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GRADED COMPLEXES OVER POWER SERIES RINGS

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A common method in studying a commutative Noetherian local ring A is to find a regular subring R contained in A so that A becomes a finitely generated R-module, and in this way one can obtain some information about the original ring by applying what is known about regular local rings. By the structure theorems of Cohen, if A is complete and contains a field, there will always exist such a subring R, and R will be a power series ring $k[[X_1, \dots, X_n]] = k[[\mathbf{X}]]$ over a field k. In this paper we show that if R is chosen properly, the ring A (or, more generally, an A-module M), will have a comparatively simple structure as an R-module. More precisely, A (or M) will have a free resolution which resembles the Koszul complex on the variables $(X_1, \dots, X_n) = (\mathbf{X})$; such a complex will be called an $(\mathbf{X})$-graded complex and will be given a precise definition below. For low dimensions ($\leqq 3$) it is possible to list all modules which have such a resolution, and there are finitely many indecomposable ones; for higher dimensions this does not appear to be possible.

Nonetheless, in any dimension $(\mathbf{X})$-graded complexes have some nice properties. The only one we will consider here is the following: if F_* is an $(\mathbf{X})$-graded complex, then it is possible to define a filtration on each module F_i so that the complex of associated graded modules one obtains is a complex of free graded modules over the associated graded ring of $k[[\mathbf{X}]]$ with homology equal to the associated graded module of a good filtration on the homology of F_*. Such complexes will be called graded complexes; again, the exact definition will be given below. Interest in "approximating" a complex by a complex of graded modules came partly from the results of Peskine and Szpiro **[2]**, where some conjectures on multiplicity are proven for graded modules by defining a sequence of invariants related to dimension and multiplicities from a complex of free graded modules. The results here allow one to define an analogous sequence of invariants in a more general situation; however, they will not have all the properties one has in the graded case, due to the inevitable dependence on the subring R. If A is Cohen-Macaulay, it is possible to produce from this a complex of modules over the graded ring of A (graded by powers of the ideal $(\mathbf{X})$), and, if the original complex is the resolution of a module M, the new complex will be a resolution of the associated

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graded module of M by modules which are not necessarily free, but can be described in terms of the ring $A/(\mathbf{X})$. However, we do not develop this in this paper.

1. (X)-graded complexes. Let $A = K[[\mathbf{X}]] = k[[X_1, \dots, X_n]]$. Before defining the concept of (**X**)-graded complex, we give a definition of the usual Koszul complex $K_*(X_1, \dots, X_n)$ which will serve as a model for the more general definition.

In what follows, the word "complex" will mean "bounded complex of finitely generated free A-modules."

The definition of the Koszul complex we give here is by induction on n.

1. If $n = 0$, we let $K_* = A = k$ in degree zero (that is, $K_i = k$ if $i = 0$ and $K_i = 0$ if $i \neq 0$).

2. Suppose $K_*(X_1, \dots, X_{n-1})$ is defined and is a complex of free $k[[X_1, \dots, X_{n-1}]]$-modules. Then $K_*(X_1, \dots, X_n)$ is the total complex of the double complex

$$K_*(X_1, \dots, X_{n-1}) \otimes k[[\mathbf{X}]] \xrightarrow{X_n} K_*(X_1, \dots, X_{n-1}) \otimes k[[\mathbf{X}]]$$

where the two copies of $K_*(X_1, \dots, X_{n-1}) \otimes k[[\mathbf{X}]]$ are given degrees 1 and 0 respectively and the tensor product is taken over $k[[X_1, \dots, X_{n-1}]]$.

We recall that if L_{**} is a double complex, then the total complex of L_{**}, denoted $\mathrm{tot}(L_{**})$, is the complex with

$$\mathrm{tot}(L_{**})_i = \bigoplus_{j+k=i} L_{jk}$$

and differentials induced by those of L_{**}.

We next give a preliminary definition.

Definition 1.1. A *projective* complex is a direct sum of complexes of the forms:

$$\dots \to 0 \to A \xrightarrow{1} A \to 0 \to \dots \quad \text{and}$$

$$\dots \to 0 \to A \to 0 \to \dots$$

where the non-zero part can occur in any degree.

For the sense in which it is reasonable to consider such a complex projective, we refer to [**3**]. We note that if A is a field, every complex is projective.

Definition 1.2. We define what it means for a complex to be (**X**)-graded by induction on n.

1. If $n = 0$, every complex is (**X**)-graded.

2. If $n > 0$, an (**X**)-graded complex is a complex which is quasi-isomorphic to a complex of the form

$$\mathrm{tot}[\,(K_* \otimes k[\,[\mathbf{X}]\,]\,) \xrightarrow{X_n \phi} (P_* \otimes k[\,[\mathbf{X}]\,]\,)\,]$$

where we have a short exact sequence of complexes

$$0 \rightarrow L_* \rightarrow K_* \xrightarrow{\phi} P_* \rightarrow 0$$

in which K_* and P_* are $(X_1, \ldots, X_{n-1})$-graded complexes and L_* is projective.

We give three examples:

Example 1. Every projective complex is (**X**)-graded.

Example 2. The Koszul complex is (**X**)-graded. More generally, if we truncate the Koszul complex by letting $K_i = 0$ for all i less than some integer j, the resulting complex is (**X**)-graded.

Example 3. The complex over $A = k[\,[X_1, X_2]\,]$ given by

$$A^2 \xrightarrow{(X_1,\, X_2)} A$$

is also a truncated Koszul complex, but it is not in the form of part 2 of the above definition. However, it is quasi-isomorphic to the complex

$$A \xrightarrow{\begin{pmatrix} -X_2 \\ X_1 \\ 1 \end{pmatrix}} A^3 \xrightarrow{(X_1\ X_2\ 0)} A$$

which is in the correct form, so it is (**X**)-graded.

We next prove a result we will use later.

