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On combinatorics of the Arthur trace formula, convex polytopes, and toric varieties

Mahdi Asgari and Kiumars Kaveh

Abstract. We explicate the combinatorial/geometric ingredients of Arthur's proof of the convergence and polynomiality, in a truncation parameter, of his noninvariant trace formula. Starting with a fan in a real, finite dimensional, vector space and a collection of functions, one for each cone in the fan, we introduce a combinatorial truncated function with respect to a polytope normal to the fan and prove the analogues of Arthur's results on the convergence and polynomiality of the integral of this truncated function over the vector space. The convergence statements clarify the important role of certain combinatorial subsets that appear in Arthur's work and provide a crucial partition that amounts to a so-called nearest face partition. The polynomiality statements can be thought of as far reaching extensions of the Ehrhart polynomial. Our proof of polynomiality relies on the Lawrence–Varchenko conical decomposition and readily implies an extension of the well-known combinatorial lemma of Langlands. The Khovanskii–Pukhlikov virtual polytopes are an important ingredient here. Finally, we give some geometric interpretations of our combinatorial truncation on toric varieties as a measure and a Lefschetz number.

1 Introduction

The Arthur Trace Formula (ATF) is a vast generalization of the Selberg Trace Formula to arbitrary rank reductive groups. The first incarnation of ATF, the noninvariant trace formula, relies on two crucial ingredients: the integral of a *truncated* kernel (of a compactly supported test function) is absolutely convergent, and the integral depends polynomially on the truncation parameter (which he has to assume is sufficiently regular). The purpose of this work is to prove two general, purely combinatorial, statements about polytopes, one on convergence and the other on polynomiality of certain integrals. These statements essentially capture, and generalize, the combinatorial aspects of Arthur's corresponding results (cf. [Ar78, Ar81]), isolating them from the analytic aspects that use reduction theory and other techniques. The long-term hope for our project, of which this work is a first step, is to aim at applications of the ATF to more general test functions [FL11, FL16, FLM11, Hoff08].

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We also give interpretations of our combinatorial results in terms of the geometry of toric varieties. We hope the present paper would shed light on the combinatorics behind ATF and its similarity with certain concepts appearing in toric geometry. The connection between polyhedral combinatorics appearing in Arthur's trace formula and in toric varieties is not quite transparent yet. In this regard, we mention the articles of Kottwitz [Kot05] and Finis and Lapid [FL11], which may be relevant.

We now briefly recall the trace formula before explaining a summary of our results and proofs.

1.1 Arthur's noninvariant trace formula

For a finite group G, the character of a representation of G (or any conjugation invariant function on G for that matter) can be written uniquely as a linear combination of characteristic functions of different conjugacy classes, as well as, a linear combination of traces of irreducible representations. The equality of these two decompositions is a special case of the Frobenius Reciprocity, which plays an important role in representation theory of finite groups. This is the prototype of many trace formulas in representation theory.

Arthur gave a far reaching trace formula for arbitrary reductive groups defined over number fields. A main problem is that in this generality, the integral representing the trace diverges. Arthur introduces an operation of *truncation* to modify this integral so that it becomes convergent.

The (noninvariant) ATF is an equality of two distributions:

$$J_{\mathrm{geom}}(f) = J_{\mathrm{spec}}(f), \quad f \in C_c^\infty(G(\mathbb{A})^1). \tag{1.1}$$

Here, G is a connected reductive linear algebraic group defined over $\mathbb{Q}$ (or any number field) whose ring of adeles we denote by $\mathbb{A}$ and $G(\mathbb{A})^1$ consists of those $x \in G(\mathbb{A})$ satisfying $|\chi(x)|_{\mathbb{A}} = 1$ for all rational characters χ of G. Both the *geometric* and the *spectral* distributions on the two sides of (1.1) are equal to the integral over $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ of a *modified kernel* $k^T(x) = k^T(x, f)$ at a certain value $T = T_0$ of a suitably regular *truncation parameter* T belonging to the positive Weyl chamber of G with respect to a fixed minimal parabolic subgroup. The space $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is in general finite volume (with respect to the Haar measure on G), but only compact when G has no proper parabolic subgroups. While the trace formula in the case of compact quotient was well understood, already the development of the trace formula in the case of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})$ led Selberg to his celebrated Selberg Trace Formula. However, Arthur realized that the presence of proper parabolic subgroups in a more general group G makes the integral of the kernel function divergent. As a result, he introduced the modified kernel $k^T(x)$. Two major properties of the modified kernel (see [Ar78, Ar81]) are the following:

(1) $\displaystyle\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} |k^T(x)|\, dx < \infty$ for suitably regular truncation parameter T.

(2) The function $T \mapsto J^T(f) = \displaystyle\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1} k^T(x)\, dx$ is a polynomial function of T.

As the truncation parameter T goes further away from the origin, the integral of $k^T(x)$ gets closer to the (divergent) integral representing the trace. Among other things,

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the proofs involve quite intricate combinatorics of convex polytopes and convex cones. Expanding the modified kernel geometrically (via conjugacy classes) and spectrally (via automorphic representations) then provides the two sides of (the truncated analogue of) the identity (1.1).

In the function field case, one also has an analogue of the ATF and the truncation parameter T. In particular, we mention the work of Laumon [Lau96, Lau97] where he develops the trace formula for certain class of test functions for which the modified kernel $k^T(\cdot)$ turns out to be equal to the usual kernel $k(\cdot)$. This makes the question of polynomiality obvious since the resulting polynomials would simply be constant. However, the convergence question still remains and indeed a similar argument as Arthur's in the number field case applies.

1.2 Main results

We introduce a notion of *combinatorial truncation* and prove two main results on its convergence and polynomiality. The idea for our results is to start with a complex-valued function on a finite dimensional real vector space whose integral over the vector space is possibly divergent. We then "truncate" this function by subtracting some other functions around some neighborhoods of infinity to arrive at a "truncated function" whose integral over the vector space is absolutely convergent. The "neighborhoods of infinity" are with respect to a toric compactification of V (in the sense of Sections 5.1 and 5.3) whose data are encoded in a polytope and its normal fan. We then prove that the integral of the truncated function, as a function of the polytope, is indeed a polynomial function.

To explain our results, we introduce some notation and refer to Sections 2.1 and 2.2 for further details on convex cones and polytopes. We first explain our convergence results.

Let $V \cong \mathbb{R}^n$ be an n-dimensional real vector space. We fix an inner product $\langle \cdot, \cdot \rangle$ on V and use it to identify V with its dual V^*. Fix a full dimensional, complete, simplicial fan Σ in V and fix a polytope $\Delta \in \mathcal{P}(\Sigma)$, the set of polytopes with normal fan Σ (see Figure 1). There is a one-to-one correspondence between the cones in Σ and the faces of Δ. For $\sigma \in \Sigma$, we let $T^-_{\Delta,\sigma}$ denote the outward-looking tangent cone of Δ at the face corresponding to σ (see Section 2.2 and Figures 3 and 4).

Suppose a function $K_0 : V \to \mathbb{C}$ is given with $\int_V K_0(x)\,dx$ possibly divergent. In fact, let K_0 be a member of a collection of functions $K_\sigma : V \to \mathbb{C}$, one for each $\sigma \in \Sigma$. We will assume that K_σ is invariant in the direction of $\mathrm{Span}(\sigma)$, i.e., $K_\sigma(x+y) = K_\sigma(x)$ for $x \in V$ and $y \in \mathrm{Span}(\sigma)$.

Associated with the collection $(K_\sigma)_{\sigma\in\Sigma}$ and the polytope Δ, we define the *truncated function*

$$k_\Delta(x) = \sum_{\sigma\in\Sigma} (-1)^{\dim\sigma} K_\sigma(x) \mathbf{1}_{T^-_{\Delta,\sigma}(x)}, \tag{1.2}$$

where $\mathbf{1}$ denotes the characteristic function. We think of $k_\Delta(x)$ as a "truncation" of K_0 by means of the polytope Δ and the functions K_σ for nonzero cones $\sigma \in \Sigma$.

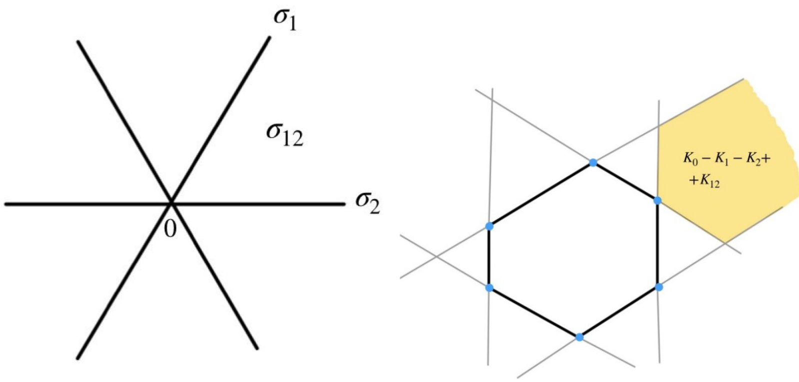

Figure 1: (Left) A complete simplicial fan in $V = \mathbb{R}^2$; we have labeled three cones in the fan. (Right) A polygon normal to the fan and regions obtained by drawing the outward face cones; the function k_Δ in the shaded region is given by $K_0 - K_1 - K_2 + K_{12}$.

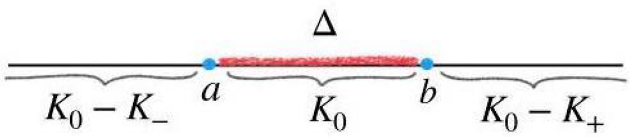

Figure 2: Illustration of the truncated function k_Δ for when Δ is a line segment.

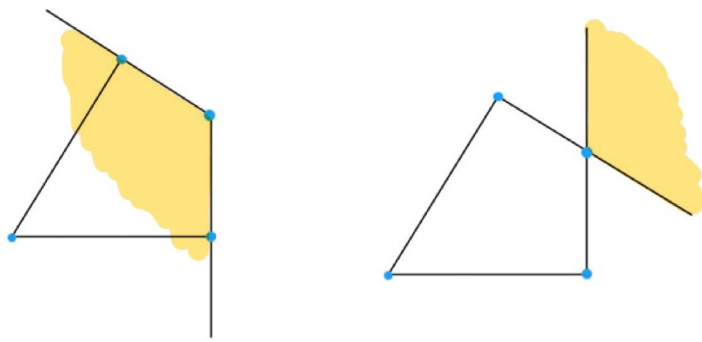

Figure 3: Inward and outward tangent cones at a vertex (left inward, right outward).

Note that the function $k_\Delta(x)$ and $K_0(x)$ coincide for $x \in \Delta$. In fact, if all the K_σ are identically equal to 1, by the classical Brianchon–Gram theorem (cf. Theorem 2.6), the function $k_\Delta(x)$ coincides with the characteristic function of Δ (see Section 1.4).

One of our main results gives a sufficient condition for $k_\Delta(x)$ to be absolutely integrable on V (see Theorem 3.4 and also Theorem 3.5).

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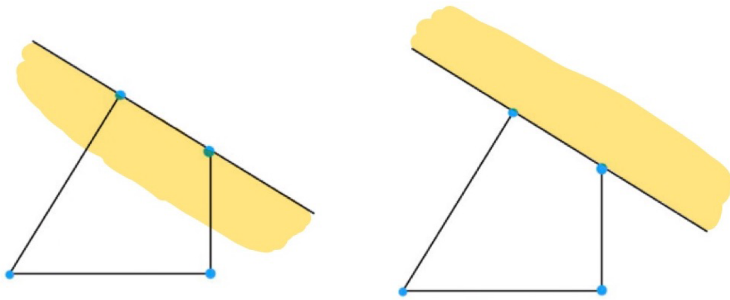

Figure 4: Inward and outward tangent cones at an edge (left inward, right outward).

For $\sigma_2 \preceq \sigma_1$ in Σ, let

$$K_{\sigma_1,\sigma_2}(x) = \sum_{\sigma_2 \preceq \tau \preceq \sigma_1} (-1)^{\dim \tau} K_\tau(x).$$

Also, let polyhedral regions $R_{\sigma_1}^{\sigma_2}$ and $S_{\sigma_1}^{\sigma_2}$ be as in Definition 3.2, i.e., $S_{\sigma_1}^{\sigma_2}$ is the cone in $\mathrm{Span}(\sigma_1)$ defined via the edge vectors and facet normals of σ_1 and σ_2 as in Definition 3.2(a) (or equivalently (3.10)) and $R_{\sigma_1}^{\sigma_2} = Q_{\sigma_1} + S_{\sigma_1}^{\sigma_2}$, where Q_{σ_1} is the face of Δ associated with the cone $\sigma \in \Sigma$.

Convergence *Assume that the fan Σ above is acute (cf. Definition 3.1). With the notation as above, suppose for any $\sigma_2 \preceq \sigma_1$, the function K_{σ_1,σ_2} is rapidly decreasing on the shifted neighborhoods of $S_{\sigma_2}^{\sigma_1}$. (See Theorem 3.5 for the precise definition.) Then for any polytope $\Delta \in \mathcal{P}(\Sigma)$, the integral*

$$J_\Sigma(\Delta) = \int_V k_\Delta(x)\, dx$$

is absolutely convergent.

We note that the conditions on K_{σ_1,σ_2} in the theorem are "local" with respect to the fan Σ in the sense that for each $\sigma \in \Sigma$, we only need to check a condition about σ and the functions K_τ, $\tau \preceq \sigma$ (and independent of other cones in the fan and their associated functions).

We also remark that the assumption that the fan Σ is acute is crucial; without it, the convergence result may fail as we show in Example 3.6 where we consider obtuse cones.

Next, we discuss our result on polynomiality. The set $\mathcal{P}(\Sigma)$ of polytopes with normal fan Σ is closed under multiplication by positive scalars and the Minkowski sum. Hence, it makes sense to talk about a polynomial function on $\mathcal{P}(\Sigma)$. In fact, if $\Sigma(1)$ denotes the set of one-dimensional cones in Σ, then a polytope $\Delta \in \mathcal{P}(\Sigma)$ has a unique representation as

$$\Delta = \{x \in V : \langle x, v_\rho \rangle \geqslant a_\rho, \forall \rho \in \Sigma(1)\},$$

where v_ρ denotes the unit vector along ρ. The numbers $(a_\rho)_{\rho \in \Sigma(1)}$ are called the *support numbers* of Δ and can be considered as coordinates on $\mathcal{P}(\Sigma)$ (see Section 2.3).

Our main polynomiality result (cf. Theorem 4.1) states that the integral of $k_\Delta(x)$ depends polynomially on $\Delta \in \mathcal{P}(\Sigma)$.

Polynomiality *The function*

$$\Delta \mapsto J_\Sigma(\Delta)$$

is a polynomial on $\mathcal{P}(\Sigma)$*, i.e., a polynomial in the support numbers of* Δ.

We remark that if all the K_σ are identically equal to 1, then $J_\Sigma(\Delta)$ coincides with the volume of Δ. Thus, our Polynomiality Theorem is a vast generalization of the classical fact that $\Delta \mapsto \mathrm{vol}(\Delta)$ is a polynomial function. The assumption that each K_σ is invariant in the direction of $\mathrm{Span}(\sigma)$ is obviously crucial in the proof of the Polynomiality Theorem. For example, one can consider examples where K_σ are not necessarily constant, but rather they are asymptotic to a constant in the direction of $\mathrm{Span}(\sigma)$. Then one can still have convergence of $J_\Sigma(\Delta)$ by our more general Theorem 3.4 on convergence, while $J_\Sigma(\Delta)$ would clearly not be a polynomial function.

The strategy to prove our Convergence Theorem is as follows. Recall that the truncated function $k_\Delta(x)$ in (1.2) is defined as an alternating sum over various outward tangent cones $T^-_{\Delta,\sigma}$. In Lemma 3.3, we prove a certain double partition of the tangent cones $T^-_{\Delta,\sigma}$ in terms of certain natural subsets that appear, associated with pairs of cones in Σ, with the smaller cone being a face of σ and the large one having σ as a face. In the double partition, the inner partition essentially amounts to the special case where σ is a full dimensional cone in Σ, whereas the outer partition amounts to a "nearest face partition" (cf. Section 2.4). This allows us to repackage the various terms appearing in k_Δ into a sum of certain alternating sums K_{σ_1,σ_2} associated with pairs of cones $\sigma_2 \preceq \sigma_1$ in Σ. As a result, we reduce the question of the absolute convergence of the integral of $k_\Delta(x)$ over V to that of absolute convergence of K_{σ_1,σ_2} on the sets we obtain out of the partition. This already gives our first, and more general, convergence result (cf. Theorem 3.4). We then go on to show that the two conditions in the above convergence theorem guarantee the convergence of the integral of K_{σ_1,σ_2} on the required sets.

The regions we mentioned above seem to show up naturally in any treatment of convergence results, including Arthur's original proof of convergence of his (non-invariant) trace formula. When σ_1 is full dimensional (corresponding to a maximal parabolic subgroup in Arthur's setting) and σ_2 is the origin, the region simply becomes the cone σ_1 shifted to the vertex of Δ corresponding to σ_1. When σ_2 is a nonzero face of σ_1, then the region is again another cone shifted to the vertex. This type of cone is precisely what Arthur has, for example, in [Ar05, Figure 8.5]. For more general σ_1, the regions are a sum (as a set) of a compact face of Δ and a somewhat simpler cone. For example, when $\dim V = 2$, these regions look like stripes.

A key step in the proof of the Polynomiality Theorem is Lemma 4.6, which is a statement concerning the polytope Δ and a cone $\sigma \in \Sigma$. As far as we know, this lemma is new and does not appear in Arthur's work. It simplifies and streamlines some of the combinatorial arguments in [Ar78, Ar81]. As a special case when $\Delta = \{0\}$, Lemma 4.6 also implies the *Langlands combinatorial lemma* (see [Ar05, equations (8.10) and (8.11)], [GKM97, Appendix]).

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When σ is full dimensional and the vertex of Δ corresponding to σ lies in σ, Lemma 4.6 gives a decomposition of the characteristic function of the polytope $\Delta \cap \sigma$ in terms of certain cones with apexes at the vertices of this polytope. We obtain Lemma 4.6 as a corollary of the Lawrence–Varchenko conical decomposition of a polytope (Theorem 2.8). In fact, we need a more general version of this decomposition that applies to virtual polytopes (Theorem 2.10). The arguments in this section rely on some key concepts and results from [KP93a, KP93b] (which we review in Section 2.6). We would like to point out that the proof of polynomiality shows that $J_\Sigma(\Delta)$ is a linear combination of volumes of certain virtual polytopes $\Gamma_{\Delta,\sigma}$, $\sigma \in \Sigma$.

In the interest of making the connections with poset theory and Möbius inversion more transparent, we show that the Langlands combinatorial lemma can be interpreted as a formula for the inverse of a certain element in the *incidence algebra* of the poset of faces of a polyhedral cone (see Corollary 4.7).

Finally, we point out that Arthur's truncation parameter T determines a polytope which is the convex hull of the Weyl group orbit of T. Thus, Arthur's combinatorics is concerned with Weyl group invariant polytopes with a vertex in each Weyl chamber. In this paper, we generalize the combinatorics to arbitrary simple polytopes.

It follows from the proof of polynomiality that

$$J_\Sigma(0) = \sum_{\sigma_2 \le \sigma_2, \dim \sigma_1 = n} \int_{S_{\sigma_1}^{\sigma_2}} K_{\sigma_1,\sigma_2}(x)dx,$$

and that, in the case of a Weyl fan Σ and a Weyl group invariant Δ, the top degree homogeneous part of the polynomial $J_\Sigma(\Delta)$ is a constant multiple of the volume of Δ.

