Glasgow Math. J. **47** (2005) 195–204. © 2005 Glasgow Mathematical Journal Trust.
DOI: 10.1017/S0017089504002162. Printed in the United Kingdom

SIMULTANEOUS LIFTING FROM IRREDUCIBLE REPRESENTATIONS OF C^*-ALGEBRAS

ROBERT ARCHBOLD

Department of Mathematical Sciences, University of Aberdeen, King's College, Aberdeen AB24 3UE
e-mail: r.archbold@maths.abdn.ac.uk

and ALDO LAZAR

Department of Pure Mathematics, School of Mathematical Sciences, Tel Aviv University
Ramat Aviv, Israel 69978
e-mail: aldo@math.tau.ac.il

(Received 5 July, 2004; accepted 21 September, 2004)

Abstract. Given a sequence (π_n) of irreducible representations of a liminal C^*-algebra A, and a sequence (b_n) of trace class operators with $b_n \in \pi_n(A)$, we investigate necessary conditions and sufficient conditions for the existence of a simultaneous lifting $a \in A$ such that, for each n, the trace of $\sigma(a)$ is bounded for irreducible representations σ in a neighbourhood of π_n.

2000 *Mathematics Subject Classification.* 46L05, 46L30.

1. Introduction. The starting point for this investigation is a result of Akemann [**1**, II.10] that strengthened an earlier result of Tomiyama [**10**, 4.2.5] concerning simultaneous lifting from irreducible representations. This states that if $(\pi_n)_{n\geq 1}$ is a sequence of distinct elements in the spectrum $\widehat{A}$ of a liminal C^*-algebra A, if (π_n) has no cluster points and if (b_n) is a null sequence with $b_n \in \pi_n(A)$, for all n, then there exists $a \in A$ such that $\pi_n(a) = b_n$, for all $n \geq 1$. In this paper, we consider the possibility of obtaining a simultaneous lifting $a \in A$ such that, for each n, if b_n has finite rank (respectively, b_n is trace-class) then there exists a neighbourhood V_n of π_n in $\widehat{A}$ such that $\{\text{rank}(\sigma(a)) : \sigma \in V_n\}$ is bounded (respectively, $\{\text{Tr}(\sigma(a)) : \sigma \in V_n\}$ is bounded).

Even in the case of a single irreducible representation π_1 and a positive element b_1, the existence of such $a \in A$ and V_1 necessarily forces the finiteness of the upper multiplicity $M_U(\pi_1)$. See Proposition 1. In view of this, it is natural to work in the context of a bounded trace C^*-algebra A, so that $M_U(\pi) < \infty$, for all $\pi \in \widehat{A}$ [**7**, 2.6]. Furthermore, motivated by [**7**, 2.5], we quantify the boundedness requirements of the first paragraph by asking that, for $\sigma \in V_n$, we have

$$\text{rank}(\sigma(a)) \leq M_U(\pi_n) \cdot \text{rank}(b_n) \tag{1}$$

and

$$|\text{Tr}(\sigma(a))| \leq M_U(\pi_n) \cdot \text{Tr}(|b_n|). \tag{2}$$

In Theorem 1, we give the following sufficient condition on $(\pi_n)_{n\geq 1}$ for the existence of $a \in A$ and $(V_n)_{n\geq 1}$ satisfying (1) and (2):

$$\phi(\pi_n) \notin \overline{\{\phi(\pi_m) : m \neq n\}} \qquad (n \geq 1), \tag{3}$$

where ϕ is the complete regularization map on $\widehat{A}$ (see below). Elementary general topology shows that condition (3) is equivalent to: $\phi(\pi_n) = \phi(\pi_m)$ if and only if $n = m$, and $\{\phi(\pi_n) : n \geq 1\}$ is discrete in the relative topology. This condition might, at first sight, appear over-strong, in that it even allows us to construct the $V_n (n \geq 1)$ so as to be pairwise disjoint and independent of the given sequence (b_n). However, we show in Theorem 2 (at least for separable, quasi-standard C^*-algebras with bounded trace) that the condition (3) is actually necessary for the existence of $a \in A$ and $(V_n)_{n\geq 1}$ satisfying (1) and (2) (given an arbitrary null sequence $(b_n)_{n\geq 1}$).

We briefly recall some properties of the complete regularization of the primitive ideal space Prim(A) of a C^*-algebra A. See [**9**, **6**] for further details. For $P, Q \in$ Prim(A) let $P \approx Q$ if and only if $f(P) = f(Q)$, for all $f \in C^b(\text{Prim}(A))$. Then $\approx$ is an equivalence relation on Prim(A) and the equivalence classes are closed subsets of Prim(A). It follows that there is a one-to-one correspondence between Prim$(A)/\approx$ and a set of closed two-sided ideals of A given by

$$[P] \to \bigcap [P] \qquad (P \in \text{Prim}(A)),$$

where $[P]$ denotes the equivalence class of P. The set of ideals obtained in this way is called Glimm(A) (in the unital case these ideals are generated by maximal ideals of the centre of A [**12**, Section 4]). The map ϕ : Prim$(A) \to$ Glimm(A) given by

$$P \to \bigcap [P] \qquad (P \in \text{Prim}(A))$$

is called the *complete regularization map*.

There are two natural Hausdorff topologies on Glimm(A): the completely regular topology τ_{cr}, that is the weakest topology for which the functions on Glimm(A) induced by $C^b(\text{Prim}(A))$ are all continuous, and the quotient topology τ_q. The second is stronger than the first, but they coincide if A is unital or if ϕ is either τ_{cr}-open or τ_q-open (and so we may speak unambiguously of ϕ being open).

