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INTEGRAL STARLIKE TREES

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Abstract

In this note we determine which of the trees homeomorphic to a star have a spectrum consisting entirely of integers. We also specify the integral double stars, and we consider the problem of trees with more complicated structure.

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1. Introduction

The term *integral graph* was introduced by Harary and Schwenk (1974) to describe a graph whose spectrum of eigenvalues consists entirely of integers. The effect of various binary operations on spectral integral properties was investigated, and several families of such integral graphs were exhibited. The identification of all integral graphs appears to be hopelessly involved. However, as with many other problems in graph theory, if we restrict our attention to trees, the prospects are much better. In Section 2 we identify all integral trees which are homeomorphic to a star $K_{1,m}$. Then, in Section 3, we consider the more difficult problem resulting when the tree is a double star. In the last section we raise the problem for trees of diameter greater than four.

2. Starlike trees

In general we follow the terminology in Harary (1969). So, for example, P_n is the path with n vertices. In addition, we use the spectral notation of Collatz and Sinogowitz (1957). In particular, when graph G has adjacency matrix A, the characteristic polynomial $\phi(G; x)$ is the determinant of $xI-A$. Even though G can be represented by more than one matrix A, the polynomial $\phi(G; x)$ is well defined because all choices for A are similar, and so yield the same polynomial. We call G an integral graph if $\phi(G; x)$ has only integer roots. We shall call a tree T *starlike* if T is homeomorphic to a star $K_{1,m}$. If $m \leqslant 2$, T is in fact a path. For $m \geqslant 3$, tree T has a unique vertex v of degree m and $T-v$ is a union of m paths. If T has n vertices, we may write

$$(1) \qquad T-v = \bigcup_{i=1}^{n-3} m_i P_i,$$

where $m = \sum m_i$ and $n = 1+\sum i m_i$. We use this notation to identify all integral starlike trees.

THEOREM 1. *The starlike tree T is integral if and only if T is one of these trees:*
(i) $T = K_1$ *and* $\phi(T; x) = x$;
(ii) $T-v = m_1 P_1$ *with* $m_1 = k^2 \geqslant 1$ *a perfect square and* $\phi(T; x) = (x^2-k^2)\,x^{m_1-1}$;
(iii) $T-v = m_2 P_2$ *with* $m_2+1 = k^2 \geqslant 4$ *a perfect square and*

$$\phi(T; x) = (x^2-k^2)\,x(x^2-1)^{m_2-1}.$$

PROOF. If T is any tree of one of these types, it is routine to verify that the characteristic polynomial given above is correct, and so the backward implication is trivial.

We assume T is integral and starlike, and we proceed to demonstrate that T must be one of the types listed. First, if T is a path, P_1, we have the trivial integral tree K_1. If $T = P_2$, then T is the first instance of type (ii). If $T = P_n$ is any longer path, then its largest eigenvalue (see Schwenk (1973), (1974)) is $2\cos \pi/(n+1)$ which is not integral. Having disposed of the easy cases, we now consider starlike trees with a unique vertex v of degree $m \geqslant 3$. Our strategy is to determine the maximum number of integral eigenvalues T may have as a function of $m_1, m_2, \ldots, m_{n-3}$. This bound will then provide restrictions on the values of the m_i's. We begin with a computational lemma:

LEMMA 1*a*. *For* $x \geqslant 2$,

$$\frac{1}{x} < \frac{\phi(P_i; x)}{\phi(P_{i+1}; x)} < \frac{1}{x-1}.$$

PROOF. In Harary *et al.* (1971) we find the recurrence relation

$$(2) \qquad \phi(P_{i+1}; x) = x\phi(P_i; x) - \phi(P_{i-1}; x).$$

Now since the largest root of $\phi(P_{i-1}; x)$ is less than 2, $\phi(P_{i-1}; x)$ is positive and so $1/x < \phi(P_i; x)/\phi(P_{i+1}; x)$ for $x \geqslant 2$. Moreover,

$$(3) \qquad \phi(P_{i+1}; x) = x\phi(P_i; x) - \phi(P_{i-1}; x) \geqslant 2\phi(P_i; x) - \phi(P_{i-1}; x).$$