1.3 The simplest example

Let Σ be the complete fan in $V = \mathbb{R}$ consisting of the origin $\sigma_0 = \{0\}$, the negative half-line σ_-, and the positive half-line σ_+. Let $\Delta \subset V^* \cong V = \mathbb{R}$ be the line segment $[a, b]$. Let K_0, K_-, and K_+ be functions on V corresponding to σ_0, σ_-, and σ_+, respectively. From definition, one computes that the truncated function $k_\Delta(x)$ is given by (see Figure 2).

$$k_\Delta(x) = \begin{cases} K_0 - K_-, & x < a, \\ K_0, & a \le x \le b, \\ K_0 - K_+, & x > b. \end{cases}$$

The assumption in Theorem 4.1 that K_σ is constant along $\mathrm{Span}(\sigma)$ implies that K_- and K_+ are constant functions. Moreover, the condition that $\int_V k_\Delta(x)dx$ is absolutely convergent means that $|K_0(x) - K_-|$ and $|K_0(x) - K_+|$ are integrable. We have

$$J_\Sigma(\Delta) = \int_{\mathbb{R}} k_\Delta(x)\, dx = \int_{-\infty}^{0} (K_0(x) - K_-)\, dx + \int_0^\infty (K_0(x) - K_+)\, dx + \int_a^0 K_-\, dx + \int_0^b K_+\, dx.$$

Note that $\int_{-\infty}^0 (K_0(x) - K_-)\, dx$ and $\int_0^\infty (K_0(x) - K_+)\, dx$ are constants independent of a and b (whose sum we denote by the constant c) and K_- and K_+ are constants. Hence, $J_\Sigma(\Delta) = c + (-a)\, K_- + b\, K_+$, a polynomial of degree 1 in a and b.

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It is easy to see that if K_+ or K_- is not a constant function, then the resulting $J_\Sigma(a,b)$ may not be a polynomial in a and b. For example let $K_0 = K_+ = K_- = e^x$. Then $K_0 - K_+ = K_0 - K_+ - = 0$, so the conditions of convergence are satisfied, and, in fact, we have $J_\Sigma(a,b) = \int_a^b e^x dx = e^b - e^a$, which is clearly not a polynomial in a and b.

1.4 Another simple example: Brianchon–Gram

If $K_\sigma \equiv 1$ for all the cones σ, then k_Δ becomes the characteristic function of the polytope Δ by the Brianchon–Gram theorem (cf. Theorem 2.6), and, as we mentioned earlier, our polynomiality result recovers the fact that the volume function $\Delta \mapsto \mathrm{vol}(\Delta)$ is a polynomial function. See Example 4.3 for details.

1.5 Discrete versions of the results

Replacing integration with summation, we obtain discrete versions of the above theorems. Given free abelian groups M and N of rank n with a perfect $\mathbb{Z}$-pairing to identify them, we let $V = N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $V^* = M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$. Then V and V^* are a pair of dual n-dimensional real vector spaces as above.

We take a fan Σ in $V = N_\mathbb{R}$ which is *rational*, i.e., all its cones are generated by rational vectors with respect to $N \subset N_\mathbb{R}$. We denote by $\mathcal{P}(\Sigma, M)$ the set of polytopes with normal fan Σ whose vertices lie in M. It is closed under the Minkowski sum. The discrete version of our convergence and polynomiality results (cf. Theorems 3.8 and 4.2) are as follows.

Convergence, discrete version *With notation as above, suppose that for any $\sigma_2 \le \sigma_1$ in Σ, the function K_{σ_1,σ_2} is rapidly decreasing on any shifted neighborhood of the cone $S_{\sigma_2}^{\sigma_1}$. Then for any polytope $\Delta \in \mathcal{P}(\Sigma, M)$, the series*

$$S_\Sigma(\Delta, M) = \sum_{x \in M} k_\Delta(x) dx$$

is absolutely convergent.

Polynomiality, discrete version *The function*

$$\Delta \mapsto S_\Sigma(\Delta, M)$$

is a polynomial on $\mathcal{P}(\Sigma, M)$.

We remark that if $K_\sigma \equiv 1$ for all nonzero cones σ in Σ, then $S_\Sigma(\Delta, M)$ coincides with the number of lattice points in Δ. Thus, the above theorem is a far reaching generalization of the classical fact that $\Delta \mapsto |\Delta \cap M|$ is a polynomial function (Ehrhart polynomial; see Theorem 2.2). It is interesting to explore whether some well-known polynomials appearing in combinatorics and representation theory, e.g., in the theory of symmetric polynomials, are instances of the polynomial $J_\Sigma(\Delta)$ or $S_\Sigma(\Delta, M)$.

1.6 Relation with toric varieties

Convex lattice polytopes are well studied in combinatorial algebraic geometry in relation to the geometry of toric varieties. In particular, there is a dictionary between algebraic geometric notions on toric varieties and convex geometric notions about lattice polytopes (see [CLS11, Fu93]). For example, the Riemann–Roch theorem for toric varieties gives beautiful formulas relating the number of lattice points in a polytope and its volume as well as volumes of its faces (see [[BV97], KP93a, KP93b]).

A complete (rational) fan Σ in $N_{\mathbb{R}}$ determines a complete toric variety X_Σ over $\mathbb{C}$. It is an equivariant compactification of the algebraic torus $T_N \cong (\mathbb{C}^*)^n$. The polytope $\Delta \in \mathcal{P}(\Sigma)$ determines a T_N-linearized ample line bundle $\mathcal{L}_\Delta$ on X_Σ (see Section 5).

In Section 5.2, we recall the well-known fact that the Brianchon–Gram theorem can be regarded as the computation of the equivariant Euler characteristic of an ample toric line bundle.

In Section 6, we give two interpretations of the function $k_\Delta(x)$ in terms of the toric variety X_Σ. In Section 6.1, we interpret it as a "truncated" measure on the toric variety X_Σ obtained by truncating a measure ω_0 on the open torus orbit $X_0 \subset X_\Sigma$ using the measures ω_σ on the torus orbits $O_\sigma \subset X_\Sigma$ (at infinity). Each tangent cone $T^-_{\Delta,\sigma}$ determines an open neighborhood $\tilde{U}_{\Delta,\sigma}$ of the torus orbit closure $\overline{O}_\sigma$. The interpretation of the tangent cones $T^-_{\Delta,\sigma}$ as neighborhoods $\tilde{U}_{\Delta,\sigma}$ justifies the assumption that the fan is acute: under the acute assumption, the neighborhood $\tilde{U}_{\Delta,\sigma}$ contains the orbit closure $\overline{O}_\sigma$.

In Section 6.2, we observe that computation of equivariant Euler characteristic of an ample toric line bundle has uncanny resemblances to the definition of truncated function $k_\Delta(x)$ and hence to Arthur's construction of the modified kernel $k^T(x)$. This leads to an interpretation of our combinatorial truncation as a Lefschetz number for computing the trace of the induced linear map of a morphism on the sheaf cohomologies of a toric variety.

We point out that the similarity between the definition of $k^T(x)$ and the Brianchon–Gram theorem about polytopes has been observed by Casselman in [Cass04].

The polynomiality of the number of lattice points in a polytope is related to the polynomiality of the Euler characteristic which is an immediate consequence of the Riemann–Roch theorem. From this point of view, it is probable that our Polynomiality Theorem (Theorem 4.2) is a special case of a more general Riemann–Roch-type theorem.

1.7 Relation with Arthur's work

As we mentioned above, Arthur's development of his noninvariant trace formula is based on the two crucial results that the integral of $k^T(x) = k^T(x, f)$ on $G(\mathbb{Q})\backslash G(\mathbb{A})^1$ is absolutely convergent for $T \in \mathfrak{a}_P^+$ sufficiently regular and $f \in C_c^\infty\left(G(\mathbb{A})^1\right)$ and it is a polynomial of T. We recall that

$$k^T(x, f) = \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(\mathbb{Q})\backslash G(\mathbb{Q})} K_P(\delta x, \delta x)\, \widehat{\tau}_P(H_P(\delta x) - T). \tag{1.3}$$

Here, the outer sum is over the standard parabolic subgroups P of G (containing a fixed minimal parabolic subgroup P_0), $H_P : G(\mathbb{A}) \longrightarrow \mathfrak{a}_P$ is the Harish–Chandra map, and $\widehat{\tau}_P(\cdot)$ is the characteristic function of $\left\{t \in \mathfrak{a}_P : \varpi(t) > 0, \varpi \in \widehat{\Delta}_P\right\}$, where $\widehat{\Delta}_P$ consists of weights ϖ_α for simple roots α corresponding to P. (We refer to [Ar05] for any unexplained notation.)

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If we take Σ to be the Weyl fan of the group G, then the parabolic subgroups of G correspond to the cones in Σ and the choice of a minimal parabolic subgroup corresponds to a choice of a full dimensional cone in Σ with the standard parabolic subgroups corresponding to the faces of this full dimensional cone. The other cones in Σ then correspond to the Weyl conjugates of the standard parabolic subgroups, and this correspondence between cones and parabolic subgroups is order reversing with respect to inclusion.

The Weyl fan Σ is a full dimensional, complete, simplicial fan that satisfies the acute assumption. The toric variety X_Σ of the fan Σ is a compactification of an algebraic torus by adding strata (orbits) at infinity for each cone $\sigma \in \Sigma$. The combinatorial truncation is an alternating sum of the K_σ times the characteristic functions of certain neighborhoods of the strata at infinity.

Similarly, one has a compactification (Mumford's *toroidal compactification*) of a reductive group G by adding strata X_P at infinity corresponding to rational parabolic subgroups P (see [KKMS73, Chapter IV, Section 1]). Arthur's truncation can be interpreted as an alternating sum of the K_P times characteristic functions of certain neighborhoods of the strata X_P at infinity.

The similarity between (1.2) and (1.3) is clear. This suggests that there is a corresponding family of functions $(K_\sigma)_{\sigma\in\Sigma}$ defined using the K_P functions. We believe that our combinatorial arguments, or a variant thereof, can be used to give convergence and polynomiality results of Arthur as follows. One would use the analytic arguments already in Arthur's work to verify the assumptions of (the variant of) our convergence and polynomiality theorems. As a consequence, one would recover Arthur's results making the combinatorial/geometric ingredients of his proofs more streamlined, at least in our view.

We expect that one can extend the geometric interpretations of truncation (e.g., as a Lefschetz number) in Section 6 to Arthur's setup by replacing the toric variety X_Σ by Mumford's toroidal compactification of a reductive algebraic group G. We hope to write the details, using Reduction Theory, in our next paper on this subject.

2 Preliminaries

We review some basic notions from the theory of polyhedral cones and fix some notations along the way. We refer to [CLS11, Section 1.2] for further details.

2.1 Cones and fans

Let V be a finite dimensional real vector space of dimension n, and let V^* denote its dual. Recall that a *(closed convex) polyhedral cone* in V is a set of the form

$$\sigma = \mathrm{Cone}(W) = \left\{ \sum_{w\in W} a_w w : a_w \geqslant 0 \right\} \subseteq V$$

with W a finite subset of V. Equivalently, there is a finite subset B of V^* such that

$$\sigma = \bigcap_{b\in B} \{x \in V : b(x) \geqslant 0\}.$$

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We say that σ is *generated* by W. Also, we write $\mathrm{Cone}(\varnothing) = \{0\}$. The dimension of σ is the dimension of its linear span. The dual cone $\sigma^\vee$ is defined as

$$\sigma^\vee := \{y \in V^* : y(x) \geqslant 0 \text{ for all } x \in \sigma\}.$$

Dual cones enjoy the property that if σ is a polyhedral cone in V, then $\sigma^\vee$ is a polyhedral cone in V^* and $\sigma^{\vee\vee} = \sigma$.

For a face τ of σ (denoted $\tau \leq \sigma$), define its dual face

$$\begin{aligned}\tau^* &= \{y \in \sigma^\vee : y(x) = 0 \text{ for all } x \in \tau\} \\ &= \sigma^\vee \cap \tau^\perp.\end{aligned}$$

Then τ^* is a face of $\sigma^\vee$, $\tau^{**} = \tau$, $\tau \leftrightarrow \tau^*$ is an inclusion-reversing bijection between faces of σ and those of $\sigma^\vee$, and $\dim \tau + \dim \tau^* = n$. One-dimensional cones, i.e., half-lines, are called *rays*. A face τ of σ is called a *facet* if $\dim \tau = \dim \sigma - 1$, and its linear span is referred to as a *wall* of σ. An *edge* is a face of dimension 1.

Define the *relative interior* σ° of σ to be the interior of σ in its span. One then checks that $x \in \sigma^\circ$ if and only if $y(x) > 0$ for all $y \in \sigma^\vee \backslash \sigma^\perp$. A polyhedral cone σ in V is *strongly convex* if the origin is a face. This is the case if and only if σ contains no positive dimensional subspace of V if and only if $\sigma \cap (-\sigma) = \{0\}$ if and only if $\dim \sigma^\vee = n$. A strongly convex polyhedral cone $\sigma \subseteq V$ is called *simplicial* if it is generated by linearly independent vectors. We note that the dual of a simplicial cone of maximal dimension is again simplicial.

For $y \in V^*$, we set

$$H_y := \{x \in V : y(x) = 0\} \subseteq V$$

and define the closed (resp. open) spaces

$$H_y^+ := \{x \in V : y(x) \geqslant 0\} \subseteq V \quad \text{and} \quad H_y^- := \{x \in V : y(x) < 0\} \subseteq V.$$

When $y \neq 0$, H_y is a hyperplane and H_y^+ and H_y^- are half-spaces in V. When $y = 0$, we have $H_y = H_y^+ = V$ while H_y^- is empty. If $\sigma \subseteq H_y^+$ for $y \neq 0$, we say H_y is a *supporting hyperplane* and H_y^+ (resp. H_y^-) is an *inward (resp. outward) supporting half-space* of σ. (When $y = 0$, we automatically have $\sigma \subseteq H_0^+ = H_0 = V$.) Note that H_y is a supporting hyperplane of σ if and only if $y \in \sigma^\vee \backslash \{0\}$. If $y_1, y_2, \dots, y_r$ generate $\sigma^\vee$, then $\sigma = H_{y_1}^+ \cap \dots \cap H_{y_r}^+$. Thus, every polyhedral cone is an intersection of finitely many closed half-spaces.

A *fan* Σ in V is a finite collection of cones $\sigma \subseteq V$ satisfying the following three properties: (a) every $\sigma \in \Sigma$ is a strongly convex polyhedral cone, (b) for all $\sigma \in \Sigma$, each face of σ also belongs to Σ, and (c) for all $\sigma_1, \sigma_2 \in \Sigma$, the intersection $\sigma_1 \cap \sigma_2$ is a face of each. The set of r-dimensional cones of Σ is denoted by $\Sigma(r)$. The *support* of Σ is defined by

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subseteq V.$$

If $|\Sigma| = V$, then Σ is called a *complete fan*. A *simplicial fan* is a fan all whose cones are simplicial. Every fan can be refined into a simplicial fan.

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Finally, for $\sigma \in \Sigma$, we let Σ/σ denote the fan in $V/\operatorname{Span}(\sigma)$ consisting of all the images of the cones $\sigma' \geq \sigma$. If we fix an inner product on V, then $V/\operatorname{Span}(\sigma)$ can be identified with $\sigma^\perp$ and Σ/σ consists of projections of $\sigma' \geq \sigma$ onto $\sigma^\perp$.

2.2 Polytopes

A *polytope* is a set in V^* of the form

$$P = \operatorname{Conv}(S) = \left\{ \sum_{u \in S} \lambda_u u : \lambda_u \geqslant 0, \sum_{u \in S} \lambda_u = 1 \right\},$$

where S is a finite subset of V^*. We say P is the *convex hull* of S. The dimension, $\dim P$, of a polytope P is the dimension of the smallest affine subspace of V^* containing P. Given $x \in V \backslash \{0\}$ and $r \in \mathbb{R}$, we have the affine hyperplane

$$H_{x,r} := \{ y \in V^* : y(x) = r \}$$

and the closed (resp. open) half-spaces

$$H^+_{x,r} := \{ y \in V^* : y(x) \geqslant r \} \quad \text{and} \quad H^-_{x,r} := \{ y \in V^* : y(x) < r \}.$$

A subset $Q \subseteq P$ is a *face* of P, denoted by $Q \preceq P$, if there is $x \in V \backslash \{0\}$ and there is $r \in \mathbb{R}$ with

$$Q = H_{x,r} \cap P \quad \text{and} \quad P \subseteq H^+_{x,r}.$$

We then say that $H_{x,r}$ is a *supporting affine hyperplane*. The polytope P is regarded as a face of itself and faces of P of dimensions 0, 1, and $(\dim P - 1)$ are called *vertices*, *edges*, and *facets*, respectively.

A polytope $P \subseteq V^*$ can be written as a finite intersection of closed half-spaces, and an intersection

$$P = \bigcap_{i=1}^{s} H^+_{x_i, r_i}$$

is a polytope provided that it is bounded. In general, an intersection of finitely many closed half-spaces is called a *polyhedron* and could be unbounded. When $\dim P = \dim V^*$ (i.e., full dimensional polytope) for each facet F, we have a *unique* supporting affine hyperplane and the corresponding closed half-space given by

$$H_F = H_{u_F^+, a_F} = \{ y \in V^* : y(u_F^+) = a_F \}$$

and

$$H_F^+ = H^+_{u_F^+, a_F} = \{ y \in V^* : y(u_F^+) \geqslant a_F \},$$

where $(u_F^+, a_F) \in V \times \mathbb{R}$ is unique up to multiplication by a positive real number. We call u_F^+ an *inward-pointing facet normal* of the facet F. Hence,

$$P = \bigcap_{F \text{ facet}} H_F^+ = \{ y \in V^* : y(u_F^+) \geqslant a_F \text{ for all proper facets } F \prec P \}. \tag{2.1}$$

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This is the so-called facet representation of P. We also have a similar representation with *outward-pointing facet normals* $u_F^- = -u_F^+$. When the facet normals $u_F^\pm$ are assumed to be unit vectors, we may call the a_F the *support numbers* of P.

Let Q be a face of P and define the *inward (resp. outward) tangent cone* $T_{P,Q}^+$ (resp. $T_{P,Q}^-$) via

$$T_{P,Q}^+ := \{y \in V^* : y(u_F^+) \geqslant a_F \text{ for all facets } F \supset Q\}, \tag{2.2}$$

$$\text{resp. } T_{P,Q}^- := \{y \in V^* : y(u_F^+) < a_F \text{ for all facets } F \supset Q\} \tag{2.3}$$
$$= \{y \in V^* : y(u_F^-) > a_F \text{ for all facets } F \supset Q\}.$$

See Figures 3 and 4 for illustrations of inward and outward tangent cones of a quadrilateral at a vertex and at an edge, respectively.

A polytope $P \subseteq V^*$ of dimension d is called a *d-simplex* (or just a *simplex*) if it has $d+1$ vertices, *simplicial* if every facet is a simplex, and *simple* if every vertex is the intersection of precisely d facets.

Given a polytope $P = \operatorname{Conv}(S)$, its multiple $rP = \operatorname{Conv}(rS)$ is also a polytope for any $r \geqslant 0$. The Minkowski sum $P_1 + P_2 = \{y_1 + y_2 : y_i \in P_i\}$ of two polytopes $P_1 = \operatorname{Conv}(S_1)$ and $P_2 = \operatorname{Conv}(S_2)$ is again a polytope, and we have the distributive law $rP + sP = (r+s)P$. The set $\mathcal{P}(V^*)$ of polytopes in V^* together with the Minkowski sum is a cancellative semigroup. The following theorem is originally due to Minkowski.