PROPOSITION 1.3. *Every* (**X**)*-graded complex is quasi-isomorphic to one of the form given in part* 2 *of Definition* 1.2 *in which* L_* *has zero differentials.*

Proof. Let

$$0 \rightarrow L_* \rightarrow K_* \xrightarrow{\phi} P_* \rightarrow 0$$

be a short exact sequence of complexes of $k[\,[X_1, \ldots, X_{n-1}]\,]$-modules as in part 2 of Definition 1.2. Suppose there exists an i such that $d_i: L_i \rightarrow L_{i-1}$ is not zero. Since L_* is projective, this means that there is a direct summand F_* of L_* isomorphic to

$$\ldots \rightarrow 0 \rightarrow A \xrightarrow{1} A \rightarrow \ldots ;$$

furthermore, since L_j is a direct summand of K_j for each j, the image of F_* in K_* is a direct summand of K_*. Denoting K_*/F_* by $\bar{K}_*$, we have a short exact sequence

$$0 \to F_* \to K_* \to \bar{K}_* \to 0$$

so that $\bar{K}_*$ is quasi-isomorphic to K_* and is thus $(X_1, \ldots, X_{n-1})$-graded. Furthermore, we have

$$0 \to F_* \otimes k[[X]] \to \operatorname{tot}(K_* \otimes k[[X]] \to P_* \otimes k[[x]])$$
$$\to \operatorname{tot}(\bar{K}_* \otimes k[[X]] \to P_* \otimes k[[x]]) \to 0$$

so that the above total complexes are also quasi-isomorphic. By continuing to remove trivial direct summands in this way we can eventually arrive at the situation in which L_* has zero differentials.

2. Graded complexes. Let A be a local ring with maximal ideal m. We define in this section an associated graded complex with respect to m for any complex of free A-modules. If the complex is minimal, so that the differentials are zero modulo m, this can be done very easily by using the m-adic filtration on each F_i shifted by i; however, we will need a more general case so we will not assume that the complex is minimal, even though this makes the definition somewhat more complicated.

Let F_* be a bounded complex of finitely generated free A-modules. We will assume, as we can, that bases are chosen for each F_i so that $d_i:F_i \to F_{i-1}$ is given by a matrix of the form

$$(1) \quad \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ M_i & N_i & P_i \end{pmatrix}$$

where I is an identity matrix and M_i, N_i, and P_i all have entries in m. We denote the corresponding decomposition of F_i into a direct sum by

$$F_i = F_i^1 \oplus F_i^2 \oplus F_i^3.$$

The fact that $d_i d_{i+1} = 0$ translates into the equations

$$N_i + P_i M_{i+1} = 0 \quad P_i N_{i+1} = 0 \quad P_i P_{i+1} = 0.$$

Thus the matrix (1) becomes

$$(2) \quad \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ M_i & -P_i M_{i+1} & P_i \end{pmatrix}$$

and the only condition is $P_i P_{i+1} = 0$.

We now wish to produce from this a complex of free graded modules over the graded ring

$$\bar{A} = \bigoplus_{i=0} m^i/m^{i+1}.$$

We use the following terminology and notation: if M is a finitely generated A-module, a *good filtration* on M is a decreasing filtration of A-submodules

$$M = \mathbf{F}_k(M) \supseteq \mathbf{F}_{k+1}(M) \supseteq \dots$$

such that $m\mathbf{F}_j(M) \subseteq \mathbf{F}_{j+1}(M)$ for all j and $m\mathbf{F}_j(M) = \mathbf{F}_{j+1}(M)$ for all but finitely many j. If M is a module with a good filtration, then $\bar{M}$, the associated graded module, is a finitely generated $\bar{A}$-module. If $f:M \to N$ satisfies

$$f(\mathbf{F}_j(M)) \subseteq \mathbf{F}_j(N) \quad \text{for all } j,$$

f induces a map from $\bar{M}$ to $\bar{N}$ which we denote $\bar{f}$.

Now let F_* be the complex above, and define a good filtration on each F_i by letting

$$\mathbf{F}_k(F_i) = m^{k-i}F_i^1 \oplus m^{k-i-1}F_i^2 \oplus m^{k-i}F_i^3.$$

The fact that $d_i(\mathbf{F}_k(F_i)) \subseteq \mathbf{F}_k(F_{i-1})$ follows from the fact that d_i is defined by the matrix (2) and M_i, M_{i+1}, and P_i have entries in m. Thus there is an associated complex $\bar{F}_*$ of free $\bar{A}$-modules; $\bar{F}_i$ is isomorphic as a graded module to

$$\bar{A}[-i]^{s_1} \oplus A[-i+1]^{s_2} \oplus \bar{A}[-i]^{s_3},$$

where s_j is the rank of F_i^j.

The filtration on F_i induces a good filtration on Ker d_i and Imd_{i+1} (by the Artin-Rees Lemma), and thus also on the homology $H_i(F_*)$. Furthermore, there is a short exact sequence

$$0 \to \overline{\mathrm{Im} d_{i+1}} \to \overline{\mathrm{Ker}\, d_i} \to \overline{H_i(F_*)} \to 0.$$

PROPOSITION 2.1. *There are natural maps*:

$$\kappa_i:\overline{\mathrm{Ker}\, d_i} \to \mathrm{Ker}\,(\bar{d}_i)$$

and

$$\mu_i:\mathrm{Im}(\bar{d}_{i+1}) \to \overline{\mathrm{Im} d_{i+1}}$$

making

$$\begin{array}{ccc} \overline{\mathrm{Im} d_{i+1}} & \longrightarrow & \overline{\mathrm{Ker}\, d_i} \\ \uparrow{\scriptstyle \mu_i} & & \uparrow{\scriptstyle \kappa_i} \\ \mathrm{Im}(\bar{d}_{i+1}) & \longrightarrow & \mathrm{Ker}(\bar{d}_i) \end{array}$$

commute.

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Proof. The maps κ_i and μ_i are induced by the identity on F_i; the fact that they are well-defined and the commutativity of the diagram are straight forward to verify.

Definition 2.2. The complex F_* is *graded* if κ_i and μ_i are isomorphisms for all i.

If F_* is graded, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Im}(\bar{d}_{i+1}) & \longrightarrow & \operatorname{Ker}(\bar{d}_i) & \longrightarrow & H_i(\bar{F}_*) & \longrightarrow & 0 \\
 & & \mu_i \downarrow \wr & & \kappa_i^{-1} \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & \overline{\operatorname{Im} d_{i+1}} & \longrightarrow & \overline{\operatorname{Ker} d_i} & \longrightarrow & \overline{H_i(F_*)} & \longrightarrow & 0
\end{array}$$

so that $H_i(\bar{F}_*)$ is isomorphic to $\overline{H_i(F_*)}$ as a graded module.

