5
Measures

The first section of this chapter is devoted to a review of basic definitions of measure theory. Among other topics, we recall basic properties of *positivity preserving operators*, which provide tools useful in constructive quantum field theory.

The rest of this chapter is devoted to measures on infinite-dimensional Hilbert spaces. It is well known that there are no Borel translation invariant measures on infinite-dimensional vector spaces. However, one can define useful measures on such spaces which are not translation invariant. In particular, the notion of a *Gaussian measure* has a natural generalization to the infinite-dimensional case.

Measures on an infinite-dimensional Hilbert space $\mathcal{X}$ is quite a subtle topic. A naive approach to this subject leads to the notion of a *weak distribution*, which is a family of measures on finite-dimensional subspaces satisfying a natural compatibility condition. It is natural to ask whether a weak distribution is generated by a measure on $\mathcal{X}$. In general, the answer is negative. In order to obtain such a measure, one has to consider a larger measure space containing $\mathcal{X}$. Many choices of such a larger space are possible. A class of such choices that we describe in detail are Hilbert spaces $B\mathcal{X}$ for a self-adjoint operator B satisfying certain conditions.

Measures on Hilbert spaces play an important role in probability theory and quantum field theory. One of them is the *Wiener measure*, used to describe Brownian motion. There are also natural representations of the Fock space as the L^2 space with respect to a Gaussian measure: the so-called *real-wave* and *complex-wave CCR representations*, which we will consider in Chap. 9.

Note that for most practical purposes many subtleties of measures in infinite dimensions can be ignored. In applications, an important role is played by such concepts as L^p spaces, the integral, the positivity a.e., etc. It is important that there exists an underlying measure space, so that we can use tools of measure theory. However, which measure space we actually take is irrelevant. Therefore, the choice of the operator B mentioned above is usually not important for applications.

5.1 General measure theory

In this section we recall basic concepts and facts of measure and integration theory.

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5.1.1 σ-algebras

Let Q be a set. Let 2^Q denote the family of its subsets. Let us introduce some useful kinds of subfamilies of 2^Q.

Definition 5.1 *Let $\mathfrak{R} \subset 2^Q$.*

(1) *We say that $\mathfrak{R}$ is a* ring *if $A, B \in \mathfrak{R} \;\Rightarrow\; A\backslash B,\; A \cup B \in \mathfrak{R}$.*
(2) *$\mathfrak{R}$ is a σ-*ring *if it is a ring and $A_1, A_2, ... \in \mathfrak{R} \;\Rightarrow\; \bigcup\limits_{n=1}^{\infty} A_n \in \mathfrak{R}$.*

Definition 5.2 *Let $\mathfrak{S} \subset 2^Q$.*

(1) *$\mathfrak{S}$ is an* algebra *if it is a ring and $Q \in \mathfrak{S}$.*
(2) *$\mathfrak{S}$ is a σ-*algebra *if it is a σ-ring and an algebra.*

Definition 5.3 *If $\mathfrak{T} \subset 2^Q$, then there exists the smallest ring, σ-ring, algebra and σ-algebra containing $\mathfrak{T}$. It is called the* ring, σ-ring, algebra, *resp.* σ-algebra generated by $\mathfrak{T}$.

Definition 5.4 *If $(Q_i, \mathfrak{S}_i)$, $i = 1, 2$, are spaces equipped with σ-algebras, we say that $F : Q_1 \to Q_2$ is* measurable *if for any $A \in \mathfrak{S}_2$, $F^{-1}(A) \in \mathfrak{S}_1$.*

5.1.2 Measures

Let $(Q, \mathfrak{S})$ be a space equipped with a σ-algebra.

Definition 5.5 *A* finite complex measure *is a function*

$$\mathfrak{S} \ni A \mapsto \mu(A) \in \mathbb{C}$$

such that $\mu(\emptyset) = 0$ and for any $A_1, A_2, ... \in \mathfrak{S}$, $A_i \cap A_j = \emptyset$, $i \neq j$,

$$\bigcup_{j=1}^{\infty} A_j = A \;\Rightarrow\; \mu(A) = \sum_{j=1}^{\infty} \mu(A_j), \tag{5.1}$$

where the above sum is absolutely convergent. A finite real, *resp.* finite positive measure *on $(Q, \mathfrak{S})$ has the same definition, except that we replace $\mathbb{C}$ with $\mathbb{R}$, resp. $[0, \infty[$. In the case of a positive measure we usually drop the word* positive. *(In this case the requirement of the absolute convergence of the series in (5.1) is automatically satisfied, and hence can be dropped from the definition).*

We say that a positive finite measure μ is a probability measure *if $\mu(Q) = 1$.*

In the positive case Def. 5.5 has a well-known generalization that allows the measure to take infinite values.

Definition 5.6 *A* (positive) measure, *is a function*

$$\mathfrak{S} \ni A \mapsto \mu(A) \in [0, \infty]$$

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such that $\mu(\emptyset) = 0$ *and for any* $A_1, A_2, ... \in \mathfrak{S}$, $A_i \cap A_j = \emptyset$, $i \neq j$,

$$\bigcup_{j=1}^{\infty} A_j = A \;\Rightarrow\; \mu(A) = \sum_{j=1}^{\infty} \mu(A_j). \tag{5.2}$$

Such a triple $(Q, \mathfrak{S}, \mu)$ *is often called a* measure space*. If in addition* μ *is a probability measure,* $(Q, \mathfrak{S}, \mu)$ *is called a* probability space*.*

A measure space $(Q, \mathfrak{S}, \mu)$ is *complete* if $B \subset A$ with $A \in \mathfrak{S}$ and $\mu(A) = 0$ implies $B \in \mathfrak{S}$. If $(Q, \mathfrak{S}, \mu)$ is a measure space, one sets

$$\mathfrak{S}^{\rm cpl} := \big\{ B \in 2^Q \;:\; \exists A_1, A_2 \in \mathfrak{S} \text{ with } A_1 \subset B \subset A_2,\; \mu(A_2 \backslash A_1) = 0 \big\},$$
$$\mu^{\rm cpl}(B) := \mu(A_1).$$

Then $(Q, \mathfrak{S}^{\rm cpl}, \mu^{\rm cpl})$ is a complete measure space called the *completion of* $(Q, \mathfrak{S}, \mu)$. It admits more measurable sets and functions and therefore is more convenient for the theory of integration.

5.1.3 Pre-measures

Generalizing Def. 5.6 to the real or complex case poses problems because the series in (5.1) could be divergent. In this case, one of the possible solutions is to use the concept of a pre-measure, which is defined only on a ring, takes finite values and is conditionally σ-additive.

Let $(Q, \mathfrak{R})$ be a space equipped with a ring.

Definition 5.7 *A* complex pre-measure on $(Q, \mathfrak{R})$ *is a function*

$$\mathfrak{R} \ni A \mapsto \nu(A) \in \mathbb{C}$$

such that $\nu(\emptyset) = 0$ *and for any* $A_1, A_2, ... \in \mathfrak{S}$, $A_i \cap A_j = \emptyset$, $i \neq j$,

$$\bigcup_{j=1}^{\infty} A_j = A \in \mathfrak{R} \;\Rightarrow\; \mu(A) = \sum_{j=1}^{\infty} \mu(A_j). \tag{5.3}$$

where the above sum is absolutely convergent. A real*, resp.* positive pre-measure on $(Q, \mathfrak{R})$ *has the same definition, except that we replace* $\mathbb{C}$ *with* $\mathbb{R}$*, resp.* $[0, \infty[$*.*

The following well-known theorem allows us to extend in a canonical way a positive pre-measure to a positive measure.

Theorem 5.8 *Suppose that* $(Q, \mathfrak{R})$ *is a space with a ring and* $\nu : \mathfrak{R} \to [0, \infty[$ *is a positive pre-measure. Let* $\mathfrak{S}$ *be a* σ*-algebra containing* $\mathfrak{R}$*. Then*

$$\mu(A) := \sup\big\{ \nu(B) \;:\; B \in \mathfrak{R},\; B \subset A \big\}, \quad A \in \mathfrak{S}, \tag{5.4}$$

is a measure on $\mathfrak{S}$ *extending* ν*. If* $\mathfrak{S}$ *coincides with the* σ*-algebra generated by* $\mathfrak{R}$*, then* μ *is the unique measure on* $\mathfrak{S}$ *extending* ν*.*

5.1.4 Borel measures and pre-measures

Let Q be a topological space. The following two families of subsets of Q play a distinguished role in measure theory:

Definition 5.9 (1) *The σ-algebra generated by the family of open sets of Q will be called the* Borel σ-algebra of Q *and denoted* $\mathfrak{B}(Q)$.

(2) *The ring that consists of pre-compact Borel sets in Q will be denoted $\mathfrak{K}(Q)$. (We say that a set is* pre-compact *if its closure is compact).*

Definition 5.10 *A* complex, real, *resp.* positive Borel pre-measure on Q *is a complex, real resp. positive pre-measure on $(Q, \mathfrak{K}(Q))$.* Meas(Q) *will denote the space of complex Borel pre-measures.*

Definition 5.11 μ *is a* positive Borel measure on Q *if it is a measure on $(Q, \mathfrak{B}(Q))$ that is finite on $\mathfrak{K}(Q)$ and*

$$\mu(A) = \sup\big\{\mu(B) \ : \ B \in \mathfrak{K}(Q), \ B \subset A\big\}, \quad A \in \mathfrak{B}(Q). \tag{5.5}$$

$\mathrm{Meas}^+(Q)$ *will denote the space of positive Borel measures on Q.*

Note that every positive Borel pre-measure possesses a unique extension to a Borel measure. Conversely, every positive Borel measure restricted to $\mathfrak{K}(Q)$ is a positive Borel pre-measure.

Definition 5.12 *Let μ be a complex Borel pre-measure on Q. The* total variation of μ *is the positive Borel measure $|\mu|$ defined for $A \in \mathfrak{B}(Q)$ by*

$$|\mu|(A) := \sup \sum_{i=1}^{\infty} |\mu(A_i)|,$$

where the supremum is taken over all families $A_1, A_2, \dots \in \mathfrak{K}(Q)$ such that $A_i \cap A_j = \emptyset$, $i \neq j$ and $A_i \subset A$. $\mathrm{Meas}^1(Q)$ *will denote the space of finite complex Borel pre-measures on Q equipped with the norm $|\mu|(Q)$, which makes it into a Banach space.*

5.1.5 Integral

Let $(Q, \mathfrak{S})$ be a space with a σ-algebra.

Definition 5.13 *Let $\mathcal{M}_+(Q, \mathfrak{S})$, resp. $\mathcal{M}(Q, \mathfrak{S})$ denote the set of $\mathfrak{S}$-measurable functions with values in $[0, \infty[$, resp. $\mathbb{C}$.*

Let $(Q, \mathfrak{S}, \mu)$ be a measure space.

We will often abbreviate $(Q, \mathfrak{S})$ to Q and $(Q, \mathfrak{S}, \mu)$ to (Q, μ).

Definition 5.14 *Let $\mathcal{N}(Q, \mu)$, denote the subset of $\mathcal{M}(Q)$ consisting of functions vanishing outside of a set of measure zero. We set $M_+(Q, \mu) := \mathcal{M}_+(Q)/\mathcal{N}(Q, \mu)$ and $M(Q, \mu) := \mathcal{M}(Q)/\mathcal{N}(Q, \mu)$.*

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Definition 5.15 *For $f \in \mathcal{M}_+(Q)$, in a standard way we define its* integral, *which is an element of $[0,\infty]$ and is denoted*

$$\int f \mathrm{d}\mu. \tag{5.6}$$

Clearly, (5.6) does not change if we add to f a function vanishing outside a set of measure zero, hence it makes sense to write $\int f \mathrm{d}\mu$ also for $f \in M_+(Q,\mu)$.

5.1.6 L^p spaces

Definition 5.16 *For $f \in \mathcal{M}_+(Q)$ we define*

$$\operatorname{ess\,sup} f := \inf \left\{ \sup f\big|_{Q\setminus N} \ : \ N \in \mathfrak{S},\ \mu(N) = 0 \right\}. \tag{5.7}$$

Clearly, (5.7) does not change if we add to f a function vanishing outside a set of measure zero, hence it makes sense to write $\operatorname{ess\,sup} f$ also for $f \in M_+(Q,\mu)$.

Definition 5.17 *For $1 \le p \le \infty$ and $f \in M(Q,\mu)$, we set*

$$\|f\|_p := \left(\int_Q |f|^p \mathrm{d}\mu \right)^{1/p},$$
$$\|f\|_\infty := \text{ess sup}|f|.$$

We also introduce in the standard way the Banach spaces $L^p(Q,\mu) \subset M(Q,\mu)$. For $f \in L^1(Q,\mu)$, we define its integral, denoted by $\int f \mathrm{d}\mu$.

If q is used as the generic variable in Q, then instead of (5.6) one can write $\int f(q)\mathrm{d}\mu(q)$. Often, especially if Q is a finite-dimensional vector space and μ is a Lebesgue measure on Q, we will write $\int f(q)\mathrm{d}q$ for (5.6).