There is another relation on Prim(A) defined by: $P \sim Q$ if and only if P and Q cannot be separated by disjoint open subsets of Prim(A). It is immediate that if $P \sim Q$ then $P \approx Q$ but the converse fails in general because $\sim$ need not be transitive. A C^*-algebra A is said to be *quasi-standard* [**6**] if $\sim$ is an open equivalence relation. In this case, $\sim$ necessarily coincides with $\approx$, ϕ is open, $\tau_{cr} = \tau_q$ and A can be represented as a continuous field of C^*-algebras over the base space Glimm(A). If A is separable then the fibre algebras are primitive for a dense subset of the base space. Thus the quasi-standard C^*-algebras may be viewed as a well-behaved class that is significantly larger than the class of C^*-algebras with Hausdorff primitive ideal space; for example, all von Neumann algebras and several group C^*-algebras are quasi-standard [**6**, **13**].

If A is a C^*-algebra of type I, then $\hat{A}$ may be identified with Prim(A) via the homeomorphism $\pi \to \ker \pi$ $(\pi \in \hat{A})$ and so we may regard ϕ as a map from $\hat{A}$ to Glimm(A) given by $\phi(\pi) = \bigcap [\ker \pi]$. For $\pi \in \hat{A}$, we write $[\pi]$ for the closed set $\phi^{-1}(\phi(\pi))$ in $\hat{A}$ (which corresponds to the closed set $[\ker \pi]$ in Prim(A)).

For $\pi \in \hat{A}$, the upper and lower multiplicities $M_U(\pi)$ and $M_L(\pi)$ are defined in [**4**]. Upper and lower multiplicities for π relative to a net in $\hat{A}$ are defined in [**8**]. (See also [**7**].) These numbers are related to the integers occurring in trace formulae and they are also related to the number of orthogonal nets of pure states that can converge to a common pure limit associated with π. A C^*-algebra A is said to have *bounded trace* [**14**, **15**] if there is a dense two-sided ideal J of A such that, for each $a \in J^+$,

$\{\mathrm{Tr}(\pi(a)) : \pi \in \hat{A}\}$ is a bounded set of non-negative real numbers. This holds if and only if $M_U(\pi) < \infty$, for all $\pi \in \hat{A}$ [**7**].

2. Results.

PROPOSITION 1. *Let A be a C^*-algebra and let π be an irreducible representation such that $\pi(A)$ contains $\mathcal{LC}(\mathcal{H}_\pi)$, the algebra of compact (linear) operators on the Hilbert space $\mathcal{H}_\pi$.*

(i) *Suppose that b is a nonzero positive operator of trace-class on $\mathcal{H}_\pi$ and that there exists a neighbourhood V of π in $\widehat{A}$ and an element $a \in A^+$ such that $\pi(a) = b$ and $\{\mathrm{Tr}(\sigma(a)) : \sigma \in V\}$ is bounded. Then $M_U(\pi) < \infty$.*

(ii) *Suppose that b is a nonzero operator of finite rank on $\mathcal{H}_\pi$ and that there exists a neighbourhood V of π in $\widehat{A}$ and an element $a \in A$ such that $\pi(a) = b$ and $\{\mathrm{rank}(\sigma(a)) : \sigma \in V\}$ is bounded. Then $M_U(\pi) < \infty$.*

Proof. (i) Suppose that $M_U(\pi) = \infty$. Since $0 \neq b \in \pi(A) \cap \mathcal{LC}(\mathcal{H}_\pi)$, $\{\pi\}$ is not open in $\widehat{A}$. (See the proof of [**4**, Proposition 4.11].) It follows from [**8**, Propositions 2.2 and 2.3] that there exists a net $\Omega = (\pi_\alpha)_{\alpha\in\Lambda}$ in $\widehat{A}\backslash\{\pi\}$ that is convergent to π and satisfies

$$M_L(\pi, \Omega) = M_U(\pi) = \infty.$$

Since $\mathrm{Tr}(b) > 0$, it follows from generalized lower semi-continuity [**8**, Theorem 4.3] that

$$\liminf \mathrm{Tr}(\pi_\alpha(a)) \geq M_L(\pi, \Omega) \cdot \mathrm{Tr}(b) = \infty.$$

This contradicts the hypothesis that $\{\mathrm{Tr}(\sigma(a)) : \sigma \in V\}$ is bounded, because $\pi_\alpha \in V$ eventually.

(ii) Since $\pi(a^*a) = b^*b \neq 0$ and $\mathrm{rank}(\sigma(a^*a)) \leq \mathrm{rank}(\sigma(a))$, for all $\sigma \in V$, we may assume that b and a are positive. Also, by scaling, we may assume that $\|b\| = 1$. Let $f : [0, \infty) \to [0, 1]$ be defined by $f(t) = t$ $(0 \leq t \leq 1)$ and $f(t) = 1$ $(t > 1)$, and let $c = f(a)$. Then $\pi(c) = b$, $0 \leq c \leq a$ and $\|c\| = 1$.

For $\sigma \in V$,

$$\mathrm{Tr}(\sigma(c)) \leq \mathrm{rank}(\sigma(c)) \leq \mathrm{rank}(\sigma(a)).$$

By part (i) of the proposition, $M_U(\pi) < \infty$. □

Let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. For vectors $\xi, \eta \in \mathcal{H}$, let $\theta_{\xi,\eta} \in \mathcal{L}(\mathcal{H})$ be the operator defined by $\theta_{\xi,\eta}(\zeta) = \langle \zeta, \eta \rangle \xi$, $\zeta \in \mathcal{H}$.

LEMMA 1. *Let A be a C^*-algebra, $\pi \in \widehat{A}$ and assume that $\pi(A) \supseteq \mathcal{LC}(\mathcal{H}_\pi)$. There is an open neighbourhood V of π such that, for every one-dimensional projection e in $\mathcal{L}(\mathcal{H}_\pi)$, there exists $a \in A^+$ with $\|a\| = 1$, $\pi(a) = e$, and $\mathrm{rank}(\sigma(a)) \leq M_U(\pi)$, for all $\sigma \in V$.*

Proof. Firstly, suppose that $M_U(\pi) = \infty$. Then we may take $V = \widehat{A}$. Given e, let $b \in A$ be any lifting and then set $a = f(b^*b)$, where f is the function used in the proof of Proposition 1. From now on, we may suppose that $M_U(\pi) < \infty$.

