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Abstract

For a proper, smooth scheme X over a p-adic field K, we show that any proper, flat, semistable $\mathcal{O}_K$-model $\mathcal{X}$ of X whose logarithmic de Rham cohomology is torsion free determines the same $\mathcal{O}_K$-lattice inside $H^i_{\mathrm{dR}}(X/K)$ and, moreover, that this lattice is functorial in X. For this, we extend the results of Bhatt–Morrow–Scholze on the construction and the analysis of an A_{inf}-valued cohomology theory of p-adic formal, proper, smooth $\mathcal{O}_{\overline{K}}$-schemes $\mathfrak{X}$ to the semistable case. The relation of the A_{inf}-cohomology to the p-adic étale and the logarithmic crystalline cohomologies allows us to reprove the semistable conjecture of Fontaine–Jannsen.

Contents

1. Introduction

1.1 Integral relations between p-adic cohomology theories

For a proper, smooth scheme X over a complete, discretely valued extension K of $\mathbb{Q}_p$ with a perfect residue field k, comparison isomorphisms of p-adic Hodge theory relate the p-adic étale, de Rham, and, in the case of semistable reduction, also crystalline cohomologies of X. For instance, they show that for $i \in \mathbb{Z}$, the $\mathrm{Gal}(\overline{K}/K)$-representation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ functorially determines the filtered K-vector space $H^i_{\mathrm{dR}}(X/K)$. Even though the integral analogues of these isomorphisms are known to fail in general, one may still consider their hypothetical consequences, for instance, one may ask the following.

- For proper, flat, semistable $\mathcal{O}_K$-models $\mathcal{X}$ and $\mathcal{X}'$ of X endowed with their standard log structures, do the images of $H^i_{\log\,\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^i_{\log\,\mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K)$ in $H^i_{\mathrm{dR}}(X/K)$ agree?

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One of the goals of the present paper is to show that the answer is positive if the logarithmic de Rham cohomology of the models $\mathcal{X}$ and $\mathcal{X}'$ is torsion free (see (8.6.2) and Theorem 8.7): in this case, both $H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^i_{\log\mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K)$ agree with the $\mathcal{O}_K$-lattice in $H^i_{\mathrm{dR}}(X/K)$ that is functorially determined by $H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)$. The good reduction case of this result may be derived from the work of Bhatt–Morrow–Scholze [BMS18] on integral p-adic Hodge theory, and our approach, as well as the bulk of this paper, is concerned with extending the framework of [BMS18] to the semistable case.

1.2 The A_{inf}-cohomology in the semistable case

To approach the question above, we set $C := \widehat{\overline{K}}$, let $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$ be the basic period ring of Fontaine, and, for a semistable $\mathcal{O}_K$-model $\mathcal{X}$ of X, similarly to the smooth case treated in [BMS18], construct the A_{inf}-cohomology object

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \in D^{[0,\,2\dim(X)]}(A_{\mathrm{inf}})$$

that is quasi-isomorphic to a bounded complex of finite free A_{inf}-modules and has finitely presented cohomology $H^i_{A_{\mathrm{inf}}}(\mathcal{X})$. We show that base changes of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ recover other cohomology theories:

$$\begin{aligned} R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(C^\flat) &\cong R\Gamma_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} W(C^\flat),\\ R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\,\theta} \mathcal{O}_C &\cong R\Gamma_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \otimes^{\mathbb{L}}_{\mathcal{O}_K} \mathcal{O}_C,\\ R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(\overline{k}) &\cong R\Gamma_{\log\mathrm{cris}}(\mathcal{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} W(\overline{k}), \end{aligned} \tag{1.2.1}$$

see § 7.2; here $R\Gamma_{\log\mathrm{cris}}$ denotes the logarithmic crystalline (that is, Hyodo–Kato) cohomology, $W(k)$ (respectively, $\mathcal{O}_K$) carries the log structure associated to $\mathbb{N}_{\geqslant 0} \xrightarrow{1\mapsto 0,\,0\mapsto 1} W(k)$ (respectively, $\mathcal{O}_K \setminus \{0\} \hookrightarrow \mathcal{O}_K$), and $\mathcal{X}_k$ is endowed with the base change of the standard log structure $\mathcal{O}_{\mathcal{X},\,\text{ét}} \cap (\mathcal{O}_{\mathcal{X},\,\text{ét}}[\frac{1}{p}])^\times$ of $\mathcal{X}$.