Therefore, we may verify recursively that for $x \geqslant 2$

(4) $$\phi(P_{i+1};\, x) - \phi(P_i;\, x) \geqslant \phi(P_i;\, x) - \phi(P_{i-1};\, x) \geqslant \ldots \geqslant \phi(P_2;\, x) - \phi(P_1;\, x) > 0.$$

In particular, since $\phi(P_{i-1};\, x) < \phi(P_i;\, x)$ we may substitute in (2) to obtain

(5) $$\phi(P_{i+1};\, x) > x\phi(P_i;\, x) - \phi(P_i;\, x)$$

and so $\phi(P_i;\, x)/\phi(P_{i+1};\, x) < 1/(x-1)$ for $x \geqslant 2$.

We return to the proof of the theorem. Suppose T has an eigenvalue $\lambda \geqslant 2$. This value cannot be a multiple eigenvalue because by the interlacing theorem (see Schwenk and Wilson (1978)) that would require $\lambda \geqslant 2$ to be an eigenvalue of $T - v$. Consider the eigenvector associated with λ. Its v-component must be nonzero (lest λ be an eigenvalue of $T - v$) and so we may normalize it to equal 1. It is routine to verify that each branch of length i, $u_1, u_2 \ldots u_i v$ produces eigenvector components of

(6) $$(1/\phi(P_i;\, \lambda), \phi(P_1;\, \lambda)/\phi(P_i;\, \lambda), \ldots, \phi(P_{i-1};\, \lambda)/\phi(P_i;\, \lambda), 1)$$

To be an eigenvector, the sum of the components neighboring v must equal λ times the v-component (see Sachs (1964)) and so

(7) $$\lambda \cdot 1 = \sum_{i=1}^{n-3} m_i \phi(P_{i-1};\, \lambda)/\phi(P_i;\, \lambda).$$

Applying the lemma, we find

(8) $$\sum_{i=1}^{n-3} \frac{m_i}{\lambda} < \lambda < \sum_{i=1}^{n-3} \frac{m_i}{\lambda - 1}.$$

Recalling that $m = \sum m_i$,we take reciprocals and multiply by $m\lambda$ to obtain

(9) $$\lambda(\lambda - 1) < m < \lambda^2 < \lambda(\lambda + 1).$$

Obviously, there is at most one integer value for λ satisfying (9). By the pairing theorem (see Sachs (1964) or Schwenk and Wilson (1978)), T can have only two eigenvalues other than 0 and ± 1.

Sachs' Theorem (see Sachs (1964) or Schwenk and Wilson (1978)) tells us that 0 is an eigenvalue with multiplicity $-1 + \sum m_{2k+1}$ unless this yields -1 whence 0 has multiplicity 1.

Similarly, 1 is an eigenvalue with multiplicity $-1 - \sum m_{3k+2}$ unless this yields -1, in which case 1 is a simple eigenvalue if $\sum m_{3k+1} = 1$, and 1 is not an eigenvalue if $\sum m_{3k+1} \neq 1$. These statements can all be verified by constructing the appropriate number of eigenvectors in each circumstance. Moreover, the multiplicity of -1 is the same as the multiplicity of 1 by the pairing theorem (see Sachs (1964)). We may now complete the proof by considering four cases:

Case 1. $\sum m_{2k+1} = 0$ and $\sum m_{3k+2} = 0$. Then we have at most five eigenvalues, but $m \geqslant 3$ and $m_1 = 0$ imply $n \geqslant 7$. Thus, no integral trees are possible.

Case 2. $\sum m_{2k+1} = 0$ and $\sum m_{3k+2} \neq 0$. Then there are $n = 1 + \sum im_i = 1 + 2\sum m_{3k+2}$ eigenvalues. This equation can be solved only if m_2 is the only nonzero term. Now $T - v = m_2 P_2$, and the largest eigenvalue for T is an integer if and only if $m_2 + 1$ is a perfect square as specified in type (iii).

Case 3*a*. $\sum m_{2k+1} \neq 0$, $\sum m_{3k+2} = 0$ and $\sum m_{3k+1} = 1$. This time

$$n = 1 + \sum im_i = 3 + \sum m_{2k+1}$$

can be solved only if $m_1 = m_3 = 1$ and all other $m_i = 0$. But then $T = P_5$ which is not integral.

