2

A charge coupled to its electromagnetic field

We plan to study the dynamics of a well-localized charge, like an electron or a proton, when coupled to its own electromagnetic field. The case of several particles is reserved for chapter 11. In a first attempt, one models the particle as a point charge with a definite mass. If its world line is prescribed, then the fields are determined through the inhomogeneous Lorentz–Maxwell equations. On the other hand, if the electromagnetic fields are given, then the motion of the point charge is governed by Newton's equation of motion with the Lorentz force as force law. While it then seems obvious how to marry the two equations, such as to have a coupled dynamics for the charge and its electromagnetic field, ambiguities and inconsistencies arise due to the infinite electrostatic energy of the Coulomb field of the point charge. Thus one is forced to introduce a slightly smeared charge distribution, i.e. an extended charge model. Mathematically this means that the interaction between particle and field is cut off or regularized at short distances, which seems to leave a lot of arbitrariness. There are also strong constraints, however. In particular, local charge conservation must be satisfied, the theory should be of Lagrangian form, and it should reproduce the two limiting cases mentioned already. In addition, as expected from any decent physical model, the theory should be well defined and empirically accurate within its domain of validity. In fact, up to the present time only two models have been worked out in some detail: (i) the semirelativistic Abraham model of a rigid charge distribution; and (ii) the Lorentz model of a relativistically covariant extended charge distribution. The aim of this chapter is to introduce and explain both models at some length. On the way we recall a few properties of the inhomogeneous Lorentz–Maxwell equations for later use.

A short preamble on units and other conventions is in order. We use the Heaviside–Lorentz units. In particular, the Coulomb potential is simply the inverse of the Laplacian with no extra factor. The vacuum susceptibilities are $\varepsilon_0 = 1 = \mu_0$, which fixes the unit of charge. c is the speed of light. Mostly we will set $c = 1$ for convenience, thereby linking the units of space and time. If needed, one can easily

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

retrieve these natural constants in the conventional way. At some parts below we will do this without notice, so as to have the dimensions right and to keep better track of the order of magnitudes. In the nonrelativistic setting we use $\nabla\times$ for rotation, but switch to the more proper exterior derivative, $\nabla_{\mathrm{g}}\wedge$, with $\mathbf{g}$ the metric tensor, in the relativistic context. We will use standard notation as often as possible. Since a fairly broad spectrum of material is covered, double meaning cannot be avoided entirely. At the risk of some repetition we strive for minimal ambiguity within a given chapter. In the classical part of the book we use boldface italic letters, $\boldsymbol{x}$, for three-vectors and boldface roman letters, $\mathbf{x}$, for four-vectors. In the quantum section such a notation tends to be cumbersome and we use lightface letters, x, throughout.

2.1 The inhomogeneous Maxwell–Lorentz equations

We prescribe a charge density, $\rho(\boldsymbol{x},t)$, and an associated current, $\boldsymbol{j}(\boldsymbol{x},t)$, linked through the law of charge conservation

$$\partial_t\,\rho(\boldsymbol{x},t)+\nabla\cdot\boldsymbol{j}(\boldsymbol{x},t)=0\,. \tag{2.1}$$

Of course, $\boldsymbol{x}\in\mathbb{R}^3$ and $t\in\mathbb{R}$, where we use $\mathbb{R}^3$ to describe physical space and $\mathbb{R}$ as the time axis. The Maxwell equations for the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{B}$ consist of the two evolution equations

$$\begin{aligned} c^{-1}\partial_t\boldsymbol{B}(\boldsymbol{x},t)&=-\nabla\times\boldsymbol{E}(\boldsymbol{x},t)\,,\\ c^{-1}\partial_t\boldsymbol{E}(\boldsymbol{x},t)&=\nabla\times\boldsymbol{B}(\boldsymbol{x},t)-c^{-1}\boldsymbol{j}(\boldsymbol{x},t) \end{aligned} \tag{2.2}$$

and the two constraints

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{x},t)=\rho(\boldsymbol{x},t)\,,\quad \nabla\cdot\boldsymbol{B}(\boldsymbol{x},t)=0\,. \tag{2.3}$$

$\diamond$ *How are the Maxwell equations written and named?* According to my survey, there seems to be no universally accepted standard. As indicated by the name "electromagnetic", the order $\boldsymbol{E},\boldsymbol{B}$ is very common and also adopted here. In the Lagrangian version $\boldsymbol{B}$ is position-like and $-\boldsymbol{E}$ is velocity-like, which would suggest the opposite order, namely $(\boldsymbol{B},-\boldsymbol{E})$. In the nineteenth century the time-derivative was written at the right side of the equation. By present standards, in evolution equations like the Boltzmann, Navier–Stokes, and Schrödinger equation, the time-derivative is always at the left, which is also our convention here.

The common practice is to call the first equation of (2.2) together with the second equation of (2.3) the "homogeneous Maxwell equations" and the remaining

Paragraphs indicated by $\diamond$ give explanations of notation and names.

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

pair the "inhomogeneous Maxwell equations". We follow here the convention used in the context of wave equations and call (2.2) with $\boldsymbol{j} = 0$ the "homogeneous Maxwell–Lorentz equations" and (2.2) with $\boldsymbol{j} \neq 0$ the "inhomogeneous Maxwell–Lorentz equations". The constraints (2.3) are always understood. "Maxwell–Lorentz equations" and "Maxwell equations" are used synonymously. ◇

We solve the Maxwell equations as a Cauchy problem, i.e. by prescribing the fields at time $t = 0$. If the constraints (2.3) are satisfied at $t = 0$, then by the continuity equation (2.1) they are satisfied at all times. Thus the initial data are

$$\boldsymbol{E}(\boldsymbol{x}, 0)\,, \quad \boldsymbol{B}(\boldsymbol{x}, 0) \tag{2.4}$$

together with the constraints

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, 0) = \rho(\boldsymbol{x}, 0)\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, 0) = 0\,. \tag{2.5}$$

The choice $t = 0$ is merely a convention. In some cases it is preferable to prescribe the fields either in the remote past or the distant future. We will only consider physical situations where the fields decay at spatial infinity and thus have the finite energy

$$\mathcal{E} = \frac{1}{2} \int \mathrm{d}^3x \left(\boldsymbol{E}(\boldsymbol{x}, t)^2 + \boldsymbol{B}(\boldsymbol{x}, t)^2 \right) < \infty\,. \tag{2.6}$$

In a thermal state at nonzero temperature, typical fields fluctuate without decay and one would be forced to consider infinite-energy solutions.

The Maxwell equations (2.2), (2.3) are inhomogeneous wave equations and are thus easy to solve. This will be done in Fourier space first, where the Fourier transform is denoted by $\widehat{\ }$ and defined through

$$\widehat{f}(\boldsymbol{k}) = (2\pi)^{-n/2} \int \mathrm{d}^n x\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} f(\boldsymbol{x})\,. \tag{2.7}$$

Then, setting $c = 1$, (2.2) becomes

$$\begin{aligned} \partial_t \widehat{\boldsymbol{B}}(\boldsymbol{k}, t) &= -\mathrm{i}\boldsymbol{k} \times \widehat{\boldsymbol{E}}(\boldsymbol{k}, t)\,, \\ \partial_t \widehat{\boldsymbol{E}}(\boldsymbol{k}, t) &= \mathrm{i}\boldsymbol{k} \times \widehat{\boldsymbol{B}}(\boldsymbol{k}, t) - \widehat{\boldsymbol{j}}(\boldsymbol{k}, t) \end{aligned} \tag{2.8}$$

with the constraints

$$\mathrm{i}\boldsymbol{k} \cdot \widehat{\boldsymbol{E}}(\boldsymbol{k}, t) = \widehat{\rho}(\boldsymbol{k}, t)\,, \quad \mathrm{i}\boldsymbol{k} \cdot \widehat{\boldsymbol{B}}(\boldsymbol{k}, t) = 0 \tag{2.9}$$

and the conservation law

$$\partial_t \widehat{\rho}(\boldsymbol{k}, t) + \mathrm{i}\boldsymbol{k} \cdot \widehat{\boldsymbol{j}}(\boldsymbol{k}, t) = 0\,. \tag{2.10}$$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

To solve the inhomogeneous equations (2.8), we rely, as usual, on the solution of the homogeneous equations,

$$\widehat{\boldsymbol{E}}_0(\boldsymbol{k},t) = \big(\cos|\boldsymbol{k}|t + (1-\cos|\boldsymbol{k}|t)\widehat{\boldsymbol{k}}\otimes\widehat{\boldsymbol{k}}\big)\widehat{\boldsymbol{E}}(\boldsymbol{k},0) + \Big(\frac{1}{|\boldsymbol{k}|}\sin|\boldsymbol{k}|t\Big)\mathrm{i}\boldsymbol{k}\times\widehat{\boldsymbol{B}}(\boldsymbol{k},0)\,,$$
$$\widehat{\boldsymbol{B}}_0(\boldsymbol{k},t) = \big(\cos|\boldsymbol{k}|t + (1-\cos|\boldsymbol{k}|t)\widehat{\boldsymbol{k}}\otimes\widehat{\boldsymbol{k}}\big)\widehat{\boldsymbol{B}}(\boldsymbol{k},0) - \Big(\frac{1}{|\boldsymbol{k}|}\sin|\boldsymbol{k}|t\Big)\mathrm{i}\boldsymbol{k}\times\widehat{\boldsymbol{E}}(\boldsymbol{k},0)\,. \tag{2.11}$$

Here $\widehat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$ is the unit vector along $\boldsymbol{k}$ and for any pair of vectors $\boldsymbol{a},\boldsymbol{b}$, $\boldsymbol{a}\otimes\boldsymbol{b}$ is the tensor of rank 2 defined through $(\boldsymbol{a}\otimes\boldsymbol{b})\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$ as acting on the vector $\boldsymbol{c}$.

We insert (2.11) in the time-integrated version of (2.8). Taking account of the constraints, making a partial integration, and using charge conservation, we arrive at

$$\begin{aligned}\widehat{\boldsymbol{E}}(\boldsymbol{k},t) &= (\cos|\boldsymbol{k}|t)\widehat{\boldsymbol{E}}(\boldsymbol{k},0) + (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t)\mathrm{i}\boldsymbol{k}\times\widehat{\boldsymbol{B}}(\boldsymbol{k},0)\\ &\quad + \int_0^t \mathrm{d}s\Big(-(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))\mathrm{i}\boldsymbol{k}\hat{\rho}(\boldsymbol{k},s) - (\cos|\boldsymbol{k}|(t-s))\widehat{\boldsymbol{j}}(\boldsymbol{k},s)\Big)\\ &= \widehat{\boldsymbol{E}}_{\mathrm{ini}}(\boldsymbol{k},t) + \widehat{\boldsymbol{E}}_{\mathrm{ret}}(\boldsymbol{k},t)\,, \qquad (2.12)\\ \widehat{\boldsymbol{B}}(\boldsymbol{k},t) &= (\cos|\boldsymbol{k}|t)\widehat{\boldsymbol{B}}(\boldsymbol{k},0) - (|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t)\mathrm{i}\boldsymbol{k}\times\widehat{\boldsymbol{E}}(\boldsymbol{k},0)\\ &\quad + \int_0^t \mathrm{d}s(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|(t-s))\mathrm{i}\boldsymbol{k}\times\widehat{\boldsymbol{j}}(\boldsymbol{k},s)\\ &= \widehat{\boldsymbol{B}}_{\mathrm{ini}}(\boldsymbol{k},t) + \widehat{\boldsymbol{B}}_{\mathrm{ret}}(\boldsymbol{k},t)\,. \qquad (2.13)\end{aligned}$$

The first terms are the initial fields propagated up to time t, while the second terms are the retarded fields. If one wanted to solve the Maxwell equations run into the past, then the retarded fields should be replaced by the advanced fields.

