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ON THE DETERMINATION OF THE MAXIMUM ORDER OF THE GROUP OF A TOURNAMENT

BY

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1. **Introduction.** Let $\mathfrak{A}(T)$ denote the automorphism group of the tournament T. Let $g(n)$ be the maximum of $|\mathfrak{A}(T)|$ taken over all tournaments of order n. It was noted in [3] that $g(n)$ is also the order of the subgroups of S_n of maximum odd order where S_n denotes the symmetric group of degree n.

THEOREM 1. *For each positive integer n we have* $g(n) \leq \sqrt{3}^{n-1}$ *with equality holding if and only if n is itself a power of* 3.

Theorem 1 was conjectured in [3] and proved by group theoretic methods in [2] and combinatorial methods in [1]. In this paper we examine the question of determining $g(n)$ for all values of n. We establish a recursive method for determining $g(n)$ and attack the problem of explicitly determining $g(n)$ from the ternary representation of n. We also give a simple proof of Theorem 1.

Throughout this paper T_n will denote a tournament of order n for which $|\mathfrak{A}(T_n)|=g(n)$ and $\mathfrak{A}_n$ will denote a subgroup of S_n for which $|\mathfrak{A}_n|=g(n)$. The wreath product of two groups G and H will be denoted by $G \wr H$.

2. **Integers n for which some $\mathfrak{A}_n$ is transitive and primitive.** In addition to being useful in the next section Theorem 2 is interesting in that it gives an upper bound for the order of $\mathfrak{A}(T)$ where T is any tournament such that $\mathfrak{A}(T)$ is primitive.

THEOREM 2. *Let T be a tournament with n vertices and suppose* $G=\mathfrak{A}(T)$ *is a primitive permutation group on T. Then n is a power of an odd prime and*

$$|G| \leq ng((n-1)/2).$$

Proof. Since $|G|$ is odd [5], we may deduce from the Feit-Thompson theorem that G is solvable. Let N be a minimal normal subgroup of G. By (a) of Satz 3.2 [4, Ch. II] we may conclude that $n=p^m$ for some prime p and integer m.

Now let $x \in T$, G_x the stabilizer of x, and $\mathscr{I}(x)$ the set of vertices in T which dominate x. By (a) of Satz 3.2 [4] for each $y \in T-\{x\}$ there is a unique $n_y \in N-\{1\}$ such that $n_y(x)=y$. We thus have a 1-1 correspondence $y \leftrightarrow n_y$ between $T-\{x\}$ and

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$N-\{1\}$. If $g\in G_x$ then $gn_yg^{-1}(x)=g(y)$, so $n_y \leftrightarrow y$ iff for all $g \in G_x$, $gn_yg^{-1} \leftrightarrow g(y)$. Thus as a permutation group on $T-\{x\}$, G_x is isomorphic to G_x considered as a permutation group (the action being conjugation) on $N-\{1\}$. Let $I(N)=\{n_y \mid y\in \mathscr{I}(x)\}$. From the above and the fact that $G_x(\mathscr{I}(x))=\mathscr{I}(x)$ we conclude that $n_y\in I(N)$ iff $gn_yg^{-1}\in I(N)$ for all $g\in G_x$. But N is abelian and, by (b) of Satz 3.2 [4, Ch. II], $G=G_xN$. Thus $I(N)$ is a union of G-conjugacy classes. As N is a minimal normal subgroup of G each G-conjugacy class in $I(N)$ contains a basis of the elementary abelian group N. It is now clear that if $g\in G_x$ and if $g(y)=y$ for each $y\in\mathscr{I}(x)$ then $g\in C_G(N)$. Since $g\in G_x$ we conclude (by (b) and (c) of Satz 3.2 [4]) that $g=1$. Hence $g\in G_x$ implies g induces a nontrivial automorphism of $\mathscr{I}(x)$. Now $\mathscr{I}(x)$ is a tournament with $(n-1)/2$ vertices. Thus $|G_x|\leq g((n-1)/2)$. As $|G|=n|G_x|$ the conclusion follows.

We now prove the following.

THEOREM 3. *$\mathfrak{A}_n$ is primitive iff $n=1, 3, 5, 7$.*

Proof. If n is even then, since $|\mathfrak{A}_n|$ is odd, $\mathfrak{A}_n$ is not primitive. Now assume n is odd. If $n=1, 3, 5, 7$ it is easy to show $\mathfrak{A}_n$ is primitive. If $7<n<17$ then, from the table in [6, p. 81], it follows that $g(n)>ng((n-1)/2)$ so $\mathfrak{A}_n$ is imprimitive. For $n=17, 19, 21$ it follows from the same table that $g((n-1)/2)>n$. If this is true for a given $n\geq 17$, then

$$g\left(\frac{(n+6)-1}{2}\right) = g\left(\frac{n-1}{2}+3\right) \geq 3g\left(\frac{n-1}{2}\right) > 3n > n+6,$$

so it is true for all $n\geq 17$. Hence, for such n,

$$g(n) \geq g\left(\frac{n-1}{2}\right)^2 > ng\left(\frac{n-1}{2}\right).$$

By Theorem 2 $\mathfrak{A}_n$ is imprimitive for all $n\geq 17$. This proves the theorem.