PROPOSITION 2.3. *F_* is graded if and only if κ_i is surjective for all i.*

Proof. We have commutative diagrams

$$\begin{array}{ccc}
\overline{\operatorname{Ker} d_i} & \hookrightarrow & \\
\kappa_i \downarrow & & \bar{F}_i \\
\operatorname{Ker} \bar{d}_i & \hookrightarrow &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\operatorname{Im} \bar{d}_{i+1} & \hookrightarrow & \\
\mu_i \downarrow & & \bar{F}_i \\
\overline{\operatorname{Im} d_{i+1}} & \hookrightarrow &
\end{array}$$

so that κ_i and μ_i are automatically injective. Thus the proposition will be proven if we can show that the surjectivity of κ_{i+1} implies the surjectivity of μ_i. We first express these conditions in terms of elements:

Surjectivity of μ_i: This says that if $\alpha \in \mathbf{F}_{i+1}$ and $d\alpha \in \mathbf{F}_k(F_i)$, then there exists a $\beta \in \mathbf{F}_k(F_{i+1})$ with $d\alpha - d\beta \in \mathbf{F}_{k+1}(F_i)$.

Surjectivity of κ_{i+1}: this says that if $\alpha \in \mathbf{F}_k(F_{i+1})$ and $d\alpha \in \mathbf{F}_{k+1}(F_i)$, then there exists $\beta \in \mathbf{F}_k(F_{i+1})$ with $\alpha - \beta \in \mathbf{F}_{k+1}(F_i)$ and $d\beta = 0$.

Suppose now that κ_{i+1} is surjective. Let $\alpha \in F_{i+1}$ with $d\alpha$ in $\mathbf{F}_k(F_i)$. If $\alpha \in \mathbf{F}_k(F_{i+1})$ we are done; if not, choose $j < k$ such that $\alpha \in \mathbf{F}_j(F_{i+1})$. Then

$$d\alpha \in \mathbf{F}_k(F_i) \subseteq \mathbf{F}_{j+1}(F_i),$$

so by the surjectivity of κ_{i+1}, there is a β in $\mathbf{F}_j(F_{i+1})$ with

$$d\beta = 0 \quad \text{and} \quad \alpha - \beta \in \mathbf{F}_{j+1}(F_{i+1}).$$

Then $d(\alpha - \beta) = d\alpha - d\beta = d\alpha$, so we can replace α by $\alpha - \beta \in \mathbf{F}_{j+1}(F_{i+1})$. This process can be continued until we find $\gamma \in \mathbf{F}_k(F_{j+1})$ with $d\gamma = d\alpha$, proving that μ_i is surjective.

The next result we wish to prove is that the property of being graded depends only on the quasi-isomorphism class of the complex F_*. Since we are only concerned here with complexes of free modules, this amounts to saying that if we represent a complex F_* as a direct sum $F_* = G_* \oplus M_*$, where G_* is chain homotopic to zero and M_* is minimal, then F_* is graded if and only if M_* is. If we represent F_* as in (2), the associated minimal complex is F_*^3 with boundary maps d_i^3 defined by the matrices P_i.

PROPOSITION 2.4. *If F_* and G_* are quasi-isomorphic, then F_* is graded if and only if G_* is graded.*

Proof. As outlined above, it suffices to show that F_* with boundary maps given by (2) is graded if and only if F_*^3 is. We use the criterion of Proposition 2.3.

Assume that F_*^3 is graded. Let

$$\eta = \begin{pmatrix} 0 \\ \bar{\alpha} \\ \bar{\beta} \end{pmatrix}$$

be in the kernel of $\bar{d}_i$ with

$$\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} \text{ in } \mathbf{F}_k(F_i) \text{ and}$$

$$d\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -PM\alpha + P\beta \end{pmatrix} \quad \text{in } \mathbf{F}_{k+1}(F_{i-1}).$$

We then have

$$-M\alpha + \beta \in \mathbf{F}_k(F_i^3) \quad \text{and} \quad P(-M\alpha + \beta) \in \mathbf{F}_{k+1}(F_{i-1}^3),$$

so, since F_*^3 is graded, there exists a $\gamma \in \mathbf{F}_k(F_i^3)$ with $P\gamma = 0$ and

$$(-M\alpha + \beta) - \gamma \in \mathbf{F}_{k+1}(F_i^3).$$

Then

$$\begin{pmatrix} 0 \\ \alpha \\ \gamma + M\alpha \end{pmatrix} \in \operatorname{Ker} d_i \text{ and}$$

$$\begin{pmatrix} 0 \\ \alpha \\ \beta \end{pmatrix} - \begin{pmatrix} 0 \\ \alpha \\ \gamma + M\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \beta - \gamma - M\alpha \end{pmatrix} \in \mathbf{F}_{k+1}(F_i).$$

Hence F_* is graded.

Conversely, assume that F_* is graded, and let $\alpha \in \mathbf{F}_k(F_i)$, be such that

$$P\alpha \in \mathbf{F}_{k+1}(F^3_{i-1}).$$

Then

$$\begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} \in \mathbf{F}_k(F_i)$$

with

$$d\begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} \in \mathbf{F}_{k+1}(F_{i-1}),$$

so there exists

$$\begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} \in \mathbf{F}_k(F_i)$$

with

$$d_i\begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} = 0 \quad \text{and} \begin{pmatrix} 0 \\ 0 \\ \alpha \end{pmatrix} - \begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} \in \mathbf{F}_{k+1}(F_i).$$

We claim that the element $\theta = \gamma - M\beta$ in F^3_i satisfies the condition of Proposition 2.3. Since

$$d_i\begin{pmatrix} 0 \\ \beta \\ \gamma \end{pmatrix} = 0$$

we have $P\theta = P\gamma - PM\beta = 0$. Furthermore, since $\beta \in \mathbf{F}_{k+1}(F^2_i)$, we have $M\beta \in \mathbf{F}_{k+1}(F^3_i)$, so

$$\alpha - \theta = \alpha + M\beta - \gamma = (\alpha - \gamma) + M\beta \in \mathbf{F}_{k+1}(F_i).$$

Thus $d^3_i(\theta) = 0$ and $\alpha - \theta \in \mathbf{F}_{k+1}(F^3_i)$, so F^3_* is graded.