Next, let us introduce the fundamental propagator, $G_t(\boldsymbol{x})$, of the wave equation which is defined as the Fourier transform of $(2\pi)^{-3/2}\,|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|t$ and satisfies

$$\partial_t^2\, G - \Delta G = \delta(\boldsymbol{x})\delta(t)\,. \tag{2.14}$$

This means $G_t(\boldsymbol{x}) = (2\pi)^{-1}\,\delta(|\boldsymbol{x}|^2 - t^2)$ and in particular for $t > 0$

$$G_t(\boldsymbol{x}) = \frac{1}{4\pi t}\delta(|\boldsymbol{x}| - t)\,. \tag{2.15}$$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

Then in physical space the solution (2.12), (2.13) of the inhomogeneous Maxwell–Lorentz equations reads as

$$\begin{aligned}\boldsymbol{E}(t) &= \partial_t G_t * \boldsymbol{E}(0) + \nabla \times G_t * \boldsymbol{B}(0) - \int_0^t \mathrm{d}s \big(\nabla G_{t-s} * \rho(s) + \partial_t G_{t-s} * \boldsymbol{j}(s)\big) \\ &= \boldsymbol{E}_{\mathrm{ini}}(t) + \boldsymbol{E}_{\mathrm{ret}}(t)\,, \qquad (2.16)\\ \boldsymbol{B}(t) &= \partial_t G_t * \boldsymbol{B}(0) - \nabla \times G_t * \boldsymbol{E}(0) + \int_0^t \mathrm{d}s \nabla \times G_{t-s} * \boldsymbol{j}(s) \\ &= \boldsymbol{B}_{\mathrm{ini}}(t) + \boldsymbol{B}_{\mathrm{ret}}(t)\,. \qquad (2.17)\end{aligned}$$

Here $*$ denotes convolution, i.e. $f_1 * f_2(\boldsymbol{x}) = \int \mathrm{d}^n y f_1(\boldsymbol{x}-\boldsymbol{y}) f_2(\boldsymbol{y})$.

For later purposes it will be convenient to have a more concise notation. In matrix form, the solution of the homogeneous Maxwell–Lorentz equations can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\boldsymbol{E}(t)\\ \boldsymbol{B}(t)\end{pmatrix} = \begin{pmatrix}0 & \nabla\times\\ -\nabla\times & 0\end{pmatrix}\begin{pmatrix}\boldsymbol{E}(t)\\ \boldsymbol{B}(t)\end{pmatrix}\,, \quad \frac{\mathrm{d}}{\mathrm{d}t}F(t) = AF(t) \qquad (2.18)$$

with the column vector $F = (\boldsymbol{E}, \boldsymbol{B})$. They have the solution

$$F(t) = U(t)F(0)\,, \quad U(t) = \mathrm{e}^{At} \qquad (2.19)$$

with $U(t)$ given explicitly by the terms with subscripts 'ini' in (2.17), (2.16). If we set $g(t) = (\boldsymbol{j}(t), 0)$ as a column vector, then

$$\frac{\mathrm{d}}{\mathrm{d}t}F(t) = AF(t) - g(t)\,, \quad F(t) = U(t)F(0) - \int_0^t \mathrm{d}s\, U(t-s)g(s)\,. \qquad (2.20)$$

The expressions (2.16), (2.17) remain meaningful even in case $\rho, \boldsymbol{j}$ are generated by the motion of a single point charge. Let us denote by $\boldsymbol{q}(t)$ the position and by $\boldsymbol{v}(t) = \dot{\boldsymbol{q}}(t)$ the velocity of the particle carrying charge e. Then

$$\rho(\boldsymbol{x}, t) = e\delta(\boldsymbol{x} - \boldsymbol{q}(t))\,, \quad \boldsymbol{j}(\boldsymbol{x}, t) = e\delta(\boldsymbol{x} - \boldsymbol{q}(t))\boldsymbol{v}(t)\,. \qquad (2.21)$$

Upon inserting this in (2.16), (2.17) one arrives at the Liénard–Wiechert fields. Since their derivation is presented in most textbooks, we do not repeat the computation here and only discuss the result. We take the world line, $t \mapsto \boldsymbol{q}(t)$, of the particle to be given for all times. Since the particle is assumed to have a relativistic kinetic energy, $|\dot{\boldsymbol{q}}(t)| < 1$. Next we prescribe the initial data for the fields at time $t = t_0$ and take the limit $t_0 \to -\infty$ in (2.16), (2.17). Then, at a fixed space-time point $(\boldsymbol{x}, t)$, the contribution from the initial fields vanishes and the retarded fields become the Liénard–Wiechert fields. To describe them we introduce the retarded

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

time $t_{\rm ret}$, depending on $\boldsymbol{x}, t$, as the unique solution of

$$t_{\rm ret} = t - |\boldsymbol{x} - \boldsymbol{q}(t_{\rm ret})| \,. \tag{2.22}$$

$t_{\rm ret}$ is then the uniquely defined time point at which the world line crosses the backward light cone with apex at $(\boldsymbol{x}, t)$. Furthermore, we introduce the unit vector

$$\widehat{\boldsymbol{n}} = \frac{\boldsymbol{x} - \boldsymbol{q}(t_{\rm ret})}{|\boldsymbol{x} - \boldsymbol{q}(t_{\rm ret})|} \,. \tag{2.23}$$

Then the electric field generated by the moving point charge is given by

$$\boldsymbol{E}(\boldsymbol{x}, t) = \frac{e}{4\pi}\Big[\frac{(1 - \boldsymbol{v}^2)(\widehat{\boldsymbol{n}} - \boldsymbol{v})}{(1 - \boldsymbol{v}\cdot\widehat{\boldsymbol{n}})^3 |\boldsymbol{x} - \boldsymbol{q}|^2} + \frac{\widehat{\boldsymbol{n}} \times [(\widehat{\boldsymbol{n}} - \boldsymbol{v}) \times \dot{\boldsymbol{v}}]}{(1 - \boldsymbol{v}\cdot\widehat{\boldsymbol{n}})^3 |\boldsymbol{x} - \boldsymbol{q}|} \Big]\Big|_{t = t_{\rm ret}} \tag{2.24}$$

and the corresponding magnetic field is

$$\boldsymbol{B}(\boldsymbol{x}, t) = \widehat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{x}, t) \,. \tag{2.25}$$

Equations (2.24) and (2.25) are less explicit than they appear to be, since $t_{\rm ret}$ depends through (2.22) on the reference point $(\boldsymbol{x}, t)$ and the particle trajectory. The first contribution in (2.24) is proportional to $|\boldsymbol{x} - \boldsymbol{q}|^{-2}$ and independent of the acceleration. This is the near field, which in a certain sense remains attached to the particle all through its motion. The second contribution is proportional to $|\boldsymbol{x} - \boldsymbol{q}|^{-1}$ as well as to the acceleration. This is the far field, which carries the information on the radiation field escaping to infinity. Whenever $\boldsymbol{q}(t)$ is smooth in t, the Liénard–Wiechert fields are also smooth functions except at $\boldsymbol{x} = \boldsymbol{q}(t)$, where they diverge as $|\boldsymbol{x} - \boldsymbol{q}(t)|^{-2}$. The corresponding potentials have a Coulomb singularity at the world line of the particle.

2.2 Newton's equations of motion

We take now the point of view that the electromagnetic fields $\boldsymbol{E}$, $\boldsymbol{B}$ are given. The motion of a charged particle, with charge e, position $\boldsymbol{q}(t)$, and velocity $\boldsymbol{v}(t)$, is then governed by Newton's equations of motion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_0 \gamma \boldsymbol{v}(t)\big) = e\big(\boldsymbol{E}(\boldsymbol{q}(t), t) + c^{-1}\boldsymbol{v}(t) \times \boldsymbol{B}(\boldsymbol{q}(t), t)\big) \,, \tag{2.26}$$

$\gamma(\boldsymbol{v}) = 1/\sqrt{1 - (\boldsymbol{v}/c)^2}$, which as an ordinary differential equation has to be supplemented with the initial conditions $\boldsymbol{q}(0)$, $\boldsymbol{v}(0)$. The force law is determined through the Lorentz force and thus (2.26) is also called the Newton–Lorentz equations. The particle is relativistic with rest mass m_0 as measured through the response to external forces. Once the particle is dynamically coupled to the Maxwell field, m_0 will attain a new meaning.

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

The $(\boldsymbol{E}, \boldsymbol{B})$ fields in (2.26) are not completely arbitrary. They are subject to the Maxwell equations with source $(\rho, \boldsymbol{j})$. In other words, we have divided all charges into a single charged particle whose motion is determined through (2.26) and the rest whose motion is taken to be known.

The Newton–Lorentz equations (2.26) are of Hamiltonian form. To see this we introduce vector potentials ϕ, $\boldsymbol{A}$ such that

$$\boldsymbol{E}(\boldsymbol{x}, t) = -\nabla\phi(\boldsymbol{x}, t) - c^{-1}\partial_t \boldsymbol{A}(\boldsymbol{x}, t)\,, \quad \boldsymbol{B}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{A}(\boldsymbol{x}, t)\,. \tag{2.27}$$

Then the Lagrangian associated with (2.26) is

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = -m_0 c^2 (1 - c^{-2}\dot{\boldsymbol{q}}^2)^{1/2} - e(\phi(\boldsymbol{q}, t) - c^{-1}\dot{\boldsymbol{q}} \cdot \boldsymbol{A}(\boldsymbol{q}, t))\,. \tag{2.28}$$

To switch to the Hamiltonian framework, one introduces the canonical momentum

$$\boldsymbol{p} = m_0 \gamma(\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{q}, t) \tag{2.29}$$

and obtains the Hamiltonian function

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \left((c\,\boldsymbol{p} - e\boldsymbol{A}(\boldsymbol{q}, t))^2 + m_0^2 c^4\right)^{1/2} + e\phi(\boldsymbol{q}, t)\,. \tag{2.30}$$

In particular, whenever the fields are time independent, the energy

$$\mathcal{E}(\boldsymbol{q}, \boldsymbol{v}) = m_0 \gamma(\boldsymbol{v}) + e\phi(\boldsymbol{q}) \tag{2.31}$$

is conserved along the solution trajectories of (2.26).

