# ON THE PROPERTIES OF AN ENTIRE FUNCTION OF TWO COMPLEX VARIABLES

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**1.** Let

$$f(z_1, z_2) = \sum_{m_1, m_2=0}^{\infty} a_{m_1 m_2} z_1^{m_1} z_2^{m_2} \tag{1.1}$$

be an entire function of two complex variables $z_1$ and $z_2$, holomorphic in the closed polydisk $P\ \{|z_j| \leqslant r_j,\ j = 1, 2\}$. Let

$$M(r_1, r_2) = M(r_1, r_2; f) = \max_{|z_j| \leqslant r_j} |f(z_1, z_2)|, \qquad j = 1, 2.$$

Following S. K. Bose (**1**, pp. 214–215), $\mu(r_1, r_2; f)$ denotes the maximum term in the double series (1.1) for given values of $r_1$ and $r_2$ and $\nu_1(m_2; r_1, r_2)$ or $\nu_1(r_1, r_2)$, $r_2$ fixed, $\nu_2(m_1; r_1, r_2)$ or $\nu_2(r_1, r_2)$, $r_1$ fixed and $\nu(r_1, r_2)$ denote the ranks of the maximum term of the double series (1.1). Let us write

$$I_\delta(r_1, r_2; f) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^\delta d\theta_1\, d\theta_2 \tag{1.2}$$

and

$$\mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f) = \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \int_0^{r_1} \int_0^{r_2} M_\delta(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1\, dx_2 \tag{1.3}$$

where $\delta \geqslant 1$ and $-1 < k_1, k_2 < \infty$, and

$$M_\delta(r_1, r_2; f) = \{I_\delta(r_1, r_2; f)\}^{1/\delta}.$$

Then $I_\delta(r_1, r_2; f)$ is an increasing function of $r_1$ and $r_2$ when one remains fixed and the other increases or both increase. The finite order $\rho$ of an entire function $f(z_1, z_2)$ is defined as (**1**, p. 219)

$$\limsup_{r_1, r_2 \to \infty} \left\{ \frac{\log \log M(r_1, r_2; f)}{\log (r_1 r_2)} \right\} = \rho. \tag{1.4}$$

Similarly, we can define the lower order $\lambda$ of $f(z_1, z_2)$ as

$$\liminf_{r_1, r_2 \to \infty} \left\{ \frac{\log \log M(r_1, r_2; f)}{\log (r_1 r_2)} \right\} = \lambda. \tag{1.5}$$

In this paper we have deduced an asymptotic relation between the two mean values defined by (1.2) and (1.3) and a number of results connecting $M(r_1, r_2; f)$ and $\mu(r_1, r_2; f)$ and $\nu(r_1, r_2; f)$ and the coefficients $a_{m_1 m_2}$.

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**2.** THEOREM 1. *Let $f(z_1, z_2)$ be an entire function of finite non-zero order $\rho$ and lower order $\lambda$ $(2\lambda < \rho)$*; then

$$\log \mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f) \sim \log M_\delta(r_1, r_2; f), \tag{2.1}$$

*where $\delta \geqslant 1$ and $-1 < k_1, k_2 < \infty$, and $f(0, 0) \neq 0$.*

*Proof.* We have

$$\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f) = \frac{1}{r_1^{k_1+1} r_2^{k_2+1}} \int_0^{r_1} \int_0^{r_2} M_\delta(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1\, dx_2 \leqslant \frac{M_\delta(r_1, r_2; f)}{(k_1+1)(k_2+1)}. \tag{2.2}$$

Further, for $0 < r_i < R_i < 2^{1/(\rho+\epsilon)} r_i$ $(i = 1, 2)$,

$$\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f) > \frac{1}{R_1^{k_1+1} R_2^{k_2+1}} \int_{r_1}^{R_1} \int_{r_2}^{R_2} M_\delta(x_1, x_2; f) x_1^{k_1} x_2^{k_2} dx_1\, dx_2 \geqslant \frac{M_\delta(r_1, r_2; f)}{(k_1+1)(k_2+1)} \frac{(R_1^{k_1+1} - r_1^{k_1+1})}{R_1^{k_1+1}} \frac{(R_2^{k_2+1} - r_2^{k_2+1})}{R_2^{k_2+1}}. \tag{2.3}$$

From (2.2) and (2.3), we get that

$$\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f) \leqslant \frac{M_\delta(r_1, r_2; f)}{(k_1+1)(k_2+1)} \leqslant \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f) \frac{R_1^{k_1+1}}{(R_1^{k_1+1} - r_1^{k_1+1})} \frac{R_2^{k_2+1}}{(R_2^{k_2+1} - r_2^{k_2+1})}. \tag{2.4}$$

