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A NOTE ON GENERALIZED POLYNOMIAL IDENTITIES

BY

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Let A be an algebra with 1 over a field F and let B be a fixed F-basis of A. Let $F\langle x\rangle = F\langle x_1, \ldots, x_n, \ldots\rangle$ be the free algebra over F in noncommutative indeterminates $x_1, \ldots, x_n, \ldots$, and denote by $A_F\langle x\rangle$ the free product of A and $F\langle x\rangle$ over F. The elements of $A_F\langle x\rangle$ of the form $a_{i_0}x_{j_1}a_{i_1}\ldots x_{j_n}a_{i_n}$ ($a_{i_k} \in B$, n varies, repetitions allowed) form an F-basis of $A_F\langle x\rangle$. They will be referred to as *basis monomials*, and the a_{i_k}'s involved in a particular basis monomial will be called the *coefficients* of that basis monomial. In case $1 \in B$ a basis monomial of the form $x_{j_1}x_{j_2}\cdots x_{j_n}$ (i.e., 1 is the only coefficient) will be called *ordinary*. If $f \in A_F\langle x\rangle$ is written $f = \sum_i \alpha_i g_i$, g_i distinct basis monomials, then those g_i for which $\alpha_i \neq 0$ are said to *belong to* f. If $f_1, f_2, \ldots, f_m$ are nonzero elements of $A_F\langle x\rangle$ such that for $i \neq j$ no basis monomial belonging to f_i belongs to f_j, then we shall say that $f_1, f_2, \ldots, f_m$ are *strongly independent*.

A nonempty subset S of A is said to satisfy a *generalized polynomial identity* $f=0$ over F if there is a nonzero element $f=f(x_1, \ldots, x_n) \in A_F\langle x\rangle$ such that $f(s_1, \ldots, s_n)=0$ for all n-tuples $(s_1, \ldots, s_n) \in S^n$. In case each basis monomial belonging to f is ordinary S is said to satisfy a *polynomial identity*.

The purpose of this note is to show that in formulating the definition of generalized polynomial identity the scalars α_i involved in the equation $f=\sum_i \alpha_i g_i=0$, g_i basis monomial, need not be constants but may be allowed to depend on the n-tuple $(s_1, \ldots, s_n)$ being substituted in. In other words, we shall show that if there exist a finite number of distinct basis monomials having the property that for each substitution of n-tuples from S^n the resulting elements of A are F-dependent, then S actually satisfies a generalized polynomial identity.

LEMMA. *Let S be a subset of A, let $f_1, f_2, \ldots, f_m$, $m>1$, be strongly independent elements of $A_F\langle x\rangle$ (involving just $x_1, x_2, \ldots, x_n$), and let C be the set of coefficients of all the basis monomials belonging to the f_j's. Suppose γ_j, $j=1, 2, \ldots, m$ are arbitrary functions of S^n into F such that $\sum_{j=1}^m \gamma_j(s_1, \ldots, s_n)f_j(s_1, \ldots, s_n)=0$ for all $s_i \in S$. Set $g_j=f_jx_{n+1}f_m-f_mx_{n+1}f_j$, $j=1, 2, \ldots, m-1$ and define $\beta_j: S^{n+1}\to F$, $j=1, 2, \ldots, m-1$, by $\beta_j(s_1, \ldots, s_n, s_{n+1})=\gamma_j(s_1, \ldots, s_n)$. Then $g_1, g_2, \ldots, g_{m-1}$ are strongly independent, with same coefficient set C, and $\sum_{j=1}^{m-1} \beta_j(s_1, \ldots, s_n, s_{n+1})g(s_1, \ldots, s_n, s_{n+1})=0$ for all $s_i \in S$.*

Proof. We make the key observation that if p_1, p_2, q_1, q_2 are basis monomials (involving just $x_1, x_2, \ldots, x_n$) such that $p_1x_{n+1}q_1=p_2x_{n+1}q_2$, then $p_1=p_2$ and

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$q_1=q_2$. This remark, in conjunction with the strong independence of the f_j's and the definition of the g_j's, shows that $g_1, g_2, \ldots, g_{m-1}$ are also strongly independent. The remaining parts of the lemma are straightforward.

THEOREM. *Let A be an algebra with 1 over a field F, let B be an F-basis of A, let S be a nonempty subset of A, and let $f_1, f_2, \ldots, f_m$ be distinct basis monomials of $A_F\langle x\rangle$ with coefficient set C and involving just $x_1, x_2, \ldots, x_n$. Suppose that for each n-tuple $(s_1, \ldots, s_n) \in S^n$ the elements $f_j(s_1, \ldots, s_n)$, $j=1, 2, \ldots, m$, are F-dependent. Then S satisfies a generalized polynomial identity $f=0$ over F, where f also has C as its coefficient set.*

Proof. We may assume that $m>1$, since otherwise $f_1=0$ would already be a generalized polynomial identity for S. We are given that for each n-tuple $(s_1, \ldots, s_n) \in S^n \sum_{j=1}^m \gamma_j(s_1, \ldots, s_n)f_j(s_1, \ldots, s_n)=0$, where some $\gamma_j(s_1, \ldots, s_n)\neq 0$. Define elements g_{ij} of $A_F\langle x\rangle$ as follows:

$$g_{1j} = f_j, \qquad j = 1, 2, \ldots, m$$

$$g_{i+1,j} = g_{ij}x_{n+i}g_{i,m-i+1} - g_{i,m-i+1}x_{n+i}g_{ij}$$
$$i = 1, 2, \ldots, m-1; \quad j = 1, 2, \ldots, m-i$$

Successive applications of the preceding lemma then show that g_{m1} is a nonzero element of $A_F\langle x\rangle$ such that for each

$$(s_1, \ldots, s_n, \ldots, s_{n+m-1}) \in S^{n+m-1} \gamma_1(s_1, \ldots, s_n)g_{m1}(s_1, \ldots, s_n, \ldots, s_{n+m-1}) = 0.$$

A repetition of this argument (by appropriate reordering of subscripts) shows that for each $j=1, 2, \ldots, m$ there exists a nonzero element $h_j=h_j(x_1, \ldots, x_n, \ldots, x_{n+m-1})$ of $A_F\langle x\rangle$ such that for each $(s_1, \ldots, s_n, \ldots, s_{n+m-1}) \in S^{n+m-1}$ we have

$$\gamma_j(s_1, \ldots, s_n)h_j(s_1, \ldots, s_n, \ldots, s_{n+m-1}) = 0.$$

The element

$$f = f(x_1, \ldots, x_n, \ldots, x_{n+m}) = h_1x_{n+m}h_2 \ldots x_{n+m}h_m$$

is clearly a nonzero element of $A_F\langle x\rangle$ whose coefficient set is again C. Let

$$(s_1, \ldots, s_n, \ldots, s_{n+m}) \in S^{n+m}.$$

For some j, $\alpha=\gamma_j(s_1, \ldots, s_n)\neq 0$. Thus $f(s_1, \ldots, s_n, \ldots, s_{n+m})=\alpha^{-1}(\alpha f)= \alpha^{-1}[h_1s_{n+m}h_2 \ldots (\alpha h_j) \ldots h_m]=0$. Therefore $f=0$ serves as a generalized polynomial identity for S.

In particular, it is immediate from the theorem that if $f_1, f_2, \ldots, f_m$ are ordinary basis monomials, than S in fact satisfies a polynomial identity.

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