It should be noted that in general the solutions to Newton's equations of motion (2.26) will have a complicated structure even for time-independent fields. This has been amply demonstrated for particular cases. Depending on how the external fields are chosen, the motion would range from regular to fully chaotic with a mixed phase space as a rule.

2.3 Coupled Maxwell's and Newton's equations

While for most practical purposes, barring a few exceptional cases, it suffices to use either Maxwell's equations with prescribed sources or Newton's equations with prescribed forces, from a more fundamental point of view such a procedure is unsatisfactory. Physically it would seem more natural to have a coupled system of equations for the time evolution of the charged particles together with their electromagnetic field and to regard the two cases discussed above as emerging limit situations. If for the moment we restrict ourselves to a single particle, it is obvious

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

how to proceed. From (2.2), (2.3) we have

$$\begin{aligned}\partial_t \boldsymbol{B}(\boldsymbol{x},t) &= -\nabla\times\boldsymbol{E}(\boldsymbol{x},t)\,,\\ \partial_t \boldsymbol{E}(\boldsymbol{x},t) &= \nabla\times\boldsymbol{B}(\boldsymbol{x},t) - e\delta(\boldsymbol{x}-\boldsymbol{q}(t))\boldsymbol{v}(t)\end{aligned} \tag{2.32}$$

with the constraints

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{x},t) = e\delta(\boldsymbol{x}-\boldsymbol{q}(t))\,,\quad \nabla\cdot\boldsymbol{B}(\boldsymbol{x},t) = 0\,. \tag{2.33}$$

Moreover, from (2.26) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_0\gamma\boldsymbol{v}(t)\big) = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{E}(\boldsymbol{q}(t),t) + \boldsymbol{v}(t)\times(\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{B}(\boldsymbol{q}(t),t))\big)\,. \tag{2.34}$$

We added the external electromagnetic fields $\boldsymbol{E}_{\mathrm{ex}}$, $\boldsymbol{B}_{\mathrm{ex}}$, which will play a prominent role later on. They are derived from potentials as

$$\boldsymbol{E}_{\mathrm{ex}} = -\nabla\phi_{\mathrm{ex}}\,,\quad \boldsymbol{B}_{\mathrm{ex}} = \nabla\times\boldsymbol{A}_{\mathrm{ex}}\,. \tag{2.35}$$

We assume the potentials to be time independent for simplicity, although a considerable part of the theory to be developed will work also for time-dependent fields. As before, (2.32)–(2.34) are to be solved as an initial value problem. Thus $\boldsymbol{E}(\boldsymbol{x},0)$, $\boldsymbol{B}(\boldsymbol{x},0)$, $\boldsymbol{q}(0)$, and $\boldsymbol{v}(0)$ are supposed to be given. Note that the continuity equation is satisfied by fiat.

Equations (2.32), (2.34) are the stationary points of a Lagrangian action, which strengthens our trust in these equations, since every microscopic classical evolution equation seems to be of that form. We continue to use the underlying electromagnetic potentials as in (2.27), (2.35). Then the action for (2.32), (2.34) reads

$$\begin{aligned}\mathcal{A}([\boldsymbol{q},\phi,\boldsymbol{A}]) = \int \mathrm{d}t\Big[&- m_0(1-\dot{\boldsymbol{q}}(t)^2)^{1/2} - e\big(\phi_{\mathrm{ex}}(\boldsymbol{q}(t)) + \phi(\boldsymbol{q}(t),t)\\ &- \dot{\boldsymbol{q}}(t)\cdot(\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{A}(\boldsymbol{q}(t),t))\big)\Big]\\ &+ \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}^3x\Big[(\nabla\phi(\boldsymbol{x},t) + \partial_t\boldsymbol{A}(\boldsymbol{x},t))^2 - (\nabla\times\boldsymbol{A}(\boldsymbol{x},t))^2\Big]\,.\end{aligned} \tag{2.36}$$

The only difficulty is that (2.32) and (2.33) taken together with (2.34) make no proper mathematical sense. As explained, the solution of the Maxwell equations is singular at $\boldsymbol{x} = \boldsymbol{q}(t)$, and in the Lorentz force we are asked to evaluate the fields precisely at that point. One might be tempted to put the blame on the mathematics which refuses to handle equations as singular as (2.32)–(2.34). However before such a drastic conclusion is drawn, the physics should be properly understood. The point charge carries along with it a potential which at short distances diverges as

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

the Coulomb potential, cf. (2.24), and which therefore has the electrostatic energy

$$\frac{1}{2}\int\limits_{\{|\boldsymbol{x}-\boldsymbol{q}(t)|\leq R\}} \mathrm{d}^3x\,\boldsymbol{E}(\boldsymbol{x},t)^2 \simeq \int\limits_0^R \mathrm{d}r r^2 (r^{-2})^2 = \int\limits_0^R \mathrm{d}r r^{-2} = \infty\,. \qquad (2.37)$$

Taken literally, such an object would have an infinite mass and hence would not respond to external forces. It would keep its velocity for ever, which is inconsistent with what is observed.

Thus we are forced to regularize at short distances the coupled system consisting of the Maxwell equations and Newton's equation of motion with the Lorentz force.

In carrying out such a program there are two, in part, complementary points of view. The first one, which we will *not* follow here, starts from the idea that regularization is a mathematical device with the sole purpose of making sense of a singular mathematical object through a suitable limiting procedure. To illustrate this approach we can think of the following prominent mathematical physics example. The free scalar field, $\phi(\boldsymbol{x})$, in Euclidean quantum field theory in $1+1$ dimensions fluctuates so wildly at short distances that an interaction such as $\int \mathrm{d}^2x\, V(\phi(\boldsymbol{x}))$ with $V(\phi) = \phi^2 + \lambda\phi^4$ cannot be properly defined. One way, not necessarily optimal, to regularize the theory is to introduce a spatial lattice with spacing a. Such a lattice field theory is well defined in any finite volume. On taking the limit $a \to 0$ along with a simultaneous readjustment of the interaction potential, $V(\phi) = V_a(\phi)$, a Euclidean-invariant, interacting quantum field theory is obtained. Ideally this limit theory should be independent of the regularization scheme. For instance one could start with the free scalar field in the continuum and regularize $\phi(\boldsymbol{x})$ as $\phi * g(\boldsymbol{x})$ with a suitable test function g concentrated at 0. Then the regularized interaction is $\int \mathrm{d}^2x\, V(\phi * g(\boldsymbol{x}))$ and in the limit $g(\boldsymbol{y}) \to \delta(\boldsymbol{y})$ a quantum field theory should be obtained identical to the one from the lattice regularization.

In the second approach one argues that there is a physical cutoff coming from a more refined theory, which is then modeled in a phenomenological way. While this is a standard procedure, it is worthwhile to illustrate it again with a concrete example. Consider a large number ($\cong 10^{23}$) of He^4 atoms in a container of adjustable size and suppose we are interested in computing their free energy according to the rules of statistical mechanics. The more refined theory is here nonrelativistic quantum mechanics which treats the electrons and nuclei as point particles carrying a spin $\frac{1}{2}$, respectively spin 0. As far as we can tell, this model approximately covers the temperature range $T = 0$ K to $T = 10^5$ K, i.e. way beyond dissociation, and the density range $\rho = 0$ to $\rho = \rho_{\mathrm{cp}}$, the density of close packing. Beyond these limits relativistic effects must be taken into account. However, there is a more limited range where we can get away with a model of classical point particles interacting through an effective potential of Lennard-Jones type. Once this

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

pair potential is specified classical statistical mechanics makes well-defined predictions at *any* T, ρ. There is no limitation in principle. Only outside a certain range of parameters would the classical model lose the correspondence with the real world. Already from the way the physical cutoff is described, there is a considerable amount of vagueness. How much error should we allow in the free energy? What about more detailed properties like density correlations? An effective potential can be defined quantum mechanically, but it is temperature dependent and never strictly a pair potential. Despite all these imprecisions and shortcomings, the equilibrium theory of fluids relies heavily on the availability of a classical model.

In the same spirit we modify the coupled Maxwell and Newton equations by introducing an extended charge distribution as a phenomenological model for the omitted quantum electrodynamics. The charge distribution is stabilized by strong interactions which act outside the realm of electromagnetic forces. On the classical level, say, an electron appears as an extended charged object with a size roughly of the order of its Compton wavelength, i.e. 4×10^{-11} cm. We impose the obvious condition that the extended charge distribution has to be adjusted such that, in the range where classical electrodynamics is applicable, the coupled Maxwell and Newton equations correctly reproduce the empirical observations.

Such general clauses seem to leave a lot of freedom in the construction of the theory. However, charge conservation and the Lagrangian form of the equations of motion severely limit the possibilities. In fact, essentially only two models of extended charge distribution have been investigated so far.

(i) *The semirelativistic Abraham model of a rigid charge distribution.* The charge e is assumed to be smeared out over a ball of radius R_φ. This means that in (2.32)–(2.34) the δ-function is replaced by a smooth charge distribution $e\varphi$. $\varphi(\boldsymbol{x})$ is taken to be radial, vanishing for $|\boldsymbol{x}| > R_\varphi$, and normalized as $\int \mathrm{d}^3x\varphi(\boldsymbol{x}) = 1$. Equivalently, having (2.32)–(2.34) recast in Fourier space, the couplings between the field modes with $|\boldsymbol{k}| \gtrsim 1/R_\varphi$ and the particle become suppressed. This particular choice for the internal structure of the charge is called the Abraham model (for a single nonrotating charge). For zero coupling the model is relativistic. However, φ is taken to be rigid, thus velocity independent in a prescribed coordinate frame, which breaks Lorentz invariance. The standard examples are that the charge is uniformly distributed either over the ball, $\varphi(\boldsymbol{x}) = (4\pi R_\varphi^2/3)^{-1}$ for $|\boldsymbol{x}| \leq R_\varphi$, $\varphi(\boldsymbol{x}) = 0$ otherwise, or over the sphere, $\varphi(\boldsymbol{x}) = (4\pi R_\varphi^2)^{-1}\delta(|\boldsymbol{x}| - R_\varphi)$. In the quantized version of the Abraham model, cf. chapter 13 below, often a sharp cutoff in Fourier space is adopted, i.e. $\widehat{\varphi}(\boldsymbol{k}) = (2\pi)^{-3/2}$ for $|\boldsymbol{k}| \leq \Lambda = R_\varphi^{-1}$, $\widehat{\varphi}(\boldsymbol{k}) = 0$ otherwise; this has the slight disadvantage of being oscillating and having slow decay in position space.

Once the charge distribution is extended, besides its center of charge, also rotational degrees of freedom must be taken into account. The Abraham model allowing for a spinning charge will be discussed in chapter 10. Since the dynamical behavior then becomes more complex, it is advisable to omit spin in the first round.

The Abraham model will be studied in considerable detail. While defined for all velocities $|\boldsymbol{v}(t)| < c$, it becomes empirically inaccurate at velocities close to c. Despite this drawback we hope that the Abraham model will serve as a blueprint towards a more realistic description of matter.