Since

$$\frac{\partial^2}{\partial R_1\, \partial R_2} \{\log(R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f))\} = \frac{\partial}{\partial R_1} \left\{ \frac{R_2^{k_2} \int_0^{R_1} x_1^{k_1} M_\delta(x_1, R_2; f) dx_1}{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)} \right\}$$

$$= \frac{R_1^{k_1} R_2^{k_2} M_\delta(R_1, R_2; f) \{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\}}{\{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\}^2} - \frac{\left\{R_1^{k_1} \int_0^{R_2} x_2^{k_2} M_\delta(R_1, x_2; f) dx_2\right\}\left\{R_2^{k_2} \int_0^{R_1} x_1^{k_1} M_\delta(x_1, R_2; f) dx_1\right\}}{\{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\}^2}$$

$$\leqslant \frac{M_\delta(R_1, R_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)} \frac{1}{R_1 R_2},$$

we have

$$\log\{R_1^{k_1+1} R_2^{k_2+1} \mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)\} \leqslant \int_0^{R_1} \int_0^{R_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2}$$

$$= \int_0^{r_1} \int_0^{r_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} + \left[ \int_0^{r_1} \int_{r_2}^{R_2} + \int_{r_1}^{R_1} \int_0^{r_2} + \int_{r_1}^{R_1} \int_{r_2}^{R_2} \right] \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1} \frac{dx_2}{x_2},$$

or

$$\text{(2.5)}\quad \log\left\{\frac{R_1^{k_1+1}R_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)}{r_1^{k_1+1}r_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)}\right\}$$
$$\leqslant \left[\int_0^{r_1}\int_{r_2}^{R_2} + \int_{r_1}^{R_1}\int_0^{r_2} + \int_{r_1}^{R_1}\int_{r_2}^{R_2}\right] \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2}$$
$$= J_1 + J_2 + J_3, \quad \text{say.}$$

For any positive number $\epsilon$, from **(2)**, we have

$$\text{(2.6)}\qquad \frac{M_\delta(r_1, r_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)} < (r_1 r_2)^{\rho+\epsilon},$$

for sufficiently large $r_1$ and $r_2$. Also

$$\text{(2.7)}\qquad \log\left(\frac{R_i}{r_i}\right) = \log\left(1 + \frac{R_i - r_i}{r_i}\right) < \frac{R_i - r_i}{r_i} \qquad \text{for } i = 1, 2.$$

Let us choose $R_i$ such that

$$\text{(2.8)}\qquad (R_i - r_i)/r_i = r_i^{-\rho-\epsilon} \qquad \text{for } i = 1, 2.$$

Then

$$J_1 = \int_0^{r_1}\int_{r_2}^{R_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2} < \log\left(\frac{R_2}{r_2}\right)\int_0^{r_1} (x_1 R_2)^{\rho+\epsilon}\frac{dx_1}{x_1} \qquad \text{from (2.6)}$$
$$< \left(\frac{R_2}{r_2}\right)^{\rho+\epsilon} \frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)}$$
$$< 2\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)},$$

since $r_i < R_i < 2^{1/(\rho+\epsilon)} r_i$ for $i = 1, 2$. Similarly,

$$J_2 < 2r_2^{\rho+\epsilon}/(\rho+\epsilon),$$

and

$$J_3 = \int_{r_1}^{R_1}\int_{r_2}^{R_2} \frac{M_\delta(x_1, x_2; f)}{\mathfrak{M}_{\delta,k_1,k_2}(x_1, x_2; f)} \frac{dx_1}{x_1}\frac{dx_2}{x_2}$$
$$< (R_1 R_2)^{\rho+\epsilon}\left(\frac{R_1 - r_1}{r_1}\right)\left(\frac{R_2 - r_2}{r_2}\right) < 4,$$

from (2.8) and for $r_i < R_i < 2^{1/(\rho+\epsilon)} r_i$ $(i = 1, 2)$. Hence (2.5) becomes

$$\text{(2.9)}\quad \log\left\{\frac{R_1^{k_1+1}R_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(R_1, R_2; f)}{r_1^{k_1+1}r_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)}\right\} < 2\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)} + 2\frac{r_2^{\rho+\epsilon}}{(\rho+\epsilon)} + 4.$$

Using (2.9) in (2.4), we get

$$\log M_\delta(r_1, r_2; f) - \log(k_1 + 1) - \log(k_2 + 1)$$
$$< \log\left\{\frac{1}{(R_1^{k_1+1} - r_1^{k_1+1})(R_2^{k_2+1} - r_2^{k_2+1})}\right\}$$
$$+ \log\{r_1^{k_1+1}r_2^{k_2+1}\mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f)\} + 2\frac{r_1^{\rho+\epsilon}}{(\rho+\epsilon)} + 2\frac{r_2^{\rho+\epsilon}}{(\rho+\epsilon)} + 4$$
$$= (1 + o(1))\log \mathfrak{M}_{\delta,k_1,k_2}(r_1, r_2; f),$$

for large $r_1$, $r_2$ and on using **(2)**

$$\limsup_{r_1, r_2 \to \infty} \left\{ \frac{\log \log \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f)}{\log(r_1 r_2)} \right\} = \rho.$$