If the measure μ is obvious from the context, we will often drop μ from our notation and we will write $L^p(Q)$, $M(Q)$ etc. for $L^p(Q,\mu)$, $M(Q,\mu)$,

Let $1 \le p, q \le \infty,\ p^{-1} + q^{-1} = 1$. If $f, g \in M(Q)$, the *Hölder's inequality* says

$$\|fg\|_1 \le \|f\|_p \|g\|_q,$$

Definition 5.18 *We will write $L^p_+(Q)$ for $L^p(Q) \cap M_+(Q)$.*

Definition 5.19 *Let $g \in M(Q)$. We say that g is* strictly positive (w.r.t. μ), *and we write $g > 0$, if $g \ge 0$ and $\mu\left(\{q \ : \ g(q) = 0\}\right) = 0$.*

Proposition 5.20 *Let $g \in L^p(Q)$, $1 \le p, q \le \infty,\ p^{-1} + q^{-1} = 1$.*

(1) *$g \ge 0$ iff*

$$\int_Q fg\mathrm{d}\mu \ge 0, \quad f \in L^q_+(Q). \tag{5.8}$$

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(2) $g > 0$ *iff*

$$\int_Q fg\mathrm{d}\mu > 0, \quad f \in L^q_+(Q), \ f \neq 0.$$

If the measure is finite, then $q \geq p$ implies $L^q(Q) \subset L^p(Q)$.

5.1.7 Operators on L^p spaces

In this subsection we recall properties of linear operators on L^p spaces.

Let μ_i be a measure on $(Q_i, \mathfrak{S}_i)$, $i = 1, 2$.

Definition 5.21 $T \in B\big(L^2(Q_1), L^2(Q_2)\big)$ *is called*

(1) positivity preserving *if* $f \geq 0 \ \Rightarrow \ Tf \geq 0$,
(2) positivity improving *if* $f \geq 0, \ f \neq 0 \ \Rightarrow Tf > 0$.

Note that T is positivity preserving (resp. improving) iff T^* is.

Let us assume in addition that μ_i, $i = 1, 2$, are probability measures.

Definition 5.22 $T \in B\big(L^2(Q_1), L^2(Q_2)\big)$ *is called* hyper-contractive *if T is a contraction and there exists $p > 2$ such that T is bounded from $L^2(Q_1)$ into $L^p(Q_2)$.*

Let μ be a probability measure on $(Q, \mathfrak{S})$. Clearly, the constant function 1 belongs to $L^2(Q)$.

Definition 5.23 $T \in B\big(L^2(Q)\big)$ *is* doubly Markovian *if T is positivity preserving and $T1 = T^*1 = 1$.*

We recall some classic results.

Proposition 5.24 *A doubly Markovian map T extends to a contraction on $L^p(Q)$ for all $1 \leq p \leq \infty$.*

Theorem 5.25 (Perron-Frobenius) *Let H be a bounded below self-adjoint operator on $L^2(Q)$, such that e^{-tH} is positivity preserving for $t \geq 0$ and $E = \inf \mathrm{spec}(H)$ is an eigenvalue. Then the following are equivalent:*

(1) $\inf \mathrm{spec}(H)$ *is a simple eigenvalue with a strictly positive eigenvector.*
(2) e^{-tH} *is positivity improving for all $t > 0$.*

5.1.8 Conditional expectations

Let μ be a measure on $(Q, \mathfrak{S})$. Let $\mathfrak{S}_0$ be a sub-σ-algebra of $\mathfrak{S}$. Let μ_0 denote the restriction of the measure μ to $\mathfrak{S}_0$.

For $1 \leq p \leq \infty$, elements of $L^p(Q, \mu)$ that are $\mathfrak{S}_0$-measurable form a closed subspace of $L^p(Q, \mu)$ that can be identified with $L^p(Q, \mu_0)$.

Definition 5.26 *We denote by* $E_{\mathfrak{S}_0}$ *the orthogonal projection from* $L^2(Q,\mu)$ *onto the subspace* $L^2(Q,\mu_0)$*.* $E_{\mathfrak{S}_0}$ *is called the* conditional expectation *w.r.t.* $\mathfrak{S}_0$*.*

The following properties are well known.

Proposition 5.27 *Let* μ *be a probability measure.*

(1) $E_{\mathfrak{S}_0}$ *extends to a contraction on* $L^p(Q,\mu)$ *for all* $1 \leq p \leq \infty$*.*
(2) $E_{\mathfrak{S}_0}$ *extends to an operator from* $M_+(Q,\mathfrak{S})$ *to* $M_+(Q,\mathfrak{S}_0)$*.*
(3) *If* $g \in L^\infty(Q,\mu)$ *is* $\mathfrak{S}_0$*-measurable, then* $E_{\mathfrak{S}_0}(gf) = gE_{\mathfrak{S}_0}(f)$ *whenever both sides are defined.*
(4) *If* $\varphi : \mathbb{R} \to \mathbb{R}$ *is convex and positive, then*

$$\varphi(E_{\mathfrak{S}_0} f) \leq E_{\mathfrak{S}_0}(\varphi(f)) \text{ a.e.}$$

(5) *If* $\mathfrak{S}_0 \subset \mathfrak{S}_1$ *are two sub-*σ*-algebras of* $\mathfrak{S}$*, then* $E_{\mathfrak{S}_0} \leq E_{\mathfrak{S}_1}$*.*
(6) *Let* $\{\mathfrak{S}_n\}_{n\in\mathbb{N}}$ *be an increasing sequence of sub-*σ*-algebras of* $\mathfrak{S}$ *such that* $\mathfrak{S}$ *is generated by* $\bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$*. Then*

$$\mathrm{s} - \lim_{n\to\infty} E_{\mathfrak{S}_n} = \mathbb{1}, \text{ in } L^p(Q,\mu),\ 1 \leq p < \infty.$$

(7) *Let* $F \in L^1(Q,\mu)$ *with* $F > 0$ *a.e. and set* $\mathrm{d}\mu_F = \left(\int_Q F\mathrm{d}\mu\right)^{-1} F\mathrm{d}\mu$*. Denote by* $E^F_{\mathfrak{S}_0}$ *the conditional expectation for the measure* μ_F*. Then*

$$E^F_{\mathfrak{S}_0}(f) = \frac{E_{\mathfrak{S}_0}(Ff)}{E_{\mathfrak{S}_0}(F)}.$$

5.1.9 Convergence in measure

Let (Q,μ) be a probability space. In this subsection we review various notions of convergence for nets of functions on a probability space.

Definition 5.28 *The topology of* convergence in measure *on* $M(Q)$ *is defined by the following family* $V(\epsilon,\delta)$ *of neighborhoods of* 0*:*

$$V(\epsilon,\delta) := \Big\{ f \in M(Q) \ :\ \mu(\{q \ :\ |f(q)| > \epsilon\}) < \delta \Big\}.$$

It is a metric topology for the distance

$$d(f,g) = \sum_{n=0}^{\infty} 2^{-n} \mu\left(\left\{q \ :\ |f(q) - g(q)| \geq 2^{-n}\right\}\right).$$

The following proposition is immediate:

Proposition 5.29 *If* $f_n \to f$ *a.e. then* $f_n \to f$ *in measure.*

We also recall the useful notion of the *equi-integrability*.

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Definition 5.30 *A family* $\{f_i\}_{i\in I}$ *in* $M(Q)$ *is* equi-integrable *if*

$$\lim_{n\to+\infty}\sup_{i\in I}\int_Q |f_i|\mathbb{1}_{[n,\infty[}(f_i)\mathrm{d}\mu = 0.$$

The following two results are well-known:

Proposition 5.31 *Let* $\{f_i\}_{i\in I}$ *belong to* $M(Q)$. *Then the following hold:*

(1) *If* $f := \sup\limits_{i\in I}|f_i|$ *is in* $L^1(Q)$, *then* $\{f_i\}_{i\in I}$ *is equi-integrable.*
(2) *If* $\sup\limits_{i\in I}\|f_i\|_p < \infty$ *for some* $p > 1$, *then* $\{f_i\}_{i\in I}$ *is equi-integrable.*

Theorem 5.32 (Lebesgue–Vitali theorem) *Let* $1 \le p < \infty$, $(f_n)_{n\in\mathbb{N}}$ *belong to* $L^p(Q)$ *and* $f \in M(Q)$. *Then the following are equivalent:*

(1) $f \in L^p(Q)$ *and* $f_n \to f$ *in* $L^p(Q)$.
(2) $\big(|f_n|^p\big)_{n\in\mathbb{N}}$ *is equi-integrable and* $f_n \to f$ *in measure.*

5.1.10 Measure preserving transformations

Let μ be a probability measure on $(Q,\mathfrak{S})$. Clearly, $L^\infty(Q)$ is a commutative W^*-algebra equipped with a faithful normal state, which we also denote by μ, that is,

$$\mu(f) := \int f\mathrm{d}\mu, \quad f \in L^\infty(Q).$$

(See Subsect. 6.2.7 for the terminology on W^*-algebras.) Conversely, every commutative W^*-algebra equipped with a faithful normal state can be represented as $L^\infty(Q)$ for some probability space $(Q,\mathfrak{S},\mu)$. However, in general there may be many non-isomorphic choices of probability spaces that lead to the same W^*-algebra and state.

Clearly, if r is a measure preserving bijection on Q, then $r_\# f := f\circ r^{-1}$ defines an isometry on $L^p(Q)$ for all $1 \le p \le \infty$. In the case of $p = \infty$, it is in addition a σ-continuous $*$-automorphism of the commutative W^*-algebra $L^\infty(Q)$ preserving the state μ. However, if we are given a σ-continuous $*$-automorphism of $L^\infty(Q)$, we have no guarantee that there exists an underlying bijection of Q. Therefore, in the following proposition we do not insist on the existence of an underlying bijection for $*$-automorphisms of $L^\infty(Q)$.

Proposition 5.33 (1) *A* $*$*-automorphism of* $L^\infty(Q)$ *that preserves the state* μ *extends to an isometry of* $L^p(Q)$ *for all* $1 \le p \le +\infty$.
(2) *Let* $\mathbb{R} \ni t \mapsto U(t)$ *be a group of* $*$*-automorphisms of* $L^\infty(Q)$ *preserving the state* μ. *Then the following statements are equivalent:*
   (i) *For some* $1 \le p < \infty$ *and all* $f \in L^p(Q)$, $\mathbb{R} \ni t \mapsto U(t)f \in L^p(Q)$ *is norm continuous.*

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(ii) *For all* $f \in L^\infty(Q)$, $\mathbb{R} \ni t \mapsto U(t)f$ *is continuous in measure.*
(iii) *For all* $1 \leq p < \infty$ *and* $f \in L^p(Q)$, $\mathbb{R} \ni t \mapsto U(t)f \in L^p(Q)$ *is norm continuous.*
(iv) *For all* $f \in L^\infty(Q)$, $\mathbb{R} \ni t \mapsto U(t)f$ *is* σ*-weakly continuous.*

Proof Let T be a $*$-automorphism of $\mathcal{M}(Q)$ as in (1). Clearly, T preserves the L^p norm of simple functions for all $1 \leq p < \infty$. Therefore, T is an isometry of L^p for $1 \leq p < \infty$. Then using that $\|f\|_\infty = \|m(f)\|_{B(L^2(Q))}$ if $m(f)$ is the operator of multiplication by f, we obtain also that T is an isometry of $L^\infty(Q)$.

We now prove (2). Since $\int |f|^p \mathrm{d}\mu \geq \epsilon^p \mu(\{|f| \geq \epsilon\})$, we obtain that (i)$\Rightarrow$(ii). Let us prove that (ii)$\Rightarrow$(iii). Using (1) it suffices by density to show that

$$\lim_{t \to 0} \int |U(t)f - f|^p \mathrm{d}\mu = 0, \text{ for } f \in L^\infty. \tag{5.9}$$

We write

$$\int |U(t)f - f|^p \mathrm{d}\mu \leq \mu\left(\{|U(t)f - f| \geq \epsilon\}\right) 2^p \|f\|_\infty^p + \epsilon^p.$$

Choosing first ϵ and then t small enough we obtain (5.9). To complete the proof of the lemma it suffices to prove that (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). Since $\int_Q fU(t)g\mathrm{d}\mu = \int_Q U(-t)fg\mathrm{d}\mu$ for $g \in L^\infty$, $f \in L^1$, we see that (iii) $\Rightarrow$ (iv). Using that $\|U(t)g - g\|_2^2 = 2\|g\|^2 - 2\mathrm{Re}\int_Q U(t)\overline{g}g\mathrm{d}\mu$ for $g \in L^\infty$, we obtain by a density argument that (iv) $\Rightarrow$ (i). □

5.1.11 Relative continuity

Let μ be a measure on $(Q, \mathfrak{S})$.

Proposition 5.34 *Let* $F \in M_+(Q)$*. Then*

$$\mathfrak{S} \ni A \mapsto \nu(A) := \int \mathbb{1}_A F \mathrm{d}\mu \tag{5.10}$$

is a measure.