Let p be a fixed one-dimensional projection in $\mathcal{L}(\mathcal{H}_\pi)$. By [**7**, Theorem 2.5], there is an open neighbourhood V of π in $\widehat{A}$ and $b \in A^+$ such that $\|b\| = 1$, $\pi(b) = p$ and

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

rank $(\sigma(b)) \leq M_U(\pi)$, $\sigma \in V$. Now let e be any one-dimensional projection in $\mathcal{L}(\mathcal{H}_\pi)$. Choose unit vectors ξ and η in the ranges of p and e respectively. By [**2**, Theorem 4.3] there is $x \in A$ such that $\|x\| = 1$ and $\pi(x) = \theta_{\xi,\eta}$. For $a = x^*bx$ we have $a \in A^+$, $\|a\| \leq 1$, $\pi(a) = \theta^*_{\xi,\eta} p \theta_{\xi,\eta} = e$ and $\operatorname{rank}(\sigma(a)) = \operatorname{rank}(\sigma(x)^*\sigma(b)\sigma(x)) \leq \operatorname{rank}(\sigma(b)) \leq M_U(\pi)$, for all $\sigma \in V$. □

LEMMA 2. *Let A be a C^*-algebra, $\pi \in \widehat{A}$ and assume that $\pi(A) \supseteq \mathcal{LC}(\mathcal{H}_\pi)$. There is a neighbourhood V of π in $\widehat{A}$ such that, for every trace class operator b on $\mathcal{H}_\pi$, there exists $a \in A$, which may be chosen to be positive if b is positive, satisfying $\|a\| = \|b\|$, $\pi(a) = b$, $|Tr(\sigma(a))| \leq M_U(\pi)Tr(|b|)$ and* $\operatorname{rank}(\sigma(a)) \leq M_U(\pi)\operatorname{rank}(b)$, *for all $\sigma \in V$.*

Proof. Firstly, suppose that $M_U(\pi) = \infty$. Then we may take $V = \widehat{A}$. Given b, there is a lifting $a \in A$ such that $\|a\| = \|b\|$, by [**2**, Theorem 4.3] (and if $b \geq 0$ we may then replace a by $|a|$). From now on, we may suppose that $M_U(\pi) < \infty$.

Let V be a neighbourhood of π as given by Lemma 1. Let b be an operator of trace class on $\mathcal{H}_\pi$. By [**16**, Theorem 1.9.3 and Lemma 2.1.2],

$$b = \sum_i \lambda_i u_i p_i, \tag{4}$$

where $\lambda_i \geq 0$, $\{p_i\}$ are mutually orthogonal one-dimensional projections, $\{u_i\}$ are partial isometries whose initial domains are the ranges of p_i, respectively, and whose final domains are mutually orthogonal, and $\operatorname{Tr}(|b|) = \sum_i \lambda_i < \infty$. If $b \geq 0$, we take $u_i = p_i$, for each i.

By Lemma 1, for each $i = 1, 2, , \ldots,$ there exists $a_i \in A^+$ such that $\|a_i\| = 1, \pi(a_i) = p_i$ and $\operatorname{rank}(\sigma(a_i)) \leq M_U(\pi)$, for every $\sigma \in V$. By [**2**, Theorem 4.3], there exists $v_i \in A$ with $\|v_i\| = 1$ and $\pi(v_i) = u_i$, for $i = 1, 2, \ldots$. If $b \geq 0$, we choose $v_i = a_i$, for each i. Put $x = \sum_i \lambda_i v_i a_i \in A$ and let $x = u|x|$, with $u \in A^{**}$, be its polar decomposition (unless $x \geq 0$ in which case we let $u = 1 \in A^{**}$). Then $|x| = u^*x$, and $\pi(x) = \sum \lambda_i \pi(v_i)\pi(a_i) = \Sigma\lambda_i u_i p_i = b$. For $\sigma \in V$, let $\overline{\sigma}$ be the unique normal extension of σ to A^{**}. Then the finite dimensional operator $\overline{\sigma}(u^* v_i a_i)$ satisfies

$$\|\overline{\sigma}(u^* v_i a_i)\|_{\mathcal{C}_1} \leq \|\overline{\sigma}(u^* v_i)\| \, \|\sigma(a_i)\|_{\mathcal{C}_1} \leq \operatorname{Tr}(\sigma(a_i)) \leq M_U(\pi)$$

and so $\sum_i \lambda_i \overline{\sigma}(u^* v_i a_i)$ is absolutely convergent in the trace class norm $\mathcal{C}_1$, and hence in the operator norm. We have,

$$\begin{aligned}\operatorname{Tr}(\sigma(|x|)) = \operatorname{Tr}(\sigma(u^*x)) &= \operatorname{Tr}\left(\sigma\left(\sum_i \lambda_i u^* v_i a_i\right)\right) \\ &= \operatorname{Tr}\left(\sum_i \lambda_i \overline{\sigma}(u^* v_i)\sigma(a_i)\right) \leq \sum_i \lambda_i |\operatorname{Tr}(\overline{\sigma}(u^* v_i)\sigma(a_i))| \\ &\leq \sum_i \lambda_i \|\overline{\sigma}(u^* v_i)\| \operatorname{Tr}(\sigma(a_i)) \leq M_U(\pi) \sum_i \lambda_i = M_U(\pi)\operatorname{Tr}(|b|).\end{aligned} \tag{5}$$