As a final result in this section, we wish to show that the $(\mathbf{X})$-graded complexes defined above are graded. Let

$$A = k[[X_1, \ldots, X_n]] \quad \text{and} \quad m = (X_1, \ldots, X_n).$$

PROPOSITION 2.5. *An* $(\mathbf{X})$*-graded complex is graded.*

Proof. The proof is by induction on n. If $n = 0$, A is a field and there is nothing to prove.

Assume the result for $n - 1$, and let F_* be an $(\mathbf{X})$-graded complex. By Proposition 1.3, we can assume that

$$F_* = \operatorname{tot}(K_* \otimes k[[\mathbf{X}]] \xrightarrow{X_n\phi} P_* \otimes k[[\mathbf{X}]]),$$

where

$$0 \to L_* \to K_* \xrightarrow{\phi} P_* \to 0$$

is exact, K_* and P_* are $(X_1, \ldots, X_{n-1})$ graded complexes, and L_* is a complex of free A-modules with zero differentials.

If the complex P_* is not minimal, we can remove a trivial summand from P_* and K_* to make P_* minimal while replacing all complexes involved by quasi-isomorphic ones. It is not always possible to make K_* minimal, but by a proper choice of splitting

$$K_i \cong P_i \oplus L_i^1 \oplus L_i^2,$$

the boundary maps of K_* can be put in the form

$$\begin{pmatrix} G_i & 0 & 0 \\ J_i & 0 & 0 \\ M_i & 0 & 0 \end{pmatrix}$$

where J_i is of the form $(I\ 0)$ for an identity matrix I and G_i and M_i have coefficients in m. The boundary map in the total complex to $X_n\phi$ will now have the form

$$\begin{pmatrix} G_{i+1} & (-1)^i X_n & 0 & 0 \\ 0 & G_i & 0 & 0 \\ 0 & J_i & 0 & 0 \\ 0 & M_i & 0 & 0 \end{pmatrix}$$

This is now in the proper form, and we can use the criterion of Proposition 2.3 to show that this total complex is graded, using the inductive hypothesis on K_* and P_*.

Let

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \text{ be in } \mathbf{F}_k(F_i) \text{ with}$$

$$d\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} G_{i+1}\alpha + (-1)^i X_n\beta \\ G_i\beta \\ J_i\beta \\ M_i\beta \end{pmatrix}$$

in $\mathbf{F}_{k+1}(F_{i-1})$. It is clear that the choices of γ and δ are arbitrary and they can be taken to be zero. We must thus find

$$\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} \bmod F_{k+1}(F_i) \quad \text{with} \quad d\begin{pmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ 0 \\ 0 \end{pmatrix} = 0.$$

Let

$$\alpha = \sum \alpha_j X_n^j \quad \text{and} \quad \beta = \sum \beta_j X_n^j$$

with α_j and β_j in $k[[X_1, \ldots, X_{n-1}]]$. Since G and M have coefficients in $k[[X_1, \ldots, X_{n-1}]]$, the condition

$$G_{i+1}\alpha + (-1)^i X_n \beta \in \mathbf{F}_{k+1}(F_{i-1})$$

becomes:

$$G_{i+1}(\alpha_0) \in \mathbf{F}_{k+1}(P_{i-1})$$

$$G_{i+1}(\alpha_j) + (-1)^i \beta_{j-1} = \theta_{j-1} \in \mathbf{F}_{k+1-j}(P_{i-1}) \quad \text{for } j > 0.$$

Since P_* is graded, we can find $\widetilde{\alpha}_0$ with $\widetilde{\alpha}_0 - \alpha_0 \in \mathbf{F}_{k+1}(P_i)$ and $d\widetilde{\alpha}_0 = 0$. Let

$$\widetilde{\alpha} = \widetilde{\alpha}_0 + \alpha_1 X_n + \alpha_2 X_n^2 + \ldots$$

and let

$$\widetilde{\beta} = \sum (\beta_j - (-1)^i \theta_j) X_n^j.$$

It is then clear from the equations above that $\widetilde{\alpha}$ and $\widetilde{\beta}$ will satisfy

$$G_{i+1}(\widetilde{\alpha}) + (-1)^i X_n \widetilde{\beta} = 0$$

and that

$$\begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \\ 0 \\ 0 \end{pmatrix} \bmod \mathbf{F}_{k+1}(F_i).$$

The fact that

$$X_n \widetilde{\beta} = G_{i+1}((-1)^i \widetilde{\alpha})$$

now implies that

$$X_n G_i(\widetilde{\beta}) = 0, \; X_n J_i(\widetilde{\beta}) = 0, \quad \text{and} \quad X_n M_i(\widetilde{\beta}) = 0,$$

so

$$G_i(\widetilde{\beta}) = J_i(\widetilde{\beta}) = M_i(\widetilde{\beta}) = 0 \quad \text{and}$$

$$d \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \\ 0 \\ 0 \end{pmatrix} = 0.$$

Thus F_* is graded.

3. General complexes over power series rings. Let $A = k[[Y_1, \dots, Y_n]]$ be a power series ring, and let F_* be a bounded complex of finitely generated A-modules. We wish to show that there is a sub-power series ring $k[[\mathbf{X}]] \subseteq k[[\mathbf{Y}]]$ such that $k[[\mathbf{Y}]]$ is a finite $k[[\mathbf{X}]]$-module and such that F_* is an $(\mathbf{X})$-graded complex. The proof is by induction; as usual, there is nothing to prove if $n = 0$. If $n > 0$, we reduce to dimension $n - 1$ by taking the dual of a Cartan-Eilenberg resolution of F_* and showing that it is quasi-isomorphic in positive degrees to a double complex of modules of dimension less than or equal to $n - 1$. One can then recover F_* up to a projective summand using a method of [**1**] and show that for proper choice of $k[[\mathbf{X}]]$, F_* is $(\mathbf{X})$-graded. For the homological results in this section which are not well-known, we refer to [**2**].