We are now able to give a simple proof of Theorem 1. We verify the theorem directly for $n=1, 2, 3$. Let $m\geq 4$ and assume the theorem to be true for all $n<m$. If $\mathfrak{A}(T_m)$ is not transitive, say $\mathfrak{A}(T_m)$ has k orbits, $k\geq 2$, then $|\mathfrak{A}(T_m)|\leq\sqrt{3}^{n-k}$. If $\mathfrak{A}(T_m)$ is transitive and primitive then the result follows from Theorem 3 and the table in [6, p. 81]. If $\mathfrak{A}(T_m)$ is transitive but imprimitive with r blocks of length s then

$$g(m) \leq g(r)g(s)^r \leq \sqrt{3}^{r-1}\sqrt{3}^{r(s-1)} = \sqrt{3}^{m-1}$$

with equality iff both r and s, and hence m, are powers of 3. If $m=3^t$, $t\geq 2$, then $|\mathfrak{A}_3 \wr \mathfrak{A}_{3^{t-1}}|=\sqrt{3}^{m-1}$ and the theorem is proved.

Although the following remarks are elementary we state them here as they are used several times in Theorems 4 and 5 without explicitly referring to them. If $\mathfrak{A}_n$ is an automorphism group of maximal order then we may consider $\mathfrak{A}_n$ as a subgroup of S_n. We know, from the general theory of permutation groups that $\mathfrak{A}_n$ is in fact a subgroup of the direct product $S_{n_1}\times\cdots\times S_{n_k}$ where $n_1+\cdots+n_k=n$

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and $\pi_i(\mathfrak{A}_n)$, the projection of $\mathfrak{A}_n$ on S_{n_i}, is a transitive subgroup of S_{n_i}. Thus, by maximality of $|\mathfrak{A}_n|$, $|\pi_i(\mathfrak{A}_n)|=g(n_i)$ and $\mathfrak{A}_n \cong \mathfrak{A}_{n_1} \times \mathfrak{A}_{n_2} \times \cdots \times \mathfrak{A}_{n_k}$. We shall refer to the $\mathfrak{A}_{n_i} (1 \leq i \leq k)$ as the *transitive constituents* of $\mathfrak{A}_n$.

THEOREM 4. *$\mathfrak{A}_n$ is transitive if and only if $n=3^k$, $5 \cdot 3^k$ or $7 \cdot 3^k$ for some integer $k \geq 0$. In this case $|\mathfrak{A}_n|=\sqrt{3}^{n-1}$, $5 \cdot \sqrt{3}^{n-5}$ or $7 \cdot \sqrt{3}^{n-5}$ and $\mathfrak{A}_n=\mathfrak{A}_{3^r}\ \mathfrak{A}_{3^{k-r}}$, $\mathfrak{A}_{3^r} \wr \mathfrak{A}_{5.3^{k-r}}$ or $\mathfrak{A}_{3^r} \wr \mathfrak{A}_{7.3^{k-r}}$ for all $1 \leq r \leq k$, respectively. In all other cases $\mathfrak{A}_n$ is intransitive, is the direct sum of its transitive constituents, and its order*

$$g(n) = \max_{1 \leq i \leq n-1} \{g(i)g(n-i)\}.$$

Proof. If $\mathfrak{A}_n$ is transitive and primitive, then $n=1$, 3, 5, or 7 and the theorem is true in these cases. Assume $\mathfrak{A}_n$ is transitive and not primitive. Then $n>7$ and there are odd integers $m>1$ and $d>1$ such that $n=md$ and $\mathfrak{A}_n=\mathfrak{A}_m \wr \mathfrak{A}_d$; thus, $g(n)=g(m)^d g(d)$ where $\mathfrak{A}_m$ and $\mathfrak{A}_d$ are transitive. By induction m and d are of the form 3^k, $5 \cdot 3^k$, or $7 \cdot 3^k$ and the orders and structure of $\mathfrak{A}_m$ and $\mathfrak{A}_d$ are as in the theorem. By Theorem 1 we may write $\mathfrak{A}_{3^k}=\mathfrak{A}_{3^r} \wr \mathfrak{A}_{3^{k-r}}$, and since wreath products are associative the necessity of the conditions follows by showing that $m=3^r$ for some $r \geq 1$. If $m=5$ and $d=3e$ then $|\mathfrak{A}_n|=|\mathfrak{A}_5 \wr \mathfrak{A}_3 \wr \mathfrak{A}_e|<|\mathfrak{A}_3 \wr \mathfrak{A}_5 \wr \mathfrak{A}_e| \leq g(n)$, a contradiction. We likewise obtain a contradiction if $m=7$ and $d=3e$. If both $m, d \in \{5, 7\}$ we obtain a contradiction by showing that $|\mathfrak{A}_{27} \oplus \mathfrak{A}_8|$ is greater than both $|\mathfrak{A}_5 \wr \mathfrak{A}_7|$ and $|\mathfrak{A}_7 \wr \mathfrak{A}_5|$, $|\mathfrak{A}_{21} \oplus \mathfrak{A}_4|>|\mathfrak{A}_5 \wr \mathfrak{A}_5|$, and $|\mathfrak{A}_{48} \oplus \mathfrak{A}_1|>|\mathfrak{A}_7 \wr \mathfrak{A}_7|$. (All these may be verified by using the table in [6, p. 81] and noting $g(48) \geq 3^{16} g(16)$.) If $m=5 \cdot 3^r$ $(r \geq 1)$ then by induction $|\mathfrak{A}_n|=|\mathfrak{A}_{3^r} \wr \mathfrak{A}_5 \wr \mathfrak{A}_d|<|\mathfrak{A}_{3^r} \wr \mathfrak{A}_{5d}| \leq g(n)$, a contradiction. A contradiction also arises if $m=7 \cdot 3^r$ $(r \geq 1)$. Thus $m=3^r$ and the necessity of the conditions holds.

