may be different) and the edges are $(x_1, x_2), \ldots, (x_i, x_{i+1}), \ldots, (x_\ell, x_1)$.

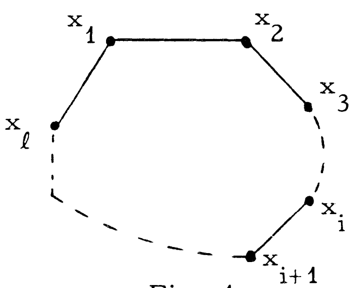

Fig. 4

One readily verifies that the adjoint of such a component is isomorphic to itself. Thus each connected component of G is isomorphic to the corresponding component of $I(G)$, i.e., G is isomorphic to $I(G)$. By induction, we see that G is isomorphic to $I^k(G)$ for every k.

2. (c) => (a). We first show that G cannot contain vertices of degree zero or one.

If possible, let $d_i = 0$ for some i, i.e., the corresponding vertex x_i is an isolated vertex. Since G and $I^k(G)$ are isomorphic, $I^k(G)$ also contains an isolated vertex. Now applying lemma 1 of section 2 repeatedly, we see that the connected component of G which gave rise to this isolated point of $I^k(G)$ must be a chain of k edges (an isolated point is a chain of zero edges), as in Fig. 5(a).

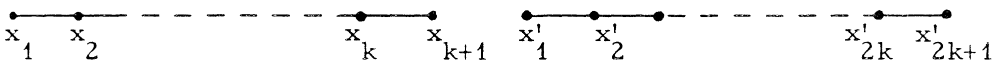

Fig. 5

But G and $I^k(G)$ are isomorphic, so $I^k(G)$ contains such a connected component (a chain of k edges). The corresponding component of G (which reduces to this component in $I^k(G)$) must be, again by repeated applications of lemma 1, a chain of $2k$ edges, as in Fig. 5(b). Proceeding thus, we see that in G there occur connected components which are chains of k, $2k$, $3k$, ... edges respectively. This contradicts the finiteness of G.

Now let $d_i = 1$ for some i, i.e., the corresponding vertex x_i is a pendant vertex. Consider the chain (of ℓ edges, say) from x_i to the first vertex of degree exceeding 2 (this chain may be of length 1), or of degree 1. If the latter applies, we can use the above argument. So we can assume that a configuration, as in Fig. 6(a) exists in G, and hence in $I^k(G)$.

By applying k times the lemma 2 of section 2, we see that G must contain the configuration of Fig. 6(b), where there are $\ell + k$ edges from the pendant vertex to the first vertex of degree > 2. Thus, as in the previous case, we can

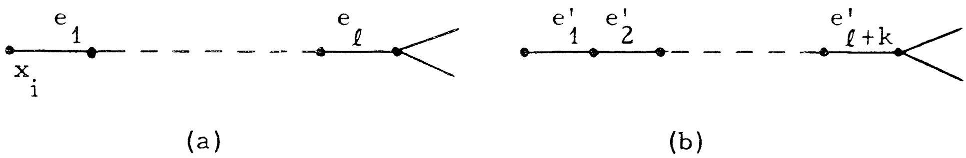

Fig. 6

show that such configurations with ℓ , $\ell + k$, $\ell + 2k, \ldots$ edges exist in G, as connected components, which is absurd since G is finite.

Hence we must have $d_i \geq 2$ for all i, in G.

Using theorem 1, we see that if G contains n vertices and m edges, and the degree of each vertex is at least 2, and if I(G) contains m_1 edges, then

i) $m_1 = \sum_{1}^{n} \frac{d_i(d_i - 1)}{2} \geq n$,

ii) the degree of each vertex of I(G) is also at least 2. The equality $m_1 = n$ holds if and only if each $d_i = 2$.

Let now n_o, m_o be respectively the number of vertices and edges of G, and let n_r, m_r be the corresponding quantities for $I^r(G)$. Then it follows that (since the degree of each vertex in $I^r(G)$ is at least 2 for all $r \geq 0$)

(1) $$m_{r+1} \geq n_r \qquad \text{for } r \geq 0 .$$

It is of course true that $n_{r+1} = m_r$, since $I^{r+1}(G) = I[I^r(G)]$, for $r \geq 0$. Now since G and $I^k(G)$ are isomorphic, they have, in particular, the same number of vertices and edges, respectively,

https://doi.org/10.4153/CMB-1965-002-7 Published online by Cambridge University Press

i.e., $n_o = n_k$

and $m_o = m_k$.

If k is even, say $k = 2r$, then using the result (1), we obtain

$$m_k = m_{2r} \geq n_{2r-1} = m_{2r-2} \geq \dots \geq n_1 = m_o$$

and equality holds if and only if each vertex is of degree 2 at all stages. But since $m_o = m_k$, we have each $d_i = 2$ for G.

If k is odd, say $2r+1$, then using the result (1), we have

$$m_k = m_{2r+1} \geq n_{2r} = m_{2r-1} \geq \dots \geq n_2 = m_1 \geq n_o$$

and

$$n_k = m_{k-1} = m_{2r} \geq m_o ,$$

whence $n_o = n_k \geq m_o = m_k \geq n_o$. Thus equality holds and hence each $d_i = 2$.

<u>Special case</u>. If we are given that G and I(G) are isomorphic, we can simplify the last stage of the proof considerably. Because if each $d_i \geq 2$, then the condition of equality of the number of edges in G and I(G) gives

$$\sum_{i=1}^{n} \frac{d_i(d_i-1)}{2} = m_1 = m_o = \frac{1}{2} \sum_{1}^{n} d_i$$

$$\text{i.e.,} \quad \sum_{i=1}^{n} d_i(d_i - 2) = 0 ,$$

whence it follows that $d_i = 2$ for all i.

https://doi.org/10.4153/CMB-1965-002-7 Published online by Cambridge University Press

Remark. We can put the condition that each $d_i = 2$, in the alternative form that the graph consists of disjoint elementary cycles.

REFERENCE

1. O. Ore, Theory of Graphs, American Mathematical Society Colloquium Publications, Vol. XXXVIII, 1962, Section 1.5, problem 5.

Indian Statistical Institute

https://doi.org/10.4153/CMB-1965-002-7 Published online by Cambridge University Press