Definition 5.35 *The measure (5.10) is called the* measure with the density F w.r.t. the measure μ *and is denoted* $\nu = F\mu$*. We will also write* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} := F$*.*

Proposition 5.36 (1) *For* F, G *measurable functions we have*

$$F = G \ \mu\text{-a.e.} \Rightarrow F\mu = G\mu.$$

(2) *If* $F\mu$ *is* σ*-finite, then the converse implication is also true.*

Definition 5.37 *Let* ν *be a measure on* $(Q, \mathfrak{S})$*.* ν *is called* continuous w.r.t. μ (or μ-continuous)*, if*

$$\mu(N) = 0 \Rightarrow \nu(N) = 0, \quad N \in \mathcal{F}.$$

Theorem 5.38 (Radon–Nikodym theorem) *Let* μ *be* σ*-finite. Let* ν *be a measure on* $(Q, \mathfrak{S})$*. Then the following conditions are equivalent:*

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(1) *there exists a positive measurable function* F *such that* $\nu = F\mu$.
(2) ν *is* μ*-continuous. The function* F *is called the* Radon–Nikodym derivative *of* ν *w.r.t.* μ *and denoted by* $\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.

Note that, in the notation of Def. 5.35, the map

$$L^2(Q,\nu) \ni f \mapsto \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^{\frac{1}{2}} f \in L^2(Q,\mu)$$

is unitary.

5.1.12 Moments of a measure

Let μ be a probability measure on $(Q,\mathfrak{S})$.

Proposition 5.39 *Let* $f : Q \to \mathbb{R}$ *be a measurable function. Let*

$$C(t) = \int \mathrm{e}^{\mathrm{i}tf} \mathrm{d}\mu, \quad t \in \mathbb{R}.$$

(1) $f \in \bigcap_{p\in\mathbb{N}} L^p(Q)$ *iff* $C(t) \in C^\infty(\mathbb{R})$, *and then*

$$\int f^p \mathrm{d}\mu = (-\mathrm{i})^p \frac{\mathrm{d}^p}{\mathrm{d}t^p} C(0).$$

(2) *Assume that* $C(t)$ *extends holomorphically to* $\{|\mathrm{Im}\, z| < R_0\}$. *Then for all* $|\mathrm{Im}\, z| < R_0$, $\mathrm{e}^{\mathrm{i}zf} \in L^1(Q)$ *and*

$$C(z) = \int \mathrm{e}^{\mathrm{i}zf} \mathrm{d}\mu.$$

Proof Let us first prove (1). The $\Rightarrow$ part is immediate by differentiating under the integral sign. It remains to prove $\Leftarrow$. It suffices to prove that $f \in L^{2n}(Q)$ for all $n \in \mathbb{N}$ by induction on n. For $\Phi \in L^2(Q)$, $f\Phi \in L^2(Q)$ iff $\Phi \in \mathrm{Dom}\, m(f)$, where $m(f)$ denotes the operator of multiplication by f on $L^2(Q)$. This is equivalent to $\|(\mathrm{e}^{\mathrm{i}tf} - \mathbb{1})\Phi\|^2 \leq Ct^2$ for $|t| \leq 1$. If $\Phi = 1$, we get

$$\begin{aligned}\|(\mathrm{e}^{\mathrm{i}tf} - \mathbb{1})\Phi\|^2 &= \int_Q (2 - \mathrm{e}^{\mathrm{i}tf} - \mathrm{e}^{-\mathrm{i}tf})\mathrm{d}\mu \\ &= 2C(0) - C(t) - C(-t) = O(t^2),\end{aligned}$$

since $C(t)$ is C^2, and hence $f \in L^2(Q)$. Assume now that $f \in L^{2n}(Q)$. We then have

$$\frac{\mathrm{d}^{2n}}{dt^{2n}} C(t) = \mathrm{i}^{2n} \int f^{2n} \mathrm{e}^{\mathrm{i}tf} \mathrm{d}\mu.$$

Applying the above remark to $\Phi = f^{2n}$, we get

$$\begin{aligned}\|(\mathrm{e}^{\mathrm{i}tf} - \mathbb{1})f^{2n}\|^2 &= \int (2 - \mathrm{e}^{\mathrm{i}tf} - \mathrm{e}^{-\mathrm{i}tf}) f^{2n} \mathrm{d}\mu \\ &= 2C^{(2n)}(0) - C^{(2n)}(t) - C^{(2n)}(-t) = O(t^2),\end{aligned}$$

since $C(t)$ is C^{2n+2}. Hence $f \in L^{2n+2}(Q)$.

To prove (2), it clearly suffices to show that $\mathrm{e}^{\pm Rf} \in L^1(Q)$ for all $0 < R < R_0$. By Cauchy's inequalities, we get for all $0 < R < R_0$

$$|C^{(n)}(0)| \leq C_R R^{-n} n!,$$

and hence

$$\int f^{2n} \mathrm{d}\mu \leq C_R R^{-2n} (2n)!,$$

$$\int |f|^{2n+1} \mathrm{d}\mu \leq \left(\int f^{2n} \mathrm{d}\mu\right)^{\frac{1}{2}} \left(\int f^{2n+2} \mathrm{d}\mu\right)^{\frac{1}{2}} \leq C_R R^{-(2n+1)} \sqrt{2n!}\sqrt{(2n+2)!}.$$

Using Stirling's formula, we see that $\sqrt{2n!}\sqrt{(2n+2)!} \sim (2n+1)!$, and hence

$$\int |f|^{2n+1} \mathrm{d}\mu \leq C'_R R^{-(2n+1)} (2n+1)!.$$

From these bounds, by expanding the exponential, we deduce that $\mathrm{e}^{\pm Rf} \in L^1(Q)$ for all $R < R_0$ □

5.2 Finite measures on real Hilbert spaces

In this section we describe the basic theory of probability measures on real Hilbert spaces.

Throughout this section, $\mathcal{X}$ will be a real *separable* Hilbert space. For $x_1, x_2 \in \mathcal{X}$ we denote their scalar product by $x_1 \cdot x_2$.

5.2.1 Cylinder sets and cylinder functions

Let $\mathcal{Y}$ be a closed subspace of $\mathcal{X}$. Recall that $P_\mathcal{Y}$ denotes the orthogonal projection on $\mathcal{Y}$. Recall also that $\mathfrak{B}(\mathcal{Y})$ stands for the σ-algebra of Borel sets in $\mathcal{Y}$. We will write $\mathfrak{B}$ for $\mathfrak{B}(\mathcal{X})$.

Definition 5.40 $\mathrm{Fin}(\mathcal{X})$ *will denote the family of finite-dimensional subspaces of* $\mathcal{X}$*. For* $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$ *and* $A \subset \mathcal{Y}$*, the set*

$$P_\mathcal{Y}^{-1}(A) := \{x \in \mathcal{X} \ : \ P_\mathcal{Y} x \in A\}$$

is called the cylinder set of base A*. Denote by* $\mathfrak{B}^\mathcal{Y}$ *the* σ*-algebra of cylinder sets of bases in* $\mathfrak{B}(\mathcal{Y})$*.*

$$\mathfrak{B}_{\mathrm{cyl}} := \bigcup_{\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})} \mathfrak{B}^\mathcal{Y}$$

is the algebra of all cylinder sets.

Clearly, $\mathfrak{B}^{\mathcal{Y}_1} \subset \mathfrak{B}^{\mathcal{Y}_2}$ if $\mathcal{Y}_1 \subset \mathcal{Y}_2$.

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Proposition 5.41 $\mathfrak{B}$ *is the* σ*-algebra generated by* $\mathfrak{B}_{\mathrm{cyl}}$.

Definition 5.42 *We say that* $F : \mathcal{X} \to \mathbb{C}$ *is* based on $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$ *if it is measurable w.r.t.* $\mathfrak{B}^{\mathcal{Y}}$. F *is called a* cylinder function *if it is based on* $\mathcal{Y}$ *for some* $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$.

Each cylinder function is of the form $F(x) = F_{\mathcal{Y}}(P_{\mathcal{Y}}x)$ for some measurable function $F_{\mathcal{Y}}$ on $\mathcal{Y}$.

5.2.2 Finite-dimensional distributions of a measure

Until the end of this section we fix a probability measure μ on $(\mathcal{X}, \mathfrak{B})$.

Definition 5.43 *If* $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$, *we define the probability measure* $\mu_{\mathcal{Y}}$ *on* $\big(\mathcal{Y}, \mathfrak{B}(\mathcal{Y})\big)$ *by*

$$\mu_{\mathcal{Y}}(A) := \mu\big(P_{\mathcal{Y}}^{-1}(A)\big), \quad A \in \mathfrak{B}(\mathcal{Y}).$$

The collection $\big\{\mu_{\mathcal{Y}} \ : \ \mathcal{Y} \in \mathrm{Fin}(\mathcal{X})\big\}$ *is called the* set of finite-dimensional distributions of the measure μ.

Finite-dimensional distributions satisfy the following *compatibility condition*:

$$\mu_{\mathcal{Y}_1}(A) = \mu_{\mathcal{Y}_2}\big(P_{\mathcal{Y}_1}^{-1}(A) \cap \mathcal{Y}_2\big), \quad A \in \mathfrak{B}(\mathcal{Y}_1), \ \ \mathcal{Y}_1 \subset \mathcal{Y}_2. \tag{5.11}$$

Proposition 5.44 *The set of finite-dimensional distributions uniquely determines the measure* μ *on the whole* $\mathfrak{B}$.

Proof Finite-dimensional distributions uniquely determine μ on $\mathfrak{B}_{\mathrm{cyl}}$. But $\mathfrak{B}_{\mathrm{cyl}}$ generates $\mathfrak{B}$. □

5.2.3 Characteristic functional of a measure

Recall that $\mathcal{X}^{\#}$ denotes the space dual to $\mathcal{X}$. Even though there exists a canonical identification of $\mathcal{X}$ and $\mathcal{X}^{\#}$, it is sometimes convenient to distinguish between $\mathcal{X}$ and $\mathcal{X}^{\#}$.

Definition 5.45 *For* $\xi \in \mathcal{X}^{\#}$, *we set*

$$\hat{\mu}(\xi) := \int_{\mathcal{X}} \mathrm{e}^{-\mathrm{i}\xi\cdot x}\mathrm{d}\mu(x).$$

The function $\hat{\mu} : \mathcal{X}^{\#} \to \mathbb{C}$ *is called the* characteristic functional of μ, *or the* Fourier transform of μ.

Proposition 5.46 *The characteristic functional of* μ *satisfies the following three conditions:*

(1) $\hat{\mu}(0) = 1$,

(2) $\sum\limits_{i,j=1}^{N} \hat{\mu}(\xi_i - \xi_j)\overline{z}_i z_j \geq 0, \quad \xi_i \in \mathcal{X}^{\#}, \quad z_i \in \mathbb{C},$

(3) $\mathcal{X}^{\#} \ni \xi \mapsto \hat{\mu}(\xi) \in \mathbb{C}$ *is sequentially continuous for the weak topology of* $\mathcal{X}^{\#}$.

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The condition (2) above is called *positive definiteness.*

Proposition 5.47 *The characteristic functional $\hat{\mu}$ uniquely determines the measure μ.*

Proof The restriction of $\hat{\mu}$ to $\mathcal{Y}^{\#}$ for $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$ is the Fourier transform of $\mu_{\mathcal{Y}}$, so $\hat{\mu}$ determines the finite-dimensional distributions of μ. By Prop. 5.44 this determines μ. □

5.2.4 Moment functions

Proposition 5.48 *Let $p_0 \geq 0$. Assume that for all $\xi \in \mathcal{X}^{\#}$, the function $x \mapsto \xi \cdot x$ belongs to $L^{p_0}(\mathcal{X}, \mathrm{d}\mu)$. Then, for $0 \leq p \leq p_0$, there exists C such that*

$$\gamma_p(\xi) := \int_{\mathcal{X}} |\xi \cdot x|^p \mathrm{d}\mu(x) \leq C\|\xi\|^p. \tag{5.12}$$

Proof For $\epsilon > 0$, set

$$\gamma_{p,\epsilon}(\xi) := \int_{\mathcal{X}} |\xi \cdot x|^p \mathrm{e}^{-\epsilon\|x\|^2} \mathrm{d}\mu(x).$$

For $n \in \mathbb{N}$, set

$$\begin{aligned} A_n &:= \big\{\xi \in \mathcal{X}^{\#} \ : \ \gamma_p(\xi) \leq n\big\}, \\ A_{n,\epsilon} &:= \big\{\xi \in \mathcal{X}^{\#} \ : \ \gamma_{p,\epsilon}(\xi) \leq n\big\}. \end{aligned}$$

Clearly, $\gamma_{p,\epsilon}(\xi) \nearrow \gamma_p(\xi)$ when $\epsilon \to 0$, hence $A_n = \bigcap_{\epsilon>0} A_{n,\epsilon}$. Since $\xi \mapsto \gamma_{p,\epsilon}(\xi)$ is norm continuous, $A_{n,\epsilon}$ is closed and so is A_n as an intersection of closed sets. Finally $\mathcal{X}^{\#} = \bigcup_{n\in\mathbb{N}} A_n$.