Case 3*b*. $\sum m_{2k+1} \neq 0$, $\sum m_{3k+2} = 0$ and $\sum m_{3k+1} \neq 1$. Then

$$n = 1 + \sum im_i = 1 + \sum m_{2k+1}$$

can be solved provided $m_i = 0$ for all $i \geqslant 2$. This yields $T = K_{1,m_1}$ which is integral if and only if m_1 is a perfect square as prescribed under type (ii).

Case 4. $\sum m_{2k+1} \neq 0$ and $\sum m_{3k+2} \neq 0$. This time

$$n = 1 + \sum im_i > -1 + \sum m_{2k+1} + 2\sum m_{3k+2}.$$

Consequently, we have found all the integral starlike trees.

3. Double stars

We would now like to examine the trees homeomorphic to a double star, that is, to a tree obtained by joining the centers of two stars with an edge. Unfortunately, the details are too involved to allow us to analyze trees in which the two vertices of high degree are nonadjacent, and so we shall limit our attention to those trees having exactly two vertices u and v of degree greater than two, and these two vertices are adjacent. Let T have m_i branches of length P_i at u and r_i branches of length P_i at v so that

$$T - u - v = \bigcup_{i=1}^{n-5} (m_i + r_i) P_i. \tag{10}$$

Furthermore, $m = 1 + \sum m_i$ and $r = 1 + \sum r_i$ are the degrees of u and v while $n = 2 + \sum i(m_i + r_i)$ is the number of vertices in T.

THEOREM 2. *If T is an integral tree and T has exactly two adjacent vertices of degree exceeding two, then T is either a double star $T-u-v=(m_1+r_1)P_1$ where the polynomial $x^4-(m_1+r_1+1)x^2+m_1r_1$ has only integral roots or*

$$T-u-v=m_1P_1+r_2P_2$$

where the polynomial $x^4-(m_1+r_2+2)x^2+m_1r_2+m_1+1$ has only integral roots.

PROOF. As in the previous theorem, we count the number of possible eigenvalues for T. For $\lambda \geqslant 2$, any eigenvector for λ cannot have a zero coordinate on v or u, for otherwise λ is also an eigenvalue of one of its branches, that is, of a path. If the eigenvector $\bar{\alpha}$ has v and u coordinates both positive and $\lambda \geqslant 2$, then every coordinate of $\bar{\alpha}$ is positive. Similarly, if $\bar{\alpha}$ is positive at v but negative at u, then $\bar{\alpha}$ is positive on all the branches at v and negative on all branches at u. Finally, if the v coordinate of $\bar{\alpha}$ is negative, $-\bar{\alpha}$ fits one of the descriptions given above. Since at most two vectors of this description can be mutually orthogonal, we conclude that T has at most two eigenvalues larger than 1 and at most two eigenvalues less than -1. It remains to specify the multiplicities of 0, 1 and -1. Let s be the common multiplicity of ± 1. We list the possibilities for each:

C_1. If $\sum m_{2k+1}>0$ and $\sum r_{2k+1}>0$, then 0 has multiplicity $-2+\sum m_{2k+1}+\sum r_{2k+1}$.

C_2. If $\sum m_{2k+1}=0$ or $\sum r_{2k+1}=0$, then 0 has multiplicity $\sum m_{2k+1}+\sum r_{2k+1}$.

D_1. If $\sum m_{3k+2}>0$ and $\sum r_{3k+2}>0$, then $s=\sum m_{3k+2}+\sum r_{3k+2}-2$.

D_2. If $\sum m_{3k+2}=\sum m_{3k+1}=\sum r_{3k+2}=\sum r_{3k+1}=0$, then $s=1$.

D_3. If $\sum m_{3k+2}=\sum r_{3k+2}=0$ and $\sum m_{3k+1}=\sum r_{3k+1}=2$, then $s=1$.

D_4. If $\sum m_{3k+2}=0$, $\sum m_{3k+1}=1$ and $\sum r_{3k+2}>0$, then $s=\sum r_{3k+2}$.

D_5. If $\sum m_{3k+2}=0$, $\sum m_{3k+1}\neq 1$ and $\sum r_{3k+2}>0$, then $s=\sum r_{3k+2}-1$.