Let

$$0 \to C_{k*} \to C_{k-1,*} \to \dots C_{1*} \to C_{0*} \to F_* \to 0$$

be a finite Cartan-Eilenberg resolution of F_*.

Let $P^{**} = \mathrm{Hom}_A(C_{**}, A)$. For each i, j we have boundary maps:

$$d_P^{ij}: P^{ij} \to P^{i+1,j}$$

$$\delta_P^{ij}: P^{ij} \to P^{i,j+1}.$$

LEMMA 3.1. *For each* $i > 0$ *and for each* j, *we have*:

a. $\dim(H^i(P^{*j})) \leqq n - 1$

b. $\dim(H^i(\mathrm{Ker}\, \delta^{*j})) \leqq n - 1.$

Proof. These homology groups are

$$\mathrm{Ext}^i(F_j, A) \quad \text{and} \quad \mathrm{Ext}^i(F_j/d_j(F_{j+1}), A)$$

respectively; since A is an integral domain of dimension n, they have dimension $\leqq n - 1$ for $i > 0$ (in fact, $\mathrm{Ext}^i(F_j, A) = 0$).

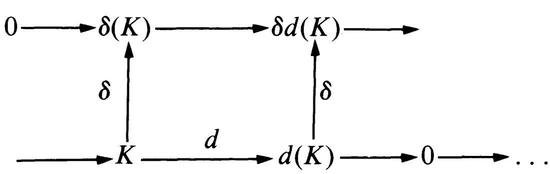
LEMMA 3.2. *There exists a sub-double complex* $K^{**} \subseteq P^{**}$ *such that*:

a. $\dim P^{ij}/K^{ij} \leqq n - 1$ *for all* i *and* j.

b. $P^{*j} \to P^{*j}/K^{*j}$ *induces isomorphisms in homology in degrees* > 0 *for all* j.

c. $\mathrm{Ker}\, \delta_p^{*j} \to \mathrm{Ker}\, \delta_{P/K}^{*j}$ *induces isomorphisms in homology in degrees* > 0 *for all* j.

Proof. We proceed step by step, dividing at each stage by a subcomplex of the form



where K is a submodule of $P^{k,l}$ such that $\dim P^{k,l}/K \leqq n - 1$ and the projection from P^{**} onto the quotient modulo this subcomplex satisfies conditions b and c of the lemma. Thus we assume that this has been done for $i < k$ and for $i = k$ and $j < l$, so that $\dim P^{i,j} \leqq n - 1$ for these indices, and show that it can then be done for $P^{k,l}$. Since there are only a finite number of non-zero modules in P^{**}, this will prove the lemma.

Fix k, l as above. The procedure is now as follows: we give several conditions on a submodule K of $P^{k,l}$ so that if K satisfies all of them, it will satisfy conditions b and c of the lemma. We show that there is a submodule K satisfying each condition with

$$\dim P^{k,l}/K \leqq n - 1.$$

Finally, each condition has the property that if K satisfies it so does any submodule of K. Then the intersection of submodules satisfying each of them will satisfy all of them and $P^{k,l}/K$ will still have the correct dimension.

CONDITION 1. *$K \xrightarrow{d} dK$ is an isomorphism if $k > 0$ and surjective if $k = 0$.*

Surjectivity is of course obvious. For injectivity, assume $k > 0$ and consider the sequence

$$P^{k-1,l} \to P^{k,l} \xrightarrow{d^{k,l}} P^{k+1,l}.$$

Since $P^{k-1,l}$ and $H^*(P^{k,l})$ have dimension $\leqq n - 1$, Ker $d^{k,l}$ must also. Thus we can choose a K with

$$K \cap \operatorname{Ker} d^{k,l} = 0 \quad \text{and} \quad \dim P^{k,l}/K \leqq n - 1.$$

CONDITION 2. *$\delta K \xrightarrow{d} d\delta K$ is an isomorphism if $k > 0$ and surjective if $k = 0$.*

As for Condition 1, we can find a submodule L of $P^{k,l+1}$ with

$$\dim P^{k,l+1}/L \leqq n - 1 \quad \text{and} \quad L \cap \operatorname{Ker} d = 0.$$

We then let $K = \delta^{-1}(L)$.

We assume henceforth that the module K under consideration satisfies Conditions 1 and 2.

CONDITION 3. *For $k > 0$, and for all j,*

$$\operatorname{Ker} \delta_K^{k,j} \xrightarrow{d} \operatorname{Ker} \delta_K^{k+1,j}$$

is an isomorphism. For $k = 0$, for all j we have

$$K \cap d^{-1}(\operatorname{Ker} \delta_P^{k+1,l}) \subseteq \operatorname{Ker} \delta_P^{k,l} + \operatorname{Ker} d_P^{k,l}.$$

This condition is trivial if $j \neq l$ or $l + 1$ and is Condition 2 if $j = l + 1$. Hence we assume $j = l$.

If $k > 0$, Condition 3 follows from Conditions 1 and 2 and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker } \delta_K^{k,l} & \longrightarrow & K & \xrightarrow{\delta} & \delta K & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow \wr & & \downarrow \wr & & \\ 0 & \longrightarrow & \text{Ker } \delta_K^{k+1,l} & \longrightarrow & dK & \longrightarrow & d\delta K & \longrightarrow & 0. \end{array}$$

Let $k = 0$. Both Ker $\delta_P^{k,l}$ and Ker $d_P^{k,l}$ are mapped by d into Ker $\delta_P^{k+1,l}$, and we have an injective map

$$0 \to d^{-1}(\text{Ker } \delta_P^{k+1,l})/(\text{Ker } \delta_P^{k,l} + \text{Ker } d_P^{k,l}) \to H^1(\text{Ker } \delta^{*,l}).$$

Since dim $H^1(\text{Ker } \delta_P^{*,l}) \leqq n - 1$, we can then find a submodule K with

$$\dim P^{k,l}/K \leqq n - 1$$

and satisfying

$$K \cap d^{-1}(\text{Ker } \delta_P^{k+1,l}) \subseteq \text{Ker } \delta_P^{k,l} + \text{Ker } d_P^{k,l}.$$

CONDITION 4. *For all* $k > 0$, *and for all* j, *the sequence*

$$0 \to \text{Ker } \delta_K^{*,j} \to \text{Ker } \delta_P^{*,j} \to \text{Ker } \delta_{P/K}^{*,j} \to 0$$

is exact.