Since $\mathcal{X}^{\#}$ has a non-empty interior, there exists by the Baire property a set A_m with a non-empty interior. Let $\xi_0 \in \mathcal{X}^{\#}$, $\delta > 0$ such that $B(\xi_0, \delta) \subset A_m$. If $\|\xi\| \leq \delta$, we write $\xi = \xi_0 + \xi_1$, $\xi_1 = \xi - \xi_0 \in A_m$. Using that

$$|\xi \cdot x|^p \leq C \sum_{p_1+p_2=p} |\xi_0 \cdot x|^{p_1} |\xi_1 \cdot x|^{p_2}$$

and the Hölder inequality, we obtain that

$$\gamma_p(\xi) \leq C, \quad \|\xi\| \leq \delta,$$

which proves (5.12). □

Definition 5.49 *Assume that the conditions of Prop. 5.48 are satisfied. The* moment functions of order $1 \leq p \leq p_0$ of the measure μ *are the maps*

$$(\xi_1, \ldots, \xi_p) \mapsto \sigma_p(\xi_1, \ldots, \xi_p) := \int_{\mathcal{X}} (\xi_1 \cdot x) \cdots (\xi_p \cdot x) \mathrm{d}\mu(x).$$

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Moment functions are well defined by the Hölder inequality.

The following proposition follows directly from Props. 5.39 and 5.48:

Proposition 5.50 (1) *The moment functions σ_p are multi-linear symmetric functionals on $\mathcal{X}^{\#}$.*

(2)

$$|\sigma_p(\xi_1, \dots, \xi_p)| \leq C\|\xi_1\| \cdots \|\xi_p\|. \tag{5.13}$$

(3) *μ admits moments of all orders iff its characteristic functional $\hat{\mu}$ is weakly infinitely differentiable. We then have*

$$\sigma_p(\xi_1, \dots, \xi_p) = (-\mathrm{i})^p \frac{\partial^p}{\partial t_1 \cdots \partial t_p} \hat{\mu}\Big(\sum_{i=1}^{p} t_i \xi_i\Big)\Big|_{t_1 = \cdots = t_p = 0}.$$

By Prop. 5.50 and the Riesz theorem, if the assumptions of Prop. 5.48 hold with $n = 1$, then there exists $q \in \mathcal{X}$ such that

$$\xi \cdot q = \int_{\mathcal{X}} (\xi \cdot x) \mathrm{d}\mu(x), \quad \xi \in \mathcal{X}^{\#}.$$

Definition 5.51 *The vector q is called the* mean of the measure *μ.*

Again by Prop. 5.50, if assumptions of Prop. 5.48 hold with $n = 2$ and q is the mean of μ, there exists a bounded positive $A \in B_{\mathrm{s}}(\mathcal{X})$ such that

$$\xi_1 \cdot A\xi_2 = \int_{\mathcal{X}} \big(\xi_1 \cdot (x - q)\big)\big(\xi_2 \cdot (x - q)\big) \mathrm{d}\mu(x), \quad \xi_1, \xi_2 \in \mathcal{X}^{\#}.$$

Definition 5.52 *The operator A is called the* covariance of the measure *μ.*

Proposition 5.53 *Assume that the measure μ has mean zero and*

$$\int_{\mathcal{X}} \|x\|^2 \mathrm{d}\mu(x) < \infty.$$

Then the covariance A of μ is trace-class and

$$\mathrm{Tr}\, A = \int_{\mathcal{X}} \|x\|^2 \mathrm{d}\mu(x).$$

Proof It suffices to let $n \to \infty$ in the equality

$$\sum_{i=1}^{n} e_i \cdot A e_i = \int_{\mathcal{X}} \sum_{i=1}^{n} (x \cdot e_i)^2 \mathrm{d}\mu(x),$$

where $(e_i)_{i \in \mathbb{N}}$ is an o.n. basis of $\mathcal{X}$. □

5.2.5 Density of exponentials

Theorem 5.54 *Let $\mathcal{D}$ be a dense subspace of $\mathcal{X}^{\#}$. Then the space*

$$\mathrm{Span}\{\mathrm{e}^{\mathrm{i}\xi\cdot x} : \quad \xi \in \mathcal{D}\}$$

is dense in $L^2(\mathcal{X})$.

Proof Let $G \in L^2(\mathcal{X})$ such that

$$\int_{\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x} G(x)\mathrm{d}\mu(x) = 0, \quad \xi \in \mathcal{D}. \tag{5.14}$$

Without loss of generality we can assume that G is real-valued. Let

$$B_1 = \big\{x \in \mathcal{X} \ : \ G(x) \geq 0\big\}, \quad B_2 = \big\{x \in \mathcal{X} \ : \ G(x) < 0\big\}.$$

We can define the finite measures

$$\mu_1(A) := \int_A 1\!\!1_{B_1}(x)G(x)\mathrm{d}\mu(x), \quad \mu_2(A) = -\int_A 1\!\!1_{B_2}(x)G(x)\mathrm{d}\mu(x),$$

where $A \in \mathfrak{B}$. From (5.14), we deduce that

$$\int_{\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\mu_1(x) = \int_{\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\mu_2(x), \quad \xi \in \mathcal{D}. \tag{5.15}$$

$\mathcal{D}$ is a dense subspace of $\mathcal{X}^{\#}$. Hence it is weakly sequentially dense in $\mathcal{X}^{\#}$. Since the characteristic functional of a measure is sequentially continuous for the weak topology, (5.15) extends to all $\xi \in \mathcal{X}^{\#}$. So μ_1 and μ_2 have the same characteristic functionals, and hence are identical, i.e. $\mu_1(A) = \mu_2(A)$ for all $A \in \mathfrak{B}$. But $\mu_i(A) = \mu_i(A \cap B_i)$, $i = 1, 2$, and $B_1 \cap B_2 = \emptyset$. Hence, $\mu_1 = \mu_2 = 0$. This implies that $G(x) = 0$ μ-a.e., and hence $G = 0$. □

5.2.6 Density of continuous polynomials

Let $\mathcal{D}$ be a subspace of $\mathcal{X}^{\#}$.

Definition 5.55 *Functions on $\mathcal{X}$ of the form $(\xi_1 \cdot x) \cdots (\xi_p \cdot x)$, for $\xi_1, \dots, \xi_n \in \mathcal{D}$, are called* monomials based on $\mathcal{D}$. *Finite linear combinations (with complex coefficients) of monomials based on $\mathcal{D}$ are called* polynomials based on $\mathcal{D}$.

Note that polynomials based on $\mathcal{X}^{\#}$ are continuous functions. Therefore, they are sometimes called *continuous polynomials.*

If the measure μ admits moments of all orders, then all continuous polynomials belong to $L^2(\mathcal{X})$. The following theorem gives a sufficient condition for the density of continuous polynomials in $L^2(\mathcal{X})$.

Theorem 5.56 *Let $\mathcal{D} \subset \mathcal{X}^{\#}$ be a dense subspace of $\mathcal{X}^{\#}$. Assume that for all $\xi \in \mathcal{D}$ there exists $R(\xi) > 0$ such that the function*

$$\mathbb{R} \ni t \mapsto \hat{\mu}(t\xi) \in \mathbb{C}$$

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extends holomorphically to $|\mathrm{Im}\, t| < R(\xi)$*. Then polynomials based on* $\mathcal{D}$ *are dense in* $L^2(\mathcal{X})$*.*

Proof Let $G \in L^2(\mathcal{X})$ be a vector orthogonal to all polynomials based on $\mathcal{D}$. Without loss of generality we can assume that G is real-valued. We then have

$$\int_{\mathcal{X}} G(x)(\xi \cdot x)^n \mathrm{d}\mu(x) = 0, \quad \xi \in \mathcal{D}, \quad n \in \mathbb{N}.$$

Let us fix $\xi \in \mathcal{D}$ and let $2R < R(\xi)$. Then by Prop. 5.39 we know that $\mathrm{e}^{2R|\xi\cdot x|} \in L^1(Q)$ and

$$\int G(x)\mathrm{e}^{\mathrm{i}R\xi\cdot x}\mathrm{d}\mu(x) = \lim_{n\to\infty}\int G(x)\sum_{k=1}^{n}\frac{(\mathrm{i}R\xi\cdot x)^k}{k!}\mathrm{d}\mu(x).$$

We can exchange sum and integral, since the integrand in the r.h.s. is less than

$$|G(x)|\mathrm{e}^{R|\xi\cdot x|} \le \frac{1}{2}\big(|G(x)|^2 + \mathrm{e}^{2R|\xi\cdot x|}\big) \in L^1(\mathcal{X}).$$

We obtain hence that

$$\int G(x)\mathrm{e}^{\mathrm{i}R\xi\cdot x}\mathrm{d}\mu(x) = 0,$$

and, by differentiating w.r.t. R,

$$\int G(x)\mathrm{e}^{\mathrm{i}R\xi\cdot x}(\xi\cdot x)^n\mathrm{d}\mu(x) = 0, \quad n \in \mathbb{N}.$$

Arguing as above with $G(x)$ replaced by $G(x)\mathrm{e}^{\mathrm{i}R\xi\cdot x}$, we obtain

$$\int G(x)\mathrm{e}^{\mathrm{i}R\xi\cdot x}\mathrm{e}^{\mathrm{i}R\xi\cdot x}\mathrm{d}\mu(x) = 0.$$

Hence, repeating this argument, we obtain

$$\int G(x)\mathrm{e}^{\mathrm{i}mR\xi\cdot x}\mathrm{d}\mu(x) = 0, \quad m \in \mathbb{N}.$$

If we choose $m \in \mathbb{N}$ and $2R < R(\xi)$ such that $mR = 1$, we finally obtain

$$\int G(x)\mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\mu(x) = 0, \quad \xi \in \mathcal{D}.$$

Applying Thm. 5.54, we obtain that $G = 0$. □

5.3 Weak distributions and the Minlos–Sazonov theorem

Throughout this section, $\mathcal{X}$ is a separable real Hilbert space.

Suppose that we have a compatible family of measures on finite-dimensional subspaces of $\mathcal{X}$. We can ask whether this family comes from a measure on a certain measure space. Often, there is no such a measure on $\mathcal{X}$ itself. However, if we enlarge $\mathcal{X}$, usually in a non-unique way, then such a measure may exist.

5.3.1 Weak distributions

Definition 5.57 *A collection* $\mu_* = \{\mu_{\mathcal{Y}} : \mathcal{Y} \in \mathrm{Fin}(\mathcal{X})\}$ *is called a* weak distribution *or a* generalized measure *if, for each* $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$, $\mu_{\mathcal{Y}}$ *is a Borel probability measure on* $\mathcal{Y}$*, and these measures satisfy the compatibility condition (5.11).*

Note that cylinder functions can be "integrated" w.r.t. a weak distribution μ_*. In fact, we can set

$$\int_{\mathcal{X}} F \mathrm{d}\mu_* := \int_{\mathcal{Y}} F_{\mathcal{Y}} \mathrm{d}\mu_{\mathcal{Y}}, \tag{5.16}$$

where $F(x) = F_{\mathcal{Y}}(P_{\mathcal{Y}}x)$. Because of the compatibility condition (5.11), the r.h.s. of (5.16) is independent of the choice of $\mathcal{Y}$ on which F is based.

For each $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$ and $1 \le p < \infty$, we can define the space $L^p(\mathcal{Y}, \mu_{\mathcal{Y}})$. For $\mathcal{Y}_1 \subset \mathcal{Y}_2$, we have natural isometric embeddings

$$L^p(\mathcal{Y}_1, \mu_{\mathcal{Y}_1}) \subset L^p(\mathcal{Y}_2, \mu_{\mathcal{Y}_2}).$$

Definition 5.58 *The* generalized L^p space *associated with a generalized measure* μ_* *is defined as the inductive limit of the spaces* $L^p(\mathcal{Y}, \mu_{\mathcal{Y}})$*, that is,*

$$\mathbf{L}^p(\mathcal{X}, \mu_*) := \left(\bigcup_{\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})} L^p(\mathcal{Y}, \mu_{\mathcal{Y}}) \right)^{\mathrm{cpl}}.$$

5.3.2 Weak distributions generated by a measure

Definition 5.59 *Let* μ *be a measure on* $(\mathcal{X}, \mathfrak{B})$*. A weak distribution* $\mu_* = \{\mu_{\mathcal{Y}} : \mathcal{Y} \in \mathrm{Fin}(\mathcal{X})\}$ *is said to be* generated by μ *if it is the set of finite-dimensional distributions of* μ*.*

The following necessary and sufficient condition for this to happen is given in Skorokhod (1974):

Theorem 5.60 *A weak distribution* μ_* *is generated by a probability measure iff*

$$\lim_{R \to \infty} \left(\sup_{\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})} \int_{\mathcal{Y}} \mathbb{1}_{[R,\infty[}(\|y\|) \mathrm{d}\mu_{\mathcal{Y}}(y) \right) = 0. \tag{5.17}$$

5.3.3 Characteristic functionals of weak distributions

The following proposition coincides with the famous Bochner theorem if $\mathcal{X}$ is finite-dimensional:

Proposition 5.61 *Let* $F : \mathcal{X} \to \mathbb{C}$ *be a function satisfying the following conditions:*

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(1) $F(0) = 1$,
(2) $\sum\limits_{i,j=1}^{n} F(\xi_i - \xi_j) z_i \overline{z_j} \geq 0, \ \ \xi_1, \ldots, \xi_n \in \mathcal{X}, \ \ z_1, \ldots, z_n \in \mathbb{C}$,
(3) $\mathcal{Y} \ni \xi \mapsto F(\xi) \in \mathbb{C}$ *is continuous for all* $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$.