Theorem 2.1 (Volume polynomial) *The map $P \mapsto \operatorname{vol}_n(P)$ is a polynomial function on $\mathcal{P}(V^*)$ in the following sense: let $P_1, \dots, P_r$ be polytopes in V^*. For any $\lambda_1, \dots, \lambda_r \geqslant 0$, we can form the polytope $\sum_i \lambda_i P_i$. Then the function $(\lambda_1, \dots, \lambda_r) \mapsto \operatorname{vol}_n(\sum_i \lambda_i P_i)$ is the restriction of a homogeneous polynomial on $\mathbb{R}^r$ to the positive orthant $\mathbb{R}^r_{\geqslant 0}$.*

There is also a discrete analogue of Theorem 2.1 which is harder and more subtle to prove. It is a generalization of the notion of the Ehrhart polynomial. Let $M \cong \mathbb{Z}^n$ be a full rank lattice in $V^* \cong \mathbb{R}^n$. Let $\mathcal{P}(M)$ denote the collection of *lattice polytopes with respect to M*, that is, all polytopes in V^* whose vertices belong to M. The set $\mathcal{P}(M)$ is closed under the Minkowski sum and multiplication by positive integers.

Theorem 2.2 (Ehrhart polynomial) *The map $P \mapsto |P \cap M|$ is a polynomial map on $\mathcal{P}(M)$.*

More generally, the polynomiality property holds for any *valuation* (also called *finitely additive measure*). A function $\Phi : \mathcal{P}(M) \to \mathbb{R}_{\geqslant 0}$ is called a *valuation* if for all $P_1, P_2 \in \mathcal{P}(M)$, the following hold:

(1) Φ is monotone with respect to inclusion, i.e., $\Phi(P_1) \leq \Phi(P_2)$ provided that $P_1 \subset P_2$.
(2) $\Phi(P_1 \cup P_2) = \Phi(P_1) + \Phi(P_2) - \Phi(P_1 \cap P_2)$.

We say Φ is *$\mathbb{Z}^n$-invariant* if $\Phi(m + P) = \Phi(P)$ for all $P \in \mathcal{P}(M)$ and $m \in M$. The following is a beautiful result of McMullen [Mc77]. It generalizes Theorem 2.2.

Theorem 2.3 *Let Φ be a $\mathbb{Z}^n$-invariant valuation on $\mathcal{P}(M)$. Then Φ is a polynomial function.*

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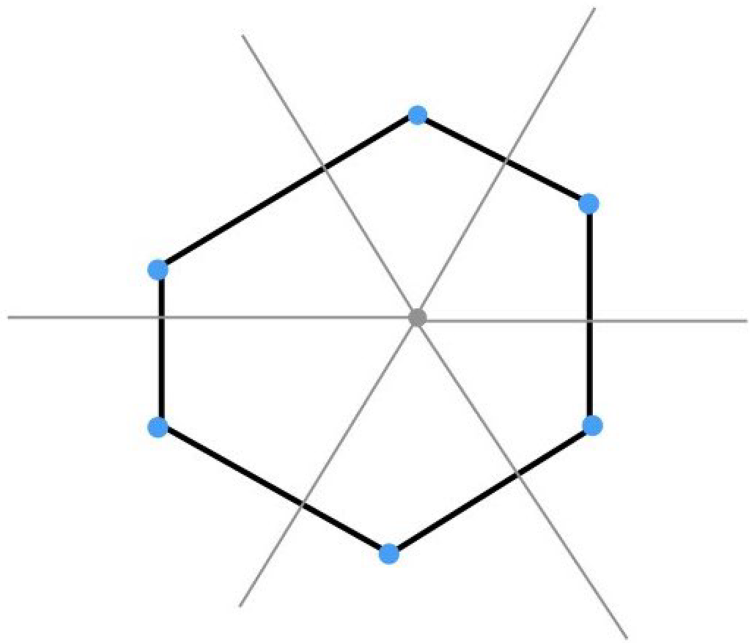

Figure 5: A polygon and its normal fan. Note that in our convention, we use *outward* facet normals to define the cones in the normal fan.

Remark 2.4 When $\Phi(P) = |P \cap M|$, one recovers Theorem 2.2. Fix a point $a \in V^*$. Theorem 2.3, in particular, implies that the function defined by $\Phi_a(P) = |P \cap (a + M)|$ is also a polynomial.

2.3 Normal fan

For $Q \preceq P$, let

$$\sigma_Q := \text{Cone}\left(u_F^- : \text{ facets } F \supset Q\right).$$

Given a full dimensional polytope $P \subseteq V^*$, the cones σ_Q fit together to form the *normal fan* of P in V given by

$$\Sigma_P = \{\sigma_Q : Q \preceq P\}.$$

Note that we have used *outward* facet normals u_F^- to define the normal fan (see Figure 5). (Some authors use inward facet normals u_F^+ instead.)

Let $\mathcal{P}(\Sigma)$ be the collection of all convex polytopes whose normal fan is Σ. This set is closed under the Minkowski sum of polytopes and multiplication by positive scalars. For $P \in \mathcal{P}(\Sigma)$, we have an inclusion-reversing bijection

$$Q = Q_\sigma \longleftrightarrow \sigma = \sigma_Q \tag{2.4}$$

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between the set of faces of P and the set of cones in the normal fan Σ. In particular, the facets F of P correspond to rays $\rho \in \Sigma(1)$. For a ray $\rho \in \Sigma(1)$, we set $a_\rho = a_F$, where F is the facet corresponding to ρ and a_F are the support numbers of P (see (2.1)). The map $P \mapsto (a_\rho)_{\rho\in\Sigma(1)}$ gives an embedding of $\mathcal{P}(\Sigma)$ into $\mathbb{R}^s$, where $s = |\Sigma(1)|$. The image is a full dimensional (open) convex polyhedral cone.

Let P be a full dimensional polytope with normal fan Σ_P. Let $Q \le P$ be a face with corresponding cone $\sigma_Q \in \Sigma_P$. Then the normal fan Σ_Q (of the polytope Q) is the fan Σ_P/σ_Q (defined at the end of Section 2.1). It consists of the images of the cones $\sigma' \ge \sigma_Q$ in the quotient vector space $V/\operatorname{Span}(\sigma_Q)$.

2.4 Nearest face partition

Fix an inner product $\langle\cdot,\cdot\rangle$ on V. Let $P \subset V$ be a convex polyhedron. To P, we can associate a partition of V into polyhedral regions V_P^Q as follows. For each face $Q \le P$, let

$$V_P^Q = \{x \in V : \text{the minimum distance from } x \text{ to } P \text{ is attained at a point in the relative interior of } Q\}.$$

The following is straightforward to verify.

Proposition 2.5 (1) *For each face $Q \le P$, the set V_P^Q is a polyhedron.*
(2) *We have a disjoint union*

$$V = \bigsqcup_{Q\le P} V_P^Q.$$

We can modify the V_P^Q to obtain a slightly different partition $\{W_P^Q : Q \le P\}$. For each face $Q \le P$, let

$$W_P^Q = \overline{V_P^Q} \setminus \left(\bigcup_{Q' \gneq Q} \overline{V_P^{Q'}} \right),$$

where $\overline{V_P^Q}$ denotes the closure of V_P^Q. The polyhedra W_P^Q and V_P^Q have the same relative interior, but they are different on the boundary.

We refer to both $\{V_P^Q : Q \le P\}$ and $\{W_P^Q : Q \le P\}$ as the *nearest face partition* of V with respect to the polyhedron P (see Figure 6). We note that if, in particular, $P = \sigma$ is a cone (with apex at the origin), then the closure of the parts in the partition with respect to σ in fact form a complete fan in V. In practice, we will also use the nearest face partition to partition a polyhedron inside V.

2.5 Conical decomposition theorems

We end this section by recalling two beautiful formulas which represent the characteristic function of a polytope as an alternating sum of characteristic functions of cones. For a nice overview of these decompositions and related topics, we refer the reader to [BHS09].

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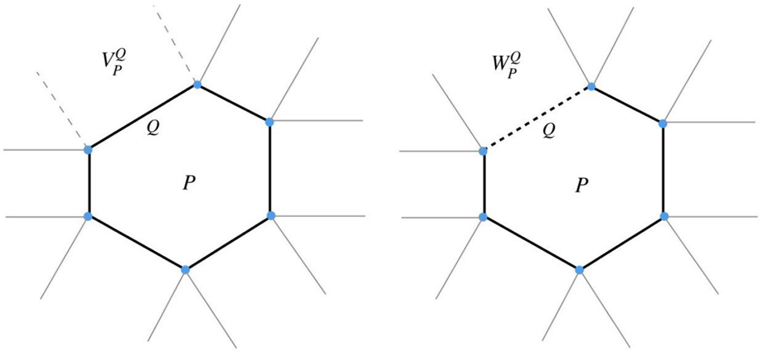

Figure 6: Nearest face partition for a polygon illustrating polyhedral regions V_P^Q and W_P^Q corresponding to an edge Q.

2.5.1 Brianchon–Gram theorem

The first conical decomposition theorem we discuss is the Brianchon–Gram theorem. It is named after Brianchon and Gram who independently proved the $n = 3$ case in 1837 and 1874, respectively ([B37, G1874]). It is the mother of all cone decompositions! See [Hass05, Section 1.1] and the references therein. Also, see [Ag06].

Theorem 2.6 (Brianchon–Gram) *Let P be a polytope in V^*. We have the following equality, where $\mathbf{1}$ denotes characteristic function:*

$$\mathbf{1}_P = \sum_{Q \preceq P} (-1)^{\dim Q} \mathbf{1}_{T^+_{P,Q}}. \tag{2.5}$$

Proof For a point $y \in P$, the right-hand side computes the Euler characteristic of P and hence is equal to 1 since P is contractible. For $y \notin P$, we have to subtract the Euler characteristic of the subcomplex that is visible from y which is again contractible. ∎

Alternatively, one can formulate Brianchon–Gram in terms of outward-looking tangent cones.

Theorem 2.7 (Brianchon–Gram, alternative version) *Let P be a polytope in V^*. We have the following equality:*

$$\mathbf{1}_P = \sum_{Q \preceq P} (-1)^{n-\dim Q} \mathbf{1}_{T^-_{P,Q}}. \tag{2.6}$$

The above version of the Brianchon–Gram formula looks similar to Arthur's definition of the modified kernel $k^T(x)$, as was observed in [Cass04]. See Figures 7 and 8 for illustrations of (2.5) and (2.6).

2.5.2 Lawrence–Varchenko theorem

The second conical decomposition due to Lawrence [Law91] and Varchenko [Vr87] represents the characteristic function of a polytope as an alternating sum of characteristic functions of certain cones associated with vertices of the polytope. It is a

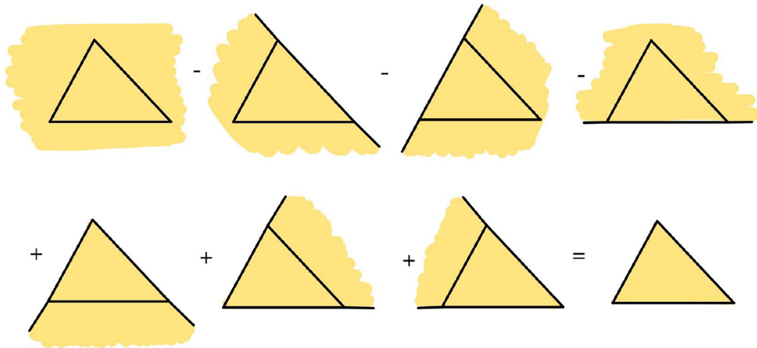

Figure 7: Illustration of the Brianchon–Gram theorem (inward-looking tangent cones) for a triangle.

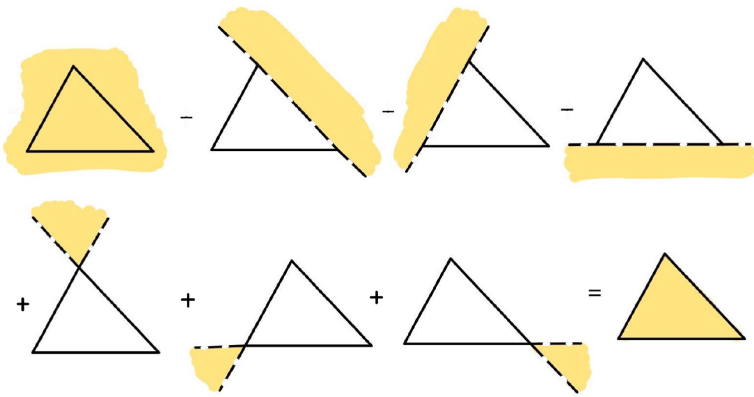

Figure 8: Illustration of the Brianchon–Gram theorem (alternative version, outward-looking tangent cones) for a triangle.

predecessor to the work of Khovanskii and Pukhlikov [KP93a, KP93b] and Brion and Vergne [Br88, BV97]. It is related to Morse theory on polytopes as well as equivariant cohomology of toric varieties. The Lawrence–Varchenko theorem follows immediately from Khovanskii–Pukhlikov results as well (see [KP93b, Section 3.2]).

Let $P \subset V$ be a simple polytope, and let v be a vertex of P. Let $w_1, \dots w_r$ be edge vectors of P at the vertex v. Fix a dual vector $\xi \in V^*$ such that $\langle w_i, \xi \rangle \neq 0$, for all i. We define vectors $w'_1, \dots, w'_r$ as follows:

$$w'_i = \begin{cases} w_i, & \text{if } \langle w_i, \xi \rangle > 0, \\ -w_i, & \text{otherwise.} \end{cases}$$

Finally, define the *polarized tangent cone* $T^{\xi}_{P,v}$ with apex at v by

$$T^{\xi}_{P,v} = \left\{ \sum_{i=1}^{r} \lambda_i w'_i : \begin{array}{l} \lambda_i \geqslant 0 \text{ if } w'_i = w_i \\ \lambda_i > 0 \text{ if } w'_i = -w_i \end{array} \right\}.$$

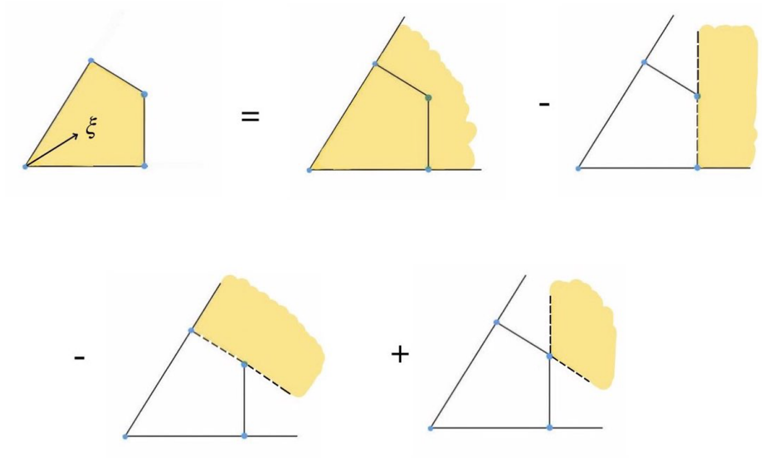

Figure 9: Illustration of the Lawrence–Varchenko theorem for a quadrangle.

Theorem 2.8 (Lawrence–Varchenko) *With notation as above, we have the following:*

$$\mathbf{1}_P = \sum_v (-1)^{n_v} \mathbf{1}_{T^{\xi}_{P,v}}, \tag{2.7}$$

where the sum is over all the vertices v of P, and $n_v = |\{i : w'_i = -w_i\}|$.

See Figure 9 for an illustration of (2.7).

2.6 Khovanskii–Pukhlikov virtual polytopes and convex chains

This is a summary of some ideas and results from [KP93a, KP93b] that we will need later. As before, $V \cong \mathbb{R}^n$ denotes an n-dimensional real vector space.

Recall that $\mathcal{P}(V^*)$ denotes the set of polytopes in the dual space V^*. The set $\mathcal{P}(V^*)$ is equipped with the operations of Minkowski sum and multiplication by positive scalars. One knows that $\mathcal{P}(V^*)$ together with the Minkowski sum is a cancellative semigroup and hence it can be extended to a real vector space $\mathcal{V}(V^*)$ consisting of formal differences $P_1 - P_2$, $P_i \in \mathcal{P}(V^*)$, where for polytopes P_1, P_2, P'_1, P'_2, we have $P_1 - P_2 = P'_1 - P'_2$ if and only if $P_1 + P'_2 = P'_1 + P_2$.

Definition 2.1 (Virtual polytope) The elements of $\mathcal{V}(V^*)$ are called *virtual polytopes* (see [KP93a]).

We note that $\mathcal{V}(V^*)$ is an infinite dimensional vector space.

Let Σ be a complete fan in V. Recall that $\mathcal{P}(\Sigma)$ denotes the set of all polytopes in V^* whose normal fan is Σ. The set $\mathcal{P}(\Sigma)$ is closed under the Minkowski sum and multiplication by positive scalars. We denote by $\mathcal{V}(\Sigma)$ the subspace of $\mathcal{V}(V^*)$ spanned by $\mathcal{P}(\Sigma)$. The elements of $\mathcal{V}(\Sigma)$ are called *virtual polytopes with normal fan* Σ. Generalizing the facet representation of a polytope $P \in \mathcal{P}(\Sigma)$, i.e., representation as an intersection of half-spaces $H^+_{u^+_\rho, a_\rho}$, $\rho \in \Sigma(1)$, each virtual polytope in $\mathcal{V}(\Sigma)$ is represented by a collection of oriented hyperplanes H_{u_ρ, a_ρ}, $\rho \in \Sigma(1)$. Note that any

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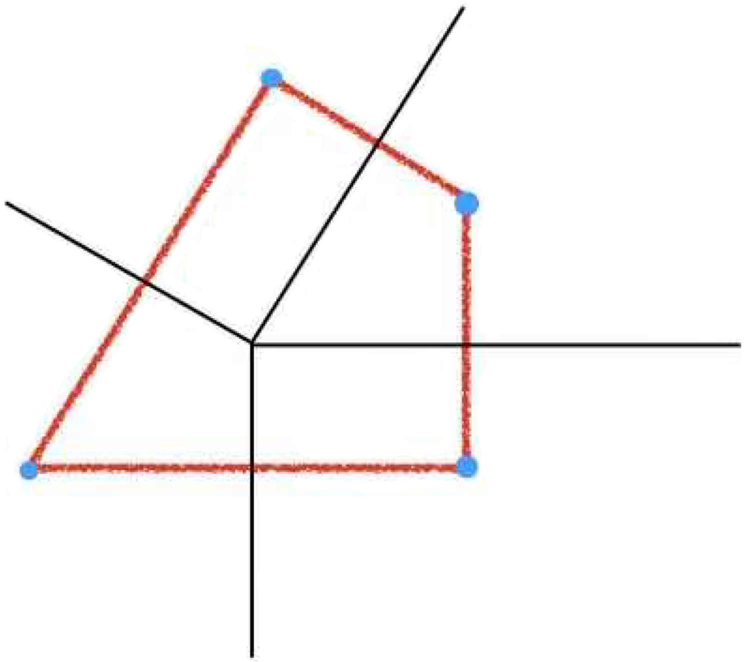

Figure 10: A usual quadrangle with its normal fan.

choice of the support numbers a_ρ yields a virtual polytope (even if the intersection of the corresponding half-spaces is empty). See Figures 10 and 11 for illustrations of a usual and virtual quadrangle with the same normal fan.

Remark 2.9 The notion of volume of a polytope extends to virtual polytopes via Theorem 2.1. For a virtual polytope $P \in \mathcal{V}(V^*)$, we defined $\mathrm{vol}_n(P)$ to be the value of the volume polynomial at P. Similarly, the notion of the number of lattice points in a polytope extends to virtual polytopes as well. Let $M \subset V^*$ be a full rank lattice. Let $\mathcal{V}(M)$ denote the collection of *lattice virtual polytopes with respect to* M, i.e., all virtual polytopes whose vertices are in M. In other words, $\mathcal{V}(M)$ is the subgroup of $\mathcal{V}(V^*)$ generated by lattice polytopes in $\mathcal{P}(M)$. By Theorem 2.2, there exists a (unique) polynomial F on $\mathcal{V}(V^*)$ such that for any lattice polytope $P \in \mathcal{P}(V^*)$, we have $F(P) = |P \cap M|$. For a virtual lattice polytope $P \in \mathcal{V}(M)$, we define the number of lattice points in P to be $F(P)$. The same applies to any valuation on the space of polytopes (see [KP93a]; see also Theorem 2.3 and the paragraph before it for the definition of a valuation).