(ii) *The Lorentz model of a relativistically rigid charge distribution.* More in accord with special relativity is to require that $e\varphi$ is the charge distribution in the momentary rest frame of the particle. While such a principle was already stated by Lorentz and Poincaré, a satisfactory *dynamical* theory has been arrived at only very recently. As we will explain in section 2.5, in a relativistic theory translational and rotational degrees of freedom are intrinsically coupled. To gain an understanding of how relativistic invariance would modify the theory, we insert some features of the Lorentz model, although our understanding of its dynamical properties is far less developed than that of the Abraham model.

We emphasize that for extended charge models the diameter R_φ of the charge distribution defines a length (and upon dividing by c also a time) scale, relative to which the approximate validity of effective theories, like the Lorentz–Dirac equation, can be addressed quantitatively. In fact, apart from the external forces, R_φ is the *only* natural length scale available.

2.4 The Abraham model

Following Abraham, we model the charged particle as a spherically symmetric, rigid body to which the charge elements are permanently attached. The charge distribution is prescribed and independent of the particle's velocity, which singles out the laboratory frame. In a relativistic theory the charge distribution would appear to be Lorentz contracted. To be specific the charge distribution $e\varphi$ is assumed to be smooth, radial, and supported in a ball of radius R_φ, and normalized to e, i.e.

Condition (C):

$$\varphi \in C^\infty(\mathbb{R}^3)\,, \quad \varphi(\boldsymbol{x}) = \varphi_{\mathrm{r}}(|\boldsymbol{x}|)\,, \quad \varphi(\boldsymbol{x}) = 0 \text{ for } |\boldsymbol{x}| \geq R_\varphi\,, \quad \int \mathrm{d}^3x \varphi(\boldsymbol{x}) = 1\,. \tag{2.38}$$

$\diamond$ $e\varphi(\boldsymbol{x})$ is the charge distribution and $\widehat{\varphi}(\boldsymbol{k})$ is the form factor, since in Fourier space it multiplies the current as $(2\pi)^{3/2}\widehat{\varphi}(\boldsymbol{k})\widehat{\boldsymbol{j}}(\boldsymbol{k}, t)$. $\diamond$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

Our goal is to set up the Abraham model as a well-defined dynamical system. Usually this point is taken for granted. Since the occurrence of ill-defined equations of motion was one of our main objections to the δ-charge, it is worthwhile to understand why this objection is no longer valid for a smeared out δ.

The equations of motion for the Abraham model are

$$\partial_t \boldsymbol{B}(\boldsymbol{x},t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x},t)\,,$$
$$\partial_t \boldsymbol{E}(\boldsymbol{x},t) = \nabla \times \boldsymbol{B}(\boldsymbol{x},t) - e\varphi(\boldsymbol{x}-\boldsymbol{q}(t))\boldsymbol{v}(t)\,, \tag{2.39}$$
$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) = e\varphi(\boldsymbol{x}-\boldsymbol{q}(t))\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x},t) = 0\,, \tag{2.40}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}}\gamma\boldsymbol{v}(t)\big) = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{E}_{\varphi}(\boldsymbol{q}(t),t) + \boldsymbol{v}(t)\times(\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{B}_{\varphi}(\boldsymbol{q}(t),t))\big)\,, \tag{2.41}$$

where we have set $c = 1$. In (2.41) we use the shorthand $\boldsymbol{E}_{\varphi}(\boldsymbol{x}) = \boldsymbol{E} * \varphi(\boldsymbol{x})$ and $\boldsymbol{B}_{\varphi}(\boldsymbol{x}) = \boldsymbol{B} * \varphi(\boldsymbol{x})$ so as to resemble (2.34). Strictly speaking also $\boldsymbol{E}_{\mathrm{ex}}$, $\boldsymbol{B}_{\mathrm{ex}}$ should be smeared over φ; however, this would only amount to a redefinition of the external potentials. In contrast to Newton's equations of motion (2.26), for the Abraham model we denote the mechanical mass of the particle by m_{b} to emphasize that this bare mass will differ from the observed mass of the compound object "particle plus surrounding Coulomb field". The external potentials ϕ_{ex}, $\boldsymbol{A}_{\mathrm{ex}}$ can be fairly arbitrary. We only require them and their derivatives to be smooth and locally bounded, to avoid too strong local oscillations. No condition on the increase at infinity is needed, since $|\boldsymbol{v}(t)| \leq 1$. However, it is convenient to have the energy, as defined in (2.44), uniformly bounded from below. To keep things simple we make the (unnecessarily strong) assumptions

Condition (P):

$$\phi_{\mathrm{ex}} \in C^{\infty}(\mathbb{R}^3)\,, \quad \boldsymbol{A}_{\mathrm{ex}} \in C^{\infty}(\mathbb{R}^3,\mathbb{R}^3), \quad \phi_{\mathrm{ex}} \geq \bar{\phi} > -\infty\,. \tag{2.42}$$

Moreover, there exists a constant C such that $|\nabla\phi_{\mathrm{ex}}| \leq C$, $|\nabla \boldsymbol{A}_{\mathrm{ex}}| \leq C$.

The Abraham model is derived from the Lagrangian

$$L = -m_{\mathrm{b}}(1-\dot{\boldsymbol{q}}^2)^{1/2} - e\big(\phi_{\mathrm{ex}}(\boldsymbol{q}) + \phi_{\varphi}(\boldsymbol{q}) - \dot{\boldsymbol{q}}\cdot\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{q}) - \dot{\boldsymbol{q}}\cdot\boldsymbol{A}_{\varphi}(\boldsymbol{q})\big) + \frac{1}{2}\int \mathrm{d}^3x\big((\nabla\phi + \partial_t\boldsymbol{A})^2 - (\nabla\times\boldsymbol{A})^2\big)\,. \tag{2.43}$$

Correspondingly, the energy

$$\mathcal{E}(\boldsymbol{E},\boldsymbol{B},\boldsymbol{q},\boldsymbol{v}) = m_{\mathrm{b}}\gamma(\boldsymbol{v}) + e\phi_{\mathrm{ex}}(\boldsymbol{q}) + \frac{1}{2}\int \mathrm{d}^3x\big(\boldsymbol{E}(\boldsymbol{x})^2 + \boldsymbol{B}(\boldsymbol{x})^2\big) \tag{2.44}$$

is conserved.

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

As for any dynamical system, the first step in dealing with (2.39)–(2.41) is to construct a suitable phase space. The dynamical variables are $(\boldsymbol{E}(\boldsymbol{x}), \boldsymbol{B}(\boldsymbol{x}), \boldsymbol{q}, \boldsymbol{v}) = Y$ which is called a state of the system. We have $\boldsymbol{q} \in \mathbb{R}^3$, $\boldsymbol{v} \in \mathbb{V} = \{\boldsymbol{v} \,|\, |\boldsymbol{v}| < 1\}$. In addition, the energy (2.44) should be bounded. Thus it is natural to introduce the (real) Hilbert space

$$L^2 = L^2(\mathbb{R}^3, \mathbb{R}^3) \tag{2.45}$$

with norm $\|\boldsymbol{E}\| = (\int \mathrm{d}^3x |\boldsymbol{E}(\boldsymbol{x})|^2)^{1/2}$ and to define $\mathcal{L}$ as the set of states satisfying

$$\|Y\|_{\mathcal{L}} = \|\boldsymbol{E}\| + \|\boldsymbol{B}\| + |\boldsymbol{q}| + |\gamma(\boldsymbol{v})\boldsymbol{v}| < \infty\,. \tag{2.46}$$

In particular for the field energy, $\frac{1}{2}(\|\boldsymbol{E}\|^2 + \|\boldsymbol{B}\|^2) < \infty$. The norm $||\cdot||_{\mathcal{L}}$ gives rise to the metric

$$d(Y_1, Y_2) = \|\boldsymbol{E}_1 - \boldsymbol{E}_2\| + \|\boldsymbol{B}_1 - \boldsymbol{B}_2\| + |\boldsymbol{q}_1 - \boldsymbol{q}_2| + |\gamma(\boldsymbol{v}_1)\boldsymbol{v}_1 - \gamma(\boldsymbol{v}_2)\boldsymbol{v}_2|\,. \tag{2.47}$$

In addition, the constraints (2.40) have to be satisfied. Thus the phase space, $\mathcal{M}$, for the Abraham model is the nonlinear submanifold of $\mathcal{L}$ defined through

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = e\varphi(\boldsymbol{x} - \boldsymbol{q})\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}) = 0\,. \tag{2.48}$$

$\mathcal{M}$ inherits its metric from $\mathcal{L}$.

On various occasions below we will need the property that the system forgets its initial field data. For this purpose it is helpful to have a little bit of smoothness and some decay at infinity. Formally we introduce the "good" subset $\mathcal{M}^\sigma \subset \mathcal{M}$, $0 \le \sigma \le 1$, consisting of fields such that componentwise and outside a ball of radius R_0, $|\boldsymbol{x}| \ge R_0$, we have

$$|\boldsymbol{E}(\boldsymbol{x})| + |\boldsymbol{B}(\boldsymbol{x})| + |\boldsymbol{x}|(|\nabla \boldsymbol{E}(\boldsymbol{x})| + |\nabla \boldsymbol{B}(\boldsymbol{x})|) \le C\,|\boldsymbol{x}|^{-1-\sigma}\,. \tag{2.49}$$

The Liénard–Wiechert fields (2.24), (2.25) are included in $\mathcal{M}^0$; moreover, $\mathcal{M}^0$ is dense in $\mathcal{M}$. However $\mathcal{M}^\sigma = \emptyset$ for $\sigma > 1$, by Gauss's law (2.40) with $e \neq 0$.

The evolution equations (2.39)–(2.41) are of the general form

$$\frac{\mathrm{d}}{\mathrm{d}t} Y(t) = F(Y(t)) \tag{2.50}$$

with $Y(0) = Y^0 \in \mathcal{M}$. We turn to the question of the existence and uniqueness of solutions of the Abraham model (2.50).

Theorem 2.1 (Existence of the dynamics for the Abraham model). *Let the conditions (C) and (P) hold and let* $Y^0 = (\boldsymbol{E}^0(\boldsymbol{x}), \boldsymbol{B}^0(\boldsymbol{x}), \boldsymbol{q}^0, \boldsymbol{v}^0) \in \mathcal{M}$. *Then the*

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

integral equation associated with (2.50),

$$Y(t) = Y^0 + \int_0^t \mathrm{d}s\, F(Y(s))\,, \tag{2.51}$$

has a unique solution $Y(t) = (\boldsymbol{E}(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t), \boldsymbol{q}(t), \boldsymbol{v}(t)) \in \mathcal{M}$*, which is continuous in* t *and satisfies* $Y(0) = Y^0$*. Along the solution trajectory*

$$\mathcal{E}(Y(t)) = \mathcal{E}(Y^0) \tag{2.52}$$

for all t*, i.e. the energy is conserved.*

For short times existence and uniqueness follow through the contraction mapping principle with constants depending only on the initial energy. For smooth initial data, energy conservation is verified directly and by continuity it extends to all finite-energy data. Thus we can construct iteratively the solution for all times.