Consider the function $f \colon [0, \infty) \to [0, \infty)$ defined by

$$f(t) = \begin{cases} t, & 0 \leq t \leq \|b\|, \\ \|b\|, & t > \|b\|. \end{cases}$$

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

Put $a = uf(|x|) \in A$. Then $\|a\| \leq \|b\|$ and

$$\pi(a) = \overline{\pi}(u)f(\pi(|x|)) = \overline{\pi}(u)f(|b|) = \overline{\pi}(u)|b| \\ = \overline{\pi}(u)\pi(|x|) = \pi(u|x|) = \pi(x) = b.$$

Note that if $b \geq 0$ then $x \geq 0$ and so, by our choice of u, $a = f(|x|) \geq 0$. Let $\sigma \in V$. Then

$$|\mathrm{Tr}(\sigma(a))| = |\mathrm{Tr}(\overline{\sigma}(u)\sigma(f(|x|))| \leq \|\overline{\sigma}(u)\| Tr(\sigma(f(|x|))) \\ \leq \mathrm{Tr}(\sigma(f(|x|))) \leq \mathrm{Tr}(\sigma(|x|)) \leq M_U(\pi)\mathrm{Tr}(|b|),$$

by (5). We also have $\mathrm{rank}(\sigma(a)) \leq M_u(\pi) \cdot \mathrm{rank}(b)$. Indeed, if the range of b is infinite dimensional there is nothing to prove. Otherwise, we may assume that the number of summands in (4) is rank(b). Then, for $\sigma \in V$,

$$\mathrm{rank}(\sigma(x)) \leq \sum_i \mathrm{rank}(\sigma(v_i a_i)) \leq \sum_i \mathrm{rank}(\sigma(a_i)) \leq M_U(\pi) \cdot \mathrm{rank}(b)$$

and so

$$\mathrm{rank}(\sigma(a)) = \mathrm{rank}(\overline{\sigma}(u)f(\sigma(|x|))) \leq \mathrm{rank}(f(\sigma(|x|))) \\ \leq \mathrm{rank}(\sigma(|x|)) = \mathrm{rank}(\sigma(x)) \leq M_U(\pi) \cdot \mathrm{rank}(b).$$ □

We shall need two elementary topological lemmas.

LEMMA 3. *Let X and Y be topological spaces, Y regular, $\varphi : X \to Y$ a continuous map and (x_n) a finite or infinite sequence in X. Suppose that $\varphi(x_n) \notin \overline{\{\varphi(x_m) : m \neq n\}}$ for each n. Then there is a sequence (V_n) of open sets in X, pairwise disjoint, such that $x_n \in V_n$ for every n.*

Proof. Let U_1 and O_1 be disjoint open sets in Y such that $\varphi(x_1) \in U_1$ and $\overline{\{\varphi(x_m) : m \neq 1\}} \subseteq O_1$. Put $V_1 = \varphi^{-1}(U_1)$, $W_1 = \varphi^{-1}(O_1)$. Then V_1, W_1 are disjoint open sets in X, $x_1 \in V_1$ and $\{x_n : n \geq 2\} \subseteq W_1$.

Suppose we have chosen open sets $\{V_i\}_{i=1}^n$, $\{W_i\}_{i=1}^n$ in X such that $\{V_i\}_{i=1}^n$ are pairwise disjoint, $V_i \cap W_i = \emptyset$, $x_i \in V_i$, $\{x_m : m \neq i\} \subseteq W_i$ for $1 \leq i \leq n$. There are disjoint open sets U_{n+1} and O_{n+1} in Y such that $\varphi(x_{n+1}) \in U_{n+1}$, $\overline{\{\varphi(x_m) : m \neq n+1\}} \subseteq O_{n+1}$. Put $V'_{n+1} = \varphi^{-1}(U_{n+1})$, $W_{n+1} = \varphi^{-1}(O_{n+1})$ and $V_{n+1} = V'_{n+1} \cap (\bigcap_{i=1}^n W_i)$. Then $\{V_i\}_{i=1}^{n+1}$, $\{W_i\}_{i=1}^{n+1}$ satisfy the induction hypothesis. □

LEMMA 4. *Let X and Y be topological spaces and let $\varphi : X \to Y$ be a continuous, open mapping. For $x \in X$, let $[x] = \varphi^{-1}(\varphi(x))$. Let S be a non-empty subset of X and let $x \in S$. The following are equivalent:*

(1) $x \notin \overline{\cup\{[y] : y \in S\backslash\{x\}\}}$,
(2) $\varphi(x) \notin \overline{\{\varphi(y) : y \in S\backslash\{x\}\}}$.

Proof. (1) ⇒ (2). Suppose that $\varphi(x) \in \overline{\{\varphi(y) : y \in S\backslash\{x\}\}}$. Let V be a neighbourhood of x. Since φ is open, $\varphi(V)$ is a neighbourhood of $\varphi(x)$. By assumption, there exists $y \in S\backslash\{x\}$ such that $\varphi(y) \in \varphi(V)$. Thus $V \cap [y] \neq \emptyset$ and so $x \in \overline{\cup\{[y] : y \in S\backslash\{x\}\}}$.