Then there exists a weak distribution $\{\mu_{\mathcal{Y}} \ : \ \mathcal{Y} \in \mathrm{Fin}(\mathcal{X})\}$ *such that, for any* $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$,

$$F(\xi) = \int_{\mathcal{Y}} \mathrm{e}^{-\mathrm{i}\xi \cdot y} \mathrm{d}\mu_{\mathcal{Y}}(y), \quad \xi \in \mathcal{Y}. \tag{5.18}$$

Note that the functions $\mathcal{X} \ni x \mapsto \mathrm{e}^{\mathrm{i}\xi \cdot x}$ are cylinder functions, hence the integral in the r.h.s. of (5.18) is well defined.

Definition 5.62 *A function F satisfying (1), (2) and (3) of Prop. 5.61 will be called a* weak characteristic functional*.*

Proof of Prop. 5.61. For any $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$, the restriction of F to $\mathcal{Y}$ satisfies the hypotheses of Bochner's theorem (see Reed–Simon (1978b)). Hence there exists a probability measure $\mu_{\mathcal{Y}}$ on $\mathcal{Y}$ such that (5.18) holds. It remains to check the compatibility condition (5.11). To check this, it suffices to show that, if $\mathcal{Y}_1 \subset \mathcal{Y}_2$, for each bounded continuous function G on $\mathcal{Y}_1$ one has

$$\int_{\mathcal{Y}_2} G \circ P_{\mathcal{Y}_1} \mathrm{d}\mu_{\mathcal{Y}_2} = \int_{\mathcal{Y}_1} G \mathrm{d}\mu_{\mathcal{Y}_1}. \tag{5.19}$$

This is clearly satisfied for $G(y) = \mathrm{e}^{\mathrm{i}\xi \cdot y}$ for $\xi \in \mathcal{Y}_1$. Next we can find a bounded sequence (G_n) of finite linear combinations of $\mathrm{e}^{\mathrm{i}\xi \cdot x}$ for $\xi \in \mathcal{Y}_1$ which converges a.e. to G, from which (5.19) follows. □

5.3.4 Minlos–Sazonov theorem

Theorem 5.63 (Minlos–Sazonov theorem) *Let $F : \mathcal{X}^{\#} \to \mathbb{C}$ be a weak characteristic functional. Then the following are equivalent:*

(1) *F is the characteristic functional of a probability measure μ on $(\mathcal{X}, \mathfrak{B})$.*
(2) *There exists a positive trace-class operator S on $\mathcal{X}$ such that $\mathcal{X} \ni \xi \mapsto F(\xi) \in \mathbb{C}$ is continuous if we equip $\mathcal{X}$ with the norm $\|\xi\|_S := (\xi|S\xi)^{\frac{1}{2}}$.*

Proof (1)⇒(2). Assume that F is the characteristic functional of a measure μ. Note that

$$|F(\xi_1) - F(\xi_2)|^2 = 2\mathrm{Re}\big(1 - F(\xi_1 - \xi_2)\big). \tag{5.20}$$

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Now, for $R > 0$,

$$\begin{aligned}\mathrm{Re}(1 - F(\xi)) &= \int_{\mathcal{X}} (1 - \cos(\xi \cdot x))\,\mathrm{d}\mu(x)\\ &\leq \frac{1}{2}\int_{\|x\|\leq R} (\xi \cdot x)^2 \mathrm{d}\mu(x) + 2\int_{\|x\|\geq R} \mathrm{d}\mu(x),\end{aligned}$$

where we used $1 - \cos\theta \leq \inf(\frac{\theta^2}{2}, 2)$. Since $\int_{\|x\|\leq R} \|x\|^2 \mathrm{d}\mu(x) < \infty$, we obtain from Prop. 5.53 that there exists a trace-class operator A_R such that

$$\int_{\|x\|\leq R} (\xi \cdot x)^2 \mathrm{d}\mu(x) = \xi \cdot A_R \xi.$$

This yields

$$\mathrm{Re}(1 - F(\xi)) \leq \xi \cdot A_R \xi + 2\mu\big(\{\|x\| \geq R\}\big).$$

Now let $\epsilon > 0$. Fixing $R_\epsilon > 0$ such that $2\mu\big(\{\|x\| \geq R_\epsilon\}\big) \leq \frac{1}{2}\epsilon$, and then taking $S_\epsilon = 2\epsilon^{-1} A_{R_\epsilon}$, we prove that for any $\epsilon > 0$ there exists a trace class S_ϵ such that $(\xi|S_\epsilon\xi) \leq 1$ implies

$$\mathrm{Re}\big(1 - F(\xi)\big) \leq \epsilon.$$

Now let $\epsilon_k \to 0$. Let S_k be positive trace-class operators such that $\mathrm{Re}(1 - F(\xi)) \leq \epsilon_k$ if $(\xi|S_k\xi) \leq 1$. We pick a sequence $(\lambda_k) > 0$ such that $\sum_k \lambda_k \mathrm{Tr}\, S_k < \infty$. Then $S = \sum_k \lambda_k S_k$ is trace-class. Moreover, if $(\xi|S\xi) \leq \lambda_k$, then $(\xi|S_k\xi) \leq 1$, and hence $\mathrm{Re}\big(1 - F(\xi)\big) \leq \epsilon_k$.

(1)$\Leftarrow$(2). Since F satisfies the conditions of Prop. 5.61, we can construct from F a weak distribution $\big\{\mu_{\mathcal{Y}} \ :\ \mathcal{Y} \in \mathrm{Fin}(\mathcal{X})\big\}$. To construct a measure from the weak distribution, we will use Thm. 5.60.

Let us fix $\delta > 0$. Let ϵ be such that $(\xi|S\xi) \leq \epsilon$ implies $\mathrm{Re}\big(1 - F(\xi)\big) \leq \delta$. Since $\mathrm{Re}\big(1 - F(\xi)\big) \leq 2$, we clearly have

$$\mathrm{Re}\big(1 - F(\xi)\big) \leq \delta + \frac{2}{\epsilon}(\xi|S\xi).$$

Let $\mathcal{Y} \in \mathrm{Fin}(\mathcal{X})$, $\alpha > 0$, $\dim \mathcal{Y} = d$. By (4.10), for $y \in \mathcal{Y}$ we have

$$\mathrm{e}^{-\frac{1}{2}\alpha\|y\|^2} = (2\pi\alpha)^{-\frac{1}{2}d} \int_{\mathcal{Y}} \mathrm{e}^{\mathrm{i}y\cdot\xi} \mathrm{e}^{-\frac{1}{2\alpha}\|\xi\|^2} \mathrm{d}\xi,$$

and hence

$$\begin{aligned}\int \Big(1 - \mathrm{e}^{-\frac{1}{2}\alpha\|y\|^2}\Big)\mathrm{d}\mu_{\mathcal{Y}}(y) &= (2\pi\alpha)^{-\frac{1}{2}d} \int_{\mathcal{Y}} \mathrm{e}^{-\frac{1}{2\alpha}\|\xi\|^2} \big(1 - F(\xi)\big)\mathrm{d}\xi\\ &= (2\pi\alpha)^{-\frac{1}{2}d} \int_{\mathcal{Y}} \mathrm{e}^{-\frac{1}{2\alpha}\|\xi\|^2} \mathrm{Re}\big(1 - F(\xi)\big)\mathrm{d}\xi\\ &\leq (2\pi\alpha)^{-\frac{1}{2}d} \int_{\mathcal{Y}} \mathrm{e}^{-\frac{1}{2\alpha}\|\xi\|^2} \Big(\delta + \frac{2}{\epsilon}\xi \cdot S\xi\Big)\mathrm{d}\xi\\ &= \delta + 2\frac{\alpha}{\epsilon}\mathrm{Tr}\, P_{\mathcal{Y}} S P_{\mathcal{Y}}\\ &\leq \delta + 2\frac{\alpha}{\epsilon}\mathrm{Tr}\, S,\end{aligned}$$

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using (4.15). Next we have

$$1-\mathrm{e}^{-\frac{1}{2}\alpha\|y\|^2} \geq (1-\mathrm{e}^{-\frac{1}{2}\alpha R^2})1\!\!1_{[R,\infty[}(\|y\|),$$

which yields

$$\begin{aligned}\int_{\mathcal{Y}} 1\!\!1_{[R,\infty[}(\|y\|)\mathrm{d}\mu_{\mathcal{Y}}(y) &\leq (1-\mathrm{e}^{-\frac{1}{2}\alpha R^2})^{-1}\int_{\mathcal{Y}}(1-\mathrm{e}^{-\frac{1}{2}\alpha\|y\|^2})\mathrm{d}\mu_{\mathcal{Y}}(y)\\ &\leq (1-\mathrm{e}^{-\frac{1}{2}\alpha R^2})^{-1}\Big(\delta+2\frac{\alpha}{\epsilon}\mathrm{Tr}\,S\Big).\end{aligned}$$

Fixing first $\delta>0$, then $\alpha>0$, and then letting $R\to\infty$, we see that condition (5.17) is satisfied. This completes the proof of the theorem. □

5.3.5 Measures on enlarged spaces

Using the Minlos–Sazonov theorem, it is possible to realize many weak characteristic functionals on $\mathcal{X}$ (and even on a dense subspace of $\mathcal{X}$) as characteristic functionals of measures on a *larger* Hilbert space.

In the theorem below the Hilbert space $B^{\frac{1}{2}}\mathcal{X}$ is defined as in Subsect. 2.3.4. We follow the usual convention for scales of real Hilbert spaces: $\mathcal{X}^{\#}$ is identified with $\mathcal{X}$, but $(B^{\frac{1}{2}}\mathcal{X})^{\#}$ is identified with $B^{-\frac{1}{2}}\mathcal{X}$ using the scalar product on $\mathcal{X}$.

Theorem 5.64 *Let $F:\mathcal{X}\to\mathbb{C}$ be a weak characteristic functional continuous for the norm of $\mathcal{X}$. Let $B>0$ be a self-adjoint operator on $\mathcal{X}$ such that B^{-1} is trace-class. Then there exists a Borel probability measure μ_B on the Hilbert space $B^{\frac{1}{2}}\mathcal{X}$ such that*

$$F(\xi)=\int_{B^{\frac{1}{2}}\mathcal{X}}\mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\mu_B(x),\quad \xi\in B^{-\frac{1}{2}}\mathcal{X}.$$

Proof Since B^{-1} is trace-class, B is bounded away from zero, and hence $B^{-\frac{1}{2}}\mathcal{X}=\mathrm{Dom}\,B^{\frac{1}{2}}\subset\mathcal{X}$. Let F_B be the restriction of the functional F to $B^{-\frac{1}{2}}\mathcal{X}$. Clearly, F_B is continuous if we equip $B^{-\frac{1}{2}}\mathcal{X}$ with the norm $(\xi|B^{-1}\xi)^{\frac{1}{2}}_{B^{-\frac{1}{2}}\mathcal{X}}=(\xi|\xi)^{\frac{1}{2}}_{\mathcal{X}}$. Hence F_B is a weak characteristic functional on $B^{-\frac{1}{2}}\mathcal{X}$.

B^{-1} can be restricted to $B^{-\frac{1}{2}}\mathcal{X}$. Interpreted in this way, it will be denoted $B^{-1}\big|_{B^{-\frac{1}{2}}\mathcal{X}}$. It is then unitarily equivalent to B^{-1} as an operator on $\mathcal{X}$. Indeed, $B^{-\frac{1}{2}}:\mathcal{X}\to B^{-\frac{1}{2}}\mathcal{X}$, $B^{\frac{1}{2}}:B^{-\frac{1}{2}}\mathcal{X}\to\mathcal{X}$ are unitary and

$$B^{-1}\big|_{B^{-\frac{1}{2}}\mathcal{X}}=B^{-\frac{1}{2}}B^{-1}B^{\frac{1}{2}}.$$

Hence, if B^{-1} is trace-class, then so is $B^{-1}\big|_{B^{-\frac{1}{2}}\mathcal{X}}$. Therefore, we can apply now Thm. 5.63, which implies that F_B is the characteristic functional of a Borel probability measure μ_B on the dual $(B^{-\frac{1}{2}}\mathcal{X})^{\#}$. By Prop. 2.60, $(B^{-\frac{1}{2}}\mathcal{X})^{\#}$ can be identified with $B^{\frac{1}{2}}\mathcal{X}$. This completes the proof of the theorem. □

Remark 5.65 *Sometimes the functional F is not continuous for the topology of $\mathcal{X}$, but for a certain norm $(\xi|A\xi)^{\frac{1}{2}}$, where $A>0$ is a self-adjoint operator on $\mathcal{X}$.*

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This case can be easily reduced to the case $A = \mathbb{1}$ *by replacing* $\mathcal{X}$ *by* $A^{-\frac{1}{2}}\mathcal{X}$. *The condition on* B *becomes that* $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ *is trace-class on* $\mathcal{X}$.