We first summarize some properties of the Maxwell equations. They follow directly from the Fourier and convolution representations (2.12), (2.13), respectively (2.16), (2.17).

Lemma 2.2 *In the Maxwell equations (2.2), (2.3), let* $e\varphi(\boldsymbol{x},t) = e\varphi(\boldsymbol{x} - \boldsymbol{q}(t))$*,* $\boldsymbol{j}(\boldsymbol{x},t) = e\varphi(\boldsymbol{x} - \boldsymbol{q}(t))\boldsymbol{v}(t)$*, with prescribed* $t \mapsto (\boldsymbol{q}(t), \boldsymbol{v}(t))$ *continuous. Then (2.2), (2.3) has a unique solution in* $C(\mathbb{R}, L^2 \oplus L^2)$*. The solution map* $(\boldsymbol{E}^0, \boldsymbol{B}^0) \mapsto (\boldsymbol{E}(t), \boldsymbol{B}(t))$ *depends continuously on* $(\boldsymbol{q}(t), \boldsymbol{v}(t))$*.*

Proof of Theorem 2.1: Let $b > 0$ be fixed and choose initial data such that $\mathcal{E}(Y^0) \leq b$.
(i) There exists a unique solution $Y(t) \in C([0,\delta], \mathcal{M})$ for $\delta = \delta(b)$ sufficiently small.

We write (2.41) in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}}\gamma\, \boldsymbol{v}(t)\big) = \boldsymbol{F}_{\mathrm{ex}}(t) + \boldsymbol{F}_{\mathrm{ini}}(t) + \boldsymbol{F}_{\mathrm{self}}(t) \tag{2.53}$$

by inserting $\boldsymbol{E}(\boldsymbol{x},t)$, $\boldsymbol{B}(\boldsymbol{x},t)$ from the Maxwell equations according to (2.16), (2.17). Let

$$\begin{aligned} W_t(\boldsymbol{x}) &= e^2 \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \frac{1}{|\boldsymbol{k}|}\, \sin|\boldsymbol{k}|t \\ &= (2\pi)^3 e^2 \int \mathrm{d}^3y \int \mathrm{d}^3y' \varphi(\boldsymbol{y})\varphi(\boldsymbol{y}') \frac{1}{4\pi t}\, \delta(|\boldsymbol{y} + \boldsymbol{x} - \boldsymbol{y}'| - t)\,. \end{aligned} \tag{2.54}$$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

Then

$$\boldsymbol{F}_{\mathrm{ex}}(t) = e^{-}\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + \boldsymbol{v}(t) \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t))\big)\,, \tag{2.55}$$

$$\begin{aligned}\boldsymbol{F}_{\mathrm{ini}}(t) = \int \mathrm{d}^3x\, e\varphi(\boldsymbol{x} - \boldsymbol{q}(t))\big[&\partial_t G_t * \boldsymbol{E}^0(\boldsymbol{x}) + \nabla \times G_t * \boldsymbol{B}^0(\boldsymbol{x}) \\ &+ \boldsymbol{v}(t) \times \partial_t G_t * \boldsymbol{B}^0(\boldsymbol{x}) - \boldsymbol{v}(t) \times (\nabla \times G_t * \boldsymbol{E}^0(\boldsymbol{x}))\big]\,,\end{aligned} \tag{2.56}$$

$$\begin{aligned}\boldsymbol{F}_{\mathrm{self}}(t) = \int_0^t \mathrm{d}s\big[&-\nabla W_{t-s}(\boldsymbol{q}(t) - \boldsymbol{q}(s)) - \boldsymbol{v}(s)\partial_t W_{t-s}(\boldsymbol{q}(t) - \boldsymbol{q}(s)) \\ &+ \boldsymbol{v}(t) \times (\nabla \times \boldsymbol{v}(s) W_{t-s}(\boldsymbol{q}(t) - \boldsymbol{q}(s)))\big]\,.\end{aligned} \tag{2.57}$$

We now integrate both sides of (2.53) over the time interval $[0, t]$. The resulting expression is regarded as a map from the trajectory $t \mapsto (\boldsymbol{q}(t), \boldsymbol{v}(t))$, $0 \le t \le \delta$, to the trajectory $t \mapsto (\bar{\boldsymbol{q}}(t), \bar{\boldsymbol{v}}(t))$ and is defined by

$$\bar{\boldsymbol{q}}(t) = \boldsymbol{q}^0 + \int_0^t \mathrm{d}s\, \boldsymbol{v}(s)\,, \tag{2.58}$$

$$m_{\mathrm{b}}\gamma(\bar{\boldsymbol{v}}(t))\bar{\boldsymbol{v}}(t) = m_{\mathrm{b}}\gamma(\boldsymbol{v}^0)\boldsymbol{v}^0 + \int_0^t \mathrm{d}s\big(\boldsymbol{F}_{\mathrm{ex}}(s) + \boldsymbol{F}_{\mathrm{ini}}(s) + \boldsymbol{F}_{\mathrm{self}}(s)\big)\,,$$

where $\boldsymbol{F}_{\mathrm{ex}}(s)$, $\boldsymbol{F}_{\mathrm{ini}}(s)$, and $\boldsymbol{F}_{\mathrm{self}}(s)$ are functionals of $\boldsymbol{q}(\cdot), \boldsymbol{v}(\cdot)$ according to (2.55)–(2.57). Since φ, W, ϕ_{ex}, and $\boldsymbol{A}_{\mathrm{ex}}$ are smooth, this map is a contraction in $C([0, t], \mathbb{R}^3 \times \mathbb{V})$, i.e.

$$\begin{aligned}&\sup_{0\le s\le t}\big(|\bar{\boldsymbol{q}}_2(s) - \bar{\boldsymbol{q}}_1(s)| + |\bar{\boldsymbol{v}}_2(s) - \bar{\boldsymbol{v}}_1(s)|\big) \\ &\quad\le c(t, b) \sup_{0\le s\le t}\big(|\boldsymbol{q}_1(s) - \boldsymbol{q}_2(s)| + |\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)|\big)\,,\end{aligned} \tag{2.59}$$

with a constant $c(t, b)$ depending on b and $c(t, b) < 1$ for sufficiently small t. Such a map has a unique fixed point which is the desired solution $(\boldsymbol{q}(t), \boldsymbol{v}(t))$. By the Maxwell equations also $\boldsymbol{B}(\boldsymbol{x}, t)$, $\boldsymbol{E}(\boldsymbol{x}, t)$ are uniquely determined.

(ii) The solution map $Y^0 \mapsto Y(t)$ is continuous in $\mathcal{M}$.

This follows from Lemma 2.2 and the continuous dependence of $(\boldsymbol{q}(t), \boldsymbol{v}(t))$ on the initial data.

(iii) The energy is conserved.

We choose smooth initial fields such that $\boldsymbol{E}, \boldsymbol{B} \in C^\infty(\mathbb{R}^3)$ and

$$|\nabla^\alpha \boldsymbol{E}(\boldsymbol{x})| + |\nabla^\alpha \boldsymbol{B}(\boldsymbol{x})| \le C(1 + |\boldsymbol{x}|)^{-(2+|\alpha|)}\,. \tag{2.60}$$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

Here $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index with $\alpha_i = 0, 1, 2, \ldots$. This subset is dense in $\mathcal{M}$. By the convolution representation (2.16), (2.17) of the solution to the Maxwell equations we have $\boldsymbol{E}(\boldsymbol{x}, t), \boldsymbol{B}(\boldsymbol{x}, t) \in C^1([0, \delta] \times \mathbb{R}^3)$ and $|\boldsymbol{E}(\boldsymbol{x}, t)| + |\boldsymbol{B}(\boldsymbol{x}, t)| \leq C(1 + |\boldsymbol{x}|)^{-2}$. Also $\boldsymbol{v}(t) \in C^1([0, \delta])$. Thus we are allowed to differentiate,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{E}(Y(t)) &= \gamma^3 \boldsymbol{v} \cdot \dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \nabla \phi_{\mathrm{ex}}(\boldsymbol{q}) + \int \mathrm{d}^3 x (\boldsymbol{E} \cdot \partial_t \boldsymbol{E} + \boldsymbol{B} \cdot \partial_t \boldsymbol{B}) \\ &= \int \mathrm{d}^3 x \big(\boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) - \boldsymbol{B} \cdot (\nabla \times \boldsymbol{E}) \big) = 0 \,, \end{aligned} \tag{2.61}$$

since the fields decay and hence the surface terms vanish. Thus $\mathcal{E}(Y(t)) = \mathcal{E}(Y^0)$ for $0 \leq t \leq \delta$. By continuity this equality extends to all of $\mathcal{M}$.

(iv) The global solution exists.

From (iii) we know that $\mathcal{E}(Y(\delta)) = \mathcal{E}(Y^0) \leq b$. Thus we can repeat the previous argument for $\delta \leq t \leq 2\delta$, etc. Backwards in time we still have the solution (2.16), (2.17) of the Maxwell equations, only the retarded fields have to be replaced by the advanced fields. Thereby we obtain the solution for all times. □

Theorem 2.1 ensures the existence and uniqueness of solutions for the Abraham model. For initial data $Y^0 \in \mathcal{M}$ the solution trajectory $t \to Y(t)$ lies in the phase space $\mathcal{M}$, is continuous in t, and its energy is conserved. We have thus established the basis for further investigations on the dynamics of the Abraham model.

2.5 The relativistically covariant Lorentz model

To improve on the semirelativistic Abraham model, following Lorentz, it is natural to assume that when viewed in a momentary inertial rest frame the charge and mass distribution of the particle remain unchanged. This is what one would call a relativistically rigid extended charge. Our requirement fixes uniquely the four-current density. The equations of motion then follow from a relativistically covariant action.

For obvious reasons we will switch to relativistic notation, where we follow the conventions of Misner, Thorne, and Wheeler. Our arena is the Minkowski spacetime $\mathbb{M}^4$. A Lorentz frame, $\mathcal{F}_{\mathrm{L}}$, in $\mathbb{M}^4$ is specified through the tetrad $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of fixed unit vectors. They have the inner product

$$\mathbf{e}_\mu \cdot \mathbf{e}_\nu = g_{\mu\nu} \,, \tag{2.62}$$

where $g_{\mu\nu}$ is the *metric tensor* with $g_{00} = -1$, $g_{\mu\mu} = 1$, $\mu = 1, 2, 3$, and $g_{\mu\nu} = 0$ otherwise. Therefore $\mathbb{M}^4$ can be identified with $\mathbb{R}^{1,3}$. In the given basis, a vector $\mathbf{x} \in \mathbb{M}^4$ is expanded as

$$\mathbf{x} = x^\mu \mathbf{e}_\mu \tag{2.63}$$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

using the Einstein summation convention over repeated indices. We group $\mathbf{x} = (t, \boldsymbol{x})$ with $t \in \mathbb{R}$ the time and $\boldsymbol{x} \in \mathbb{R}^3$ the space coordinate. The scalar product is $\mathbf{x} \cdot \mathbf{y} = g_{\mu\nu} x^\mu y^\nu$ and $|\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{x}$.