Each polytope $P \in \mathcal{P}(V^*)$ is determined by its characteristic function $\mathbf{1}_P : V^* \to \{0,1\}$. We would like to extend the assignment $P \mapsto \mathbf{1}_P$ to virtual polytopes. The natural extension of the set of characteristic functions of convex polytopes (to a vector space) is the set of convex chains (defined by Khovanskii and Pukhlikov).

Definition 2.2 (Convex chain) A *convex chain* Z is a finite linear combination (with real coefficients) of characteristic functions of convex polytopes in V^*, that is,

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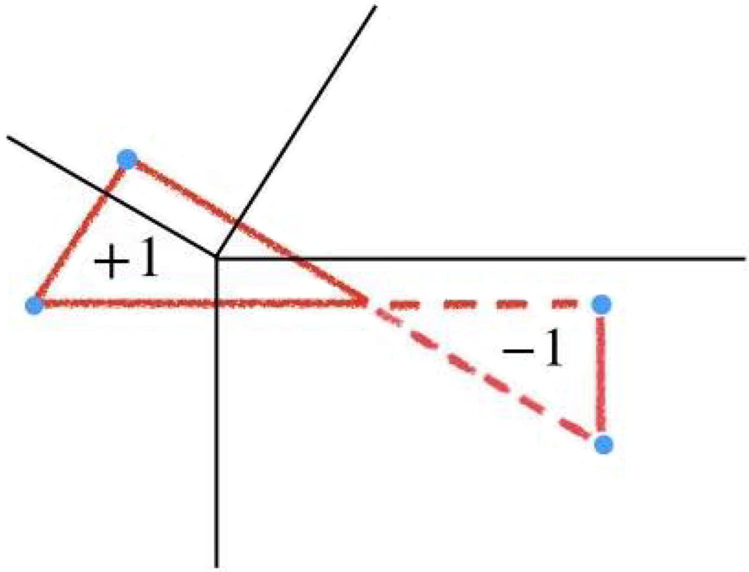

Figure 11: A virtual quadrangle with the same normal fan.

$Z = \sum_i \lambda_i \mathbf{1}_{\Delta_i}$, where the Δ_i are convex polytopes in V^* and $\lambda_i \in \mathbb{R}$. We denote the set of convex chains by $\mathcal{Z}(V^*)$. It is an infinite dimensional vector space with addition and scalar multiplication of functions.

Moreover, in general, one can consider the characteristic functions of convex polyhedral cones.

Definition 2.3 (Conical convex chain) A *conical convex chain* C is a finite linear combination (with real coefficients) of characteristic functions of shifted convex cones in V^*, that is, $C = \sum_i \lambda_i \mathbf{1}_{a_i + C_i}$, where the C_i are convex polyhedral cones in V^* (with apex at the origin), $a_i \in V^*$, and $\lambda_i \in \mathbb{R}$. We denote the set of convex conical chains by $\mathcal{CZ}(V^*)$.

A remarkable construction in [KP93a] is a "convolution" operation $*$ on $\mathcal{Z}(V^*)$ which makes it a commutative algebra (together with addition and scalar multiplication of functions). It has the property that for any two polytopes P_1 and P_2, we have

$$\mathbf{1}_{P_1} * \mathbf{1}_{P_2} = \mathbf{1}_{P_1 + P_2}.$$

In particular, the identity element for the $*$ operation is $\mathbf{1}_{\{0\}}$, the characteristic function of the origin.

For a polytope P, it is shown in [KP93a] that the inverse (with respect to $*$) of $\mathbf{1}_P$ is the convex chain $(-1)^{\dim P} \mathbf{1}_{P^\circ}$, where P° denotes the relative interior of P. In other words,

$$\mathbf{1}_P * (-1)^{\dim P} \mathbf{1}_{P^\circ} = \mathbf{1}_{\{0\}}.$$

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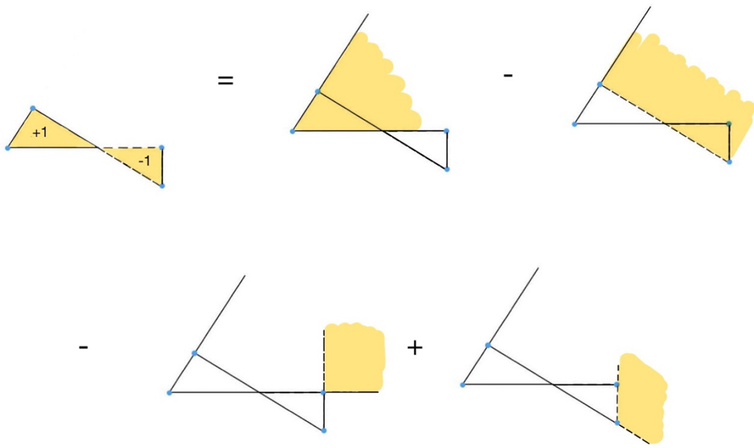

Figure 12: Illustration of the Lawrence–Varchenko theorem for a virtual quadrangle.

One verifies that

$$(-1)^{\dim P}\mathbf{1}_{P^\circ} = \sum_{Q \le P} (-1)^{\dim Q}\mathbf{1}_Q,$$

and hence $(-1)^{\dim P}\mathbf{1}_{P^\circ}$ is indeed a convex chain. It follows that

$$\iota : P_1 - P_2 \mapsto \mathbf{1}_{P_1} * (-1)^{\dim P_2}\mathbf{1}_{P_2^\circ} = \sum_{Q \le P_2} (-1)^{\dim Q}\mathbf{1}_{P_1+Q} \tag{2.8}$$

defines a natural embedding of the group of virtual polytopes (with the Minkowski sum) into the semigroup of convex chains (with convolution $*$). We refer to the right-hand side of (2.8) as the *convex chain associated with* the *characteristic function of* the virtual polytope $P_1 - P_2$. In fact, it is shown in [KP93a] that the image of ι coincides with the set of $*$-invertible convex chains.

We can talk about vertices of a virtual polytope. For a virtual polytope $P \in \mathcal{V}(\Sigma)$, the vertices are in one-to-one correspondence with the full dimensional cones in Σ. Similarly, the notion of a tangent cone of a polytope extends to virtual polytopes. The tangent cones of $P \in \mathcal{V}(\Sigma)$ are in one-to-one correspondence with $\sigma \in \Sigma$.

There is a generalization of the Brianchon–Gram theorem to convex chains (see [KP93a, Section 4, Proposition 2]). The Lawrence–Varchenko theorem also extends to simple virtual polytopes.

Theorem 2.10 (Lawrence–Varchenko for virtual polytopes) *Let P be a virtual polytope in V^*, and let $\pi : V^* \to \mathbb{R}$ be the corresponding convex chain. Then*

$$\pi = \sum_v (-1)^{n_v}\mathbf{1}_{T^\xi_{P,v}}, \tag{2.9}$$

where the sum is over all the vertices v of P and $T^\xi_{P,v}$ and n_v are as in Theorem 2.8.

See Figure 12 for an illustration of (2.9).

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2.7 Incidence algebra of a poset and Möbius inversion

For a nice reference about incidence algebra and Möbius inversion, see [St12, Sections 3.6 and 3.7]. Let $\mathcal{P}$ be a finite poset with partial order $<$. Let R be a commutative ring with 1 which we take as the ring of scalars. Let $\tilde{\mathcal{P}} = \{(\tau, \sigma) : \tau \leq \sigma\} \subset \mathcal{P} \times \mathcal{P}$ be the collection of all intervals in $\mathcal{P}$. Let $I(\mathcal{P}) = \{F : \tilde{\mathcal{P}} \to R\}$ be the set of functions from $\tilde{\mathcal{P}}$ to R. Clearly, $I = I(\mathcal{P})$ is an abelian group with addition of functions. One defines a *convolution* operation $*$ on I as follows. For $F, G \in I$ define $F * G \in I$ by

$$(F * G)(\tau, \sigma) = \sum_{\tau \leq \tau' \leq \sigma} F(\tau, \tau')G(\tau', \sigma).$$

It can be verified that $(I, +, *)$ is an algebra over R, called *incidence algebra of the poset* $\mathcal{P}$. In general, $I(\mathcal{P})$ is not commutative.

The identity (for the convolution operation $*$) is the function δ defined by

$$\delta(\tau, \sigma) = \begin{cases} 1, & \tau = \sigma, \\ 0, & \tau \neq \sigma. \end{cases}$$

A distinguished element of the incidence algebra is the constant function $\zeta(\tau, \sigma) = 1$, for any interval $\tau \leq \sigma$. The *Möbius inversion formula* states that the function ζ is invertible and its inverse is the *Möbius* function μ. For the general poset $\mathcal{P}$, the Möbius function is constructed/defined inductively, but in specific examples, it can be defined/computed explicitly.

Example 2.11 (Poset of subsets of a finite set) Let $\mathcal{P}$ be the poset of all subset of $\{1, \ldots, d\}$ ordered by inclusion. It can be shown that the Möbius function in this case is given by

$$\mu(I, J) = (-1)^{|I|-|J|}, \quad J \subset I,$$

and the Möbius inversion formula recovers the inclusion–exclusion principle.

The following is the main example of a poset that we will be concerned with in the paper.

Example 2.12 (Poset of faces of a convex polyhedral cone) Let $\mathcal{P}$ be the poset of all faces of a given convex polyhedral cone $C \subset \mathbb{R}^n$. If σ is simplicial of dimension d, then this poset is the same as the poset of all subsets of $\{1, \ldots, d\}$ above. It can be shown that the Möbius function in this case is given by

$$\mu(\tau, \sigma) = (-1)^{\dim \sigma - \dim \tau}, \quad \tau \leq \sigma.$$

3 Convergence

In this section, we give some combinatorial/geometric results that contain the combinatorial ingredients of Arthur's result on the convergence and polynomiality (in a truncation parameter T) of the truncated trace $J^T(f)$ in his noninvariant trace formula. See [Ar78, Section 7] and [Ar81, Section 2] as well as the survey [Ar05, Sections 8 and 9].

We continue to denote the n-dimensional real vector space we fixed in Section 2 by V. We choose an inner product $\langle\cdot,\cdot\rangle$ on V and use it to identify V with its dual. Our results in this section depend on the choice of this inner product. In particular, we view the dual cone $\sigma^\vee$ as a subset of V itself,

$$\sigma^\vee := \{x \in V : \langle x, y\rangle \geqslant 0, \text{for all } y \in \sigma\}.$$

Our starting point is a full dimensional, complete, simplicial fan Σ in V. Let $\Delta \in \mathcal{P}(\Sigma)$ be a convex polytope whose normal fan is Σ. Suppose that we are given a collection of continuous functions

$$K_\sigma : V \longrightarrow \mathbb{C}, \quad \sigma \in \Sigma. \tag{3.1}$$

To these data, we associate the *truncated function* $k_\Delta : V \longrightarrow \mathbb{C}$ defined by

$$k_\Delta(x) = \sum_{\sigma\in\Sigma} (-1)^{\dim\sigma} K_\sigma(x)\, \mathbf{1}_{T^-_{\Delta,\sigma}}(x), \tag{3.2}$$

where $T^-_{\Delta,\sigma} = T^-_{\Delta,Q_\sigma}$ is the outward tangent cone, as in (2.3), of the face Q_σ of Δ that stands in bijection with σ as in (2.4). The main result of this section is to prove that if the functions K_σ satisfy certain assumptions, then the integral of k_Δ over V is absolutely convergent. In particular, these assumptions hold when the functions K_σ satisfy certain growth conditions as we explain below. The latter is the setting in which ATF appears.

For a cone $\sigma \in \Sigma$, let $W(\sigma) = \{w_i \in V : i \in I\}$ be a set of unit edge vectors of σ. We also let $B(\sigma) = \{b_i \in V : i \in I\}$ denote the set of unit, inward, facet normals in $\mathrm{Span}(\sigma)$ to the facets of σ. Note that the b_i form a basis of $\mathrm{Span}(\sigma)$ dual to the w_i, i.e.,

$$\langle w_i, b_j\rangle = \delta_{i,j}, \quad i, j \in I.$$

When σ is full dimensional, $B(\sigma)$ is the set of edge vectors of the dual cone $\sigma^\vee$.

Definition 3.1 (Acute cone and acute fan) We say that a convex cone σ in V is acute if $\sigma \subseteq \sigma^\vee$. We call the fan Σ acute if all its cones are acute.

Notice that our definition of acute allows for right angles. We also remark that the notion of acute depends on the inner product we have chosen in V. Indeed, the acute assumption will be crucial for the convergence results below to hold as Example 3.6 shows.

Observe that

$$\sigma \text{ is acute} \iff \langle w_i, w_j\rangle \geqslant 0, \quad i, j \in I. \tag{3.3}$$

It follows from Definition 3.1 that if σ is acute, then for $x \in \mathrm{Span}(\sigma)$,

$$\langle x, b_i\rangle > 0 \text{ for all } i \in I \implies \langle x, w_i\rangle > 0 \text{ for all } i \in I. \tag{3.4}$$

Next, fix a pair of cones $\sigma_2 \le \sigma_1$ in Σ. Write $W(\sigma_1) = \{w_i \in V : i \in I_1\}$ and $B(\sigma_1) = \{b_i \in V : i \in I_1\}$ as above. Then $W(\sigma_2) = \{w_i : i \in I_2\}$ for some $I_2 \subseteq I_1$ and the set $\{b_j : j \in I_1\backslash I_2\}$ consists of vectors normal to σ_2. (However, $B(\sigma_2)$ is not $\{b_j : j \in I_2\}$ as the latter depends on σ_1.)

Define

$$C_{\sigma_1} = C^0_{\sigma_1} := \left\{x \in \mathrm{Span}(\sigma_1) : \langle x, b_j\rangle > 0, \text{ for all } j \in I_1\right\}, \tag{3.5}$$

and similarly, define

$$\widehat{C}_{\sigma_1} = \widehat{C}^0_{\sigma_1} := \{x \in \mathrm{Span}(\sigma_1) : \langle x, w_i \rangle > 0, \text{ for all } i \in I_1\}. \tag{3.6}$$

More generally, we define

$$C^{\sigma_2}_{\sigma_1} := \left\{x \in \mathrm{Span}(\sigma_1) : \langle x, b_j \rangle > 0, \text{ for all } j \in I_1 \backslash I_2\right\}, \tag{3.7}$$

and

$$\widehat{C}^{\sigma_2}_{\sigma_1} := \{x \in \mathrm{Span}(\sigma_1) : \langle x, w_i \rangle > 0, \text{ for all } i \in I_1 \backslash I_2\}. \tag{3.8}$$

Next, we define the following subsets of V which play a crucial role in our results.

Definition 3.2 Let Σ be a full dimensional, complete, simplicial, acute fan in V. Assume that $\sigma_2 \preceq \sigma_1$ are two cones in Σ with unit edge vectors indexed by $I_2 \subset I_1$ as above.

(a) Define $S^{\sigma_2}_{\sigma_1}$ to be the set of $x \in \mathrm{Span}(\sigma_1) \cap \sigma_1^\vee$ such that the face of σ_1 that is nearest to x is the cone generated by $\{w_i : i \in I_1 \backslash I_2\}$. Also, let $\mathbf{1}_{S^{\sigma_2}_{\sigma_1}}$ denote its characteristic function. (See Section 2.4.)

(b) Define the "shifted" subset

$$R^{\sigma_2}_{\sigma_1} := Q_{\sigma_1} + S^{\sigma_2}_{\sigma_1} = \left\{x_0 + x \in V : x_0 \in Q_{\sigma_1} \text{ and } x \in S^{\sigma_2}_{\sigma_1}\right\}. \tag{3.9}$$

We also note that while the subsets $S^{\sigma_2}_{\sigma_1}$ may have smaller dimensions, the subsets $R^{\sigma_2}_{\sigma_1}$, when nonempty, are always full dimensional because the dimension of Q_{σ_1} (as an affine space) and that of $S^{\sigma_2}_{\sigma_1}$ add up to $n = \dim V$.

As Lemma 3.1 below shows, the $S^{\sigma_2}_{\sigma_1}$ are the analogues of the subsets appearing in [Ar78, Lemma 6.1], which also appear to play a similar crucial role in Arthur's results on convergence and polynomiality.

Lemma 3.1 *With $\sigma_2 \preceq \sigma_1$ in Σ, the vectors w_i and b_i, and $I_2 \subset I_1$ as above, we have*

$$S^{\sigma_2}_{\sigma_1} = \left\{ x \in \mathrm{Span}(\sigma_1) : \begin{array}{ll} \langle x, b_j \rangle > 0, & j \in I_1 \backslash I_2 \\ \langle x, b_j \rangle \leqslant 0, & j \in I_2 \\ \langle x, w_i \rangle > 0, & i \in I_1 \end{array} \right\}. \tag{3.10}$$

Proof Write $\tau = \mathrm{Cone}\,(w_i : i \in I_1 \backslash I_2)$. Fix $x \in \mathrm{Span}(\sigma_1) \cap \sigma_1^\vee$. Now, x belongs to $S^{\sigma_2}_{\sigma_1}$ if and only if among all the faces of σ_1 the face τ is the unique face that is nearest to x. Note that the distances to the faces of σ_1 are controlled by the normal vectors b_j and for τ to be the unique nearest face, we must have $\langle x, b_j \rangle > 0$ for $j \in I_1 \backslash I_2$ while $\langle x, b_j \rangle \leqslant 0$ for $j \in I_2$. This implies that $x \in S^{\sigma_2}_{\sigma_1}$ satisfies the first two sets of inequalities on the right-hand side of (3.10). Also, x satisfies the third set of inequalities on the right-hand side of (3.10) by (3.4) because $x \in \sigma_1^\vee$, a cone whose edge vectors are the b_i's.

Next, assume that x belongs to the right-hand side of (3.10). The first two sets of inequalities imply that σ_2 is the unique nearest face of σ_1 to x and the third set of inequalities means that $x \in \sigma_1^\vee$. ■

Remark 3.2 Even though we start with simplicial cones $\sigma_2 \preceq \sigma_1$, the cone $S^{\sigma_2}_{\sigma_1}$ may not be simplicial. As an example, consider $V = \mathbb{R}^3$, and let $w_1 = e_1, w_2 = e_2$, and $w_3 = e_1 + e_2 + e_3$. Take $\sigma_2 = \mathrm{Cone}(w_3) \preceq \sigma_1 = \mathrm{Cone}(w_1, w_2, w_3)$. We then have $b_1 = e_1 - e_3$,

$b_2 = e_2 - e_3$, and $b_3 = e_3$. A simple calculation then shows that $S_{\sigma_1}^{\sigma_2} = \mathrm{Cone}(w_1, w_2, b_1, b_2)$, which is not simplicial.

The following is a type of double nearest face partition that will help us prove our convergence results.

Lemma 3.3 *Let Σ be a full dimensional, complete, simplicial fan in V which is assumed to be acute. Let $\Delta \in \mathcal{P}(\Sigma)$ be a convex polytope whose normal fan is Σ. Then, for any $\sigma \in \Sigma$, the outward tangent cone $T_{\Delta,\sigma}^-$ has the partition*

$$T_{\Delta,\sigma}^- = \bigsqcup_{\{\sigma_1 \in \Sigma \,:\, \sigma \preceq \sigma_1\}} \; \bigsqcup_{\{\sigma_2 \in \Sigma : \sigma_2 \preceq \sigma\}} R_{\sigma_1}^{\sigma_2}. \tag{3.11}$$

Proof Consider the inner disjoint union in (3.11) first. Fix σ_1 in Σ with $\sigma \preceq \sigma_1$. Write $W(\sigma_1) = \{w_i \in V : i \in I_1\}$, and assume that $I_2 \subseteq I \subseteq I_1$ are such that $W(\sigma) = \{w_i \in V : i \in I\}$ and similarly for $W(\sigma_2)$. Also, write $B(\sigma_1) = \left\{b_j \in V : j \in I_1\right\}$. Notice that b_j is normal to σ for $j \in I_1 \backslash I$ and b_j is normal to σ_2 for $j \in I_1 \backslash I_2$.