With no assumptions on K, we have an exact sequence

$$0 \to \text{Ker } \delta_P^{*,j} \to \text{Ker } \delta_P^{*,j} \to \text{Ker } \delta_{P,K}^{*,j} \to \text{Coker } \delta_K^{*,j}.$$

Thus we must make the map from Ker $\delta_{P/K}^{*,j}$ to Coker $\delta_K^{*,j}$ zero. Since Coker $\delta_K^{*,j} = 0$ unless $j = l - 1$, we assume $j = l - 1$. In this case, the map looks like:

$$\begin{array}{ccccccc} \text{Coker } \delta_K^{*,l-1} & = \ldots \to 0 \to & K & \xrightarrow{d} & dK & \to 0 \to \ldots \\ \uparrow & & \uparrow{\scriptstyle \delta^{k,l-1}} & & \uparrow{\scriptstyle \delta^{k+1,l-1}} & \\ \text{Ker } \delta_{P/K}^{*,l-1} & = \ldots \quad 0 \to & \delta^{-1}(K) & \xrightarrow{d} & \delta^{-1}(dK) & \to 0 \to \ldots \end{array}$$

We must now choose K so that both vertical caps are zero.

To make $\delta^{k,l-1} = 0$ in this diagram, it suffices to make

$$K \cap \delta(P^{k,l-1}) = 0.$$

Now dim $P^{k,l-1} \leqq n - 1$ by hypothesis, so

$$\dim \delta(P^{k,l-1}) \leqq n - 1$$

and we can do this.

To make $\delta^{k+1,l-1}$ zero, we need

$$\delta P^{k+1,l-1} \cap dK = 0.$$

First, the short exact sequence of complexes:

$$0 \to \text{Ker } \delta^{*,l-1} \to P^{*,l-1} \to \text{Im } \delta^{*,l-1} \to 0$$

together with Lemma 3.1 imply that Im $\delta^{*,l-1}$ has homology of dimension $\leqq n - 1$ in degrees $\geqq 1$. Since $d(\text{Im } \delta^{k,l-1})$ has dimension $\leqq n - 1$, we deduce that

$$\text{Im } \delta^{k+1,l-1} \cap \text{Ker } d^{k+1,l}$$

has dimension $\leqq n - 1$. Hence we can find a submodule L of $P^{k+1,l}$ with

$$\dim P^{k+1}/L \leqq n - 1$$

and such that

$$L \cap \text{Im } \delta^{k+1,l-1} \cap \text{Ker } d^{k+1,l} = 0.$$

It then suffices to take $K = d^{-1}(L)$.

We now show that if Conditions 1-4 are satisfied, the map $P^{**} \to P^{**}/K^{**}$ satisfies Conditions b and c of the lemma.

Condition b. Conditions 1 and 2 imply that for each j, and each $i > 0$, we have $H^i(K^{*,j}) = 0$. Thus the long exact sequence associated to

$$0 \to K^{*,j} \to P^{*,j} \to (P/K)^{*,j} \to 0$$

implies Condition b.

Condition c. We divide this into two cases.

First assume $k > 0$. Then Condition 3 says that

$$H^i(\text{Ker } \delta_K^{*,j}) = 0 \quad \text{for all } i,$$

and Condition 4 says that

$$0 \to \text{Ker } \delta_K^{*,j} \to \text{Ker } \delta_P^{*,j} \to \text{Ker } \delta_{P/K}^{*,j} \to 0$$

is exact. Hence the long exact sequence implies Condition c.

Now assume $k = 0$. We must check that

$$H^1(\text{Ker } \delta_P^{*,j}) \to H^1(\text{Ker } \delta_{P/K}^{*,j})$$

is an isomorphism.

Surjectivity follows from Condition 4 as in the case when $k > 0$. We now show that Condition 3 is enough to imply that this map is injective. This is non-trivial only if $j = l$.

Let $\eta \in H^1(\text{Ker } \delta^{*,l}_{P/K})$ be such that its image in $H^1(\text{Ker } \delta^{*,l}_{P/K})$ is zero. Represent η by x in $P^{k+1,l}$ with $\delta x = dx = 0$. Then, since the image of η is zero in $H^1(\text{Ker}\delta^{*,l}_{P/K})$, there exists $\overline{y}$ in $P^{k,l}/K$ with $\overline{\delta y} = 0$ in $P^{k,l+1}/\delta K$ and $\overline{dy} = \overline{x}$ in $P^{*,l}/dK$. In other words, there exists $y \in P^{k,l}$ and k, k' in K with

$$\delta y = \delta k'$$

$$dy = x + dk.$$

Replacing y by $y - k'$ and k by $k - k'$, we can replace the first equation by

$$\delta y = 0.$$

We have

$$\delta(dk) = \delta(dy - x) = d\delta y - \delta x = 0,$$

so

$$dk \in \text{Ker } \delta_P^{k+1,l}.$$

Thus, by Condition 3 we can write $k = s + t$, with

$$s \in \text{Ker } \delta_P^{k,l} \quad \text{and} \quad t \in \text{Ker } d_P^{k,l}.$$

Now let $y' = y - s$. Then:

$$\delta y' = \delta y - \delta s = 0$$

$$dy' = dy - ds = x + dk - dk = x.$$

Thus $x \in d(\text{Ker } \delta^{k,l})$, $\eta = 0$ in $H^1(\text{Ker } \delta_P^{*,l})$. This completes the proof of the lemma.

We now return to the inductive proof that there is a power series subring R of A such that F_* is an $(\mathbf{X})$-graded complex of R-modules. Let K^{**} be as in Lemma 3.5, and let $M^{**} = P^{**}/K^{**}$. Then the dimension of M^{ij} is less than n for all i and j, so there is a non-zero element $X_n \in A$ such that $X_nM^{ij} = 0$ for all i, j. Choose $\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}$ in A so that A/X_nA is a finitely generated $k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}]]$-module. We note that A is then a finitely generated $k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}, X_n]]$-module.