The motion of a particle is specified through its world line $\tau \mapsto \mathbf{q}(\tau)$ parametrized in terms of the eigentime τ, $\mathrm{d}\tau^2 = -\mathrm{dx} \cdot \mathrm{dx}$. Denoting by $\dot{\mathbf{q}}$ differentiation of $\mathbf{q}(\tau)$ with respect to τ, the four-velocity is $\mathbf{u}(\tau) = \dot{\mathbf{q}}(\tau)$. $\mathbf{u}$ is time-like, $\mathbf{u} \cdot \mathbf{u} = -1$, and $u_0 > 0$ for a particle moving forward in time. In the given Lorentz frame we have

$$\mathbf{u} = (\gamma, \gamma \boldsymbol{v}) \,, \quad \gamma = (1 - |\boldsymbol{v}|^2)^{-1/2} \tag{2.64}$$

with $\boldsymbol{v}$ the usual three-velocity.

If the charged particle is at rest, then, as before, its charge is smeared according to the charge distribution $e\varphi$. In addition we assume that now the bare mass, m_b, is smeared also according to φ. In principle, one should distinguish between the charge and mass form factor. We suppress such a distinction, since it can be unambiguously recovered from the prefactors e and m_b. By the definition of a rigid charge, we require that in any momentary rest frame the mass, respectively charge, distribution are given by $m_\mathrm{b}\varphi$, respectively $e\varphi$.

Since our charged body is extended, in its kinematical description, besides $\mathbf{q}(\tau)$ and the velocity $\mathbf{u}(\tau) = \dot{\mathbf{q}}(\tau)$, we have to specify its state of rotation. Let us introduce the (noninertial) body frame $\mathcal{F}_\mathrm{body}$ through the tetrad $\{\mathbf{e}'_\mu\}_{\mu=0,\ldots,3}$ of unit vectors. $\mathcal{F}_\mathrm{body}$ is fixed in the charged body and thus comoving and corotating. We set $\mathbf{e}'_0 = \mathbf{u}(\tau)$. $\{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ gives then the spatial orientation of $\mathcal{F}_\mathrm{body}$ in the momentary rest frame. In the course of time $\mathcal{F}_\mathrm{body}$ evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \mathbf{e}'_\mu = -\boldsymbol{\Omega} \cdot \mathbf{e}'_\mu \,, \quad \mu = 0, \ldots, 3 \,, \tag{2.65}$$

where $\boldsymbol{\Omega}$ is the antisymmetric tensor of the instantaneous rate of four-gyration of $\mathcal{F}_\mathrm{body}$ as seen in the Lorentz frame $\mathcal{F}_\mathrm{L}$.

Even if there is no external torque acting on the rigid charged body, the frame $\mathcal{F}_\mathrm{body}$ rotates. This is the famous Thomas precession, determined by the Fermi–Walker transport equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \bar{\mathbf{e}}_\mu = -\boldsymbol{\Omega}_\mathrm{FW} \cdot \bar{\mathbf{e}}_\mu \,, \quad \mu = 0, \ldots, 3 \,, \tag{2.66}$$

where

$$\boldsymbol{\Omega}_\mathrm{FW} = \dot{\mathbf{u}} \wedge \mathbf{u} \,. \tag{2.67}$$

Here the exterior product of two vectors is defined by $\mathbf{a} \wedge \mathbf{b} = \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}$ or, as acting on a vector $\mathbf{c}$, $(\mathbf{a} \wedge \mathbf{b}) \cdot \mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{c})$. Together with the initial

conditions $\bar{\mathbf{e}}_0(0) = \mathbf{u}(0)$, $\bar{\mathbf{e}}_\mu(0) = \mathbf{e}_\mu$, $\mu = 1, 2, 3$, (2.66) defines the noninertial frame $\mathcal{F}_{\mathrm{FW}}$.

If there is an external torque acting, then $\mathcal{F}_{\mathrm{body}} \neq \mathcal{F}_{\mathrm{FW}}$ and it is natural to introduce the intrinsic (Eulerian) four-gyration by

$$\mathbf{\Omega}_{\mathrm{E}} = \mathbf{\Omega} - \mathbf{\Omega}_{\mathrm{FW}} \,. \tag{2.68}$$

As $\mathbf{\Omega}$, $\mathbf{\Omega}_{\mathrm{FW}}$, also $\mathbf{\Omega}_{\mathrm{E}}$ is antisymmetric and satisfies

$$\mathbf{\Omega}_{\mathrm{E}} \cdot \mathbf{u} = 0 \,. \tag{2.69}$$

Therefore $\mathbf{\Omega}_{\mathrm{E}}$ has only three independent components and is dual to a space-like four-vector $\mathbf{w}_{\mathrm{E}}$ which satisfies

$$\mathbf{\Omega}_{\mathrm{E}} \cdot \mathbf{w}_{\mathrm{E}} = 0 \,, \quad \mathbf{w}_{\mathrm{E}} \cdot \mathbf{u} = 0 \,. \tag{2.70}$$

In $\mathcal{F}_{\mathrm{FW}}$, $\mathbf{w}_{\mathrm{E}}$ is of the form $(0, \boldsymbol{\omega}_{\mathrm{E}})$, where $\boldsymbol{\omega}_{\mathrm{E}}$ is the usual angular velocity vector which points along the instantaneous axis of body gyration in the space-like three-slice of $\mathcal{F}_{\mathrm{FW}}$. For zero torque $\boldsymbol{\omega}_{\mathrm{E}} = 0$.

We conclude that relative to $\mathcal{F}_{\mathrm{FW}}$ the rotational state is either given by $\mathbf{\Omega}_{\mathrm{E}}(\tau)$ or by $\mathbf{w}_{\mathrm{E}}(\tau)$. $\mathbf{w}_{\mathrm{E}}(\tau)$ is space-like, $|\mathbf{w}_{\mathrm{E}}(\tau)|^2 \geq 0$.

2.5.1 The four-current density

Our task is to construct a relativistically covariant current density, which will serve both as the source term in Maxwell's equations and as the force, respectively torque, term in Newton's equations of motion.

For a given world line let $\mathcal{F}'_{\mathrm{L}}$ be the momentary rest frame at time τ centered at $\mathbf{q}(\tau)$ with spatial axes oriented as in $\mathcal{F}_{\mathrm{L}}$. In the coordinates of $\mathcal{F}'_{\mathrm{L}}$, by definition, the four-current density is given by

$$\mathbf{j}'(t', \boldsymbol{x}') = e\varphi_{\mathrm{r}}(|\boldsymbol{x}'|)\delta(t')(1, 0) \,. \tag{2.71}$$

Transformed to our laboratory frame $\mathcal{F}_{\mathrm{L}}$ the current density becomes

$$\mathbf{j}(\mathbf{x}) = e\varphi_{\mathrm{r}}(|\mathbf{x} - \mathbf{q}(\tau_0)|)\mathbf{u}(\tau_0)|_{\sigma(\tau_0)} \,. \tag{2.72}$$

Here $\sigma(\tau)$ is the hyperplane defined by $\sigma(\tau) = \{\mathbf{y} | \mathbf{u}(\tau) \cdot (\mathbf{y} - \mathbf{q}(\tau)) = 0\}$ and the subscript in (2.72) means that for given $\mathbf{x}$ we have to choose τ_0 such that $\mathbf{x} \in \sigma(\tau_0)$, see figure 2.1. In general, there will be several such planes, see figure 2.2. Of course, they contribute to the current only if $\mathbf{x} - \mathbf{q}(\tau_0)$ is space-like and the distance $|\mathbf{x} - \mathbf{q}(\tau_0)|$ satisfies $|\mathbf{x} - \mathbf{q}(\tau_0)| \leq R_\varphi$. Let us assume for the moment

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

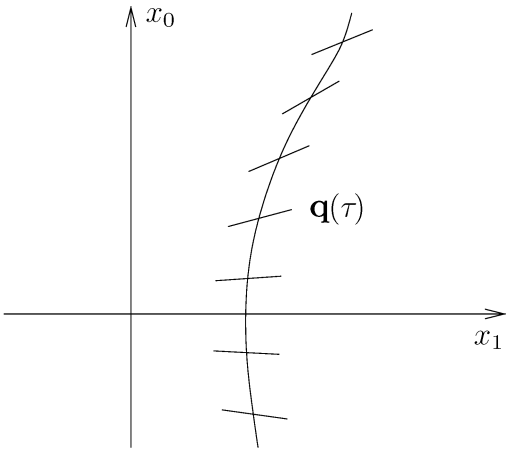

Figure 2.1: World line of an extended charge and the associated current density.

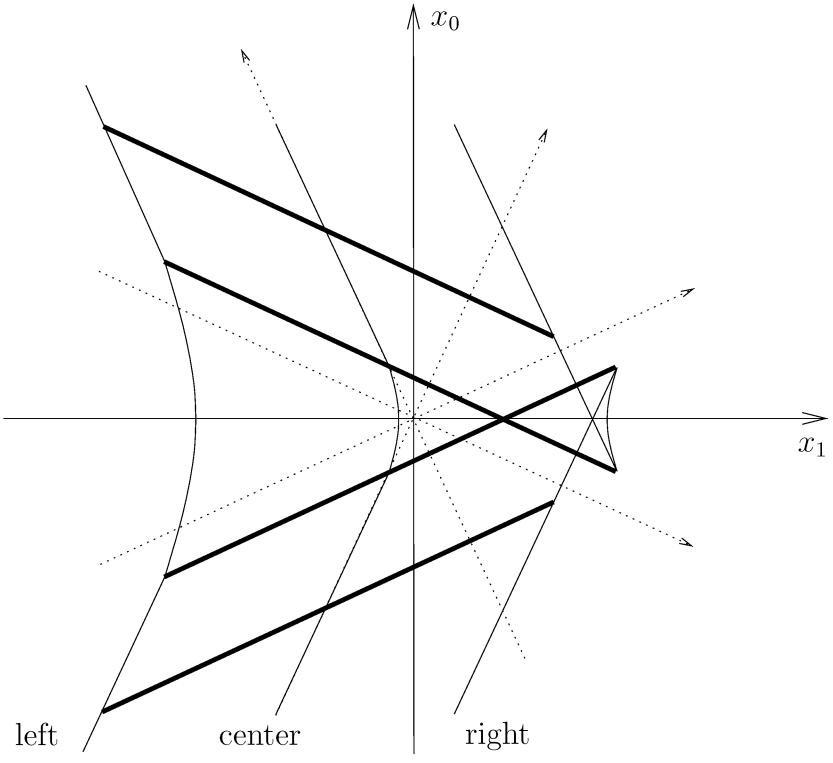

Figure 2.2: World line of an extended charge with large acceleration and backward currents.