Simply considering all the subsets of I, we see that

$$A_{\sigma_1}^{\sigma} := \bigsqcup_{\sigma_2 : \sigma_2 \preceq \sigma \preceq \sigma_1} R_{\sigma_1}^{\sigma_2} = \left\{ x \in V : \begin{array}{ll} \langle x - q, b_i \rangle > 0, & i \in I_1 \backslash I, \\ \langle x - q, w_i \rangle > 0, & i \in I_1, \end{array} \text{ for some } q \in Q_{\sigma_1} \right\}.$$

This is because, for $q \in Q_{\sigma_1}$, the set $q + S_{\sigma_1}^{\sigma_2}$ is, by (3.10), given by

$$\begin{array}{ll} \langle x - q, b_i \rangle > 0, & i \in I_1 \backslash I_2 = (I_1 \backslash I) \sqcup (I \backslash I_2), \\ \langle x - q, b_i \rangle \leqslant 0, & i \in I_2, \\ \langle x - q, w_i \rangle > 0, & i \in I_1. \end{array}$$

In the disjoint union over all subsets I_2 of I corresponding to the faces σ_2 of σ, the first set of inequalities for $i \in I_1 \backslash I$ are common for all the subsets I_2 and the remaining inequalities along with the second set of inequalities cover all possible signs for $\langle x - q, b_i \rangle$ for all $i \in I$. Moreover, we have $\langle x - q, w_i \rangle > 0$ for $i \in I_1$. This proves our claim about the inner union and, in fact, already proves the lemma for the case when σ is full dimensional since we only have the inner union in that case.

Next, we consider the outer union. The assertion of the lemma now amounts to a nearest face partition. The set $T_{\Delta,\sigma}^-$ consists of $x \in V$ satisfying $\langle x - q, w_i \rangle > 0, i \in I$ for every $q \in Q_\sigma$. Fix one such x. There is a unique face Q_{σ_1} of Δ with $\sigma \preceq \sigma_1$ such that the distance from x to Q_{σ_1} is smallest among all the faces contained in Q_σ. Note that the distances are controlled by the normal vectors b_j and for the smallest distance to occur for the face Q_{σ_1} of Q_σ, we must have $\langle x - q, b_j \rangle > 0$ for $j \in I_1 \backslash I$ and $\langle x - q, b_j \rangle \leqslant 0$ for $j \in I_0 \backslash I_1$ for any $I_0 \supset I$ with $\sigma_0 \in \Sigma$ for some $q \in Q_{\sigma_1}$. Therefore, among the $A_{\sigma_1'}^{\sigma}$ with $\sigma \preceq \sigma_1'$, only $A_{\sigma_1}^{\sigma}$ contains x. Hence, (3.11) holds. ∎

Let us also fix the following notation. For $\sigma_2 \preceq \sigma_1$ in Σ, define the functions

$$K_{\sigma_1,\sigma_2}(x) = \sum_{\{\tau \in \Sigma \,:\, \sigma_2 \preceq \tau \preceq \sigma_1\}} (-1)^{\dim(\tau)} K_\tau(x), \quad x \in V. \tag{3.12}$$

We are now prepared to state our first convergence result.

Theorem 3.4 (Absolute convergence) *Let Σ be a full dimensional, complete, simplicial fan in V which is assumed to be acute. Let $\Delta \in \mathcal{P}(\Sigma)$ be a simple full dimensional polytope*

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in V whose normal fan is Σ. Suppose that a collection of functions $(K_\sigma)_{\sigma\in\Sigma}$ is given as in (3.1) and k_Δ is defined as in (3.2).

For each pair $\sigma_2 \preceq \sigma_1$ in Σ, assume that the function K_{σ_1,σ_2} is absolutely integrable on the set $R_{\sigma_1}^{\sigma_2}$. Then

$$J_\Sigma(\Delta) := \int_V k_\Delta(x)\,dx \tag{3.13}$$

is absolutely convergent. Recall that $R_{\sigma_1}^{\sigma_2}$ is defined by (3.9) and K_{σ_1,σ_2} by (3.12).

Proof Recall that $k_\Delta(x)$ is defined in terms of outward tangent cones $T^-_{\Delta,\sigma}$. It follows from Lemma 3.3 that

$$\begin{aligned} k_\Delta(x) &= \sum_{\sigma\in\Sigma}(-1)^{\dim(\sigma)}K_\sigma(x)\,\mathbf{1}_{T^-_{\Delta,\sigma}}(x) \\ &= \sum_{\sigma\in\Sigma}(-1)^{\dim(\sigma)}K_\sigma(x)\Bigg(\sum_{\sigma_1:\sigma\preceq\sigma_1}\ \sum_{\sigma_2:\sigma_2\preceq\sigma}\mathbf{1}_{R_{\sigma_1}^{\sigma_2}}(x)\Bigg) \\ &= \sum_{\sigma_2\preceq\sigma_1}K_{\sigma_1,\sigma_2}(x)\mathbf{1}_{R_{\sigma_1}^{\sigma_2}}(x). \end{aligned}$$

Hence,

$$\int_V |k_\Delta(x)|\,dx \leqslant \sum_{\{\sigma_1,\sigma_2\in\Sigma\,:\,\sigma_2\preceq\sigma_1\}}\ \int_{R_{\sigma_1}^{\sigma_2}}|K_{\sigma_1,\sigma_2}(x)|\,dx,$$

and each of the integrals on the right-hand side is finite by assumption. Therefore, the integral on the left-hand side is finite. ∎

A special case of Theorem 3.4 is particularly suitable for applications to Arthur's non-invariant trace formula. To state it, we review the following standard notions of growth.

Let σ be a cone in V. A function $K : V \to \mathbb{C}$ is said to be *of order N* in σ if there is a constant $C = C_{K,N}$ such that

$$|K(x)| \le C\,|x|^N$$

for x in σ with $|x|$ sufficiently large. In other words, $K(x) = O(|x|^N)$ as x tends to ∞ in σ. We say K is *rapidly decreasing* on σ if, for every $N > 0$, we have $K(x) = O(|x|^{-N})$ as x tends to ∞ in σ.

Theorem 3.5 *Let Σ be a full dimensional, complete, simplicial fan in V which is assumed to be acute, and let $(K_\sigma)_{\sigma\in\Sigma}$ be a collection of continuous functions as in (3.1). Assume that the following two assumptions are satisfied:*

- (i) *For all $\sigma \in \Sigma$, the function K_σ is constant in the direction of* $\mathrm{Span}(\sigma)$ *(i.e., a function on $\sigma^\perp$).*
- (ii) *For all pairs of cones $\sigma_2 \preceq \sigma_1$ in Σ with the subset $S_{\sigma_1}^{\sigma_2}$ nonempty, the function K_{σ_1,σ_2} is of order $N = -(n_1+\varepsilon)$ for some $\varepsilon > 0$ in every shifted neighborhood $B(y,\delta) + S_{\sigma_1}^{\sigma_2}$ for all $y \in V$ where $B(y,\delta)$ is a (small) ball in V of positive radius δ around y, and $n_1 = \dim\sigma_1$. In particular, this condition is satisfied if K_{σ_1,σ_2} is rapidly decreasing on the shifted neighborhoods.*

Then, for $\Delta \in \mathcal{P}(\Sigma)$, the integral (3.13) defining $J_\Sigma(\Delta)$ converges absolutely.

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Proof By Theorem 3.4, it is enough to prove that the two assumptions in the statement imply that

$$\int_{R_{\sigma_1}^{\sigma_2}} |K_{\sigma_1,\sigma_2}(x)|\, dx < \infty$$

for all pairs $\sigma_2 \leq \sigma_1$ in Σ.

We may replace the domain of integration by its closure. Also, recall that the closure of $R_{\sigma_1}^{\sigma_2}$ is equal to closure of Q_{σ_1}, which is compact, plus the closure of $S_{\sigma_1}^{\sigma_2}$, which can be given by making all the inequalities in (3.10) nonstrict. Note that $S_{\sigma_1}^{\sigma_2}$ is a cone, even though it may be nonsimplicial.

To estimate the integral above, we apply Fubini's theorem to break the integral as three iterated integrals: an integral over Q_{σ_1}, an integral over $A = \sigma_2^{\perp} \cap \mathrm{Span}(\sigma_1)$, and a third integral in the direction of σ_2.

Note that $\mathrm{Span}(\sigma_2)$ does not intersect $S_{\sigma_1}^{\sigma_2}$ because, for any $x \in \mathrm{Span}(\sigma_2)$, the third set of inequalities in (3.1) for $i \in I_2$ and (3.4) imply that x cannot satisfy the second set of inequalities in (3.1). This observation and our first assumption imply that the contribution of the integral over σ_2 is bounded, up to a constant, by the product of the integrand with $|x|^{n_2}$, where $n_2 = \dim \sigma_2$. Hence, the integral above is bounded, up to a constant, by

$$\int_{Q_{\sigma_1}} \int_{A} |K_{\sigma_1,\sigma_2}(x)|\, |x|^{n_2} dx.$$

Next, using the second assumption and the fact that Q_{σ_1} is compact, we may cover the domain of integration by a finite number of shifted neighborhoods. Therefore, up to a constant, the integral over A, which is a cone of dimension $n_1 - n_2$, is bounded by

$$\int_{A} |x|^{N+n_2}\, dx.$$

The volume element on A involves $|x|^{\dim A-1}$ and $\dim A = n_1 - n_2$ which implies that the original integral is convergent if $N + n_2 + (n_1 - n_2 - 1) + 1 = -\varepsilon < 0$ which is clear. This proves the theorem. ∎

We will give several examples of the convergence theorems later in Section 4. At the moment, we mention the following example, which shows that the acute assumption in our convergence results is crucial.

Example 3.6 Consider the complete fan Σ in $V = \mathbb{R}^2$ pictured in Figure 13. In addition to zero, Σ contains three one-dimensional cones σ_x, σ_y, and σ_z, as well as three two-dimensional cones σ_{xy}, σ_{xz}, and σ_{yz}. Also, let Δ be a polytope whose normal fan is Σ as indicated.

For convenience, let us write $z = x + y$. Define the collection of functions $(K_\sigma)_{\sigma \in \Sigma}$ as follows.

- $K_{xy} = K_{xz} = K_{yz} = 1$.
- $K_x = K_x(y) = 1 + e^{-|y|}$; $K_y = K_y(x) = 1 + e^{-|x|}$; $K_z = K_z(x, y) = 1 + e^{-|z|}$.
- $K_0 = K_0(x, y) = e^{-|z|} + e^{-|x|} + e^{-|y|}$.

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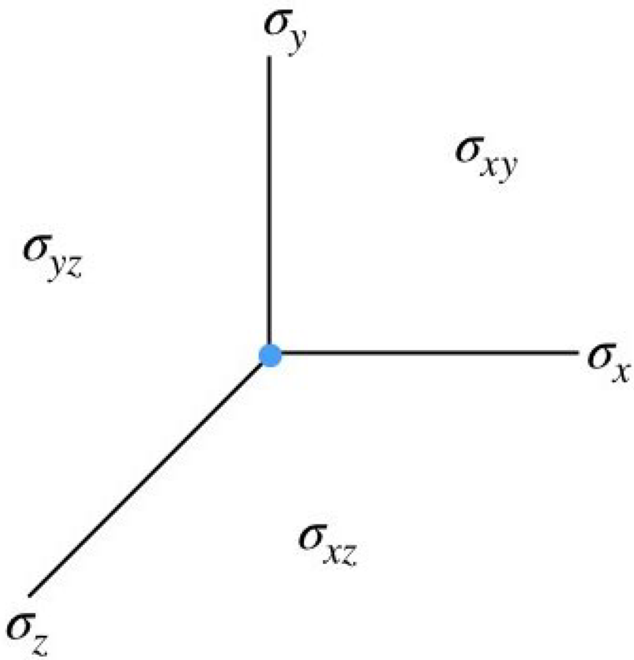

Figure 13: An example of an obtuse fan, it is the normal fan of a right triangle.

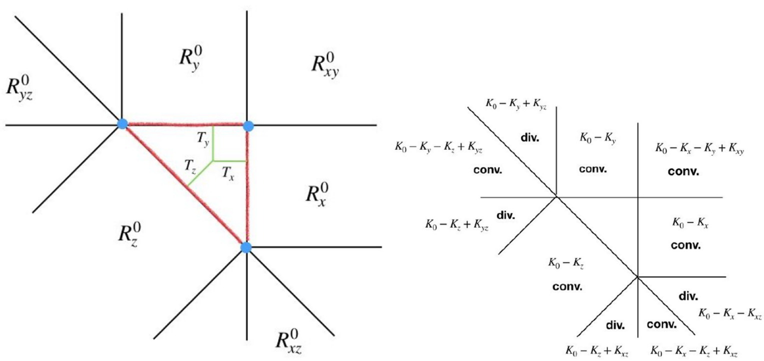

Figure 14: The regions $R_{\sigma_1}^{\sigma_2}$ and their corresponding K_{σ_1,σ_2} functions.

In Figure 14, we have indicated all the nonempty $R_{\sigma_1}^{\sigma_2}$. The truncated function k_Δ is the sum of the functions in the various regions indicated. A simple calculation shows that there are four regions where the integral of $|k_\Delta|$ is divergent. These regions are precisely those that are not of the form $R_{\sigma_1}^{\sigma_2}$ in this example, whereas on the other regions, the hypotheses of Theorem 3.5 clearly hold. As it is evident from this example, the crucial Lemma 3.3 fails, which leads to the failure of Theorem 3.5 without the acute assumption.

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We also prove the following lemma for later use in Section 4. Let τ be a cone in Σ. Recall from Section 2.3 that Σ/τ denotes the fan consisting of all the images of the cones $\sigma \geq \tau$ in the quotient vector space $V/\operatorname{Span}(\tau) \cong \tau^\perp$. For $\sigma \geq \tau$, let us denote the image of σ in $V/\operatorname{Span}(\tau)$ by $\bar{\sigma}$. Note that by assumption, for any $\sigma \geq \tau$, the function K_σ is constant along $\operatorname{Span}(\tau)$ and hence induces a well-defined function $\bar{K}_{\bar{\sigma}}$ on $V/\operatorname{Span}(\tau)$.

Lemma 3.7 *Suppose the conditions in Theorem 3.5 for convergence are satisfied for the K_σ, $\sigma \in \Sigma$. Then, for any $\tau \in \Sigma$, these conditions are also satisfied for the $\bar{K}_{\bar{\sigma}}$, $\bar{\sigma} \in \Sigma/\tau$, and hence $J_{\Sigma/\tau}(0)$ is convergent as well.*

Proof This is an immediate corollary of the following two observations. Let $\tau \leq \sigma_2 \leq \sigma_1$. Then we have that (1) the cone $\mathcal{S}_{\bar{\sigma}_2}^{\bar{\sigma}_1}$ (as in the proof of Theorem 3.4) coincides with the image of $\mathcal{S}_{\sigma_2}^{\sigma_1}$ in $V/\operatorname{Span}(\tau)$ and (2) the function $K_{\bar{\sigma}_1,\bar{\sigma}_2}$ (as in the statement of Theorem 3.4) is rapidly decreasing on a shifted neighborhood $\mathcal{S}_{\bar{\sigma}_2}^{\bar{\sigma}_1}$ because K_{σ_1,σ_2} is rapidly decreasing on a shifted neighborhood of $\mathcal{S}_{\sigma_2}^{\sigma_1}$. ∎

Finally, we give a discrete version of Theorem 3.5. As usual, let N and M be dual lattices, and let $V = N_\mathbb{R} = N \otimes \mathbb{R}$ and $V^* = M_\mathbb{R} = M \otimes \mathbb{R}$ be the corresponding vector spaces, respectively. We fix a perfect pairing $N \times N \to \mathbb{Z}$ and use it to identify N and M as well as $N_\mathbb{R}$ and $M_\mathbb{R}$.

Theorem 3.8 *With the notations and assumptions as in Theorem 3.5, the sum*

$$S_\Sigma(\Delta, M) = \sum_{m \in M} k_\Delta(m)$$

is absolutely convergent.

Proof In the proof of Theorem 3.5, replace all integrals $\int_A f(x)dx$ with sums $\sum_{m \in A \cap M} f(m)$. ∎

We should note that the discrete analogue of Lemma 3.7 also holds with the same proof.

4 Polynomiality

In this section, we prove the following theorems.

Theorem 4.1 (Polynomiality) *Let Σ be a full dimensional, complete, simplicial fan in V which is assumed to be acute. Let $(K_\sigma)_{\sigma \in \Sigma}$ be a collection of continuous functions satisfying the assumptions* (i) *and* (ii) *in Theorem 3.5. Then*

$$J_\Sigma(\Delta) = \int_V k_\Delta(x)dx$$

is a polynomial function on $\mathcal{P}(\Sigma)$, i.e., a polynomial in the support numbers of Δ.

We also prove a discrete version of the above polynomiality result. Let N and M be dual lattices with $V = N_\mathbb{R}$ and $V^* = M_\mathbb{R}$ the corresponding vector spaces. We fix a perfect $\mathbb{Z}$-pairing $N \times N \to \mathbb{Z}$ and use it to identify N and M. Recall that $\mathcal{P}(\Sigma, M)$ denotes the collection of polytopes with normal fan Σ whose vertices lie in M.

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Theorem 4.2 *Let the notations and assumptions be as in Theorem 4.1. Then*

$$S_\Sigma(\Delta) = \sum_{m \in M} k_\Delta(m)$$

is a polynomial function on $\mathcal{P}(\Sigma, M)$.

A key step in the proof of Theorem 4.1 is a combinatorial lemma (Lemma 4.6) which we deduce as a corollary of the Lawrence–Varchenko conical decomposition (Theorem 2.10). The notion of a virtual polytope naturally appears here (see Section 2.6). The proof of Theorem 4.2 is a slight modification of the proof of Theorem 4.1. We give the proofs in Section 4.2 below after some preparation. Let us give some examples first.

Example 4.3 (Brianchon–Gram) Let Σ be a simplicial fan in V with $\Delta \in \mathcal{P}(\Sigma)$ a polytope normal to Σ. Let $K_\sigma \equiv 1$ and $\forall \sigma \in \Sigma$. The combinatorial truncation k_Δ in this case is given by

$$k_\Delta = \sum_{\sigma \in \Sigma} (-1)^{\dim(\sigma)} \mathbf{1}_{T^-_{\Delta,\sigma}}.$$

By the Brianchon–Gram theorem (Theorem 2.7), we have

$$k_\Delta = \mathbf{1}_\Delta.$$

For any pair of cones $\sigma_1 \preceq \sigma_2$ in Σ, we have

$$K_{\sigma_1,\sigma_2} = \sum_{\{\tau \in \Sigma \,:\, \sigma_2 \subseteq \tau \subseteq \sigma_1\}} (-1)^{\dim(\tau)} = 0$$

by the binomial identity $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$. Thus, the conditions in Theorem 3.5 are satisfied. Moreover, the K_σ are constant, and hence the assumptions in the polynomiality theorem are also satisfied. Thus, we recover the polynomiality of the volume function $\Delta \mapsto \mathrm{vol}(\Delta)$ (see Theorem 2.1).

Example 4.4 (Rectangle) We consider the fan Σ in $V = \mathbb{R}^2$ as in Figure 15, consisting of one-dimensional cones σ_x and σ_y and their opposites, as well as the two-dimensional cone σ_{xy} and its counterparts for the other three quadrants. We also have the cone $\{0\}$. The fan Σ is normal to the rectangle Δ with support numbers T_1, T_2, T_1', T_2' as indicated.