Taking the limit, we get

$$\lim_{r_1, r_2 \to \infty} \log M_\delta(r_1, r_2; f) / \log \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f) \leqslant 1,$$

and from (2.4)

$$\lim_{r_1, r_2 \to \infty} \log M_\delta(r_1, r_2; f) / \log \mathfrak{M}_{\delta, k_1, k_2}(r_1, r_2; f) \geqslant 1,$$

which gives the result.

**3.** THEOREM 2. *Let $f(z_1, z_2)$ be an entire function of order zero and suppose that*

$$\lim_{k \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left[ \frac{\log k}{\log\left\{ \frac{1}{k} \log \frac{1}{|a_{k_1 k_2}|} \right\}} \right] = \begin{matrix} A_2 \\ a_2 \end{matrix}, \tag{3.1}$$

$$\lim_{r_1, r_2 \to \infty} \begin{matrix} \sup \\ \inf \end{matrix} \left\{ \frac{\log \log M(r_1, r_2; f)}{\log \log (r_1 r_2)} \right\} = \begin{matrix} P_2 \\ p_2 \end{matrix}, \tag{3.2}$$

*then if* $1 < A_2 < l < \infty$,

$$P_2 \leqslant \{(l+1)/l\} A_2 \tag{3.3}$$

*and a fortiori*

$$p_2 < (l+1) \tag{3.4}$$

*where* $k = k_1 + k_2$.

*Proof.* We know (**1**, p. 218) that

$$M(r_1, r_2) < \mu(r_1, r_2)[3\nu(2r_1, 2r_2) + 3].$$

Taking the logarithms of both the sides, we get

$$\begin{aligned} \log M(r_1, r_2) &< \log \mu(r_1, r_2) + \log \nu(2r_1, 2r_2) + O(1) \\ &= \log \mu(r_1, r_2) + \log \log \mu(2r_1, 2r_2) + o(\log(r_1 r_2)), \end{aligned}$$

from (**1**, (5.1), p. 218 and (5.2), p. 219), for large values of $r_1$ and $r_2$. This gives

$$\log M(r_1, r_2) < \log \mu(r_1, r_2) + \log \log \mu(2r_1, 2r_2) + o(\log \mu(r_1, r_2));$$

hence

$$\log M(r_1, r_2) < (1 + o(1)) \log \mu(r_1, r_2). \tag{3.5}$$

For any positive $\epsilon$, there must exist a sufficiently large positive number $k_0(A_2)$ such that

$$|a_{k_1 k_2}| < \exp(-k^{1+1/(A_2+\epsilon)}), \tag{3.6}$$

for every $k > k_0(A_2)$. Let us suppose that

$$r_{1,k_1} = \exp\{k_1^{1/A_2} + k_1^{1/l}\} \tag{3.7}$$

and

$$r_{2,k_2} = \exp\{k_2^{1/A_2} + k_2^{1/l}\}, \tag{3.7a}$$

where $A_2 < l < \infty$ and $r_{1,k_1} \leqslant r_1 < r_{1,k_1+1}$, $r_{2,k_2} \leqslant r_2 < r_{2,k_2+1}$. Now, for $r_1 < r_{1,k_1+1}$ and $r_2 < r_{2,k_2+1}$,

$$\begin{aligned} \log \mu(r_1, r_2) &= \log\{|a_{k_1k_2}|r_1^{k_1}r_2^{k_2}\} \\ &< \log\{|a_{k_1k_2}|r_{1,k_1+1}^{k_1}r_{2,k_2+1}^{k_2}\}, \end{aligned} \tag{3.7b}$$

where the maximum term $\mu(r_1, r_2)$ has the position $(k_1, k_2)$ in the square representation of the double series (1.1) of $f(z_1, z_2)$. We obtain

$$\begin{aligned} \log \mu(r_1, r_2) < -(k_1 + k_2)^{1+1/(A_2+\epsilon)} &+ k_1\{(k_1 + 1)^{1/A_2} + (k_1 + 1)^{1/l}\} \\ &+ k_2\{(k_2 + 1)^{1/A_2} + (k_2 + 1)^{1/l}\}, \end{aligned}$$

from (3.6), (3.7), and (3.7a). Hence,

$$\log \mu(r_1, r_2) < (1 + o(1))(k_1 + k_2)^{1+1/l},$$

for $k_1$, $k_2$ sufficiently large and $A_2 < l$. Hence for sufficiently large $k_1$ and $k_2$,

$$\log\log \mu(r_1, r_2) < (1 + o(1))(1 + 1/l)\log k. \tag{3.8}$$

Also for $r_{1,k_1} \leqslant r_1$ and $r_{2,k_2} \leqslant r_2$,

$$\begin{aligned} \log\log(r_1 r_2) &\geqslant \log\log(r_{1,k_1} r_{2,k_2}) \\ &= \log\{k_1^{1/A_2} + k_1^{1/l} + k_2^{1/A_2} + k_2^{1/l}\} \quad \text{from (3.7) and (3.7a)} \\ &\sim \log\{(k_1 + k_2)^{1/A_2}\} \quad \text{for } 1 < A_2 < l \\ &\sim \frac{1}{A_2}\log k. \end{aligned} \tag{3.9}$$