Remark 5.66 *Note that we still use the notation* x *for the generic variable in the enlarged space* $B^{\frac{1}{2}}\mathcal{X}$.

5.3.6 Comparison of enlarged spaces

Proposition 5.67 *Let* F *be as in Thm. 5.64 and let* $B_i > 0$, $i = 1, 2$, *be two self-adjoint operators on* $\mathcal{X}$. *Assume that* B_1^{-1} *is trace-class and* $B_1 \leq B_2$. *Then* B_2^{-1} *is trace-class. Let* μ_i *be the associated probability measures on* $B_i^{\frac{1}{2}}\mathcal{X}$. *Then* $B_1^{\frac{1}{2}}\mathcal{X}$ *is a Borel subset of* $B_2^{\frac{1}{2}}\mathcal{X}$ *and*

$$\mu_2(C) = \mu_1(C \cap B_1^{\frac{1}{2}}\mathcal{X}), \quad C \in \mathfrak{B}(B_2^{\frac{1}{2}}\mathcal{X}).$$

For the proof we will use the following lemma:

Lemma 5.68 *Let* $\mathcal{X}$ *be a real Hilbert space and* $A \in B(\mathcal{X})$. *Then* $\mathrm{Ran}\, A \in \mathfrak{B}(\mathcal{X})$.

Proof We use the polar decomposition $A = U|A|$ of A, where U is a partial isometry. It is clear that partial isometries map Borel sets onto Borel sets. Therefore, it suffices to show that $\mathrm{Ran}\,|A|$ is Borel. By the spectral theorem,

$$\begin{aligned}\mathrm{Ran}\,|A| &= \left\{x \in \mathcal{X},\ \sup\nolimits_{n\in\mathbb{N}} \left\| \left(|A| + n^{-1}\right)^{-1} x \right\|_{\mathcal{X}} < \infty \right\} \\ &= \bigcup_{m\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \left\{x \in \mathcal{X},\ \left\| \left(|A| + n^{-1}\right)^{-1} x \right\|_{\mathcal{X}} < m \right\}.\end{aligned}$$

This proves that $\mathrm{Ran}|A| \in \mathfrak{B}(\mathcal{X})$. □

Proof of Prop. 5.67. $B_1^{\frac{1}{2}}\mathcal{X}$ equals $AB_2^{\frac{1}{2}}\mathcal{X}$, where $A = B_1^{\frac{1}{2}}B_2^{-\frac{1}{2}} \in B(B_2^{\frac{1}{2}}\mathcal{X})$. Hence, by Lemma 5.68, $B_1^{\frac{1}{2}}\mathcal{X} \in \mathfrak{B}(B_2^{\frac{1}{2}}\mathcal{X})$.

Recall from Subsect. 2.3.4 that we have a natural embedding $I : B_1^{\frac{1}{2}}\mathcal{X} \to B_2^{\frac{1}{2}}\mathcal{X}$. Its adjoint is an embedding $I^{\#} : B_2^{-\frac{1}{2}}\mathcal{X} \to B_1^{-\frac{1}{2}}\mathcal{X}$. Both $B_2^{-\frac{1}{2}}\mathcal{X}$ and $B_1^{-\frac{1}{2}}\mathcal{X}$ are embedded in $\mathcal{X}$. Thus, for $\xi \in B_2^{-\frac{1}{2}}\mathcal{X}$ treated as an element of $\mathcal{X}$, we can write $I^{\#}\xi = \xi$.

Define a measure $\tilde{\mu}_2$ on $\mathfrak{B}(B_2^{\frac{1}{2}}\mathcal{X})$ by

$$\tilde{\mu}_2(C) = \mu_1(I^{-1}C) = \mu_1(C \cap B_1^{\frac{1}{2}}\mathcal{X}), \quad C \in \mathfrak{B}(B_2^{\frac{1}{2}}\mathcal{X}).$$

For $\xi \in B_2^{-\frac{1}{2}}\mathcal{X}$, we have

$$\begin{aligned}\int_{B_2^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{-\mathrm{i}\xi\cdot x_2}\,\mathrm{d}\tilde{\mu}_2(x_2) = \int_{B_1^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{-\mathrm{i}\xi\cdot Ix_1}\,\mathrm{d}\mu_1(x_1) \ &= \int_{B_1^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{-\mathrm{i}I^{\#}\xi\cdot x_1}\,\mathrm{d}\mu_1(x_1) \\ = F(I^{\#}\xi) = F(\xi) \ &= \int_{B_2^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{-\mathrm{i}\xi\cdot x_2}\,\mathrm{d}\mu_2(x_2).\end{aligned}$$

This implies that the characteristic functionals of $\tilde{\mu}_2$ and μ_2 are equal. Hence $\mu_2 = \tilde{\mu}_2$. This completes the proof of the proposition. □

5.4 Gaussian measures on real Hilbert spaces

Let $\mathcal{X}$ be a real Hilbert space. We would like to discuss *Gaussian measures* on real Hilbert spaces and the corresponding $\mathbf{L}^2$ *spaces*. This section has a natural continuation in Sect. 9.3, where we discuss the *real-wave representation of CCR*.

5.4.1 Gaussian measures

Proposition 5.69 *Let A be a positive self-adjoint operator on $\mathcal{X}$ and q be a bounded linear functional on $A^{-\frac{1}{2}}\mathcal{X}$.*

(1) *The function*

$$\operatorname{Dom} A \ni \xi \mapsto F(\xi) = \mathrm{e}^{\mathrm{i}q\cdot\xi - \frac{1}{2}\xi\cdot A\xi} \tag{5.21}$$

is a weak characteristic functional.

(2) *It is the characteristic functional of a probability measure μ on $\mathcal{X}$ iff A is trace-class.*

Proof (1) To prove the conditions of Prop. 5.61 we can assume that $\mathcal{X}$ is finite-dimensional. Setting $\mathcal{X}_1 = \operatorname{Ker} A$, we decompose $\mathcal{X}$ as $\mathcal{X}_1 \oplus \mathcal{X}_2$ and $q = (q_1, q_2)$. Let A_2 be A restricted to $\mathcal{X}_2$. Using (4.10) we see that F is the Fourier transform of the probability measure $\mathrm{d}\mu = \mathrm{d}\mu_1 \otimes \mathrm{d}\mu_2$ for

$$\begin{aligned}
\mathrm{d}\mu_1(x_1) &= \delta(x_1 - q_1)\mathrm{d}x_1,\\
\mathrm{d}\mu_2(x_2) &= (2\pi)^{-\frac{1}{2}\dim \mathcal{Y}_2} \det A_2^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}(x_2-q_2)\cdot A_2^{-1}(x_2-q_2)}\mathrm{d}x_2.
\end{aligned}$$

(2) Let us prove $\Leftarrow$. We have

$$\begin{aligned}
\operatorname{Re}(1 - F(\xi)) &= (1 - \mathrm{e}^{-\frac{1}{2}\xi\cdot A\xi}) + \mathrm{e}^{-\frac{1}{2}\xi\cdot A\xi}(1 - \cos(q\cdot\xi))\\
&\leq \tfrac{1}{2}\xi\cdot A\xi + c|q\cdot\xi|^2.
\end{aligned}$$

Since q is bounded on $A^{-\frac{1}{2}}\mathcal{X}$ we obtain that $|\operatorname{Re}(1 - F(\xi))| \leq C\xi\cdot A\xi$. By (5.20) this proves the continuity of F for the norm given by A, which is trace-class. So we can apply the Minlos–Sazonov theorem.

Let us now prove $\Rightarrow$. Let us assume that F is the characteristic functional of a measure μ. By translating the measure μ we can assume that $q = 0$. Splitting $\mathcal{X}$ as $\operatorname{Ker} A \oplus \operatorname{Ker} A^{\perp}$, we may assume that A is non-degenerate. If A is not compact, we can find a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\mathrm{w}-\lim\limits_{n\to\infty}\xi_n = 0$ and $\lim\limits_{n\to\infty}\xi_n\cdot A\xi_n = \lambda \neq 0$. This contradicts the weak continuity of F. Hence A is a compact operator.

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Now let $(e_j)_{j\in\mathbb{N}}$ be an o.n. basis of eigenvectors of A for the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$. Let $\mathcal{Y}_n = \mathrm{Span}\{e_1,\dots,e_n\}$, P_n be the orthogonal projection on $\mathcal{Y}_n$ and $A_n = P_n A P_n$. Let μ_n denote the measure $\mu_{\mathcal{Y}_n}$ on $\mathcal{Y}_n$, y_n the generic variable on $\mathcal{Y}_n$ and $\mathrm{d}y_n$ the Lebesgue measure on $\mathcal{Y}_n$. By (4.10), we know that

$$\mathrm{d}\mu_n(y_n) = (2\pi)^{-\frac{n}{2}} \det A_n^{-\frac{1}{2}} \mathrm{e}^{-\frac{1}{2}y_n\cdot A^{-1}y_n}\mathrm{d}y_n.$$

Hence, for $\epsilon > 0$,

$$\int_{\mathcal{X}} \mathrm{e}^{-\frac{\epsilon}{2}\|P_n x\|^2}\mathrm{d}\mu(x) = \int_{\mathcal{Y}_n} \mathrm{e}^{-\frac{\epsilon}{2}\|y_n\|^2}\mathrm{d}\mu_n(y_n) = \prod_{j=1}^{n}(1+\epsilon\lambda_j)^{-\frac{1}{2}}.$$

Now

$$1 = \lim_{\epsilon\searrow 0}\lim_{n\to\infty}\int_{\mathcal{X}} \mathrm{e}^{-\frac{\epsilon}{2}\|P_n x\|^2}\mathrm{d}\mu(x) = \lim_{\epsilon\searrow 0}\prod_{j=1}^{\infty}(1+\epsilon\lambda_j)^{-\frac{1}{2}}.$$

This implies that $\prod_{j=1}^{\infty}(1+\epsilon\lambda_j) < \infty$ for small enough $\epsilon > 0$, and hence the series $\sum_{j=1}^{\infty}\lambda_j$ is convergent and A is trace-class. □

Definition 5.70 *The measure defined in Prop. 5.69 will be called the* Gaussian measure on $\mathcal{X}$ of mean q and covariance A *and will be denoted by*

$$C\delta(x_1 - a_1)\mathrm{e}^{-\frac{1}{2}(x_2-q_2)\cdot A_2^{-1}(x_2-q_2)}\mathrm{d}x_1\mathrm{d}x_2, \tag{5.22}$$

or, if $\mathrm{Ker}\, A = 0$, *by*

$$C\mathrm{e}^{-\frac{1}{2}(x-q)\cdot A^{-1}(x-q)}\mathrm{d}x. \tag{5.23}$$

Note that C in (5.22) and (5.23) has the meaning of the "normalizing constant" that makes (5.22) a probability measure.

Remark 5.71 *Prop. 5.69 provides an example of a weak distribution on $\mathcal{X}$ which is not generated by a probability measure on $\mathcal{X}$.*

5.4.2 Gaussian measures on enlarged spaces

In this subsection we consider the case of a covariance for which (5.21) is only a weak characteristic functional.

Let A be a positive self-adjoint operator on $\mathcal{X}$. Consider the function

$$\mathcal{X} \ni \xi \mapsto \mathrm{e}^{-\frac{1}{2}\xi\cdot A\xi}. \tag{5.24}$$

It is a weak characteristic functional. It is not a characteristic functional of a measure unless A is trace-class.

Definition 5.72 *The generalized measure given by the weak characteristic functional (5.24) will be called the* generalized Gaussian measure on $\mathcal{X}$ with

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covariance A. *We will denote by*

$$\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot A^{-1}x}\mathrm{d}x)$$

the corresponding $\mathbf{L^2}$ *space. We will call it the* Gaussian $\mathbf{L^2}$ space over $\mathcal{X}$ with covariance A.

If B is a positive self-adjoint operator B on $\mathcal{X}$ such that $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ is trace-class, then $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{\frac{1}{2}xA^{-1}x}\mathrm{d}x)$ is naturally isomorphic to $L^2(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$, where

$$\int_{B^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\mu_B(x) = \mathrm{e}^{-\frac{1}{2}\xi\cdot A\xi}, \quad \xi \in B^{-\frac{1}{2}}\mathcal{X}.$$

Note that there is no canonical choice of the operator B.