D_6. If $\sum r_{3k+2}=0$, $\sum r_{3k+1}=1$ and $\sum m_{3k+2}>0$, then $s=\sum m_{3k+2}$.

D_7. If $\sum r_{3k+2}=0$, $\sum r_{3k+1}\neq 1$ and $\sum m_{3k+2}>0$, then $s=\sum m_{3k+2}-1$.

D_8. Otherwise, $s=0$.

These possibilities can now be combined to form sixteen possible cases labeled C_iD_j. Most of these cases produce impossible equations. For example, in C_1D_2, if we let c denote the number of eigenvalues with $|\lambda| \geqslant 2$, then the total number of vertices equals the number of eigenvalues, and so

$$2+\sum im_i+\sum ir_i=c+(-2+\sum m_{2k+1}+\sum r_{2k+1})+2(1). \tag{11}$$

This reduces to

$$(12) \qquad 2+2(m_2+r_2)+2(m_3+r_3)+4(m_4+r_4)+\ldots = c.$$

If $c = 2$, this forces $m_i = r_i = 0$ for $i \geqslant 2$. But $m_1 \leqslant \sum m_{3k+1} = 0$, which makes the degree of u equal to one! Thus, $c = 2$ is impossible. Hence, c must equal 4, but now only one of m_2, r_2, m_3 and r_3 can be nonzero. Without loss of generality, say $r_2 = r_3 = 0$. Then $r_1 \leqslant \sum r_{3k+1} = 0$, and so v has degree one. This last contradiction permits us to deduce that no integral trees satisfy case $C_1 D_2$.

The other cases are treated similarly, and all but four provide negative results. The only solutions occur in cases $C_1 D_3$, $C_1 D_8$, $C_2 D_5$ and $C_2 D_7$.

In case $C_1 D_3$, we find equation (12) is satisfied, and the unique tree is the double star with $m_1 = r_1 = 2$.

Case $C_1 D_8$ requires

$$(13) \qquad 4+2(m_2+r_2+m_3+r_3)+\ldots = c.$$

This can only be solved if m_i and r_i equal zero for all $i \geqslant 2$, and then, by Sachs' Theorem (Sachs (1964)).

$$(14) \qquad \phi(T;\, x) = x^n-(m_1+r_1+1)\, x^{n-2}+m_1 r_1\, x^{n-4}.$$

This polynomial has only integral roots if and only if $x^4-(m_1+r_1+1)\, x^2+m_1 r_1$ factors into $(x^2-a^2)(x^2-b^2)$ with $a \leqslant b$ both positive integers.

In case $C_2 D_5$, the eigenvalue count yields

$$(15) \qquad 4+2(m_2+m_3+r_3)+\ldots = c.$$

This can only be solved if all m_i's are 0 except m_1 and all r_i's are 0 except r_2. Again Sachs' Theorem yields

$$(16) \qquad \phi(T;\, x) = (x^2-1)^{r_2-1}\, x^{m_1}(x^4-(m_1+r_2+2)\, x^2+m_1 r_2+m_1+1).$$

Case $C_2 D_7$ is analogous, but replaces m_1 and r_2 with r_1 and m_2. This completes the proof of the theorem.

It is intriguing to try to specify all solutions to equation (14) with $r_1 \leqslant m_1$. One family of solutions occurs when $m_1 = r_1 = a(a+1)$ for any positive integer a, for which

$$(17) \qquad \phi(T;\, x) = (x^2-a^2)(x^2-(a+1)^2)\, x^{2a^2+2a-2}$$

The first two instances ($a = 1$ and 2) give the integral double stars displayed in Harary and Schwenk (1974). The difficult problem of finding all solutions of (14)

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in integers has been solved by Graham (1978). He finds that

(18)
$$m_1 = \tfrac{1}{4}(A^2+B^2-2)+C,$$
$$r_1 = \tfrac{1}{4}(A^2+B^2-2)-C,$$

where the integers A, B, C satisfy

(19)
$$(A^2-1)(B^2-1) = C^2.$$

The values for A, B, C can be found in terms of Chebyshev polynomials. Similarly, in equation (16) we find

(20)
$$m_1 = \tfrac{1}{4}(A^2+B^2-6)+C,$$
$$r_2 = \tfrac{1}{4}(A^2+B^2-2)-C,$$

where this time $(A^2-1)(B^2-1) = C^2-4$. These integral solutions can be found recursively by the method developed by Graham.

