This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to G. D. Findlay, Department of Mathematics, McGill University, Montreal, P. Q.

A NOTE ON A THEOREM OF MOSER AND WHITNEY

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In a recent paper [1], L. Moser and E. L. Whitney have proved the following.

THEOREM: The number of compositions of n into parts $\equiv 1, 2, 4$ or $5 \pmod 6$ and involving an even number of parts $\equiv 4$ or $5 \pmod 6$ exceeds by n the number of compositions of n into parts $\equiv 1, 2, 4$ or $5 \pmod 6$ and involving an odd number of parts $\equiv 4$ or $5 \pmod 6$.

Their method of proof utilizes the notion of weighted compositions and the method of generating series. They remark that they have not been able to find a direct combinatorial proof. The purpose of this note is to give a direct proof of a more general result.

Let $r > 1$ be a fixed positive integer. In what follows we shall be concerned with compositions of the positive integer n into parts congruent to $1, 2, \ldots, r-1, r+1, \ldots, 2r-1$ modulo $2r$, i.e., we exclude all compositions of n which involve some part which is a multiple of r. Let $g(n;r)$ denote the number of such compositions of n. Further, let $g^E(n;r)$ and $g^O(n;r)$ denote the number of such compositions which involve an even or odd number of parts $\equiv r+1, \ldots, 2r-1 \pmod{2r}$. Clearly, $g^E(n;r) + g^O(n;r) = g(n;r)$. Let us

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define $h(n;r) = g^E(n;r) - g^O(n;r)$. Then we may prove the following.

THEOREM:

$$g(n;r) = \begin{cases} 2^{n-1} & \text{for } n < r \\ 2^{n-1} - 1 & \text{for } n = r \\ g(n-1;r) + g(n-2;r) + \dots + g(n-r;r) & \text{for } n > r \end{cases}$$

$$h(n;r) = \begin{cases} g(n;r) & \text{for } n \le r \\ h(n-1;r) + h(n-2;r) + \dots + h(n-r+1;r) - h(n-r;r) & \text{for } n > r \end{cases}$$

Proof: The result for $n \le r$ follows from our definitions. We shall restrict ourselves to the case where $n > r$.

Consider the set $S = S(n;r)$ of all compositions of n into parts not involving multiples of r. Let $S^E(S^O)$ be the subset of S which consists of all compositions involving an even (odd) number of parts $\equiv r+1, \dots, 2r-1 \pmod{2r}$. Let S_j be the subset of S which consists of all compositions whose first part is j. Clearly, S_j is empty if $j > n$ or $j = kr$. Also $S_i \cap S_j$ is empty if $i \ne j$, so that the S_j constitute a partition of S.

To prove the recurrence relation for $g(n;r)$, we note that

$$S = S_1 + S_2 + \dots + S_{r-1} + S_>$$

where $S_>$ denotes the union of the S_j with $j > r$, and we use a + sign rather than $\cup$ to denote a disjoint union. If a composition of S belongs to S_j, $j < r$, let us agree to suppress the first part, namely, j; if it belongs to $S_>$, let us subtract r from the first part. It is then immediately seen that

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$$g(n;r) = g(n-1;r) + g(n-2;r) + \ldots + g(n-r;r) .$$

We use essentially the same argument to prove the recurrence for $h(n;r)$. Using an obvious notation for the cross-partition of the two partitions introduced,

$$S^E = S_1^E + S_2^E + \ldots + S_{r-1}^E + S_>^E$$

$$S^O = S_1^O + S_2^O + \ldots + S_{r-1}^O + S_>^O$$

$$\therefore g^E(n;r) = g^E(n-1;r) + \ldots + g^E(n-r+1;r) + g^O(n-r;r)$$

$$g^O(n;r) = g^O(n-1;r) + \ldots + g^O(n-r+1;r) + g^E(n-r;r)$$

By subtraction,

$$h(n;r) = h(n-1;r) + h(n-2;r) + \ldots + h(n-r+1;r) - h(n-r;r) ,$$

thus completing the proof.

Let us now consider the special case where $r = 3$. From the initial conditions $h(1;3) = 1$, $h(2;3) = 2$, and $h(3;3) = 3$, and the recurrence $h(n;3) = h(n-1;3) + h(n-2;3) - h(n-3;3)$, it is easily shown (either by solving the recurrence or simply by induction) that $h(n;3) = n$ for all n. Thus we have a direct proof of the result of Moser and Whitney.

The recurrence relation for the numbers $g(n;r)$ is the same as the one for the generalized Fibonacci numbers defined by E. P. Miles. In his recent paper [2], he shows that the auxiliary equation

$$x^r - x^{r-1} - \ldots - x - 1 = 0$$

of the r-th order difference equation has distinct roots $Z_1, \ldots, Z_r$, so that the general solution has the form

$$g(n;r) = C_1 Z_1^n + C_2 Z_2^n + \ldots + C_r Z_r^n .$$

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The simple case for $r = 2$ leads to a well-known expression for the ordinary Fibonacci numbers (see, for example, [2], equation (2)). For the case $r = 3$, the expression for $g(n;r)$ is quite unwieldy.

REFERENCES

1. L. Moser and E. L. Whitney, "Weighted Compositions", Canad. Math. Bull., Vol. 4 (1961), pp. 39-43.

2. E. P. Miles, Jr., "Generalized Fibonacci Numbers and Associated Matrices", Amer. Math. Monthly, Vol. 67 (1960), pp. 745-752.

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