Definition 5.73 *Following (5.23), the measure* μ_B *will often be denoted*

$$C\mathrm{e}^{-\frac{1}{2}x\cdot A^{-1}x}\mathrm{d}x.$$

(Note that this notation hides the dependence on B, which plays only an auxiliary technical role.)

Consider in particular the case of covariance $\mathbb{1}$. $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$ can be realized as an L^2 space over $\mathcal{X}$ iff $\mathcal{X}$ is finite-dimensional. $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$ is then equal to $L^2(\mathcal{X}, (2\pi)^{-\frac{1}{2}d}\mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$, where $d = \dim\mathcal{X}$ and $\mathrm{d}x$ is the Lebesgue measure on $\mathcal{X}$ compatible with the Euclidean structure.

Remark 5.74 *(5.24) is a weak characteristic functional even if the positive operator* A *has a non-zero kernel. If this is the case, then the corresponding Gaussian* $\mathbf{L}^2$ *space can be identified with* $\mathbf{L}^2(\mathcal{X}_1, \mathrm{e}^{-\frac{1}{2}x_1\cdot A_1^{-1}x_1}\mathrm{d}x_1)$, *where* $\mathcal{X}_1 := (\mathrm{Ker}\, A)^{\perp}$, A_1 *is the restriction of* A *to* $\mathcal{X}_1$ *and* x_1 *is the generic variable of* $\mathcal{X}_1$.

5.4.3 Exponential law for Gaussian spaces

In this subsection, for simplicity we restrict ourselves to covariance $\mathbb{1}$.

Proposition 5.75 *Let* $\mathcal{X}_1$, $\mathcal{X}_2$ *be two real Hilbert spaces. Set* $\mathcal{X} := \mathcal{X}_1 \oplus \mathcal{X}_2$. *Then the map*

$$\begin{aligned} U : \mathbb{C}\mathrm{Pol}(\mathcal{X}_1) \otimes \mathbb{C}\mathrm{Pol}(\mathcal{X}_2) &\to \mathbb{C}\mathrm{Pol}(\mathcal{X}) \\ P_1(x_1) \otimes P_2(x_2) &\mapsto P(x_1)P(x_2) \end{aligned}$$

extends to a unitary map

$$U : \mathbf{L}^2(\mathcal{X}_1, \mathrm{e}^{-\frac{1}{2}x_1^2}\mathrm{d}x_1) \otimes \mathbf{L}^2(\mathcal{X}_2, \mathrm{e}^{-\frac{1}{2}x_2^2}\mathrm{d}x_2) \to \mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x).$$

Proof Let us choose two operators B_1, B_2 such that B_i^{-1} is trace-class on $\mathcal{X}_i$, and use $L^2(B_i^{\frac{1}{2}}\mathcal{X}_i, \mathrm{d}\mu_{B_i})$ as representatives for $\mathbf{L}^2(\mathcal{X}_i, \mathrm{e}^{-\frac{1}{2}x_i^2}\mathrm{d}x_i)$. Then the map U extends to a unitary map from $L^2(B_1^{\frac{1}{2}}\mathcal{X}_1, \mathrm{d}\mu_{B_1}) \otimes L^2(B_2^{\frac{1}{2}}\mathcal{X}_2, \mathrm{d}\mu_{B_2})$ into

$L^2(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$ for $B = B_1 \oplus B_2$. We have

$$\int_{B^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{\mathrm{i}\xi\cdot x}\mathrm{d}\mu_B(x) = \mathrm{e}^{-\frac{1}{2}\xi_1^2 - \frac{1}{2}\xi_2^2} = \mathrm{e}^{-\frac{1}{2}\xi^2},$$

which shows that $L^2(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$ is a representative of $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$. $\square$

5.4.4 Polynomials in Gaussian spaces

Let A, B be positive operators with $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ trace-class. We identify $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x\cdot A^{-1}x}\mathrm{d}x)$ with $L^2(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$.

Proposition 5.76 *Polynomials based on $B^{-\frac{1}{2}}\mathcal{X}$ are dense in $\mathbf{L}^2(\mathcal{X}, \mathrm{e}^{-\frac{1}{2}x^2}\mathrm{d}x)$.*

Proof Clearly, for $\xi \in B^{-\frac{1}{2}}\mathcal{X}$, the function

$$\mathbb{C} \ni t \mapsto \hat{\mu}_B(t\xi) = \int_{B^{\frac{1}{2}}\mathcal{X}} \mathrm{e}^{-\mathrm{i}t\xi\cdot x}\mathrm{d}\mu_B(x) = \mathrm{e}^{-\frac{t^2}{2}\xi\cdot A\xi}$$

is entire. Hence the statement follows from Thm. 5.56. $\square$

Clearly, we have the inclusion $B^{-\frac{1}{2}}\mathcal{X} \subset A^{-\frac{1}{2}}\mathcal{X}$. If we regard $B^{\frac{1}{2}}\mathcal{X}$ as the underlying space, then only polynomials based on $B^{-\frac{1}{2}}\mathcal{X}$ are continuous functions. Those based on $A^{-\frac{1}{2}}\mathcal{X}$ do not have to be continuous. However, they are L^p integrable, as the following proposition shows.

Proposition 5.77 *Polynomials based on $A^{-\frac{1}{2}}\mathcal{X}$ belong to $\bigcap\limits_{1\leq p<\infty} L^p(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$ and, for $\xi \in A^{-\frac{1}{2}}\mathcal{X}$, we have*

$$\begin{aligned} \int_{B^{\frac{1}{2}}\mathcal{X}} (\xi\cdot x)^{2n+1}\mathrm{d}\mu_B(x) &= 0, \\ \int_{B^{\frac{1}{2}}\mathcal{X}} (\xi\cdot x)^{2n}\mathrm{d}\mu_B(x) &= \frac{2n!}{2^n n!}(\xi\cdot A\xi)^n. \end{aligned} \tag{5.25}$$

Proof Using Prop. 5.50, we obtain (5.25) for $\xi \in B^{-\frac{1}{2}}\mathcal{X}$.

Using (5.25), we see that if $(\xi_n)_{n\in\mathbb{N}}$ is a sequence in $B^{-\frac{1}{2}}\mathcal{X}$ converging to some $\xi \in \mathcal{X}$, then the sequence of functions $(\xi_n\cdot x)^m$ is Cauchy in $\bigcap\limits_{1\leq p<\infty} L^p(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$. Hence we can define the function

$$(\xi\cdot x)^m := \lim_{n\to\infty}(\xi_n\cdot x)^m,$$

which belongs to $\bigcap\limits_{1\leq p<\infty} L^p(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu_B)$. $\square$

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5.4.5 Relative continuity of Gaussian measures

Let A_i, $i = 1,2$, be two bounded positive operators on $\mathcal{X}$. For simplicity we assume that $A_i > 0$, i.e. $\operatorname{Ker} A_i = \{0\}$. Let B^{-1} be trace-class. Consider the Gaussian measures μ_i with the covariances A_i, $i = 1,2$, on the space $B^{\frac{1}{2}}\mathcal{X}$.

Theorem 5.78 (Feldmann–Hajek theorem) *The measures μ_1 and μ_2 are absolutely continuous w.r.t. one another iff $A_1^{-\frac{1}{2}} A_2 A_1^{-\frac{1}{2}} - \mathbb{1} \in B^2(\mathcal{X})$.*

Let us now discuss the Radon–Nikodym derivative $\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}(x)$ under the hypotheses of Thm. 5.78. For simplicity we assume that $A_1 = \mathbb{1}$ and denote A_2 by A, μ_1 by μ and μ_2 by $\tilde{\mu}$. It is easy to obtain the corresponding statements in the general case by replacing $\mathcal{X}$ by $A_1^{-\frac{1}{2}}\mathcal{X}$ (see Subsect. 11.4.6).

Proposition 5.79 *Assume that $\mathbb{1} - A \in B^2(\mathcal{X})$. Then the following hold:*

(1) *Let $\{\pi_n\}_{n\in\mathbb{N}}$ be an increasing sequence of finite rank orthogonal projections in $\mathcal{X}$ with $\mathrm{s}-\lim \pi_n = \mathbb{1}$. Set*

$$F_n(x) := (\det \pi_n A \pi_n)^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2} x\cdot\pi_n(\mathbb{1}-A^{-1})\pi_n x}, \; n \in \mathbb{N}.$$

Then $\{F_n\}_{n\in\mathbb{N}}$ converges in $L^1(B^{\frac{1}{2}}\mathcal{X}, \mathrm{d}\mu)$ to a positive function F with $\int F \mathrm{d}\mu = 1$.

(2) *If $\mathbb{1} - A \in B^1(\mathcal{X})$, then*

$$F(x) = (\det A)^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2} x\cdot(\mathbb{1}-A^{-1})x}.$$

(3) *One has $\frac{\mathrm{d}\tilde{\mu}}{\mathrm{d}\mu}(x) = F(x)$.*

Remark 5.80 *Statement (3) of Prop. 5.79 shows that F is independent on the choice of $\{\pi_n\}$. Note also that $x \mapsto x\cdot(\mathbb{1} - A^{-1})x$ is continuous on $B^{\frac{1}{2}}\mathcal{X}$, hence $x \mapsto \mathrm{e}^{\frac{1}{2}x\cdot(\mathbb{1}-A^{-1})x}$ is measurable on $B^{\frac{1}{2}}\mathcal{X}$, although not integrable if $\mathbb{1} - A \notin B^1(\mathcal{X})$. Therefore, a convenient notation for F is*

$$F(x) = C\mathrm{e}^{\frac{1}{2}x\cdot(\mathbb{1}-A^{-1})x},$$

where C is the "normalizing constant", as in Def. 5.70.

The proof of this theorem will be given later on; see Subsect. 11.4.6.

5.5 Gaussian measures on complex Hilbert spaces

Let $\mathcal{Z}$ be a separable (complex) Hilbert space. We denote by $\overline{z}_1 \cdot z_2$ the scalar product of $z_1, z_2 \in \mathcal{Z}$.

We will discuss *Gaussian L^2 spaces of anti-holomorphic functions on $\mathcal{Z}$*. This section has a natural continuation in Sect. 9.2, where we discuss the *complex-wave representation of CCR*.

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5.5.1 Holomorphic and anti-holomorphic functions

Recall from Subsect. 3.5.6 that inside the space of all complex polynomials $\mathbb{C}\mathrm{Pol}(\mathcal{Z}_\mathbb{R})$ we have the subspace $\mathrm{Pol}(\mathcal{Z})$, resp. $\mathrm{Pol}(\overline{\mathcal{Z}})$ of holomorphic, resp. anti-holomorphic polynomials spanned by $\prod\limits_{i=1}^{p} \overline{w_i} \cdot z$, resp. $\prod\limits_{i=1}^{p} w_i \cdot \overline{z}$, for $w_i \in \mathcal{Z}$.

The following definition generalizes the notion of a holomorphic function to an arbitrary dimension.

Definition 5.81 *A function $F : \mathcal{Z} \to \mathbb{C}$ is* holomorphic*, resp.* anti-holomorphic *if its restriction to any finite-dimensional complex subspace of $\mathcal{Z}$ is holomorphic, resp. anti-holomorphic.*

5.5.2 Measures on complex Hilbert spaces

Recall from Subsect. 3.6.9 that, in the context of the integration, a complex space $\mathcal{Z}$ is often identified with $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ by the map

$$\mathcal{Z} \ni z \mapsto (z, \overline{z}) \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}). \tag{5.26}$$

This suggests adoption of the following convention for characteristic functionals on complex spaces:

Definition 5.82 *If μ is a Borel probability measure on $\mathcal{Z}$, its characteristic functional is defined by*

$$\mathcal{Z} \ni w \mapsto \hat{\mu}(w) := \int_{\mathcal{Z}} \mathrm{e}^{-2\mathrm{i}\mathrm{Re}\overline{w}\cdot z} \mathrm{d}\mu(z) = \int_{\mathcal{Z}} \mathrm{e}^{-\mathrm{i}\overline{w}\cdot z - \mathrm{i}w\cdot\overline{z}} \mathrm{d}\mu(z).$$

5.5.3 Gaussian measures on complex spaces

Now let $A > 0$ be a trace-class self-adjoint operator on $\mathcal{Z}$. There exists a unique measure μ on $\mathcal{Z}$ such that

$$\hat{\mu}(w) = \mathrm{e}^{-\overline{w}\cdot Aw}, \quad w \in \mathcal{Z}. \tag{5.27}$$

This follows from Prop. 5.69, if we consider $\mathcal{Z}$ as the real Hilbert space $\mathcal{Z}_\mathbb{R}$ equipped with the scalar product $\mathrm{Re}\,\overline{z}_1 \cdot z_2$.

Definition 5.83 *The measure μ defined by (5.27) will be denoted $C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z$ and called the* Gaussian measure of covariance A*.*

Let $\mathcal{Z}$ be finite-dimensional of complex dimension d with a fixed (complex) volume form $\mathrm{d}z$. By Subsect. 4.1.9, we then have

$$C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z = \det A^{-1}(2\pi\mathrm{i})^{-d}\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z. \tag{5.28}$$

(The notation $\mathrm{i}^{-d}\mathrm{d}\overline{z}\mathrm{d}z$ is explained in Subsect. 3.6.9.)