Let $f(x, y)$ be an absolutely integrable function on $\mathbb{R}^2$ with f_{++} denoting the value of its integral over the first quadrant. Similarly, let the values of its integral over the other quadrants be denoted by f_{+-}, f_{-+}, f_{--}. Also, let $g(x)$ and $h(y)$ be absolutely integrable functions on $\mathbb{R}$ with their integrals over $[0, \infty)$ denoted by g_+ and h_+ and their integrals over $(-\infty, 0]$ denoted by g_- and h_-, respectively. Finally, let k denote a constant.

We assign the following functions to the cones in Σ:

- $K_0(x, y) = f(x, y) + g(x) + h(y) + k$,
- $K_{\sigma_{\pm x}}(x, y) = h(y) + k$,
- $K_{\sigma_{\pm y}}(x, y) = g(x) + k$, and
- $K_\sigma(x, y) = k$ for all two-dimensional cones σ in Σ.

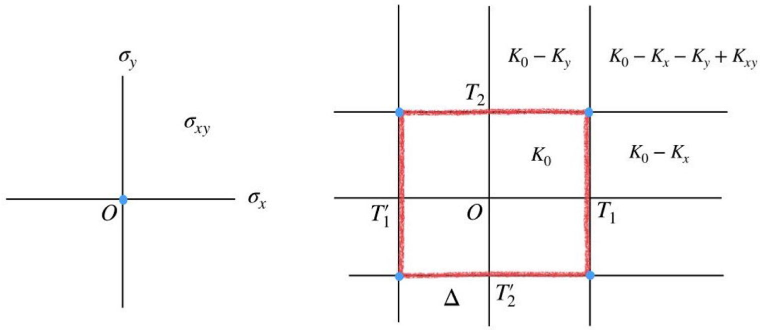

Figure 15: Illustration of the truncated function k_Δ for when Δ is a rectangle.

Notice that the conditions (i) and (ii) of Theorem 3.5 are clearly satisfied.

Let us calculate $J_\Sigma(\Delta)$. Because of the symmetry in this example, it is enough to consider a quarter of the picture. We have

$$\begin{aligned}&\int_0^{T_1}\int_0^{T_2}(f(x,y)+g(x)+h(y)+k)\,dy\,dx+\int_0^{T_1}\int_{T_2}^{\infty}(f(x,y)+h(y))\,dy\,dx\\&\quad+\int_{T_1}^{\infty}\int_0^{T_2}(f(x,y)+g(x))\,dy\,dx+\int_{T_1}^{\infty}\int_{T_2}^{\infty}f(x,y)\,dy\,dx\\&=f_{++}+g_+T_2+h_+T_1+kT_1T_2,\end{aligned}$$

which is a polynomial of degree 2 in T_1 and T_2. Adding similar contributions from the other three quadrants, we arrive at

$$\begin{aligned}J_\Sigma(\Delta)&=k\,(T_1+T_1')(T_2+T_2')+h_+\,T_1+g_-\,T_1'+g_+\,T_2+g_-\,T_2'\\&\quad+(f_{++}+f_{+-}+f_{-+}+f_{--}).\end{aligned}$$

4.1 An extension of the Langlands combinatorial lemma

As before, V is an n-dimensional real vector space. We fix an inner product $\langle\cdot,\cdot\rangle$ on V and identify V with its dual space V^*. Let Σ be a full dimensional, complete, simplicial fan in V, and let $\Delta\in\mathcal{P}(\Sigma)$ be a full dimensional simple polytope with normal fan Σ. Since we identified V and V^*, we take both Σ and Δ to lie in V.

Let $\sigma\in\Sigma$ be a cone. First, we consider the case where σ is full dimensional. Let v_σ be the corresponding vertex of Δ. Let $W=\{w_1,\ldots,w_n\}$ (resp. $B=\{b_1,\ldots,b_n\}$) be the set of edge vectors of σ (resp. of $\sigma^\vee$). Then the b_i (resp. the w_j) are the inward facet normals to σ (resp. $\sigma^\vee$), and the cone σ is given by inequalities as

$$\sigma=\{x:\langle x,b_i\rangle\geqslant 0,\ i=1,\ldots,n\}.$$

Also, the inward-looking tangent cone $T^+_{\Delta,\sigma}$ at the vertex v_σ is given by

$$T^+_{\Delta,\sigma}=\{x:\langle x,w_i\rangle\leqslant\langle v_\sigma,w_i\rangle,\ i=1,\ldots,n\}.$$

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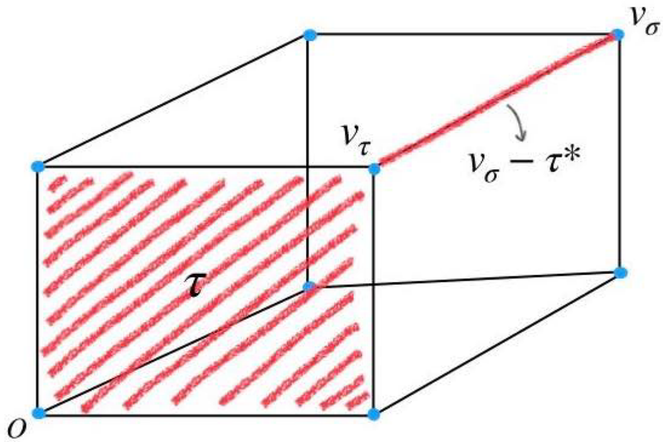

Figure 16: A three-dimensional example where $\Gamma_{\Delta,\sigma}$ is a cube. A face τ (of σ) and its corresponding dual face τ^* (of $\sigma^\vee$) and the vertex v_τ (of $\Gamma_{\Delta,\sigma}$) are illustrated.

We consider the oriented hyperplanes corresponding to the union of these two sets of inequalities:

$$H_{b_i,0} = \{x : \langle x, b_i \rangle = 0\}, \qquad i = 1, \dots, n,$$
$$H_{w_i,\langle v_\sigma, w\rangle} = \{x : \langle x, w_i \rangle = \langle v_\sigma, w \rangle\}, \qquad i = 1, \dots, n. \tag{4.1}$$

If v_σ lies in σ, then the hyperplanes in (4.1) are the facets of the polytope $\Delta \cap \sigma$ oriented outward. In general, v_σ may not lie in σ.

Definition 4.1 We denote the virtual polytope in V determined by the oriented hyperplanes in (4.1) by $\Gamma_{\Delta,\sigma}$. We denote the convex chain corresponding to $\Gamma_{\Delta,\sigma}$ by $\gamma_{\Delta,\sigma}$.

See Section 2.6 for a review of the notions of virtual polytope and convex chain. Also, see Figure 16 for a three-dimensional example of $\Gamma_{\Delta,\sigma}$ and Figure 17 for a pair of two-dimensional examples of the virtual polytope $\Gamma_{\Delta,\sigma}$ and its convex chain $\gamma_{\Delta,\sigma}$.

In this section, we consider the Lawrence–Varchenko conical decomposition for the virtual polytope $\Gamma_{\Delta,\sigma}$ (Theorem 2.10). We will see that this recovers and extends some of the key combinatorial lemmas appearing in Arthur's work (e.g., [Ar81]). As a special case, we immediately recover the Langlands combinatorial lemma (see [Ar05, Section I.8, p. 46], [GKM97, Appendix B]). In addition, we interpret the Langlands combinatorial lemma as a formula for the inverse of a distinguished element in the incidence algebra of poset of faces of σ (see Section 2.7).

Recall that for $\tau \preceq \sigma$, the largest face of $\sigma^\vee$ orthogonal to τ is denoted by τ^* and we have $\dim \tau + \dim \tau^* = n$ (Section 2.1). It follows that the intersection $\mathrm{Span}(\tau) \cap (v_\sigma + \mathrm{Span}(\tau^*))$ is a single point which can be shown to be a vertex v_τ of $\Gamma_{\Delta,\sigma}$. In fact, we will see below that $\tau \mapsto v_\tau$ gives a one-to-one correspondence between the faces of σ and the vertices of $\Gamma_{\Delta,\sigma}$. The vertex corresponding to the zero-dimensional face 0 is

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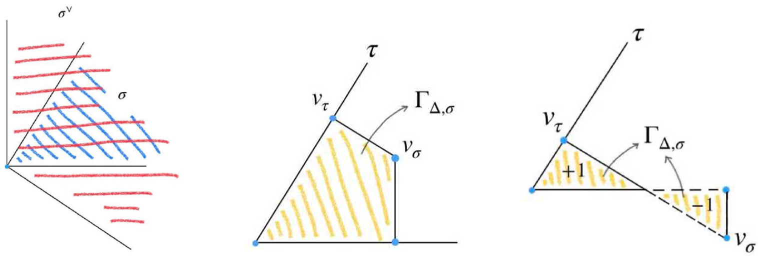

Figure 17: Two examples of the virtual polytopes $\Gamma_{\Delta,\sigma}$. In the first example, the vertex v_σ lies in the cone σ and $\Gamma_{\Delta,\sigma}$ is an actual polytope (a quadrangle). The convex chain $\gamma_{\Delta,\sigma}$ is the characteristic function of the quadrangle. In the second example, v_σ lies outside σ and $\Gamma_{\Delta,\sigma}$ is a virtual quadrangle. The convex chain $\gamma_{\Delta,\sigma}$ is the function which has values 1 and −1 in the two shaded regions, respectively.

0 itself. On the other hand, the vertex corresponding to the whole σ is the vertex v_σ of Δ.

For a face $\tau \preceq \sigma$, let $W(\tau) \subset W$ (resp. $B(\tau) \subset B$) be the subset of edge vectors of τ (resp. τ^*). Thus,

$$\tau = \{x \in \sigma : \langle x, b\rangle = 0,\ b \in B(\tau)\}.$$

The vertex v_τ is then the unique solution of the system of equations

$$\begin{cases} \langle x, w\rangle = \langle v_\sigma, w\rangle, & \forall w \in W(\tau), \\ \langle x, b\rangle = 0, & \forall b \in B(\tau). \end{cases}$$

Moreover, the inward tangent cone $T^+_{\Gamma_{\Delta,\sigma}, v_\tau}$ at the vertex v_τ is given by the inequalities

$$T^+_{\Gamma_{\Delta,\sigma}, v_\tau} = \left\{ x \in V : \begin{array}{ll} \langle x, w\rangle \leqslant \langle v_\sigma, w\rangle, & \forall w \in W(\tau) \\ \langle x, b\rangle \geqslant 0, & \forall b \in B(\tau) \end{array} \right\}.$$

Thus, the set of outward facet normals of $\Gamma_{\Delta,\sigma}$ at v_τ is $W(\tau) \cup -B(\tau)$. In other words, the cone in the normal fan of $\Gamma_{\Delta,\sigma}$ corresponding to the vertex v_τ is generated by the set of vectors $W(\tau) \cup -B(\tau)$ (Section 2.3).

Consider the nearest face partition corresponding to σ (Section 2.4). That is, for each face τ, let V_σ^τ be the set of points $x \in V$ whose shortest distance to σ is attained at a point in the relative interior of τ. Since σ is a cone, each V_σ^τ is a full dimensional cone. Moreover, the closures of the cones V_σ^τ, $\tau \preceq \sigma$, are the maximal cones of a complete simplicial fan in V which we call the *nearest face fan* of σ. The following is straightforward to verify.

Proposition 4.5 *In the nearest face fan of σ, the cone corresponding to a face $\tau \preceq \sigma$ is the convex cone generated by the set of vectors $W(\tau) \cup -B(\tau)$.*

Since the V_σ^τ partition of the whole space V, the above proposition shows that the union of the cones generated by $W(\tau) \cup -B(\tau)$, $\tau \preceq \sigma$, is V. This then implies that the normal fan of $\Gamma_{\Delta,\sigma}$ coincides with the nearest fan of σ. In particular, the v_τ are all of the

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vertices of $\Gamma_{\Delta,\sigma}$. In other words, $\tau \mapsto v_\tau$ gives a one-to-one correspondence between the faces of σ and the vertices of $\Gamma_{\Delta,\sigma}$.

Now, take a vector ξ in $\sigma^\circ \cap (\sigma^\vee)^\circ$, that is,

$$\langle \xi, b\rangle > 0, \ \forall b \in B,$$

$$\langle \xi, w\rangle > 0, \ \forall w \in W.$$

Note that since $\sigma \neq V$, we know $(\sigma^\circ)^\vee + \sigma^\circ \neq V$ and hence $\sigma^\circ \cap (\sigma^\vee)^\circ = ((\sigma^\circ)^\vee + \sigma^\circ)^\vee \neq \varnothing$.

Let $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ be the polarized tangent cone at the vertex v_τ appearing in the Lawrence–Varchenko decomposition of $\Gamma_{\Delta,\sigma}$ relative to the vector ξ (see Section 2.5). By construction, the edge vectors of $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ are $\pm$ the edge vectors of the tangent cone of $\Gamma_{\Delta,\sigma}$ at v_τ so that the minimum of $\langle \xi, \cdot\rangle$ on $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ is attained at the vertex v_τ. Since the inner product of ξ with any vector in $W \cup B$ is positive, it follows that the set of inward facet normals of $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ is exactly $W(\tau) \cup B(\tau)$. More precisely, $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ is defined by the inequalities

$$T^\xi_{\Gamma_{\Delta,\sigma},v_\tau} = \left\{ x \in V : \begin{array}{ll} \langle x, w\rangle > \langle v_\sigma, w\rangle, & \forall w \in W(\tau) \\ \langle x, b\rangle \geqslant 0, & \forall b \in B(\tau) \end{array} \right\}. \tag{4.2}$$

On the other hand, let C^τ_σ be the inward-looking tangent cone of σ at τ. It is the cone defined as

$$C^\tau_\sigma = \{x \in V : \langle x, b\rangle \geqslant 0, \ \forall b \in B(\tau)\}.$$

It follows from (4.2) that $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ can be written as

$$T^\xi_{\Gamma_{\Delta,\sigma},v_\tau} = C^\tau_\sigma \cap T^-_{\Delta,\tau}.$$

If σ is not full dimensional, we can repeat the above, replacing Δ with $\Delta \cap \mathrm{Span}(\sigma)$. Then $\gamma_{\Delta,\sigma}$ is a convex chain supported on $\mathrm{Span}(\sigma)$. We extend $\gamma_{\Delta,\sigma}$ to the whole V by requiring it to be constant along $\sigma^\perp$. Now, applying the Lawrence–Varchenko theorem to the virtual polytope $\Gamma_{\Delta,\sigma}$ and the vector ξ as above, we obtain the following conical decomposition for $\Gamma_{\Delta,\sigma}$.

Lemma 4.6 *With notation as above, let $\gamma_{\Delta,\sigma}$ be the convex chain associated with the virtual polytope $\Gamma_{\Delta,\sigma}$. We have*

$$\gamma_{\Delta,\sigma} = \sum_{\tau \preceq \sigma} (-1)^{\dim \tau} \mathbf{1}_{C^\tau_\sigma} \mathbf{1}_{T^-_{\Delta,\tau}}. \tag{4.3}$$

Proof First, we note that the number n_{v_τ} of the edges flipped in the polarized tangent cone $T^\xi_{\Gamma_{\Delta,\sigma},v_\tau}$ is equal to $|W(\tau)| = \dim \tau$. The above discussion then proves the case where σ is full dimensional. If σ is not full dimensional, all the cones considered in the right-hand side of (4.3) above should be extended in the orthogonal direction $\sigma^\perp$. This finishes the proof. ■

Letting $\Delta = \{0\}$, we recover a combinatorial lemma of Langlands.

Corollary 4.7 (Langlands combinatorial lemma) *Let $\sigma \subset V$ be a convex polyhedral cone. The following identities hold.*

$$\sum_{\tau \le \tau' \le \sigma} (-1)^{\dim \tau + \dim \tau'} \mathbf{1}_{C_\sigma^{\tau'}} \mathbf{1}_{C_{\tau^*}^{\tau'^*}} = \begin{cases} 1, & \textit{if } \tau = \sigma, \\ 0, & \textit{if } \tau \ne \sigma, \end{cases} \tag{4.4}$$

$$\sum_{\tau \le \tau' \le \sigma} (-1)^{\dim \tau' + \dim \tau} \mathbf{1}_{C_\tau^{\tau'}} \mathbf{1}_{C_{\sigma^*}^{\tau'^*}} = \begin{cases} 1, & \textit{if } \tau = \sigma, \\ 0, & \textit{if } \tau \ne \sigma. \end{cases} \tag{4.5}$$

Alternatively, consider the incidence algebra of the poset of faces of σ with ring of scalars R being the ring of all real-valued functions on V (see Section 2.7 and Example 2.12). Define the elements F and G of the incidence algebra by

$$F(\tau, \tau') = (-1)^{\dim \tau} \mathbf{1}_{C_\tau^{\tau'}},$$
$$G(\tau, \tau') = (-1)^{\dim \tau} \mathbf{1}_{C_{\tau^*}^{\tau'^*}}.$$

Equations (4.4) and (4.5) state that F and G are inverses of each other in the incidence algebra, that is,

$$(F * G)(\tau, \sigma) = (G * F)(\tau, \sigma) = \delta(\tau, \sigma). \tag{4.6}$$

Proof First, to prove (4.4), we can assume without loss of generality that $\tau = 0$. Equation (4.4) is then an immediate consequence of (4.3) when we let $\Delta = \{0\}$. To obtain (4.5), we apply (4.4) to $\sigma^\vee$ in place of σ. Finally, (4.6) is a rewriting of (4.4) and (4.5) using the language of incidence algebra. ∎

Corollary 4.8 *With notation as before, we have*

$$\mathbf{1}_{T_{\Delta,\sigma}^-} = \sum_{\tau \le \sigma} (-1)^{\dim \tau} \mathbf{1}_{C_\sigma^\tau} \gamma_{\Delta,\tau}. \tag{4.7}$$

Proof Let H and L be elements of the incidence algebra such that $H(0,\tau) = \mathbf{1}_{T_{\Delta,\sigma}^-}$ and $L(0,\tau) = \gamma_{\Delta,\tau}$, $\forall \tau \le \sigma$. Then (4.3) states that $L(0,\tau) = (H * F)(0,\tau)$. Convolution of both sides from right with G gives $(L * G)(0,\tau) = H(0,\tau)$, which is exactly (4.7). ∎

4.2 Proof of polynomiality

Proof of Theorem 4.1 In the definition of $J_\Sigma(\Delta)$, we use Corollary 4.8 to write $T_{\Delta,\sigma}^-$ as $\sum_{\tau \le \sigma} (-1)^{\dim \tau} \mathbf{1}_{C_\sigma^\tau} \gamma_{\Delta,\tau}$. We have

$$\begin{aligned} J_\Sigma(\Delta) &= \int_V \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} K_\sigma(x) \mathbf{1}_{T_{\Delta,\sigma}^-}(x) dx \\ &= \int_V \sum_{\sigma \in \Sigma} (-1)^{\dim \sigma} K_\sigma(x) \left(\sum_{\tau : \tau \le \sigma} (-1)^{\dim \tau} \mathbf{1}_{C_\sigma^\tau}(x) \gamma_{\Delta,\tau}(x) \right) dx \\ &= \sum_{\tau \in \Sigma} (-1)^{\dim \tau} \int_V \left(\sum_{\sigma : \tau \le \sigma} (-1)^{\dim \sigma} K_\sigma(x) \mathbf{1}_{C_\sigma^\tau}(x) \gamma_{\Delta,\tau}(x) \right) dx. \end{aligned}$$

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Now, we use the assumption that $K_\sigma(x)$ is invariant along σ and $\gamma_{\Delta,\tau}$ is invariant along $\tau^\perp$ (by definition of $\gamma_{\Delta,\tau}$) to write the above as

$$\sum_\tau (-1)^{\dim\tau}\left(\int_{\tau^\perp}\sum_{\sigma:\tau\preceq\sigma}(-1)^{\dim\sigma}K_\sigma(x_2)\,\mathbf{1}_{C^\tau_\sigma}(x_2)dx_2\right)\cdot\left(\int_{\mathrm{Span}(\tau)}\gamma_{\Delta,\tau}(x_1)dx_1\right).$$