Hence from (3.5), (3.8), and (3.9), we have

$$\limsup_{r_1, r_2 \to \infty}\left\{\frac{\log\log M(r_1, r_2; f)}{\log\log(r_1 r_2)}\right\} \leqslant \left(\frac{l+1}{l}\right)A_2,$$

i.e.

$$P_2 \leqslant \left(\frac{l+1}{l}\right)A_2,$$

and *a fortiori*

$$p_2 < (l + 1).$$

THEOREM 3. *Let $f(z_1, z_2)$ be an entire function of order zero and suppose that*

$$\lim_{r_1, r_2 \to \infty} \begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log \nu(r_1, r_2; f)}{\log\log(r_1 r_2)}\right\} = \begin{matrix} r \\ \eta \end{matrix}, \tag{3.10}$$

*if* $1 < \alpha_2 < \beta < \infty$ *($\alpha_2$ as defined in Theorem 2); then*

(3.11) $$\gamma \leqslant \{(\beta + 1)/\beta\}\alpha_2$$

*and*

(3.12) $$\eta < (\beta + 1).$$

*Proof.* For any positive $\epsilon$, there must exist a sufficiently large $k_0(\alpha_2)$ such that

(3.13) $$|a_{k_1k_2}| < \exp\{-k^{1+1/(\alpha_2-\epsilon)}\} \quad \text{for every } k > k_0(\alpha_2).$$

Let us suppose that

(3.14) $$r_{1,k_1} = \exp\{k_1^{1/\alpha_2} + k_1^{1/\beta}\}$$

and

(3.14a) $$r_{2,k_2} = \exp\{k_2^{1/\alpha_2} + k_2^{1/\beta}\},$$

where $\alpha_2 < \beta < \infty$ and $r_{1,k_1} \leqslant r_1 < r_{1,k_1+1}$ and $r_{2,k_2} \leqslant r_2 < r_{2,k_2+1}$. From **(1**, (5.4), p. 220**)**, we have

(3.14b) $$\log \nu(r_1, r_2) \leqslant (1 + o(1)) \log\log M(r_1, r_2),$$

for $r_1$ and $r_2$ sufficiently large. Using (3.5) in (3.14b), we get

(3.15) $$\log \nu(r_1, r_2) < (1 + o(1)) \log\log \mu(r_1, r_2).$$

Now using (3.13), (3.14) and (3.14a) in (3.7b), we obtain

$$\begin{aligned}\log \mu(r_1, r_2) &< -(k_1 + k_2)^{1+1/(\alpha_2-\epsilon)} + k_1\{(k_1 + 1)^{1/\alpha_2} + (k_1 + 1)^{1/\beta}\} \\ &\qquad + k_2\{(k_2 + 1)^{1/\alpha_2} + (k_2 + 1)^{1/\beta}\} \\ &< (1 + o(1))(k_1 + k_2)^{1+1/\beta},\end{aligned}$$

for $1 < \alpha_2 < \beta$ and $k_1$, $k_2$ sufficiently large. Hence

(3.16) $$\log\log \mu(r_1, r_2) < (1 + o(1))\left(\frac{\beta + 1}{\beta}\right) \log k.$$

Also, for $r_{1,k_1} \leqslant r_1$ and $r_{2,k_2} \leqslant r_2$

(3.17) $$\begin{aligned}\log\log(r_1 r_2) &\geqslant \log\log(r_{1,k_1} r_{2,k_2}) \\ &= \log\{k_1^{1/\alpha_2} + k_1^{1/\beta} + k_2^{1/\alpha_2} + k_2^{1/\beta}\} && \text{from (3.14) and (3.14a)} \\ &\sim (1/\alpha_2)\log k && \text{for } 1 < \alpha_2 < \beta.\end{aligned}$$

Hence from (3.15), (3.16), and (3.17), we have

$$\limsup_{r_1, r_2 \to \infty}\left\{\frac{\log \nu(r_1, r_2; f)}{\log\log(r_1 r_2)}\right\} \leqslant \left(\frac{\beta + 1}{\beta}\right)\alpha_2,$$

from which the required result follows.

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