(2) ⇒ (1). This is immediate from the continuity of φ. □

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

THEOREM 1. *Let A be a bounded trace C^*-algebra and $\varphi : \widehat{A} \to \mathrm{Glimm}(A)$ be the complete regularization map, the latter space being considered with its τ_{cr} topology. Let (π_n) be a finite or infinite sequence in $\widehat{A}$ satisfying*

$$\varphi(\pi_n) \notin \overline{\{\varphi(\pi_m) : m \neq n\}}\,, \quad n = 1, 2, \ldots \tag{6}$$

in Glimm(A). *Then there are pairwise disjoint open neighbourhoods V_n of π_n $(n \geq 1)$ such that for each sequence (b_n) of trace class operators $b_n \in \mathcal{L}(\mathcal{H}_{\pi_n})$ with $\lim_n \|b_n\| = 0$ (if the sequence (π_n) is infinite) there exists $a \in A$ (which may be chosen to be positive if all b_n are positive) such that for $n \geq 1$: $\pi_n(a) = b_n$ and, for all $\sigma \in V_n$, $\sigma(a)$ is of trace class, $|\mathrm{Tr}(\sigma(a))| \leq M_U(\pi_n) \cdot \mathrm{Tr}(|b_n|)$ and $\mathrm{rank}(\sigma(a)) \leq M_U(\pi_n) \cdot \mathrm{rank}(b_n)$.*

Proof. By Lemma 3, there is a sequence (U_n) of pairwise disjoint open sets in X such that $\pi_n \in U_n$ for $n = 1, 2, \ldots$. Let I_n be the closed two-sided ideal of A corresponding to the open subset U_n of $\widehat{A}$, $n = 1, 2, \ldots$ and let I be the closed two-sided ideal of A for which $\widehat{I} = \bigcup_n U_n$, so that I is the (restricted) direct sum of the I_n. For each n, we may apply Lemma 2 to I_n to obtain an open neighbourhood V_n of π_n such that $V_n \subseteq U_n$ and for each $b_n \in \mathcal{L}(\mathcal{H}_{\pi_n})$ of trace class there exists $a_n \in I_n$ (which may be chosen to be positive if b_n is positive) such that $\|a_n\| = \|b_n\|$, $\pi_n(a_n) = b_n$, and for all $\sigma \in V_n$, $\sigma(a_n)$ is of trace class, $|\mathrm{Tr}(\sigma(a_n))| \leq M_U(\pi_n) \cdot \mathrm{Tr}(|b_n|)$, and $\mathrm{rank}(\sigma(a_n)) \leq M_U(\pi_n) \cdot \mathrm{rank}(b_n)$. Given (b_n) as in the statement of the theorem, there exists (a_n) as above, and then $a = \Sigma a_n \in I \subseteq A$ has the required properties. □

REMARKS 1. If we suppose that the bounded trace C^*-algebra A is quasi-standard, then φ is open and so, by Lemma 4, the sequence (π_n) satisfies (6) if, for every n, $\pi_n \notin \overline{\cup\{[\pi_m] : m \neq n\}}$. Thus, in particular, if (π_n) is a sequence of separated points in $\widehat{A}$ such that $\pi_n \notin \overline{\{\pi_m : m \neq n\}}$ for each n, then (6) will be satisfied because $[\pi_m] = \overline{\{\pi_m\}} = \{\pi_m\}$ for each m. For instance, any sequence (π_n) of distinct separated points that has no cluster points in $\widehat{A}$ will satisfy (6).

The strong hypothesis (6) on (π_n) is justified for quasi-standard C^*-algebras in Theorem 2 below. Nevertheless, one may ask if it is always implied by (π_n) being a sequence of distinct points of $\widehat{A}$ that has no cluster points (which is all that is required for Akemann's result quoted in the introduction). The negative answer is illustrated by the following example.

EXAMPLE 1. Let A be the C^*-algebra of all continuous functions $f : [0, 1] \to M_2(\mathbb{C})$ such that $f(\frac{1}{n}) = \begin{pmatrix} \lambda_n(f) & 0 \\ 0 & \mu_n(f) \end{pmatrix}$, $f(0) = \begin{pmatrix} \lambda(f) & 0 \\ 0 & 0 \end{pmatrix}$, where $\lambda(f) \in \mathbb{C}$, $\lambda_n(f) \in \mathbb{C}$ and $\mu_n(f) \in \mathbb{C}$ for $n \geq 1$. Then A is a quasi-standard, bounded trace C^*-algebra (in fact, it is a Fell C^*-algebra). The sequence $\lambda, \mu_1, \mu_2, \ldots$ has no cluster points in $\widehat{A}$. However, since $\lambda_n \to \lambda$ in $\widehat{A}$ and $[\mu_n] = \{\lambda_n, \mu_n\}$, for $n \geq 1$, we have $\lambda \in \overline{\{[\mu_n] : n \geq 1\}}$. Since φ is continuous, condition (6) fails for the sequence $\lambda, \mu_1, \mu_2, \ldots$. Furthermore, the conclusion of Theorem 1 fails for this sequence: this will follow from Theorem 2 but also can be easily seen directly.

The next example shows that, for separable, bounded trace C^*-algebras, condition (6) is not necessary for the conclusion of Theorem 1 to hold. It follows that, in Theorem 2 below, the hypothesis of quasi-standardness cannot be deleted.

EXAMPLE 2. Let A be the C^*-algebra of all the continuous functions

$$f : \{(x, i) : 0 \leq x \leq 1, i = 0, 1\} \to M_2(\mathbb{C})$$

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

such that

$$f(0,1)=0, \quad f\left(\frac{1}{n},1\right)=\begin{pmatrix}\lambda_n(f) & 0\\ 0 & \mu_n(f)\end{pmatrix}, \quad f(0,0)=\begin{pmatrix}0 & 0\\ 0 & \lambda(f)\end{pmatrix},$$

$f(\frac{1}{n},0)=\begin{pmatrix}\mu_n(f) & 0\\ 0 & \nu_n(f)\end{pmatrix}$, where $\lambda_n(f)$, $\mu_n(f)$, $\nu_n(f)$, $\lambda(f)$ are scalars for $n\geq 1$. Then A is a separable, bounded trace C^*-algebra. However, the relation $\sim$ is not transitive and so A is not quasi-standard. It is easily checked that the conclusion of Theorem 1 holds for the sequence $\lambda, \lambda_1, \lambda_2, \ldots$. Nevertheless, $\lambda \in \overline{\cup\{[\lambda_n] : n\geq 1\}}$ because $[\lambda_n]=\{\lambda_n, \mu_n, \nu_n\}$ for each n and $\nu_n \to \lambda$ in $\widehat{A}$. Since φ is continuous, $\varphi(\lambda)\in\overline{\{\varphi(\lambda_n) : n\geq 1\}}$, which shows that condition (6) fails.