4. Trees of large diameter

It happens that all the integral trees found so far have diameter at most four. In this section we construct one more family of integral trees, which also happen to have diameter four, but are not starlike. Let $T = T(r,m)$ be formed by joining the centers of r copies of $K_{1,m}$ to a new vertex v. Thus, $T-v = rK_{1,m}$. Moreover, T has rm endpoints.

THEOREM 3. *$T(r,m)$ is integral if and only if both m and $r+m$ are perfect squares.*

PROOF. From Theorem 2 of Schwenk (1974), we have

(21)
$$\phi(T(r,m);\, x) = x\phi(T-v;\, x) - \sum_{u \text{ adj } v} \phi(T-v-u;\, x).$$

This reduces to

(22)
$$\phi(T(r,m);\, x) = (x^2-m)^{r-1}(x^2-m-r)\,x^{mr-r+1},$$

and so the conclusion of the theorem is evident.

REMARKS. We note that the case $m = 1$ is identical to Theorem 1 (iii). The smallest member of this family with $m > 1$ is shown in Fig. 1.

Christopher Godsil has observed that one can construct integral trees of diameter 6 by attaching t new endpoints to each vertex of the trees $T(r,m)$ in

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Theorem 3. The parameters t, r, m must be chosen so that m, $m+r$, t, $m+4t$ and $m+r+4t$ are all perfect squares. To accomplish this, select

$$m = (a^2-b^2)^2, \quad r = (c^2-d^2)^2-(a^2-b^2)^2 \quad \text{and} \quad t = a^2b^2 = c^2d^2.$$

For example, $a = 3$, $b = 2$, $c = 6$, $d = 1$ gives an integral tree of diameter 6 with 1 123 236 vertices. Integral trees of larger diameter should also be constructable.

Acknowledgement

We are grateful to Brendan McKay for providing us with an example which led us to correct an error in our earliest version of Theorem 2.

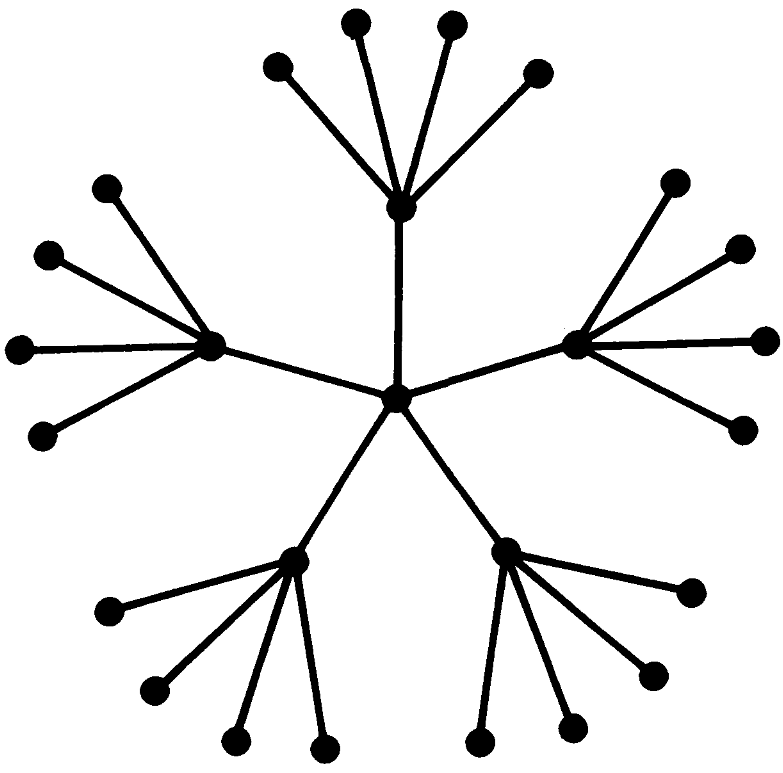

FIG. 1. The smallest integral tree of the form $T(r, m)$ with $m = 4$, $r = 5$.

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