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Note on the Perpendicular Generators of a Hyperboloid of One Sheet.—The necessary and sufficient condition that two generators of the H_1 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ should be at right angles is that their point of intersection should lie on the director sphere $x^2+y^2+z^2=a^2+b^2-c^2$. Let $a^2+b^2-c^2=r^2$ and $a^2>b^2$. Then if $r^2>a^2$, $b^2>c^2$, and in this case the curve of intersection of the H_1 and sphere consists of two closed ovals lying on opposite sides of the plane XOY. Thus every generator of the H_1 meets the sphere in two real points. This result is easy to obtain analytically. For the generator

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm c}(=k)$$

meets the sphere in points given by

$$k^2(a^2\sin^2\alpha+b^2\cos^2\alpha+c^2)+2k\sin\alpha\cos\alpha(a^2-b^2)$$
$$-(a^2\sin^2\alpha+b^2\cos^2\alpha-c^2)=0.$$

The roots of this equation are real, equal, or imaginary, according as

$$(a^4-b^4)\sin^2\alpha+b^4-c^4 \gtreqless 0, \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and hence if $a^2>b^2>c^2$, the roots are real for every value of α.

Suppose that a generator, of the λ-system, say, meets the sphere in P and Q, then the generators of the μ-system through P and Q are at right angles to PQ, which is therefore their shortest distance. Hence if $a^2>b^2>c^2$, every generator of the H_1 is the shortest distance between two generators of the opposite system.

Again, if $a^2>r^2>b^2$, then $a^2>c^2>b^2$. In this case the H_1 and sphere intersect in two closed ovals lying on opposite sides of the plane YOZ, and we have an infinite number of generators meeting the sphere in two real points, an infinite number meeting it in two imaginary points, and four generators of each system touching it. This result is obtained analytically from equation (1), which shows that the generators through ($a\cos\alpha$, $b\sin\alpha$, 0) touch the sphere if $\sin^2\alpha = (c^4 - b^4)/(a^4 - b^4)$ and intersect the sphere in real or imaginary points according as $\sin^2\alpha \gtreqless (c^4 - b^4)/(a^4 - b^4)$. If P is the point of contact of one of the generators which touch the sphere, that generator is the shortest distance between two coincident generators of the opposite system, and thus P lies on a line of striction.

Lastly, if $b^2 > r^2$, then $c^2 > a^2 > b^2$. In this case the H_1 and sphere have no real common points, and hence no two of the generators of the H_1 intersect at right angles. Since the generators of the H_1 and its asymptotic cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ are parallel, the greatest angle between a pair of generators is $2\tan^{-1}b/c$.

R. J. T. BELL.

Note on the Principal Axes of a Normal Section of a Cylinder which Envelopes an Ellipsoid.—This note deals with methods of determining the lengths and direction-cosines of the axes. If the lengths are 2α and 2β, then α^{-2} and β^{-2} are the two non-zero roots of the discriminating cubic for the cylinder, and thus are easily found. The following methods apply geometrical considerations and determine both the lengths and the direction-cosines.

Let the ellipsoid be $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, and let the generators of the cylinder be parallel to $x/l = y/m = z/n$. Consider the normal section of the cylinder through the centre of the ellipsoid. The plane of the section is $lx + my + nz = 0$. The perpendiculars from the centre to the common tangent planes of the ellipsoid and cylinder lie in the plane $lx + my + nz = 0$, and are the perpendiculars to the tangents to the section of the cylinder by this plane. Let a common tangent plane be $\lambda x + \mu y + \nu z = \sqrt{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}$, and let the perpendicular from the centre to this plane be of length r.

Then
$$r^2 = \frac{a^2\lambda^2 + b^2\mu^2 + c^2\nu^2}{\lambda^2 + \mu^2 + \nu^2},$$

and hence the perpendicular lies on the cone

$$(a^2 - r^2)x^2 + (b^2 - r^2)y^2 + (c^2 - r^2)z^2 = 0. \quad \text{.............}(1)$$

It also lies in the plane $lx + my + nz = 0$. ....(2)