The next two lemmas will be needed in the proof of Theorem 2.

Lemma 5. *Let A be a separable, quasi-standard, liminal C^*-algebra. For each $\pi\in\widehat{A}$ there is a sequence $(\rho_n)_{n\geq 1}$ in $\widehat{A}$ such that*

$$M_L(\pi,(\rho_n)_{n\geq 1})=M_U(\pi,(\rho_n)_{n\geq 1})=M_U(\pi)$$

and $(\rho_n)_{n\geq 1}$ converges to each $\rho\in[\pi]$.

Proof. Let $\pi\in\widehat{A}$. The set of separated points of $\widehat{A}$ is dense in A by [**10**, Proposition 2]. By [**5**, Lemma 1.2], there is a sequence $(\rho_n)_{n\geq 1}$ of separated points in $\widehat{A}$ that converges to π and satisfies

$$M_L(\pi,(\rho_n)_{n\geq 1})=M_U(\pi,(\rho_n)_{n\geq 1})=M_U(\pi).$$

Now let $\rho\in[\pi]$. For each open neighbourhood N of ρ, $\varphi(N)$ is an open neighbourhood of $\varphi(\rho)=\varphi(\pi)$. Hence $\varphi(\rho_n)\in\varphi(N)$ for n sufficiently large. Since A is quasi-standard and liminal, $\varphi^{-1}(\varphi(\rho_n))$ is the singleton $\{\rho_n\}$, for each n, and so $\rho_n\in N$ eventually. □

Lemma 6. *Let m be a positive integer and let $\xi_1,\ldots,\xi_{m+1}$ be unit vectors in a Hilbert space such that $|\langle\xi_i,\xi_j\rangle|<\frac{1}{m}$, for $1\leq i<j\leq m+1$. Then $\{\xi_1,\ldots,\xi_{m+1}\}$ is linearly independent.*

Proof. Suppose that $\sum_{i=1}^{m+1}\alpha_i\xi_i=0$, where not all of the complex coefficients $\alpha_1,\ldots,\alpha_{m+1}$ are zero. Choose j such that $|\alpha_j|\geq|\alpha_i|$ for all i. Then

$$1=\langle\xi_j,\xi_j\rangle=\left|\sum_{i\neq j}\left\langle\frac{\alpha_i}{\alpha_j}\xi_i,\xi_j\right\rangle\right|<m\cdot\frac{1}{m}=1,$$

a contradiction. □

Theorem 2. *Let A be a separable, quasi-standard C^*-algebra with bounded trace and let (π_n) be a finite or infinite sequence in $\widehat{A}$. Suppose that, for every sequence (b_n), where b_n is a positive operator of finite rank in $\pi_n(A)$ and $\lim_{n\to\infty}\|b_n\|=0$ (if the sequence (π_n) is infinite), there exist $a\in A$ and a sequence (V_n) of open subsets of $\hat{A}$ such that*

(i) *for all n, $\pi_n\in V_n$ and $\pi_n(a)=b_n$,*

(ii) *either, for all n and for all $\sigma\in V_n$, $\sigma(a)$ is a positive operator and*

$$\mathrm{Tr}(\sigma(a))\leq M_U(\pi_n)\cdot\mathrm{Tr}(b_n)$$

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

or, for all n and all $\sigma \in V_n$,

$$\operatorname{rank}(\sigma(a)) \leq M_U(\pi_n) \cdot \operatorname{rank}(b_n).$$

Then $\varphi(\pi_n) \notin \overline{\{\varphi(\pi_m) : m \neq n\}}$ for every n.

Proof. First of all, we show that $\varphi(\pi_n) \neq \varphi(\pi_m)$ whenever $m \neq n$. Suppose, on the contrary, that $\varphi(\pi_n) = \varphi(\pi_m)$ for some distinct m and n. Since A is quasi-standard, $\pi_n \sim \pi_m$ (that is, π_n and π_m cannot be separated by disjoint open subsets of $\hat{A}$) and so there is a net (σ_α) in $\hat{A}$ that is convergent to both π_m and π_n. Define b_n to be a nonzero operator of norm one in $\pi_n(A)$ and define $b_j = 0$ for $j \neq n$. By hypothesis, there exists $a \in A$ and a neighbourhood V_m of π_m such that $\pi_n(a) = b_n$ and $\sigma(a) = 0$, for all $\sigma \in V_m$. Eventually, $\sigma_\alpha \in V_m$ and then $\sigma_\alpha(a) = 0$. By lower semi-continuity [**11**, 3.3.2],

$$1 = \|b_n\| = \|\pi_n(a)\| \leq \liminf \|\sigma_\alpha(a)\| = 0,$$

a contradiction.