The procedure is now to reconstruct F_* from M^{**}. Let $Q^{**} \rightarrow M^{**}$ be a Cartan-Eilenberg resolution of the complex of complexes of $k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}]]$-modules:

$$0 \rightarrow M^{0*} \rightarrow M^{1*} \rightarrow \ldots \rightarrow M^{k*} \rightarrow 0.$$

Let S^{**} be the total complex (or mapping cone) of the map of complexes of complexes of $k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}, X_n]]$ modules given by

$$\widetilde{Q}^{**} \xrightarrow{X_n} \widetilde{Q}^{**},$$

where

$$\widetilde{Q}^{**} = Q^{**} \otimes_{k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}]]} k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}, X_n]].$$

Thus for each i, we have

$$S^{i*} = \widetilde{Q}^{i*} \oplus \widetilde{Q}^{i+1,*},$$

with differentials induced by those of $\widetilde{Q}^{**}$ and by X_n.

Then S^{**} is a resolution of M^{**} over $k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}, X_n]]$. Since P^{**} is a resolution of M^{**} in degrees $\geqq 1$ (by the construction in Lemma 3.2), S^{**} and P^{**} agree in degrees $\geqq 1$ up to trivial direct summands and in degree zero up to a projective direct summand. Let T^{**} denote S^{**} truncated by omitting everything in degrees < 0. The complex T^{**} is constructed from $\widetilde{Q}^{**}$ as follows: let $\widetilde{Q}^{**}_{\geqq 0}$ denote $\widetilde{Q}^{**}$ truncated by omitting $\widetilde{Q}^{i*}$ for $i < 0$ (i.e., replacing these $\widetilde{Q}^{i*}$ with zeros), and let $\widetilde{Q}^{**}_{\geqq 1}$ by $\widetilde{Q}^{**}$ truncated by omitting $\widetilde{Q}^{i*}$ for $i < 1$; similarly define $Q^{**}_{\geqq 1}$ and $Q^{**}_{\geqq 0}$. Then if

$$\widetilde{Q}_{\geqq 1} \xrightarrow{\psi} \widetilde{Q}^{**}_{\geqq 0}$$

is the inclusion, we have

$$T^{**} = \mathrm{tot}(\widetilde{Q}^{**}_{\geqq 1} \xrightarrow{X_n\psi} \widetilde{Q}^{**}_{\geqq 0}).$$

We now dualize to get back to C_{**}. Let

$$R = k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}, X_n]].$$

We have:

$$C_{**} \cong \mathrm{Hom}_A(P^{**}, A).$$

We also know that

$$\mathrm{Hom}_R(T^{**}, R) \cong \mathrm{Hom}_R(P^{**}, R)$$

up to trivial summands in positive degrees and a projective summand in degree zero. We wish to show that the total complexes of all these complexes are isomorphic as complexes of R-modules up to a projective direct summand, and to do this it will suffice to show that

$$\mathrm{Hom}_A(P^{**}, A) \cong \mathrm{Hom}_R(P^{**}, R)$$

as complexes of R-modules.

To see this last isomorphism, we note that since P^{**} is a complex of free modules, we can decompose it into summands isomorphic to A and maps given by multiplication by elements of A. Hence it suffices to find an isomorphism

$$\alpha : \mathrm{Hom}_A(A, A) \xrightarrow{\sim} \mathrm{Hom}_R(A, R)$$

such that for each $a \in A$, the diagram

$$\begin{array}{ccc} \operatorname{Hom}_A(A, A) = A & \xrightarrow{a} & A \\ \downarrow \alpha & & \downarrow \alpha \\ \operatorname{Hom}_R(A, R) & \xrightarrow{a} & \operatorname{Hom}_R(A, R) \end{array}$$

commutes. But this is the same as an isomorphism of A-modules:

$$\operatorname{Hom}_R(A, R) \xrightarrow{\sim} A,$$

and this exists whenever A is a Gorenstein ring.

Putting these isomorphisms together, we deduce that

$$F_* \cong \operatorname{tot}(\operatorname{Hom}_R(T^{**}, R))$$

up to a projective direct summand. Let

$$R' = k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}]].$$

By changing the order in which we tensor with R and take total complexes, we have that $\operatorname{tot}(\operatorname{Hom}_R(T^{**}, R))$ is isomorphic to

$$\operatorname{tot}[\operatorname{Hom}(\operatorname{tot} Q^{**}_{i \geqq 0}, R')$$

$$\otimes_{R'} R \xrightarrow{X_n' \operatorname{Hom}(\operatorname{tot} \psi, R') \otimes R} \operatorname{Hom}(\operatorname{tot} Q^{**}_{i \geqq 1}, R') \otimes_{R'} R].$$

By the induction hypothesis, we can find a subring

$$k[[X_1, \ldots, X_{n-1}]] \subseteq k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}]]$$

such that the complexes $\operatorname{Hom}(\operatorname{tot} Q^{**}_{i \geqq 0}, R')$ and $\operatorname{Hom}(\operatorname{tot} Q^{**}_{i \geqq 1}, R')$ are $(X_1, \ldots, X_{n-1})$-graded and such that $k[[\widetilde{X}_1, \ldots, \widetilde{X}_{n-1}]]$ is a finitely generated $k[[X_1, \ldots, X_{n-1}]]$-module. It then follows that A is a finitely generated $k[[X_1, \ldots, X_n]]$-module and $\operatorname{Hom}(T^{**}, R)$, so also F_*, is an $(X_1, \ldots, X_n)$-graded $k[[X_1, \ldots, X_n]]$-module, as was to be shown.

4. (X)-graded complexes in low dimension. If the dimension R is one or two, it is possible to write down a list of all indecomposable (**X**)-graded complexes. If the dimension is three, this does not appear to be possible; however, it is still possible to list all modules whose free resolutions are of this type.

In dimension one, R is a discrete valuation ring, so there is a structure theorem for all complexes; every complex is a direct sum of free modules and complexes of the form

$$\ldots \to 0 \to R \xrightarrow{X^n} R \to 0 \to \ldots.$$

We remark that the complex is (**X**)-graded if and only if $n = 1$ in every direct summand of this form; we will use this in the next example.