Here, $x = x_1 + x_2$ where $x_1 \in \mathrm{Span}(\tau)$ and $x_2 \in \tau^\perp$, and dx_1 and dx_2 are the Lebesgue measures on $\mathrm{Span}(\tau)$ and $\tau^\perp$, respectively, so that $dx = dx_1dx_2$. By Theorem 2.1 and Remark 2.9, we know that

$$\mathrm{vol}(\Gamma_{\Delta,\tau}) = \int_{\mathrm{Span}(\tau)}\gamma_{\Delta,\tau}(x_1)\,dx_1$$

is a polynomial in the support numbers of $\Gamma_{\Delta,\tau}$ of degree $\dim\tau$. By definition (see (4.1)), these support numbers either correspond to the b_i in which case they are 0, or they correspond to the w_i in which case they are equal to the a_i, the corresponding support numbers of Δ. It follows that $\mathrm{vol}(\Gamma_{\Delta,\tau})$ is a polynomial in the support numbers of Δ of degree $\dim\tau$. Recall that the normal fan of the face of Δ corresponding to τ is the fan Σ/τ consisting of all the images of the cones $\sigma \succeq \tau$ in the quotient vector space $V/\mathrm{Span}(\tau) \cong \tau^\perp$. One then observes that $\int_{\tau^\perp}\sum_{\sigma:\tau\preceq\sigma}(-1)^{\dim\sigma}K_\sigma(x)\,\mathbf{1}_{C^\tau_\sigma}(x)\,dx_2$ is exactly $J_{\Sigma/\tau}(0)$. In summary,

$$J_\Sigma(\Delta) = \sum_{\tau\in\Sigma}(-1)^{\dim\tau}J_{\Sigma/\tau}(0)\,\mathrm{vol}(\Gamma_{\Delta,\tau}).$$

This shows that $J_\Sigma(\Delta)$ is a linear combination of the polynomials $\mathrm{vol}(\Gamma_{\Delta,\tau})$ and hence is a polynomial itself. It remains to show that $J_{\Sigma/\tau}(0)$ is convergent. But this is the content of Lemma 3.7, and the proof is finished. ∎

Proof of Theorem 4.2 In the proof of Theorem 4.1, replace any integral $\int_A f(x)dx$ with a sum $\sum_{m\in A\cap M} f(m)$. In particular, replace vol with the number of lattice points. For $\tau \in \Sigma$, let $M_1 = \mathrm{Span}(\tau) \cap M$ and $M_2 = \tau^\perp \cap M$. Note that it is possible that $M_1 + M_2 \neq M$. Nevertheless, $M_1 + M_2$ is a subgroup of finite index in M. Let $M' \subset M$ be a system of coset representatives for $M/(M_1 + M_2)$. Then every $m \in M$ can be uniquely written as $m' + m_1 + m_2$ where $m' \in M'$ and $m_i \in M_i$. Then, similar to the proof of Theorem 4.1, we write

$$\begin{aligned}
S_\Sigma(\Delta, M) &= \sum_{m\in M}\sum_{\sigma\in\Sigma}(-1)^{\dim\sigma}K_\sigma(m)\,\mathbf{1}_{T^-_{\Delta,\sigma}}(m)\\
&= \sum_{m\in M}\sum_{\sigma\in\Sigma}(-1)^{\dim\sigma}K_\sigma(m)\left(\sum_{\tau:\tau\preceq\sigma}(-1)^{\dim\tau}\mathbf{1}_{C^\tau_\sigma}(m)\,\gamma_{\Delta,\tau}(m)\right)\\
&= \sum_{\tau\in\Sigma}(-1)^{\dim\tau}\sum_{m\in M}\left(\sum_{\sigma:\tau\preceq\sigma}(-1)^{\dim\sigma}K_\sigma(m)\,\mathbf{1}_{C^\tau_\sigma}(m)\,\gamma_{\Delta,\tau}(m)\right)
\end{aligned}$$

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$$= \sum_{\tau}(-1)^{\dim\tau} \sum_{m'\in M'} \left(\sum_{m_2\in M_2} \sum_{\sigma:\tau\preceq\sigma} (-1)^{\dim\sigma} K_\sigma(m'+m_2)\, \mathbf{1}_{C_\sigma^\tau}(m'+m_2) \right) \cdot \left(\sum_{m_1\in M_1} \gamma_{\Delta,\tau}(m'+m_1) \right).$$

One shows that, for fixed $m' \in M'$, the quantity $\sum_{m_2\in M_2} \sum_{\sigma:\tau\preceq\sigma} (-1)^{\dim\sigma} K_\sigma(m'+m_2)$ is equal to $S_{\Sigma/\tau}(0)$ with respect to the functions $K_\sigma(m'+x)$ (instead of $K_\sigma(x)$). By the discrete version of Lemma 3.7, we know that $S_{\Sigma/\tau}(0)$ is convergent. Let us see that the other term $\sum_{m_1\in M_1} \gamma_{\Delta,\tau}(m'+m_1)$ depends polynomially on Δ. Let $\pi : V \to \mathrm{Span}(\tau)$ be the orthogonal projection. Since $\gamma_{\Delta,\tau}$ is invariant in the $\tau^\perp$ direction, we have $\gamma_{\Delta,\tau}(m'+m_1) = \gamma_{\Delta,\tau}(\pi(m')+m_1)$. Now, the polynomiality of $\sum_{m_1\in M} \gamma_{\Delta,\tau}(\pi(m')+m_1)$ follows from Remark 2.9 (see also Theorem 2.3 and Remark 2.4). Thus, $S_\Sigma(\Delta, M)$ is a finite sum (over $m' \in M'$) of polynomials and hence a polynomial itself. This finishes the proof. ■

5 Toric varieties

5.1 Background on toric varieties

In this section, we review some basic facts about toric varieties. Common references on toric varieties are [CLS11, Fu93]. Let $T = T_N \cong (\mathbb{C}^*)^n$ be an algebraic torus of dimension n over $\mathbb{C}$, with character lattice $M \cong \mathbb{Z}^n$ and cocharacter lattice $N \cong \mathbb{Z}^n$. We denote the corresponding vector spaces $N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M \otimes_{\mathbb{Z}} \mathbb{R}$ by $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively. For $m \in M$, we denote the corresponding character/irreducible representation by $\chi_m : T \to \mathbb{C}^*$.

Let $\sigma \subset N_{\mathbb{R}}$ be a rational strongly convex polyhedral cone. Recall that σ is rational if it is generated as a cone by vectors from N. To σ, one associates an affine toric variety U_σ defined by

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M].$$

Here, $\mathbb{C}[\sigma^\vee \cap M]$ is the semigroup algebra of the semigroup of all lattice points in the dual cone $\sigma^\vee$. If $\tau \preceq \sigma$, then we have natural inclusion $U_\tau \hookrightarrow U_\sigma$. The variety U_0 associated with the origin is just the algebraic torus T itself. The M-grading on the algebra $\mathbb{C}[\sigma^\vee \cap M]$ induces a T-action on the variety U_σ with open orbit U_0.

Recall that a fan Σ in $N_{\mathbb{R}}$ is *rational* if all the cones in Σ are generated by vectors in N. Let X_Σ be the toric variety corresponding to a complete rational fan Σ (see [CLS11, Chapter 3] for more details). The (abstract) variety X_Σ is obtained by gluing all the affine toric varieties U_σ, $\sigma \in \Sigma$, with respect to inclusion maps $U_\tau \hookrightarrow U_\sigma$ and $\tau \preceq \sigma$.

There is an inclusion-reversing correspondence between the cones in Σ and the T-orbits in X_Σ. For $\sigma \in \Sigma$, let the corresponding T-orbit be O_σ.

For a ray $\rho \in \Sigma(1)$, we denote the corresponding T-orbit closure $\overline{O_\rho}$ by D_ρ. The D_ρ for $\rho \in \Sigma(1)$ are T-invariant prime divisors on X_Σ. For each ray $\rho \in \Sigma(1)$, let $v_\rho \in N$ be the primitive vector along ρ, i.e., shortest lattice vector on ρ. Let $\xi \in \sigma \cap N$ be a

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cocharacter. One knows that for $x \in U_0$, $\lim_{t\to 0} \xi(t) \cdot x$ exists and is a point in the orbit O_σ.

Let us assume that X_Σ is a projective variety. This is equivalent to the set $\mathcal{P}(\Sigma)$, of polytopes with normal fan Σ, being nonempty. Let $\Delta \subset M_\mathbb{R}$ be a lattice polytope with normal fan Σ. The faces of Δ are in one-to-one correspondence with cones in Σ. For $\sigma \in \Sigma$, let Q_σ be the corresponding face of Δ. We note that $\dim Q_\sigma = \operatorname{codim} \sigma$. The polytope Δ can be represented as

$$\Delta = \{x \in M_\mathbb{R} : \langle x, v_\rho \rangle \leqslant -a_\rho, \forall \rho \in \Sigma(1)\}, \tag{5.1}$$

where the a_ρ are the support numbers of Δ (see Section 2). Recall that for $\sigma \in \Sigma$, we let $T^+_{\Delta,\sigma}$ (resp. $T^-_{\Delta,\sigma}$) be the inward-looking (resp. outward-looking) tangent cone of the corresponding face Q_σ in Δ (see equations (2.2) and (2.3)).

To Δ, one associates a T-invariant (Cartier) divisor

$$D_\Delta = \sum_{\rho \in \Sigma(1)} -a_\rho D_\rho.$$

It can be shown that D_Δ is an ample divisor. We denote the corresponding line bundle on X_Σ by $\mathcal{L}_\Delta$. Since D_Δ is T-invariant, the line bundle $\mathcal{L}_\Delta$ comes with a natural T-linearization. The divisor D_Δ defines a sheaf of rational functions $\mathcal{O}(D_\Delta)$ by

$$H^0(U, \mathcal{O}(D_\Delta)) = \{f \in \mathbb{C}(X_\Sigma) : (f) + D_\Delta > 0 \text{ on } U\} \subset \mathbb{C}[U_0], \tag{5.2}$$

$$= \{f \in \mathbb{C}(X_\Sigma) : \operatorname{ord}_{D_\rho}(f) \geqslant a_\rho,\ \forall \rho \in \Sigma(1) \text{ such that } D_\rho \cap U \neq \varnothing\}. \tag{5.3}$$

In particular, for an open affine chart U_σ, the subspace $H^0(U_\sigma, \mathcal{O}(D_\Delta))$ is T-invariant and hence decomposes into one-dimensional T-modules. Let $m \in M$. One verifies that for any ray $\rho \in \Sigma(1)$, the order of zero/pole of the character χ_m, regarded as a rational function on $U_0 \cong T$, along the divisor D_ρ is given by

$$\operatorname{ord}_{D_\rho}(\chi_m) = -\langle m, v_\rho \rangle.$$

It follows that, for any $\sigma \in \Sigma$, the irreducible representation χ_m appears in $H^0(U_\sigma, \mathcal{O}(D_\Delta))$ if and only if $\langle m, v_\rho \rangle \leqslant -a_\rho$, for all $\rho \in \sigma(1)$. Since $\mathbb{C}[U_0]$, the coordinate ring of the algebraic torus, is multiplicity-free as a T-module, it follows that $H^0(U_\sigma, \mathcal{O}(D_\Delta))$ is also multiplicity-free. Thus, the T-module $H^0(U_\sigma, \mathcal{O}(D_\Delta))$ decomposes into one-dimensional irreducible representation as

$$H^0(U_\sigma, \mathcal{O}(D_\Delta)) = \bigoplus_{m \in T^+_{\Delta,\sigma} \cap M} \chi_m, \tag{5.4}$$

where as before $T^+_{\Delta,\sigma}$ denotes the inward-looking tangent cone of Δ at the face corresponding to σ. Similarly, χ_m appears in the space of global sections $H^0(X_\Sigma, \mathcal{O}(D_\Delta))$ if and only if $\langle m, v_\rho \rangle \leqslant -a_\rho$, for all $\rho \in \Sigma(1)$, and we have

$$H^0(X_\Sigma, \mathcal{O}(D_\Delta)) = \bigoplus_{m \in \Delta \cap M} \chi_m. \tag{5.5}$$

This implies that $\dim(H^0(X_\Sigma, \mathcal{O}(D_\Delta)) = |\Delta \cap M|$, the number of lattice points in M.

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5.2 Brianchon–Gram theorem and equivariant Euler characteristic

Let $\mathcal{F}$ be a T-linearized sheaf (of rational functions) on X_Σ, that is, for any T-invariant open set U, the space of sections $H^0(U, \mathcal{F})$ is a T-module and the restriction maps are T-equivariant. For $m \in M$ and V a T-module, let V_m denote the m-isotypic component of V. By the *equivariant Euler characteristic* of $\mathcal{F}$, we mean the function $\chi_T(X_\Sigma, \mathcal{F}) : M \to \mathbb{Z}_{\geq 0}$ given by

$$\chi_T(X_\Sigma, \mathcal{F})(m) = \sum_{i=0}^{n} (-1)^i \dim(H^i(X_\Sigma, \mathcal{F})_m).$$

Let us compute the equivariant Euler characteristic of the T-linearized sheaf $\mathcal{O}(D_\Delta)$. As explained above, for each cone $\sigma \in \Sigma$, the T-module $H^0(U_\sigma, \mathcal{O}(D_\Delta))$ decomposes as

$$H^0(U_\sigma, \mathcal{O}(D_\Delta)) = \bigoplus_{m \in T^+_{\Delta,\sigma} \cap M} \chi_m.$$

Recall that $T^+_{\Delta,\sigma}$ denotes the inward tangent cone of Δ at the face corresponding to σ (see Section 2.2).

From above, it follows that the equivariant Euler characteristic $\chi_T(X_\Sigma, \mathcal{O}(D_\Delta))$, computed using Čech cohomology, can be written as:

$$\chi_T(X_\Sigma, \mathcal{O}(D_\Delta)) = \sum_{\sigma \in \Sigma} (-1)^{\dim(Q_\sigma)} \mathbf{1}_{T^+_{\Delta,\sigma} \cap M}, \tag{5.6}$$

where as usual $\mathbf{1}_A$ denotes the characteristic function of a set A.

One knows that $\mathcal{O}(D_\Delta)$ is ample and hence $H^i(X_\Sigma, \mathcal{O}(D_\Delta)) = 0$ for $i > 0$. Thus, we also obtain

$$\chi_T(X_\Sigma, \mathcal{O}(D_\Delta))(m) = \dim(H^0(X_\Sigma, \mathcal{O}(D_\Delta))_m), \quad \forall m \in M. \tag{5.7}$$

And hence, from (5.5), we have

$$\chi_T(X_\Sigma, \mathcal{O}(D_\Delta)) = \mathbf{1}_{\Delta \cap M}. \tag{5.8}$$

Comparing with (5.6), one recovers the Brianchon–Gram theorem (Theorem 2.6).

The alternative version of the Brianchon–Gram theorem using outward face cones (Theorem 2.7) can also be obtained in a similar fashion. Let Δ' be the polytope with support numbers $a_\rho + 1$ and $D' = D_{\Delta'} = \sum_{\rho \in \Sigma(1)} -(a_\rho + 1) D_\rho$ the corresponding Cartier divisor. Note that $\langle x, v_\rho \rangle \leqslant -(a_\rho + 1)$ if and only $\langle -x, v_\rho \rangle > a_\rho$. Thus, for all $m \in M$, we have

$$\chi_T(X_\Sigma, \mathcal{O}(-D'))(m) = \sum_{\sigma \in \Sigma} (-1)^{n - \dim \sigma} \mathbf{1}_{T^-_{\Delta,\sigma} \cap M}(-m) \tag{5.9}$$

(recall (2.3) for defining inequalities of outward tangent cone $T^-_{\Delta,\sigma}$). On the other hand, the Khovanskii–Pukhlikov formula for inverse of the polytope Δ with respect to the convolution $*$ (see Section 2.6) tells us that:

$$\chi_T(X_\Sigma, \mathcal{O}(-D'))(m) = (-1)^n \chi_T(X_\Sigma, \mathcal{O}(D_\Delta))(-m) = (-1)^n \mathbf{1}_{\Delta \cap M}(-m). \tag{5.10}$$

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Putting together (5.9) and (5.10), we obtain

$$(-1)^n \mathbf{1}_{\Delta\cap M} = \sum_{\sigma\in\Sigma} (-1)^{n-\dim\sigma} \mathbf{1}_{T^-_{\Delta,\sigma}\cap M},$$

which immediately implies Theorem 2.7.

Remark 5.1 (A symplectic interpretation of the Brianchon–Gram theorem) We can also give a symplectic geometric interpretation of the Brianchon–Gram theorem, namely as an identity between Liouville measures. Let X be a symplectic manifold with a Hamiltonian S^1-action with moment map $\mu : X \to \mathbb{R}$. This means that the Hamiltonian vector field of μ generates the S^1-action. Let ε be a regular value of the moment map μ. Then $\mu^{-1}([\varepsilon,\infty))$ is a manifold with boundary. The *symplectic cut* $\overline{X}_{\mu\geq\varepsilon}$ is the manifold obtained by collapsing each S^1-orbit in the boundary $\mu^{-1}(\varepsilon)$ to a point.

We can decompose $T = (\mathbb{C}^*)^n$ as $T = (S^1)^n \times \mathbb{R}^n_{>0}$. Equip T with the standard symplectic form from $\mathbb{C}^n$. Each ray $\rho \in \Sigma(1)$ defines a Hamiltonian function $\mu_\rho : U_0 \to \mathbb{R}$ on $U_0 \cong T$ by

$$\mu_\rho(x) = |x|^{v_\rho} := |x_1|^{r_1}\cdots|x_n|^{r_n},$$

where $x = (x_1,\ldots,x_n)$ and $v_\rho = (r_1,\ldots,r_n)$. One verifies that the Hamiltonian vector field of μ_ρ generates the $\mathbb{C}^*$-action on T corresponding to the cocharacter $v_\rho \in N$. Let Σ be a smooth fan, let Δ be a rational polytope with normal fan Σ, and let a_ρ, $\rho \in \Sigma(1)$, be its support numbers. Starting with $(\mathbb{C}^*)^n$, doing repeated symplectic cuts with respect to the $\mu = \mu_\rho$ and $\varepsilon = a_\rho$, $\rho \in \Sigma(1)$, one arrives at the toric variety X_Σ. One can show that the open affine chart U_σ is the symplectic manifold obtained by symplectic cuts using rays of σ. Moreover, the image of the moment map of U_σ is the inward tangent cone $T^+_{\Delta,\sigma}$.

The Brianchon–Gram equality (2.5) can be thought of as an equality involving pushforwards (to $N_\mathbb{R} = \mathbb{R}^n$) of Liouville measures on all the symplectic manifolds U_σ and X_Σ.

5.3 Positive part of a toric variety and logarithm map

As before, let X_Σ be the toric variety associated with a rational fan Σ in $N_\mathbb{R}$. Take $\sigma \in \Sigma$. By definition, the set $U_\sigma(\mathbb{C})$ of points of U_σ defined over $\mathbb{C}$ is the set of maximal ideals of the semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$. This set then can be identified with $\mathrm{Hom}(\sigma^\vee \cap M, \mathbb{C})$, where Hom denotes the semigroup homomorphisms. This observation enables us to construct X^+_Σ, the points of X_Σ over the semigroup $\mathbb{R}_{\geq 0}$ (see [Fu93, Section 4.1]). We think of X^+_Σ as the "positive" part of $X_\Sigma(\mathbb{C})$. It is constructed as follows. For each $\sigma \in \Sigma$, let $U^+_\sigma = \mathrm{Hom}(\sigma^\vee \cap M, \mathbb{R}_{\geq 0})$. Then, as before, the U^+_σ glue together to give X^+_Σ. One has natural inclusion $X^+_\Sigma \hookrightarrow X_\Sigma(\mathbb{C})$. Moreover, the absolute value $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ induces a retraction map $X_\Sigma(\mathbb{C}) \to X^+_\Sigma$. Let $T_K = (S^1)^n$ denote the usual compact torus which is the maximal compact subgroup of T. One verifies that the retraction map induces a homeomorphism between the quotient $X_\Sigma(\mathbb{C})/T_K$ and X^+_Σ.