Since $\varphi(\pi_m) \neq \varphi(\pi_n)$ for $m \neq n$ and Glimm(A) is Hausdorff, the conclusion of the theorem is now clear if the sequence (π_n) is finite. From now on we assume that (π_n) is an infinite sequence. Suppose that the conclusion of the theorem fails. By renumbering, we may as well suppose that $\varphi(\pi_1) \in \overline{\{\varphi(\pi_n) : n \geq 2\}}$. Since A is quasi-standard, φ is open and so, by Lemma 4,

$$\pi_1 \in \overline{\cup_{n\geq 2}[\pi_n]}. \tag{7}$$

Since A is separable, there exists a decreasing base $(U_k)_{k\geq 2}$ of open neighbourhoods of π_1 in $\hat{A}$. By (7), there exists $n_2 \geq 2$ such that there is $\sigma_2 \in U_2 \cap [\pi_{n_2}]$. Since $\varphi(\pi_1) \neq \varphi(\pi_n)$ for $n \geq 2$, we have $\pi_1 \notin [\pi_n]$ for $n \geq 2$ and so $U_3 \setminus \cup_{r=2}^{n_2} [\pi_r]$ is an open neighbourhood of π_1. Hence there is $n_3 > n_2$ such that there is $\sigma_3 \in U_3 \cap [\pi_{n_3}]$. Proceeding in this way, we may construct a strictly increasing sequence of integers $(n_k)_{k\geq 2}$ (with $n_2 \geq 2$) and $\sigma_k \in U_k \cap [\pi_{n_k}]$ for all $k \geq 2$. Since (U_k) is decreasing, $\sigma_k \to \pi_1$ as $k \to \infty$.

For each $n \geq 1$, let p_n be a projection of rank one in $\pi_n(A)$. Let $(\lambda_n)_{n\geq 1}$ be a strictly decreasing null sequence in $\mathbb{R}$ with $\lambda_1 = 1$, and let $b_n = \lambda_n p_n$ $(n \geq 1)$. By hypothesis, there exists $a \in A$ and a sequence $(V_n)_{n\geq 1}$ of open subsets of $\hat{A}$ such that (i) and (ii) hold. The set $\{\sigma \in \hat{A} : \|\sigma(a)\| > \frac{1}{2}\}$ is an open neighbourhood of π_1 [**11**, 3.3.2] and so, since $\|\pi_{n_k}(a)\| = \lambda_{n_k} \to 0$ as $k \to \infty$, there exists $K \geq 1$ such that $\sigma_K \in V_1$, $\|\sigma_K(a)\| > \frac{1}{2}$ and $\sigma_K \neq \pi_{n_K}$. By Lemma 5, there is a sequence $(\rho_k)_{k\geq 1}$ in $\hat{A}$ that is convergent to both σ_K and π_{n_K} and satisfies

$$M_U\big(\pi_{n_K}, (\rho_k)\big) = M_L\big(\pi_{n_K}, (\rho_k)\big) = m,$$

where $m = M_U(\pi_{n_K})$. Since $\rho_k \to \pi_{n_K}$ as $k \to \infty$, there exists $L \geq 1$ such that $\rho_k \in V_{n_K}$ for all $k \geq L$.

We have to consider the two possibilities for a and (V_n) in (ii). Firstly, suppose that a and (V_n) satisfy the tracial condition. Then

$$\operatorname{Tr}(\sigma(a)) \leq m\operatorname{Tr}\big(b_{n_K}\big) = m\lambda_{n_K} \qquad \big(\sigma \in V_{n_K}\big)$$

and, since $\sigma_K \in V_1$, $\sigma_K(a)$ is positive. It follows that $\operatorname{Tr}(\rho_k(a)) \leq m\lambda_{n_K}$ for all $k \geq L$. Hence, by generalised lower semi-continuity [**8**, Theorem 4.3] and the fact that $\sigma_K(a)$

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

is a nonzero positive operator,

$$m\lambda_{n_K} \geq \liminf \mathrm{Tr}(\rho_k(a)) \geq M_L\big(\pi_{n_K}, (\rho_k)\big) \cdot \mathrm{Tr}\big(\pi_{n_K}(a)\big) + M_L(\sigma_K, (\rho_k)) \cdot \mathrm{Tr}(\sigma_K(a)) > m\lambda_{n_K},$$

a contradiction.

Secondly, suppose that a and (V_n) satisfy the condition on rank. Then

$$\mathrm{rank}(\sigma(a)) \leq m\, \mathrm{rank}\big(b_{n_K}\big) = m \qquad \big(\sigma \in V_{n_K}\big)$$

and so $\mathrm{rank}(\rho_k(a)) \leq m$ for all $k \geq L$. Let ξ be a unit vector in the range of the projection p_{n_K} and let η be a unit vector in the Hilbert space for σ_K such that $\|\sigma_K(a)\eta\| > \frac{1}{2}$. Let ψ be the pure state of A defined by $\psi(x) = \langle \pi_{n_K}(x)\xi, \xi\rangle$ for $x \in A$. Since A is separable, a simple adaptation of the proof of [**7**, Lemma 5.2(i)] shows that there is a subsequence $(\rho_{k_r})_{r\geq 1}$ of (ρ_k) and an orthonormal set $\{\xi_r^i : 1 \leq i \leq m\}$ in the Hilbert space for ρ_{k_r} $(r \geq 1)$ such that, for $x \in A$ and $1 \leq i \leq m$,

$$\lim_{r\to\infty} \big\langle \rho_{k_r}(x)\xi_r^i, \xi_r^i\big\rangle = \psi(x). \tag{8}$$

Hence, for $1 \leq i \leq m$,

$$\lim_{r\to\infty} \big\|\rho_{k_r}(a)\xi_r^i\big\|^2 = \lim_{r\to\infty} \big\langle \rho_{k_r}(a^*a)\xi_r^i, \xi_r^i\big\rangle = \psi(a^*a) = \lambda_{n_K}^2$$

and

$$\lim_{r\to\infty} \big\|\rho_{k_r}(a)\xi_r^i - \lambda_{n_K}\xi_r^i\big\|^2 = \lim_{r\to\infty} \Big[\big\|\rho_{k_r}(a)\xi_r^i\big\|^2 + \lambda_{n_K}^2 - \lambda_{n_K}\big\langle \xi_r^i, \rho_{k_r}(a)\xi_r^i\big\rangle - \lambda_{n_K}\big\langle \rho_{k_r}(a)\xi_r^i, \xi_r^i\big\rangle\Big] = 0.$$

Thus

$$\lim_{r\to\infty} \big\langle \rho_{k_r}(a)\xi_r^i, \rho_{k_r}(a)\xi_r^j\big\rangle = 0 \qquad (1 \leq i < j \leq m). \tag{9}$$

(This also follows from (8) by [**8**, Lemma 2.5].)