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

that with this restriction there is only a single hyperplane intersecting $\boldsymbol{x}$. Then

$$\begin{aligned}\mathbf{j}(\mathbf{x}) &= \int \mathrm{d}\tau e\varphi_{\mathrm{r}}(|\mathbf{x}-\mathbf{q}(\tau)|)\mathbf{u}(\tau)\delta(\tau-\tau_0)|_{\sigma(\tau_0)} \\ &= \int \mathrm{d}\tau e\varphi_{\mathrm{r}}(|\mathbf{x}-\mathbf{q}(\tau)|)\mathbf{u}(\tau)\big(1+\dot{\mathbf{u}}(\tau)\cdot(\mathbf{x}-\mathbf{q}(\tau))\big)\delta(\mathbf{u}(\tau)\cdot(\mathbf{x}-\mathbf{q}(\tau)))\,.\end{aligned} \tag{2.73}$$

The additional term comes from the change in the volume element, since

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{u}\cdot(\mathbf{x}-\mathbf{q}) = \dot{\mathbf{u}}\cdot(\mathbf{x}-\mathbf{q}) - \mathbf{u}\cdot\mathbf{u} = 1+\dot{\mathbf{u}}\cdot(\mathbf{x}-\mathbf{q})\,. \tag{2.74}$$

Note that, because of $\delta(\mathbf{u}\cdot(\mathbf{x}-\mathbf{q}))$, the factor $\mathbf{u}(1+\dot{\mathbf{u}}\cdot(\mathbf{x}-\mathbf{q}))$ in (2.73) may be replaced by $\mathbf{u}-\boldsymbol{\Omega}_{\mathrm{FW}}\cdot(\mathbf{x}-\mathbf{q})$. The Thomas precession generates a current in addition to that due to translations.

In general, the body-fixed frame will be rotated by $\boldsymbol{\Omega}$ and we arrive at the final form of the four-current density as

$$\mathbf{j}(\mathbf{x}) = \int \mathrm{d}\tau e\varphi_{\mathrm{r}}(|\mathbf{x}-\mathbf{q}|)\delta(\mathbf{u}\cdot(\mathbf{x}-\mathbf{q}))(\mathbf{u}-\boldsymbol{\Omega}\cdot(\mathbf{x}-\mathbf{q}))\,. \tag{2.75}$$

One readily verifies the charge conservation

$$\nabla_g\cdot\mathbf{j}(\mathbf{x}) = 0\,, \tag{2.76}$$

where $\nabla_g f = (-\partial_{x_0} f, \nabla f)$.

Before proceeding to the action for the dynamics, we should understand whether the current (2.75) conforms with naive physical intuition. An instructive example is a uniformly accelerated charge, the so-called hyperbolic motion. We assume that the particle is accelerated along the positive 1-axis starting from rest at the origin. In the orthogonal direction the current traces out a tube of diameter $2R_\varphi$ and it suffices to treat the two-dimensional space-time problem. The center, C, of the charge moves along the orbit

$$C = \Big(t,\, g^{-1}\Big(\sqrt{1+g^2t^2}-1\Big)\Big),\ t\geq 0\,, \tag{2.77}$$

where $g>0$ is the acceleration. The curves traced by the right and left ends, C_+ and C_-, are determined from (2.73) and are given in parameter form as

$$C_\pm = \Big((1\pm R_\varphi g)t,\, g^{-1}\Big((1+R_\varphi g)\sqrt{1+g^2t^2}-1\Big)\Big),\ t\geq 0\,. \tag{2.78}$$

The equal-time distance between the center and C_+ is $t^{-1}((R_\varphi g)^2+2R_\varphi g)/(2g^2(1+R_\varphi g))$ for large t and is thus well bounded. However the left end motion depends crucially on the magnitude of $R_\varphi g$. If $R_\varphi g<1$, then the distance to the center is $t^{-1}((R_\varphi g)^2-2R_\varphi g)/(2g^2(1-R_\varphi g))$ for large t. On the other hand, for

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

$R_\varphi g > 1$, the left end moves into the past and the current density looks strangely distorted. To gain a feeling for the order of magnitudes involved we insert the classical electron radius. Then

$$g > \frac{c^2}{R_\varphi} = 10^{31}\,[\mathrm{m\ s^{-1}}]\,, \tag{2.79}$$

which is far beyond the domain of the validity of the theory. Of course, one would hope that for reasonable initial data such accelerations can never be reached. But the mere fact that charge elements may move backwards in time is an extra difficulty.

2.5.2 *Relativistic action, equations of motion*

For given current density, $\boldsymbol{j}$, the Maxwell equations read

$$\nabla_g \cdot^* \mathbf{F} = 0\,, \quad \nabla_g \cdot \mathbf{F} = \mathbf{j}\,, \tag{2.80}$$

where $\mathbf{F}$ is the antisymmetric electromagnetic field tensor of rank 2 and ${}^*\mathbf{F}$ its star dual. Equations (2.80) can be regarded as the Euler–Lagrange equations of an action functional $\mathcal{A}_\mathrm{f}$, which most conveniently is written in terms of a Lagrange density $\mathcal{L}_\mathrm{f}(\mathbf{x}) + \mathcal{L}_\mathrm{int}(\mathbf{x})$. The field part of the Lagrangian is given by

$$\mathcal{L}_\mathrm{f}(\mathbf{x}) = -\frac{1}{4}\mathrm{tr}[\mathbf{F}(\mathbf{x}) \cdot \mathbf{F}(\mathbf{x})]\,. \tag{2.81}$$

The interaction Lagrangian, $\mathcal{L}_\mathrm{int}(\mathbf{x})$, is defined through minimal coupling. We recall that (2.80) implies that $\mathbf{F}$ is the exterior derivative of a vector potential $\mathbf{A}$, $\mathbf{F} = \nabla_g \wedge \mathbf{A}$. If we adopt the Lorentz gauge $\nabla_g \cdot \mathbf{A} = 0$, then

$$\mathcal{L}_\mathrm{int}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \cdot \mathbf{j}(\mathbf{x})\,. \tag{2.82}$$

The variation of

$$\mathcal{A}_\mathrm{f} = \int (\mathcal{L}_\mathrm{f}(\mathbf{x}) + \mathcal{L}_\mathrm{int}(\mathbf{x}))\mathrm{d}^4\mathrm{x} \tag{2.83}$$

with respect to $\mathbf{A}$ yields indeed (2.80).

Thus we are left with writing down the particle Lagrangian. One might be tempted to simply take $-m_\mathrm{b}\int \mathrm{d}\tau$ from the relativistic mechanics of a single particle. This cannot be correct, unless all mass is concentrated at the center, i.e. $\varphi(\boldsymbol{x}) = \delta(\boldsymbol{x})$, since $-m_\mathrm{b}\int \mathrm{d}\tau$ ignores the energy stored in the inner rotation.

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

Including rotation the Lagrangian density for the particle becomes

$$\mathcal{L}_{\mathrm{p}}(\mathbf{x}) = -\int_{\tau_1}^{\tau_2} (1 - |\boldsymbol{\Omega}_{\mathrm{E}} \cdot (\mathbf{x} - \mathbf{q})|^2)^{1/2} m_{\mathrm{b}} \varphi_{\mathrm{r}}(|\mathbf{x} - \mathbf{q}|) \delta(\mathbf{u} \cdot (\mathbf{x} - \mathbf{q})) \mathrm{d}\tau, \quad (2.84)$$

where $\mathbf{q} = \mathbf{q}(\tau)$, $\mathbf{u} = \mathbf{u}(\tau)$, and $\boldsymbol{\Omega}_{\mathrm{E}} = \boldsymbol{\Omega}_{\mathrm{E}}(\tau)$ along the world line of the particle.

Let us check that (2.84) yields the physically correct equations of motion when $\mathbf{A}(\mathbf{x})$ is taken to be given. We have

$$\mathcal{A}_{\mathrm{p}} = \int (\mathcal{L}_{\mathrm{p}}(\mathbf{x}) + \mathcal{L}_{\mathrm{int}}(\mathbf{x})) \mathrm{d}^4\mathrm{x} \quad (2.85)$$

and must work out the variation of the world line $\tau \mapsto \mathbf{q}(\tau)$ at fixed end points, $\delta\mathbf{q}(\tau_1) = 0 = \delta\mathbf{q}(\tau_2)$, which induces also a change in the Fermi–Walker frame. The second independent variation is the body-fixed frame $\mathcal{F}_{\mathrm{body}}$ relative to $\mathcal{F}_{\mathrm{FW}}$. Thereby we obtain two equations of motion, which we write as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{p}(\tau) = \mathbf{f}(\tau), \quad (2.86)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{s}(\tau) + \boldsymbol{\Omega}_{\mathrm{FW}} \cdot \mathbf{s}(\tau) = \mathbf{t}(\tau)\,. \quad (2.87)$$

Let us discuss each equation separately. $\mathbf{p}$ is the momentum of the particle, related to the velocity by

$$\mathbf{p} = m_{\mathrm{g}}\mathbf{u}\,. \quad (2.88)$$

m_{g} depends on $|\boldsymbol{\omega}_{\mathrm{E}}|$ and is defined by

$$m_{\mathrm{g}} = \int_{\mathbb{R}^{1,3}} (1 - |\boldsymbol{\Omega}_{\mathrm{E}} \cdot \mathbf{x}|^2)^{-1/2} m_{\mathrm{b}} \varphi_{\mathrm{r}}(|\mathbf{x}|) \delta(\mathbf{u} \cdot \mathbf{x}) \mathrm{d}^4\mathrm{x}\,. \quad (2.89)$$

m_{g} is the *bare gyrational mass*, a Lorentz scalar. For small gyration frequency it can be expanded as

$$m_{\mathrm{g}} = m_{\mathrm{b}} + \frac{1}{2} I_{\mathrm{nr}} |\boldsymbol{\omega}_{\mathrm{E}}|^2 + \mathcal{O}(|\boldsymbol{\omega}_{\mathrm{E}}|^4) \quad (2.90)$$

with

$$I_{\mathrm{nr}} = m_{\mathrm{b}} \frac{2}{3} \int \mathrm{d}^3 x \varphi(\boldsymbol{x}) \boldsymbol{x}^2\,, \quad (2.91)$$

the moment of inertia in the nonrelativistic limit. $\mathbf{f}(\tau)$ in (2.86) is the Minkowski force

$$\mathbf{f}(\tau) = \int_{\mathbb{R}^{1,3}} \mathbf{F}(\mathbf{x}) \cdot (\mathbf{u} - \boldsymbol{\Omega} \cdot (\mathbf{x} - \mathbf{q})) e \varphi_{\mathrm{r}}(|\mathbf{x} - \mathbf{q}|) \delta(\mathbf{u} \cdot (\mathbf{x} - \mathbf{q})) \mathrm{d}^4\mathrm{x}\,. \quad (2.92)$$

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

It reduces to the Lorentz force, $e\mathbf{F}\cdot\mathbf{u}$, in the case where $\mathbf{F}(\mathbf{x})$ is slowly varying on the scale of R_φ.