Another way to look at X_Σ^+ is as follows. Consider the *logarithm map*

$$\mathrm{Log} : T_N = (\mathbb{C}^*)^n \longrightarrow N_{\mathbb{R}} = \mathrm{Hom}(M, \mathbb{R})$$

defined as follows. For $z \in T$ and $m \in M$, let

$$\mathrm{Log}(z)(m) = \log(|\chi_m(z)|). \tag{5.11}$$

In the standard coordinates for $(\mathbb{C}^*)^n$, the logarithm map is given by

$$\mathrm{Log}(z_1, \ldots, z_n) = (\log|z_1|, \ldots, \log|z_n|). \tag{5.12}$$

For each $\sigma \in \Sigma$, the orbit O_σ can be identified with T/T_σ where T_σ is the T-stabilizer of O_σ. Let N_σ denote the cocharacter lattice of T_σ. It follows from the definitions that $N_\sigma \otimes \mathbb{R} = \mathrm{Span}(\sigma)$. The logarithm map then induces a map $\mathrm{Log}_\sigma : T/T_\sigma \to N_{\mathbb{R}}/\mathrm{Span}(\sigma)$. In the same way, that $X_\Sigma(\mathbb{C})$ is a disjoint union of the tori O_σ, $\sigma \in \Sigma$, the positive part X_Σ^+, is a disjoint union of the real vector spaces $N_{\mathbb{R}}/\mathrm{Span}(\sigma)$, $\sigma \in \Sigma$.

Finally, X_Σ^+ is actually homeomorphic to a polytope (in a nonunique way). Given a polytope Δ with normal fan Σ, one can construct explicitly a T_K-invariant continuous map $\mu : X_\Sigma \to \Delta$ such that the induced map $\bar{\mu} : X_\Sigma/T_K \to \Delta$ is a homeomorphism and the following diagram is commutative (see [Fu93, Section 4.2]).

$$\begin{array}{ccccccc} & & (\mathbb{C}^*)^n \cong U_0 & \hookrightarrow & X_\Sigma(\mathbb{C}) & & \\ & \swarrow^{\mathrm{Log}} & \downarrow \mathrm{Log} & & \downarrow \mathrm{Log} & \searrow^{\mu} & \\ N_{\mathbb{R}} & \xleftarrow{\cong} & \mathbb{R}^n \cong U_0^+ & \hookrightarrow & X_\Sigma^+ & \xrightarrow[\cong]{\bar{\mu}} & \Delta \end{array} \tag{5.13}$$

Moreover, the bottom row gives a homeomorphism between $N_{\mathbb{R}}$ and the interior Δ° of Δ. The map μ is a special case of the notion of *momentum map* from the theory of Hamiltonian group actions in symplectic geometry.

6 Geometric interpretations of combinatorial truncation

We propose two geometric interpretations of our combinatorial truncation in terms of geometric notions on toric varieties. The same ideas should extend to give geometric interpretations of Arthur's truncation and modified kernel. We expect that in this case one should replace a toric variety X_Σ by Mumford's compactification of a reductive algebraic group as in [KKMS73, Section IV.2].

6.1 Combinatorial truncation as a complex measure on a toric variety

In this section, we propose that combinatorial truncation can be interpreted as a "truncated" complex measure on a projective toric variety, obtained from the data of prescribed measures on each torus orbit as well as choice of a polytope normal to the fan which determines certain neighborhoods of the torus orbits.

As usual, let X_Σ be the toric variety associated with a (rational) fan Σ in $N_{\mathbb{R}}$. Recall that the starting data of combinatorial truncation are a collection of functions $\{K_\sigma : N_{\mathbb{R}} \to \mathbb{C} : \sigma \in \Sigma\}$, where each K_σ is invariant in the direction of $\mathrm{Span}(\sigma)$.

As before, let $T_K = (S^1)^n$ denote the compact torus in $T = (\mathbb{C}^*)^n$, which is the maximal compact subgroup of T. Suppose we are given a T_K-invariant complex

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measure $\omega_0 = f_0 d\mu_0$ on $U_0 = T$ where f_0 is a continuous function on U_0 and $d\mu_0$ denotes a Haar measure on U_0. Moreover, suppose, for each $\{0\} \neq \sigma \in \Sigma$, we have a T_K-invariant complex measure $\omega_\sigma = f_\sigma d\mu_\sigma$ on the torus orbit O_σ, the T-orbit in X_Σ associated with σ. Here, f_σ is a continuous function on O_σ, and $d\mu_\sigma$ is the Haar measure on O_σ induced from $d\mu_0$. Recall that $O_\sigma \cong T/T_\sigma$ is itself isomorphic to a torus, where $T_\sigma \subset T$ is the stabilizer of any point in O_σ. Since ω_σ, and hence f_σ, are T_K-invariant, the function f_σ induces a continuous function $k_\sigma : N_\mathbb{R}/\operatorname{Span}(\sigma) \longrightarrow \mathbb{C}$.

The projection $N_\mathbb{R} \to N_\mathbb{R}/\operatorname{Span}\sigma$ maps the cone σ to $\{0\}$. This gives us an equivariant morphism π_σ from the T-toric variety U_σ to the (T/T_σ)-toric variety O_σ (see [CLS11, Section 3.3]). We can use $\pi_\sigma : U_\sigma \to O_\sigma$ to extend the measure ω_σ to a measure Ω_σ on the affine toric chart $U_\sigma \subset X_\Sigma$ (and, in particular, on the open orbit $U_0 \cong T$) by defining

$$\Omega_\sigma = \pi_\sigma^*(\omega_\sigma).$$

The measure Ω_σ then gives a continuous function $K_\sigma : N_\mathbb{R} \to \mathbb{C}$ which is invariant in the direction of $\operatorname{Span}(\sigma)$.

Now, fix an inner product $\langle\cdot,\cdot\rangle$ on $N_\mathbb{R}$ and identify $M_\mathbb{R}$ with $N_\mathbb{R}$ via $\langle\cdot,\cdot\rangle$. As usual, take a polytope $\Delta \subset M_\mathbb{R} \cong N_\mathbb{R}$ with normal fan Σ. Recall that $\operatorname{Log} : T \to N_\mathbb{R}$ denotes the logarithm map on the torus, which extends to $\operatorname{Log} : X_\Sigma \to X_\Sigma^+$ (see (5.11) and the diagram (5.13)). Consider the tangent cone $T^-_{\Delta,\sigma}$. We regard it as an open subset of $U_0^+ \cong N_\mathbb{R} \cong \mathbb{R}^n$ and hence as an open subset of X_Σ^+. We have

$$U_{\Delta,\sigma} = \operatorname{Log}^{-1}(T^-_{\Delta,\sigma}).$$

We can also define the subset $U_\Delta \subset U_0$ by

$$U_\Delta = \operatorname{Log}^{-1}(\Delta).$$

We think of $\Omega_\sigma \mathbf{1}_{U_{\sigma,\Delta}}$ as an extension of the measure ω_σ to the neighborhood $U_{\Delta,\sigma}$. Finally, we can define a complex measure Ω_Δ on X_Σ by

$$\Omega_\Delta = \sum_{\sigma\in\Sigma} (-1)^{\dim\sigma} \Omega_\sigma \mathbf{1}_{U_{\Delta,\sigma}}.$$

It is a T_K-invariant complex measure on X_Σ and corresponds to the function k_Δ on $N_\mathbb{R}$. We think of it as a *truncation* of ω_0 with respect to the measures ω_σ at infinity. From Theorems 3.5 and 4.1, we have the following.

Proposition 6.1 *Under the assumptions in Theorem 3.5 on the functions K_σ, the total measure of Ω_Δ is finite and is a polynomial in the support numbers of Δ.*

Remark 6.2 In fact, each tangent cone $T^-_{\Delta,\sigma}$ gives us an open neighborhood of the orbit closure $\overline{O}_\sigma$ in X_Σ. To construct this open neighborhood, we complete $T^-_{\Delta,\sigma} \subset N_\mathbb{R}$ to an open subset $\tilde{T}_{\Delta,\sigma} \subset X_\Sigma^+$ containing the closure $\overline{O_\sigma^+}$ by

$$\tilde{T}_{\Delta,\sigma} = \bigcup_{\sigma':\sigma\preceq\sigma'} \bigcup_{\tau:\tau\preceq\sigma'} T^-_{Q_\tau,\sigma'} \subset X_\Sigma^+ := \bigsqcup_{\sigma\in\Sigma} N_\mathbb{R}/\operatorname{Span}(\sigma).$$

One verifies that $\tilde{T}_{\Delta,\sigma}$ is indeed an open subset of X_Σ^+ containing $\overline{O_\sigma^+}$. It follows that $\tilde{U}_{\Delta,\sigma} = \operatorname{Log}^{-1}(\tilde{T}_{\Delta,\sigma})$ is an open neighborhood of the orbit closure $\overline{O}_\sigma$ in the toric

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variety X_Σ. We note that $T^-_{\Delta,\sigma}$ is open dense in $\tilde{T}_{\Delta,\sigma}$, and hence, for the purposes of truncation, it does not matter whether we work with $T^-_{\Delta,\sigma}$ or $\tilde{T}_{\Delta,\sigma}$.

6.2 Combinatorial truncation as a Lefschetz number

In this section, we give an interpretation of the combinatorial truncation as a Lefschetz number.

6.2.1 Lefschetz number

Let X be a topological space such that all its cohomology groups $H^i(X,\mathbb{R})$ are finite dimensional and for some $n \geqslant 0$, $H^i(X,\mathbb{R}) = 0$, $\forall i > n$. Let $\Phi : X \to X$ be a continuous map. Recall that the *Lefschetz number* of Φ is defined to be

$$\Lambda(\Phi) = \sum_{i=0}^{n} (-1)^i \ \mathrm{Tr}(\Phi^* : H^i(X,\mathbb{R}) \to H^i(X,\mathbb{R})).$$

The Lefschetz number of the identity map is, by definition, equal to the Euler characteristic of X. The Lefschetz number appears in the Lefschetz fixed point theorem which states that if X is a compact triangulable space and $\Lambda(\Phi) \neq 0$, then Φ has at least one fixed point.

Let us define an analogue of the notion of Lefschetz number for morphisms of sheaves. Let $\mathcal{F}$ be a sheaf of vector spaces on X such that all the cohomology groups of $(X,\mathcal{F})$ are finite dimensional and for some n, $H^i(X,\mathcal{F}) = 0$, $\forall i > n$. By a *morphism of sheaves* $\Psi : \mathcal{F} \to \mathcal{F}$, we mean a collection of linear maps $\{\Psi_U : \mathcal{F}(U) \to \mathcal{F}(U) : U \subset X \text{ open }\}$ which are compatible with the restriction maps. That is, for $U \subset V$, we have

$$\Psi_U \circ \mathrm{rest}_{V,U} = \mathrm{rest}_{V,U} \circ \Psi_V.$$

Clearly, Ψ induces linear maps $\Psi^* : H^i(X,\mathcal{F}) \to H^i(X,\mathcal{F})$ between the cohomology groups of $(X,\mathcal{F})$. Extending the above notion of Lefschetz number, we make the following definition.

Definition 6.1 (Lefschetz number for morphisms of sheaves) The *Lefschetz number* $\Lambda(\Psi,\mathcal{F})$ is defined to be

$$\Lambda(\Psi,\mathcal{F}) = \sum_{i=0}^{n} (-1)^i \ \mathrm{Tr}(\Psi^* : H^i(X,\mathcal{F}) \to H^i(X,\mathcal{F})).$$

Remark 6.3 When Ψ is the identity morphism, i.e., all the maps Ψ_U are identities, then $\Lambda(\Psi,\mathcal{F})$ is just the *Euler characteristic* of the sheaf $\mathcal{F}$.

Let $\mathcal{U}$ be a finite open cover of X. Suppose $\mathcal{U}$ is a good open cover with respect to $\mathcal{F}$, that is, $\mathcal{F}$ is acyclic on any intersection of the open sets in $\mathcal{U}$. It is a standard result in topology that the Čech cohomology groups of $(\mathcal{U},\mathcal{F})$ are independent of the choice of the good open cover and coincide with the sheaf cohomology groups of $(X,\mathcal{F})$.

Suppose the vector spaces in the Čech cochain complex $C^\bullet(\mathcal{U},\mathcal{F})$ are finite dimensional. In other words, for any collection of open sets $U_1,\dots,U_k \in \mathcal{U}$, we have $\dim H^0(U_1 \cap \dots \cap U_k,\mathcal{F}) < \infty$. In this case, the Lefschetz number can be computed in terms of the traces of the vector spaces in the cochain complex $C^\bullet(\mathrm{U},\mathcal{F})$ as well. This straightforward result is sometimes referred to as the Hopf trace formula.

Proposition 6.4 *With assumptions as above, the Lefschetz number can be computed as*

$$\Lambda(\Psi) = \sum_{i=0}^{n}(-1)^i \operatorname{Tr}(\Psi^* : C^i(\mathcal{U},\mathcal{F}) \to C^i(\mathcal{U},\mathcal{F})),$$

where $C^i(\mathcal{U},\mathcal{F})$ denotes the vector space of ith Čech cochains of $\mathcal{U}$ with coefficients in $\mathcal{F}$.

Similarly, suppose X is equipped with a measure and $\mathcal{F}$ a sheaf of L^2-functions on X, and let $\Psi : \mathcal{F} \to \mathcal{F}$ be a morphism of sheaves. Moreover, suppose for every open set U, the linear operator $\Psi : \mathcal{F}(U) \to \mathcal{F}(U)$ is a trace class operator with kernel function K_U. Then, for each i, the induced map $\Psi^* : H^i(X,\mathcal{F}) \to H^i(X,\mathcal{F})$ is also a trace class operator. We denote its kernel by T_i.

Definition 6.2 (Lefschetz number for morphisms of sheaves of L^2-functions) We define the *Lefschetz number* $\Lambda(\Psi,\mathcal{F})$ by

$$\Lambda(\Psi,\mathcal{F}) = \int_X \sum_{i=0}^{n}(-1)^i T_i(x)\,dx. \tag{6.1}$$

As above, let $\mathcal{U}$ be a finite open cover of X which is a good cover with respect to $\mathcal{F}$. Suppose, for each i, the operator $\Psi^* : C^i(\mathcal{U},\mathcal{F}) \to C^i(\mathcal{U},\mathcal{F})$ is trace class with kernel K_i. Similarly to Proposition 6.4, the Lefschetz number $\Lambda(\Psi,\mathcal{F})$ can be computed as

$$\Lambda(\Psi,\mathcal{F}) = \int_X \sum_{i=0}^{n}(-1)^i K_i(x)\,dx.$$

The observation in this section is that when $X = X_\Sigma$ is a toric variety, the Lefschetz number is given by a combinatorial truncation $J_\Sigma(\Delta)$. As usual, let Σ be a (rational) fan in $N_\mathbb{R}$, and let $\Delta \in \mathcal{P}(\Sigma)$ be a polytope with normal fan Σ. As in Section 5.1, let X_Σ be the toric variety of the fan Σ and $\mathcal{O}(D_\Delta)$ be the sheaf of sections of the (Cartier) divisor D_Δ associated with Δ. Let the a_ρ, $\rho \in \Sigma(1)$, be the support numbers of Δ. Let Δ' be the polytope whose support numbers are the $a_\rho - 1$. Let $\Psi : \mathcal{O}(-D_{\Delta'}) \to \mathcal{O}(-D_{\Delta'})$ be a morphism of sheaves.

Recall that the characters χ_m, $m \in M$, form a vector space basis for $\mathbb{C}[U_0]$. Moreover, a subset of this basis is a basis for $\mathcal{O}(-D_{\Delta'})$. For $m \in M$, let $K_\sigma(m)$ be the (m,m)-entry of the matrix of the linear operator $\Psi_\sigma : \mathcal{O}(-D_{\Delta'})(U_\sigma) \to \mathcal{O}(-D_{\Delta'})(U_\sigma)$. The following follows from Section 5.2 and in particular (5.9).

Proposition 6.5 (Combinatorial truncation as a Lefschetz number on a toric variety) *With notation as above, the Lefschetz number $\Lambda(\Psi,\mathcal{O}(-D_{\Delta'}))$ is equal to the truncated sum $S_\Sigma(\Delta,M)$:*

$$\Lambda(\Psi,\mathcal{O}(-D_{\Delta'})) = S_\Sigma(\Delta,M) := \sum_{m\in M}\sum_{\sigma\in\Sigma}(-1)^{\dim\sigma}K_\sigma(m)\,\mathbf{1}_{T^-_{\Delta,\sigma}\cap M}(m).$$

Remark 6.6 The reason for the appearance of the polytope Δ' instead of Δ is that we defined the outward tangent cones $T^-_{\Delta,\sigma}$ using strict inequalities. If we change the convention and use nonstrict inequalities in the definition of $T^-_{\Delta,\sigma}$, then Proposition 5.9 holds with D in place of D'.

Finally, as a side remark, we also mention an example of a presheaf that is reminiscent of Arthur's construction of the kernels K_P (see [Ar05, Section 4]).

Example 6.7 (A sheaf of W-invariant sections on the toric variety of Weyl fan) Suppose Σ is the Weyl fan and hence the Weyl group acts on Σ. Note that by definition W acts on the character lattice M. For $\sigma \in \Sigma$, let W_σ be the W-stabilizer of σ. Let $\mathcal{O}(\Delta)$ be the invertible sheaf associated with a W-invariant polytope Δ. We define the sheaf $\mathcal{O}(\Delta)^W$ by

$$H^0(U_\sigma, \mathcal{O}(\Delta)^W) := H^0(U_\sigma, \mathcal{O}(\Delta))^{W_\sigma}, \quad \forall \sigma \in \Sigma.$$

Let $\tau \subset \sigma$ be cones in Σ. Note that $W_\sigma \subset W_\tau$ and hence if $f \in H^0(U_\sigma, \mathcal{O}(\Delta))^{W_\sigma}$, then, in general, $f_{|U_\tau}$ may not be W_τ-invariant and hence may not lie in $H^0(U_\tau, \mathcal{O}(\Delta))^{W_\tau}$. We remedy this by defining the restriction map $i_{\sigma\tau} : H^0(U_\sigma, \mathcal{O}(\Delta)^W) \to H^0(U_\tau, \mathcal{O}(\Delta)^W)$ by:

$$i_{\sigma\tau}(f) = \sum_{w \in W_\tau / W_\sigma} (w \cdot f)_{|U_\tau}.$$

Let us verify that the above restriction maps $i_{\sigma\tau}$ give a well-defined presheaf on X_Σ. Suppose we have cones $\gamma \subset \tau \subset \sigma$ in Σ with corresponding affine charts $U_\gamma \subset U_\tau \subset U_\sigma$. We need to show $i_{\tau\gamma} \circ i_{\sigma\tau} = i_{\sigma\gamma}$. Let $f \in H^0(U_\sigma, \mathcal{O}(\Delta)^W)$. We have

$$i_{\tau\gamma}(i_{\sigma\tau}(f)) = \sum_{w \in W_\gamma / W_\tau} \sum_{w' \in W_\tau / W_\sigma} (ww') \cdot f.$$

As w (resp. w') runs over a set of representatives for W_γ / W_τ (resp. W_τ / W_σ), the product ww' runs over a set of representatives for W_γ / W_σ. This proves the claim.

It is interesting to compute the Euler characteristic and Čech cohomologies of the above presheaf.

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Department of Mathematics, Oklahoma State University, Stillwater, OK, USA
e-mail: asgari@math.okstate.edu

Department of Mathematics, University of Pittsburgh, Pittsburgh, PA, USA
e-mail: kaveh@pitt.edu