Since A is separable, the w^*-topology on A^* is first countable. Hence, since the canonical mapping from the set of pure states of A to $\hat{A}$ is open, we may assume, by passing to a subsequence of (ρ_{k_r}) if necessary, that there exists a unit vector η_r in the Hilbert space for ρ_{k_r} $(r \geq 1)$ such that

$$\lim_{r\to\infty} \langle \rho_{k_r}(x)\eta_r, \eta_r\rangle = \langle \sigma_K(x)\eta, \eta\rangle \qquad (x \in A).$$

In particular,

$$\lim_{r\to\infty} \|\rho_{k_r}(a)\eta_r\|^2 = \lim_{r\to\infty} \langle \rho_{k_r}(a^*a)\eta_r, \eta_r\rangle = \langle \sigma_K(a^*a)\eta, \eta\rangle = \|\sigma_K(a)\eta\|^2 > 1/4\,.$$

Thus for r large enough, $\|\rho_{k_r}(a)\eta_r\| > 1/2$.

Since $\pi_{n_K} \neq \sigma_K$, the pure states ψ and $\langle \sigma_K(\cdot)\eta, \eta\rangle$ are inequivalent and so Lemma 2 of [**3**] implies that for $x \in A$

$$\lim_{r\to\infty} \big\langle \rho_{k_r}(x)\eta_r, \xi_r^i\big\rangle = 0 \qquad (1 \leq i \leq m).$$

https://doi.org/10.1017/S0017089504002162 Published online by Cambridge University Press

In particular,

$$\lim_{r\to\infty} \langle \rho_{k_r}(a)\eta_r, \rho_{k_r}(a)\xi_r^i \rangle = 0 \qquad (1 \le i \le m). \tag{10}$$

Since $\lim_{r\to\infty} \|\rho_{k_r}(a)\xi_r^i\| = \lambda_{n_K}$ $(1 \le i \le m)$ and $\|\rho_{k_r}(a)\eta_r\| > \frac{1}{2}$ eventually, there exists $R \ge 1$ such that, for $r \ge R$, $u_r^i = \rho_{k_r}(a)\xi_r^i/\|\rho_{k_r}(a)\xi_r^i\|$ $(1 \le i \le m)$ and $u_r^{m+1} = \rho_{k_r}(a)\eta_r/\|\rho_{k_r}(a)\eta_r\|$ are well-defined unit vectors for which (using (9) and (10))

$$\lim_{r\to\infty} \langle u_r^i, u_r^j \rangle = 0 \qquad (1 \le i < j \le m)$$

and

$$\lim_{r\to\infty} \langle u_r^i, u_r^{m+1} \rangle = 0 \qquad (1 \le i \le m).$$

It follows from Lemma 6 that, for r sufficiently large, the set $\{u_r^1, \ldots, u_r^{m+1}\}$ is linearly independent. This contradicts the fact that $\operatorname{rank}(\rho_k(a)) \le m$ for all $k \ge L$. □

ACKNOWLEDGEMENTS. The research reported in this paper was done while Aldo Lazar visited the Department of Mathematical Sciences of the University of Aberdeen. He wishes to thank Professor R. Archbold for his kind hospitality.

REFERENCES

1. C. A. Akemann, Left ideal structure of C^*-algebras, *J. Funct. Anal.* **6** (1970), 305–317.

2. C. A. Akemann, G. K. Pedersen and J. Tomiyama, Multipliers of C^*-algebras, *J. Funct. Anal.* **13** (1973), 277–301.

3. R. J. Archbold, Limits of pure states, *Proc. Edinburgh Math. Soc.(2)* **32** (1989), 249–254.

4. R. J. Archbold, Upper and lower multiplicity for irreducible representations of C^*-algebras, *Proc. London Math. Soc. (3)* **69** (1994), 121–143.

5. R. J. Archbold and E. Kaniuth, Upper and lower multiplicity for irreducible representations of SIN-groups, *Illinois J. Math.* **43** (1999), 692–706.

6. R. J. Archbold and D. W. B. Somerset, Quasi-standard C^*-algebras, *Math. Proc. Camb. Phil. Soc.* **107** (1990), 349–360.

7. R. J. Archbold, D. W. B. Somerset and J. S. Spielberg, Upper multiplicity and bounded trace ideals in C^*-algebras, *J. Funct. Anal.* **146** (1997), 430–463.

8. R. J. Archbold and J. S. Spielberg, Upper and lower multiplicity for irreducible representations of C^*-algebras, II, *J. Operator Theory* **36** (1996), 201–231.

9. J. Dauns and K. H. Hofmann, Representations of rings by continuous sections, *Mem. Amer. Math. Soc.* **83** (1968).

10. J. Dixmier, Points séparés dans le spectre d'une C^*-algèbre, *Acta Sci. Math. (Szeged)* **22** (1961), 115–128.

11. J. Dixmier, *C^*-algebras* (North-Holland, 1977).

12. J. Glimm, A Stone-Weierstrass theorem for C^*-algebras, *Ann. of Math.* **72** (1960), 216–244.

13. E. Kaniuth, G. Schlichting and K. F. Taylor, Minimal primal and Glimm ideal spaces of group C^*-algebras, *J. Funct. Anal.* **130** (1995), 43–76.

14. D. Milicic, On C^*-algebras with bounded trace, *Glasnik Mat.* **8** (1973), 7–21.

15. F. Perdrizet, Topologie et traces sur les C^*-algèbres, *Bull. Soc. Math. France* **99** (1971), 193–239.

16. J. R. Ringrose, *Compact non-self-adjoint operators* (Van Nostrand, 1971).