Suppose $n=3^k$, $5 \cdot 3^k$, or $7 \cdot 3^k$. If $\mathfrak{A}_n$ is intransitive then it is the direct sum of at least three transitive constituents for n is odd and a transitive constituent has odd degree. Hence $|\mathfrak{A}_n| \leq \sqrt{3}^{n-3}$, by Theorem 1. However $|\mathfrak{A}_3 \wr \mathfrak{A}_{3^{k-1}}|=\sqrt{3}^{n-1}$, $|\mathfrak{A}_{3^k}| \wr \mathfrak{A}_5|=5 \cdot \sqrt{3}^{n-5}$, and $|\mathfrak{A}_{3^k} \wr \mathfrak{A}_7|=7 \cdot \sqrt{3}^{n-5}$, all of which exceed $\sqrt{3}^{n-3}$. Hence $\mathfrak{A}_n$ must be transitive and the theorem is proved.

We now state some consequences of Theorem 4.

THEOREM 5. *The following statements are true:*

(1) *If $n=3r$ then $g(n)=3^r g(r)$*

(2) *If $n \equiv 1$, 2, or 4(9) then $g(n)=g(n-1)$*

(3) *If $n \equiv 8(9)$ then $g(n)=g(n-1)$ or $g(n)=g(5)g(n-5)$*

(4) *If $n \equiv 5(9)$ then $g(n)=g(2)g(n-2)$, $g(5)g(n-5)$ or $g(n)=g(7)g(n-7)$*

(5) *If $n \equiv 7(9)$ then $g(n)=g(7)g(n-7)$*

Proof. We shall prove (1) here, the proofs of the others using the same method. We note (1) is true for $r=1$ and suppose it proved for all r', $1 \leq r' < r$. If $n=3r$ and $\mathfrak{A}_n$ is transitive then (1) follows for $n=3r$ immediately from Theorem 1. Assume

$\mathfrak{A}_n$ is intransitive and $n_1, \ldots, n_k$ are the degrees of its transitive constituents. Then each n_i has the form 3^e, $5 \cdot 3^e$, or $7 \cdot 3^e$. If each $n_i \equiv 0(3)$ then the induction hypothesis applies to each n_i and a short calculation shows the theorem then holds for $n=3r$. If some $n_i \not\equiv 0(3)$ then $n=n_1+\cdots+n_k \equiv 0(3)$ implies there are three n_i (say n_1, n_2, n_3) which are members of the set $\{1, 5, 7\}$ such that $n_1+n_2+n_3 \equiv 0(3)$. Using the table of [6, p. 81] it is easy to show that, for any such n_1, n_2, n_3,

$$g(n_1+n_2+n_3) > g(n_1)g(n_2)g(n_3).$$

This contradicts the maximality of $|\mathfrak{A}_n|$, so each $n_i \equiv 0(3)$ and (1) is now proved.

Our final result generalizes Theorem 1.

COROLLARY 1. *If the ternary expansion of n involves only* 1'*s and* 0'*s and if the number of* 1'*s in this expansion is* n_1 *then* $g(n)=\sqrt{3}^{n-n_1}$.

Proof. We induct on n, the result being true if $n=1$ or 3. If the ternary expansion of n ends in 0 we may write $n=3m$, where the ternary expansion of m involves only 1's and 0's and $m_1=n_1$. Then $g(n)=3^m g(m)=\sqrt{3}^{2m}\sqrt{3}^{m-m_1}=\sqrt{3}^{n-n_1}$. If the ternary expansion of n ends in a 1 then $n\equiv 1, 4 \pmod 9$ and Theorem 5 implies $g(n)=g(n-1)$. But $(n-1)_1=n_1-1$ and $n-1$ satisfies the first hypothesis of the corollary. Hence $g(n)=g(n-1)=\sqrt{3}^{(n-1)-(n-1)_1}=\sqrt{3}^{n-n_1}$.

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