In the rotational equation (2.87), $\mathbf{s}$ is the four-vector of *spin angular momentum* and is related to the four-gyration by

$$\mathbf{s}_\mathrm{b} = I_\mathrm{b}\mathbf{w}_\mathrm{E} \tag{2.93}$$

with I_b the *relativistic moment of inertia* relative to $\mathbf{q}$,

$$I_\mathrm{b}(|\boldsymbol{\omega}_\mathrm{E}|)\mathbf{g} = \int_{\mathbb{R}^{1,3}} (|\mathbf{x}|^2\mathbf{g} - \mathbf{x}\otimes\mathbf{x})(1 - |\boldsymbol{\Omega}_\mathrm{E}\cdot\mathbf{x}|^2)^{-1/2} m_\mathrm{b}\varphi_\mathrm{r}(|\mathbf{x}|)\delta(\mathbf{u}\cdot\mathbf{x})\mathrm{d}^4\mathrm{x}\,. \tag{2.94}$$

In (2.87) $\boldsymbol{s}$ is kinematically Fermi–Walker transported by $\boldsymbol{\Omega}_\mathrm{FW}$ and changed through the external Minkowski torque $\mathbf{t}(\tau)$. From the variation of (2.85) we obtain

$$\mathbf{t}(\tau) = \int_{\mathbb{R}^{1,3}} (\mathbf{x}-\mathbf{q})\wedge(\mathbf{F}(\mathbf{x})\cdot(\mathbf{u}-\boldsymbol{\Omega}\cdot(\mathbf{x}-\mathbf{q})))^\perp e\varphi_\mathrm{r}(|\mathbf{x}-\mathbf{q}|)\delta(\mathbf{u}\cdot(\mathbf{x}-\mathbf{q}))\mathrm{d}^4\mathrm{x}\,, \tag{2.95}$$

where by definition $\boldsymbol{a}^\perp = (\mathbf{g}+\mathbf{u}\otimes\mathbf{u})\cdot\mathbf{a}$. In the case of slow variation of $\mathbf{F}$, (2.95) becomes the BMT equation, cf. section 10.1.

We remark that through (2.86), (2.87) the translational and rotational motion are coupled in a rather complicated way with some simplification for a slowly varying external potential $\mathbf{A}_\mathrm{ex}$.

Having discussed the action (2.83) for the field at prescribed currents and the action (2.85) for the particle at prescribed fields, the action for the Lorentz model of an extended charge is inevitable. The Lagrangian density reads

$$\mathcal{L}(\mathbf{x}) = \mathcal{L}_\mathrm{p}(\mathbf{x}) + \mathcal{L}_\mathrm{int}(\mathbf{x}) + \mathcal{L}_\mathrm{f}(\mathbf{x}) \tag{2.96}$$

with the corresponding action

$$\mathcal{A} = \int_\Xi \mathcal{L}(\mathbf{x})\mathrm{d}^4\mathrm{x}\,. \tag{2.97}$$

To include an external potential, $\mathcal{L}_\mathrm{int}$ from (2.82) has to be merely modified to $\mathcal{L}_\mathrm{int}(\mathbf{x}) = \mathbf{A}(\mathbf{x})\cdot\mathbf{j}(\mathbf{x}) + \mathbf{A}_\mathrm{ex}(\mathbf{x})\cdot\mathbf{j}(\mathbf{x})$.

One has to be careful with the domain of integration, Ξ. It is a region of $\mathbb{M}^4$, which is bordered by two space-like surfaces, $\partial\Xi_i$, $i=1,2$. One first fixes an interval $[\tau_1,\tau_2]$ of eigentimes. Restricted to a ball of radius R_φ, $\partial\Xi_i = \{\mathbf{y}|\,\mathbf{u}(\tau_i)\cdot(\mathbf{y}-\mathbf{q}(\tau_i)) = 0\}$, $i=1,2$. $\partial\Xi_1$, $\partial\Xi_2$ are then smoothly extended to hypersurfaces such that they do not intersect each other, see figure 2.3. The variation is carried out at fixed end points, which means that $\mathbf{q}(\tau_1)$, $\mathbf{q}(\tau_2)$, $\boldsymbol{\Omega}_\mathrm{E}(\tau_1)$, $\boldsymbol{\Omega}_\mathrm{E}(\tau_2)$, and $\mathbf{A}$ on the hypersurfaces $\partial\Xi_i$, $i=1,2$, are prescribed. In addition we require a properly

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

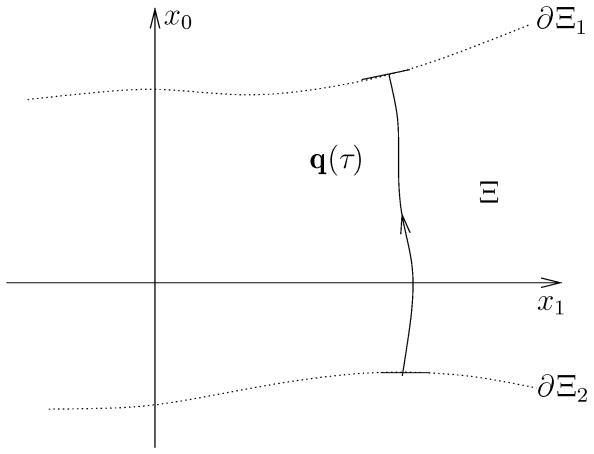

Figure 2.3: Space-like boundary surfaces in the variation of the action.

time-ordered history of momentary charge slices. Then the Euler–Lagrange equations for (2.97) are given by Maxwell's equations (2.80), by Newton's equations (2.86) for the translational degrees of freedom together with (2.88), (2.89), (2.92), and by Newton's equations (2.87) for the rotational degrees of freedom together with (2.93), (2.94), (2.95), as a coupled set of equations for the extended charge and the Maxwell field.

As for the Abraham model we should discuss the existence and uniqueness of solutions. This project is hampered by the fact that we have two constraints. The equator must have a subluminal speed of gyration, which is ensured by $|\boldsymbol{\omega}_{\mathrm{E}}|R_\varphi < 1$. In addition, the charge slices have to move forward in time, which is ensured by $|\ddot{\mathbf{q}}|R_\varphi < 1$. The difficulty is that, even if these conditions are met initially, there seems to be no mechanism which ensures their validity later on. At present, the general Cauchy problem is known to have a solution only for a finite interval of time, whose duration depends on the initial data.

Notes and references

Sections 2.1 and 2.2

The material discussed can be found in most textbooks. I find Landau and Lifshitz (1959), Panofsky and Phillips (1962), Jackson (1999), and Scharf (1994) particularly useful.

Section 2.3

In our history chapter, chapter 3, we discuss the Wheeler–Feynman approach which cannot be subsumed under short distance regularization. In the literature

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

the size of a classical electron, r_{cl}, is usually determined through equating the rest mass with the Coulomb energy, $m_{\mathrm{e}}c^2 = e^2/r_{\mathrm{cl}}$, which gives $r_{\mathrm{cl}} = 3 \times 10^{-13}$cm. This is really a lower bound in the sense that an even smaller radius would be in contradiction to the experimentally observed mass of the electron (assuming a positive bare mass, cf. the discussion in section 6.3). Milonni (1994) argues that due to quantum fluctuations the electron appears to have a classical spread, which is given by its Compton wavelength $\lambda_{\mathrm{c}} = r_{\mathrm{cl}}/\alpha$, with α the fine structure constant. Renormalization in Euclidean quantum field theory is covered by Glimm and Jaffe (1987) and Huang (1998). Effective potentials for classical fluids are discussed, e.g., in Huang (1987).

Section 2.4

The Abraham model was very popular in the early 1900s as studied by Abraham (1903, 1905), Lorentz (1892, 1915), Sommerfeld (1904a, 1904b, 1904c, 1905), and Schott (1912), among others. The extension to a rigid charge with rotation was already introduced in Abraham (1903) and further investigated by Herglotz (1903) and Schwarzschild (1903); compare with chapter 10. The dynamical systems point of view is stressed in Galgani *et al.* (1989). The proof of existence and uniqueness of the dynamics is taken from Komech and Spohn (2000), where a much wider class of external potentials is allowed. A somewhat different technique is used by Bauer and Dürr (2001). They also cover the case of a negative bare mass and discuss the smoothness of solutions in terms of the smoothness of initial data.

Section 2.5

This section is based on Appel and Kiessling (2001). Amongst many other results they explain the somewhat tricky variation of the action (2.97). Global existence of solutions is available in the case where the charge moves with constant velocity (Appel and Kiessling 2002). Appel and Kiessling (2001) rely on the monumental work of Nodvik (1964), but differ in one crucial aspect. Nodvik assumes that the mass of the extended body is concentrated in its center, which implies $I_{\mathrm{b}} = 0$. Newton's equations for the torque degenerate then into a constraint, which makes the Cauchy problem singular. A discussion of the Nodvik model can be found in Rohrlich (1990), chapter 7-4. The relativistic Thomas precession is discussed in Thomas (1926, 1927), Møller (1952), and in Misner, Thorne and Wheeler (1973), which is an excellent source on relativistic electrodynamics. Another informative source is Thirring (1997).

Of course, relativistic theories were studied much earlier, e.g. Born (1909). I refer to Yaghjian (1992) for an exhaustive discussion. The early models use a

https://doi.org/10.1017/9781009402286.003 Published online by Cambridge University Press

continuum description of the extended charge where each charge element has a velocity. They are not dynamical models in our sense, simply because there are more unknowns than equations. Also inner rotation is neglected, which, as we discussed, is not admissible in a relativistic theory.

The current generated by a point charge can be written as

$$\mathbf{j}(\mathbf{x}) = e \int_{-\infty}^{\infty} \mathrm{d}\tau \mathbf{u}(\tau) \delta(\mathbf{x} - \mathbf{q}(\tau)) \,. \tag{2.98}$$

McManus (1948) proposes to smear out the δ-function as

$$\mathbf{j}(\mathbf{x}) = e \int_{-\infty}^{\infty} \mathrm{d}\tau \mathbf{u}(\tau) \varphi_{\mathrm{MM}}((\mathbf{x} - \mathbf{q}(\tau))^2) \,, \tag{2.99}$$

which is to be inserted in the Lagrange density (2.82). He does not identify the conserved four-momentum, see also Peierls (1991) for illuminating explanations. Schwinger (1983) discusses the structure of the electromagnetic energy–momentum tensor in the case of rectilinear motion of the charge.

A more radical approach to a fully relativistic theory is to give up the notion of a material charged object and to regard electrons as point singularities of the Maxwell–Lorentz field. The guiding example are point vortices in a two-dimensional ideal Euler fluid, whose motion is governed by a closed set of differential equations which are of Hamiltonian form with the 1- and 2-component of the position as a canonically conjugate pair. In electrodynamics such a program was launched by Born (1933) and Born and Infeld (1933) and has not lost in attraction even now, mostly through activities in high-energy physics and string theory. Still, to have meaningful Newtonian equations of motion for the singularities is not so readily achieved. A recent proposal, based on the Hamilton–Jacobi equation, has been made by Kiessling (2003). He also provides a coherent overview of earlier attempts.