Definition 5.84 *We denote by* $L^2_{\mathbb{C}}(\mathcal{Z}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$*, resp.* $L^2_{\mathbb{C}}(\overline{\mathcal{Z}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$ *the closure in* $L^2(\mathcal{Z}_{\mathbb{R}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$ *of* $\mathrm{Pol}(\mathcal{Z})$*, resp.* $\mathrm{Pol}(\overline{\mathcal{Z}})$*.*

Theorem 5.85 *The space* $L^2_{\mathbb{C}}(\mathcal{Z}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$*, resp.* $L^2_{\mathbb{C}}(\overline{\mathcal{Z}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$ *coincides with the space of holomorphic, resp. anti-holomorphic functions in* $L^2(\mathcal{Z}_{\mathbb{R}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$*.*

Proof It suffices to consider the holomorphic case.

Let $\mathcal{Y} \subset \mathcal{Z}$ be a finite-dimensional complex subspace. If G is a function on $\mathcal{Z}$, let $G_{|\mathcal{Y}}$ be its restriction to $\mathcal{Y}$. Let $F \in L^2_{\mathbb{C}}(\mathcal{Z}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$, and (P_n) a sequence in $\mathrm{Pol}(\mathcal{Z})$ converging to F in $L^2(\mathcal{Z}_{\mathbb{R}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$. If $\mathcal{Y}$ is finite-dimensional then $(P_n)_{|\mathcal{Y}}$ converges to $F_{|\mathcal{Y}}$ in $L^2(\mathcal{Y}_{\mathbb{R}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$, hence in $\mathcal{D}'(\mathcal{Y}_{\mathbb{R}})$. By Prop. 4.12 it follows that $F_{|\mathcal{Y}}$ is holomorphic.

Conversely, let $F \in L^2(\mathcal{Z}_{\mathbb{R}}, C\mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$ be a holomorphic function, and assume that F is orthogonal to all holomorphic polynomials. Let $(e_j)_{j\in\mathbb{N}}$ be an o.n. basis of eigenvectors of A for the eigenvalues $(\lambda_j)_{j\in\mathbb{N}}$. We fix d and restrict F to $\mathrm{Span}\{e_1, \dots, e_d\}$. If we identify $\mathbb{C}^d$ with $\mathrm{Span}\{e_1, \dots, e_d\}$ by the map

$$(z_1, \dots, z_d) \mapsto \sum_{i=1}^{d} \frac{z_j}{\sqrt{\lambda_j}} e_j,$$

we are reduced to considering a holomorphic function G on $\mathbb{C}^d$, which is orthogonal to all holomorphic polynomials for the measure $(2\pi\mathrm{i})^{-d}\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z$.

For $\vec{n} = (n_1, \dots, n_d) \in \mathbb{N}^d$ we recall that $\vec{n}! := n_1! \dots n_d!$, $\partial_z^n = \partial_{z_1}^{n_1} \dots \partial_{z_d}^{n_d}$. From Cauchy's formula, we get

$$\partial_z^n G(0) = \frac{\vec{n}!}{(2\pi)^d} \int_{[0,2\pi]^d} G(r_1\mathrm{e}^{\mathrm{i}\theta_1}, \dots, r_d\mathrm{e}^{\mathrm{i}\theta_d}) \prod_{j=1}^{d} \mathrm{e}^{\mathrm{i}n_j\theta_j} r_j^{-n_j} \mathrm{d}\theta_1 \dots \mathrm{d}\theta_d.$$

If $C(n) = \prod\limits_{j=1}^{d} \int_0^{+\infty} r^{2n_j+1}\mathrm{e}^{-r^2}\mathrm{d}r$, we obtain

$$C(n)\partial_z^n G(0) = \vec{n}!2^{-d} \int G(z_1, \dots, z_d) \prod_{j=1}^{d} \overline{z}_j^{n_j} \mathrm{e}^{-\overline{z}\cdot z}(2\mathrm{i}\pi)^{-d}\mathrm{d}\overline{z}\mathrm{d}z.$$

Hence, if $G \in L^2(\mathbb{C}^d, (2\mathrm{i}\pi)^{-d}\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ is holomorphic and orthogonal to the holomorphic polynomials, we have $\partial_z^{\vec{n}} G(0) = 0$ for all $\vec{n}$ and hence $G(z) \equiv 0$.

This implies that the restriction of F to $\mathrm{Span}\{e_1, \dots, e_d\}$ is equal to 0 for all d. In particular, F is orthogonal to all real polynomials generated by $\mathrm{Re}(\overline{e}_j \cdot z)$ and $\mathrm{Im}(\overline{e}_j \cdot z)$. Since these polynomials are dense in $L^2(\mathcal{Z}_{\mathbb{R}}, \mathrm{e}^{-\overline{z}\cdot A^{-1}z}\mathrm{d}\overline{z}\mathrm{d}z)$, we have $F \equiv 0$. □

5.5.4 Generalized Gaussian measures on complex spaces

We now extend Def. 5.84 to generalized Gaussian measures that cannot be realized as measures on $\mathcal{Z}$. For simplicity, we assume that the covariance of the measure is given by the scalar product of the underlying (complex) Hilbert space.

Definition 5.86 *Denote by* $\mathbf{L}^2_{\mathbb{C}}(\mathcal{Z}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$*, resp.* $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *the closure in* $\mathbf{L}^2(\mathcal{Z}_{\mathbb{R}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *of the space of holomorphic, resp. anti-holomorphic polynomials on* $\mathcal{Z}$*. The space* $\mathbf{L}^2_{\mathbb{C}}(\mathcal{Z}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$*, resp.* $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *will be called the* holomorphic*, resp.* anti-holomorphic Gaussian $\mathbf{L}^2$ space with covariance $\mathbb{1}$.

Proposition 5.87 *Let* $B \geq 0$ *be an operator such that* B^{-1} *is trace-class. Identify* $\mathbf{L}^2(\mathcal{Z}_{\mathbb{R}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *with* $L^2(B^{\frac{1}{2}}\mathcal{Z}_{\mathbb{R}}, C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *in the usual way. Then* $\mathbf{L}^2_{\mathbb{C}}(\mathcal{Z}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$*, resp.* $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$ *coincide with* $L^2_{\mathbb{C}}(B^{\frac{1}{2}}\mathcal{Z}, C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$*, resp.* $L^2_{\mathbb{C}}(\overline{B}^{\frac{1}{2}}\overline{\mathcal{Z}}, C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$*.*

5.5.5 Isomorphism with modified Fock spaces

Recall the modified Fock space $\Gamma_{\mathrm{s}}^{\mathrm{mod}}(\mathcal{Z})$, defined as the completion of $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{Z})$ with the scalar product given by $(\Phi|\Psi)_{\Gamma_{\mathrm{s}}^{\mathrm{mod}}(\mathcal{Z})} := \big(\Phi|\frac{1}{N!}\Psi\big)_{\Gamma_{\mathrm{s}}(\mathcal{Z})}$. Moreover, we recall from Subsect. 3.5.1 that $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{Z})$ can be identified with $\mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}})$, which is dense in $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. It turns out that this identification extends to a unitary map:

Theorem 5.88 *The map*

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{Z}) \ni \Phi \mapsto \Phi(\cdot) \in \mathrm{Pol}_{\mathrm{s}}(\overline{\mathcal{Z}})$$

given by

$$\Phi(\overline{z}) := \sum_{n=0}^{\infty} (z^{\otimes n}|\Phi)$$

extends by continuity to a unitary map

$$\Gamma_{\mathrm{s}}^{\mathrm{mod}}(\mathcal{Z}) \ni \Phi \mapsto \Phi(\cdot) \in \mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z). \tag{5.29}$$

The proof of the above theorem for $\dim \mathcal{Z} = 1$ follows immediately from the following simple computation:

Lemma 5.89 *Let* $z \in \mathbb{C}$*. Then*

$$(2\pi\mathrm{i})^{-1} \int_{\mathbb{C}} \mathrm{e}^{-\overline{z}\cdot z} z^m \overline{z}^n \mathrm{d}z\mathrm{d}\overline{z} = n!\delta_{n,m}. \tag{5.30}$$

Proof We identify $\mathbb{C}$ with $\mathbb{R}^2$. In the polar coordinates $z = r\mathrm{e}^{\mathrm{i}\phi}$, the l.h.s. of (5.30) equals

$$\pi \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} \mathrm{d}r \mathrm{e}^{\mathrm{i}\phi(m-n)} r^{m+n+1} \mathrm{e}^{-r^2}. \tag{5.31}$$

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For $n \neq m$ the integral w.r.t. ϕ yields zero. For $n = m$ we get

$$\frac{1}{2}\int_0^\infty r^{2m+1}\mathrm{e}^{-r^2}\mathrm{d}r = \int_0^\infty r^{2m}\mathrm{e}^{-r^2}\mathrm{d}r^2 = m!.$$

Alternatively, we can rewrite (5.30) as

$$\begin{aligned}\mathrm{i}^{n+m}\partial_t^n\partial_{\overline{t}}^m(2\pi\mathrm{i})^{-1}\int \mathrm{e}^{-\overline{z}\cdot z}\mathrm{e}^{-\mathrm{i}z\overline{t}-\mathrm{i}\overline{z}t}\mathrm{d}z\mathrm{d}\overline{z}\big|_{t=0} &= \mathrm{i}^{n+m}\partial_t^n\partial_{\overline{t}}^m\mathrm{e}^{-|t|^2}\big|_{t=0}\\ &= n!\delta_{nm}.\end{aligned}$$

□

Proof of Thm. 5.88. For notational simplicity assume that $\dim\mathcal{Z} < \infty$. Let $(e_1, \dots, e_n)$ be an o.n. basis of $\mathcal{Z}$. Recall that $\{e_{\vec{k}} : \vec{k} \in \mathbb{N}^n\}$ is an o.n. basis of $\Gamma_{\mathrm{s}}^{\mathrm{mod}}(\mathcal{Z})$, where

$$e_{\vec{k}} := \frac{1}{\sqrt{\vec{k}!}}e_1^{\otimes k_1}\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}}e_n^{\otimes k_n},$$

$e_{\vec{0}} = \Omega$ and $\vec{k}! = k_1!\cdots k_n!$. The vector $e_{\vec{k}}$ is mapped onto the polynomial

$$e_{\vec{k}}(\overline{z}) = \frac{1}{\sqrt{\vec{k}!}}\prod_{i=1}^n(e_i\cdot\overline{z})^{k_i}.$$

Using Lemma 5.89 we see that $\{e_{\vec{k}}(\cdot) : \vec{k} \in \mathbb{N}^n\}$ form an o.n. basis of $L^2_{\mathbb{C}}(\overline{\mathcal{Z}}, C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. □

The following proposition is an illustration of the formalism of Gaussian complex spaces.

Proposition 5.90 *Let $F \in \mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. Then*

$$F(\overline{z}_0) = \int F(\overline{z})\mathrm{e}^{\overline{z}\cdot z_0}C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z, \quad z_0 \in \mathcal{Z}.$$

Proof The integral on the r.h.s. is well defined, since $\overline{z} \mapsto \mathrm{e}^{\overline{z}\cdot z_0}$ belongs to $\mathbf{L}^2_{\mathbb{C}}(\overline{\mathcal{Z}}, \mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z)$. By density and linearity it suffices to check the identity for monomials. We have

$$\begin{aligned}\int_{\mathcal{Z}}\prod_{i=1}^p(e_i\cdot\overline{z})^{n_i}\mathrm{e}^{z\cdot\overline{z}_0}C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z &= \int_{\mathcal{Z}}\prod_{i=1}^p\partial_{t_i}^{n_i}\exp\Big(z\cdot\overline{z}_0 + \sum_{i=1}^p t_i e_i\cdot\overline{z}\Big)\\ &\qquad\times C\mathrm{e}^{-\overline{z}\cdot z}\mathrm{d}\overline{z}\mathrm{d}z\big|_{t=0}\\ &= \prod_{i=1}^p\partial_{t_i}^{n_i}\exp\Big(\sum_{i=1}^p t_i e_i\cdot\overline{z}_0\Big)\big|_{t=0}\\ &= \prod_{i=1}^p(e_i\cdot\overline{z}_0)^{n_i}.\end{aligned}$$

This completes the proof of the proposition. □

5.6 Notes

General measure theory is studied e.g. in the monographs by Halmos (1950) and Bauer (1968).

Properties of positivity preserving maps are discussed e.g. in Reed–Simon (1978b).

The notion of equi-integrability and the Lebesgue–Vitali theorem can be found in Kallenberg (1997). Measures on Hilbert spaces is the subject of a monograph by Skorokhod (1974). The proof of Prop. 5.41 can be found e.g. in Chap. I.1 of Skorokhod (1974).

The Feldman–Hajek theorem about relative continuity of Gaussian measures was proved independently by Feldman (1958) and Hajek (1958).