Assume now that the dimension of R is 2. Here there are numerous infinite families of indecomposable modules, but we will show that there are exactly five distinct indecomposable (**X**)-graded complexes. For convenience, we will write $R = k[[X, Y]]$, and we assume that we have a map $\phi:K_* \to P_*$ of (**X**)-graded complexes over $k[[X]]$ which fits into a short exact sequence

$$(*) \quad 0 \to L_* \to K_* \to P_* \to 0$$

where L_* is a complex of free modules with zero differentials. We now wish to describe the complex

$$\mathrm{tot}(K_* \otimes k[[X, Y]] \xrightarrow{Y\phi} P_* \otimes k[[X, Y]]).$$

We will do this by modifying the complex (*) by row and column operations on its matrices to split off direct summands of various types. Let $S = k[[X]]$. Let n be the highest degree for which L_n, K_n, or P_n is not zero. We can then throw out L_n and the highest two degrees of the sequence will look like:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & S^r & \longrightarrow & S^r & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \left(\frac{M}{N}\right)} & & \downarrow{\scriptstyle N} & & \\
0 & \longrightarrow & S^s & \longrightarrow & S^s + S^m & \longrightarrow & S^m & \longrightarrow & 0
\end{array}$$

where M and N are matrices with coefficients in S.

If we have a column of zeros in the matrix $\left(\frac{M}{N}\right)$, we can split off a summand of the form

$$S \xrightarrow{1} S$$

from the first row, which gives the complex

$$(1) \quad R \xrightarrow{Y} R.$$

If we have a row of zeros, we can split off a summand from the second row and deal with it when considering the map from degree $n - 1$ to degree $n - 2$.

We now wish to reduce the matrix $\left(\frac{M}{N}\right)$. Note that we are not allowed to add a row of M to N or interchange rows when one is in M and the other in N, as this will not preserve the subcomplex L_*. Other row and column operations are allowed.

We first reduce N to obtain the form

$$\begin{pmatrix} 0 & M' & \\ 0 & & \\ \hline I & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where we use X to denote X times an identity matrix of the appropriate size. This gives summands of the form

$$\begin{array}{ccc} S & \xrightarrow{1} & S \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ S & \xrightarrow{1} & S \end{array}$$

which give rise to trivial complexes.

Now reduce the part of M' lying above zeros:

$$\begin{pmatrix} 0 & I & 0 & 0 \\ M'' & 0 & X & 0 \\ & 0 & 0 & 0 \\ \hline X & 0 & 0 & 0 \end{pmatrix}$$

The I gives direct summands of the form

$$\begin{array}{ccc} S & \xrightarrow{1} & S \\ {\scriptstyle 1}\downarrow & & \\ S & & \end{array}$$

which produce the complex

(2) $$R \xrightarrow{\begin{pmatrix} Y \\ 1 \end{pmatrix}} R^2 = R.$$

We now have:

$$\begin{pmatrix} M'' & X \\ & 0 \\ X & 0 \end{pmatrix}$$

We note that a column operation on M'' can be undone in X by an appropriate row operation, and that any multiple of X in M'' can be removed by subtracting a multiple of a row below the dotted line, so that we can reduce the part of M'' to the left of zero and obtain:

$$\begin{pmatrix} 0 & M''' & X \\ I & 0 & 0 \\ 0 & 0 & 0 \\ \hline X & 0 & 0 \\ 0 & X & 0 \end{pmatrix}$$

This produces summands of the form

$$\begin{array}{ccc} S & \xrightarrow{\;1\;} & S \\ \Big\downarrow{\scriptstyle \begin{pmatrix} 1 \\ X \end{pmatrix}} & & \Big\downarrow{\scriptstyle X} \\ S^2 & \xrightarrow{(0,\,1)} & S \end{array}$$

which give

$$(3) \qquad R \xrightarrow{\begin{pmatrix} -Y \\ 1 \\ X \end{pmatrix}} R^3 \xrightarrow{(X\ 0\ Y)} R = R^2 \xrightarrow{(X\ Y)} R.$$

We can now reduce M''' to get

$$\begin{pmatrix} I & 0 & X & 0 \\ 0 & 0 & 0 & X \\ \hline X & 0 & 0 & 0 \\ 0 & X & 0 & 0 \end{pmatrix}$$

This gives three types of direct summand. The first is:

$$\begin{pmatrix} 1 & X \\ X & 0 \end{pmatrix}$$

however, in diagonal form this becomes

$$\begin{pmatrix} 1 & 0 \\ 0 & X^2 \end{pmatrix}$$

which cannot occur since K_* is $(\mathbf{X})$-graded. The others are:

$$\begin{array}{ccc} S & \xrightarrow{\;1\;} & S \\ \Big\downarrow{\scriptstyle X} & & \Big\downarrow \\ S & \longrightarrow & 0 \end{array}$$

which produces

$$(4) \qquad R \xrightarrow{\begin{pmatrix} X \\ Y \end{pmatrix}} R^2$$

and

$$\begin{array}{ccc} S & \xrightarrow{1} & S \\ \downarrow{\scriptstyle X} & & \downarrow{\scriptstyle X} \\ S & \xrightarrow{1} & S \end{array}$$

which gives rise to the Koszul complex

$$(5) \qquad R \xrightarrow{\begin{pmatrix} -Y \\ X \end{pmatrix}} R^2 \xrightarrow{(X\ Y)} R.$$

Thus there are five (**X**)-graded complexes over $k[[X, Y]]$; it is clear that these are distinct (up to quasi-isomorphism).

If $R = k[[X, Y, Z]]$, there does not appear to be a simple classification of (**X**)-graded complexes of this sort. However, if F_* is the resolution of a module, we can take its dual (as in Section 3); this will give an (**X**)-graded complex over $k[[X, Y]]$, and, using the above classification, one can see that there are eight modules which arise in this way. Adding a free module of rank one, this gives nine (**X**)-graded modules in dimension three.

References

1. G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc *14* (1964), 689-713.

2. C. Peskine and L. Szpiro, *Syzygies et multiplicités*, C. R. Acad. Sci. *278* (1974), 1421-1424.

3. P. Roberts, *Some remarks on the homological algebra of multiple complexes* (to appear).

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