If the cohomology of $R\Gamma_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ is torsion free, then each $H^i_{A_{\mathrm{inf}}}(\mathcal{X})$ is A_{inf}-free and the base changes (1.2.1) hold in each individual cohomological degree (see § 7.6 and Proposition 7.7). In this case, the Fargues equivalence for Breuil–Kisin–Fargues $\mathrm{Gal}(\overline{K}/K)$-modules allows us to prove that

$$\text{the } \mathrm{Gal}(\overline{K}/K)\text{-representation} \quad H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p) \quad \text{determines} \quad H^i_{A_{\mathrm{inf}}}(\mathcal{X})$$

(see Theorem 8.7). Then $H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)$ also determines[1] $H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ (and $H^i_{\log\mathrm{cris}}(\mathcal{X}_k/W(k))$) and, since the same reasoning applies to another model $\mathcal{X}'$, the result claimed in § 1.1 follows.

The base changes (1.2.1) also allow us to extend the cohomology specialization results obtained in the good reduction case in [BMS18]. Qualitatively, in Proposition 7.7 we show that $H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ is torsion free if and only if $H^i_{\log\mathrm{cris}}(\mathcal{X}_k/W(k))$ is torsion free, in which case $H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)$ is torsion free. Quantitatively, in Theorems 7.9 and 7.12 we show that for every $n \geqslant 0$, we have

$$\begin{aligned} \mathrm{length}_{\mathbb{Z}_p}((H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leqslant \mathrm{length}_{W(k)}((H^i_{\log\mathrm{cris}}(\mathcal{X}_k/W(k))_{\mathrm{tors}})/p^n),\\ \mathrm{length}_{\mathbb{Z}_p}((H^i_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leqslant \frac{1}{\mathrm{length}_{\mathcal{O}_K}(\mathcal{O}_K/p)} \cdot \mathrm{length}_{\mathcal{O}_K}((H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)_{\mathrm{tors}})/p^n). \end{aligned}$$

[1] The implicit functor is nonexact, as it must be: there exists a nonexact sequence of abelian schemes over $\mathbb{Z}_2$ that is short exact over $\mathbb{Q}_2$ (see [BLR90, 7.5/8]), so there is no *exact* functor F with $F(H^1_{\text{ét}}((-)_{\overline{\mathbb{Q}}_2},\mathbb{Z}_2)) = H^1_{\mathrm{dR}}(-/\mathbb{Z}_2)$.

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1.3 The semistable comparison isomorphism

The analysis of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$, specifically, its relation to the p-adic étale and the logarithmic crystalline cohomologies, permits us to reprove in Theorem 9.5 the semistable conjecture of Fontaine–Jansen [Kat94a, Conjecture 1.1]:

$$R\Gamma_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)\otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}} \cong R\Gamma_{\log\,\mathrm{cris}}(\mathcal{X}_k/W(k))\otimes^{\mathbb{L}}_{W(k)} B_{\mathrm{st}}. \tag{1.3.1}$$

Other proofs of this conjecture have been given in [Tsu99], [Fal02], [Niz08], [Bha12], [Bei13a], and [CN17], whereas [BMS18] used $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ to reprove the crystalline conjecture. Similarly to [CN17], we establish (1.3.1) for p-adic *formal* $\mathcal{O}_K$-schemes $\mathcal{X}$ that are proper, flat, and 'semistable'.

A key result that leads to (1.3.1) is the absolute crystalline comparison isomorphism

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \cong R\Gamma_{\log\,\mathrm{cris}}(\mathcal{X}_{\mathcal{O}_{\overline{K}}/p}/A_{\mathrm{cris}}) \tag{1.3.2}$$

of Corollary 5.43, whose construction in § 5 forms the technical core of this paper. This construction is based on an 'all possible coordinates' technique that is a variant of its analogue used to establish (1.3.2) in the smooth case in [BMS18, § 12]. The presence of singularities and log structures creates additional complications that do not appear in the smooth case and are examined in § 5.

Using the absolute crystalline comparison isomorphism, in Theorem 6.6 we compare the A_{inf}-cohomology of $\mathcal{X}$ with the B^{+}_{dR}-cohomology of X defined by Bhatt–Morrow–Scholze in [BMS18, § 13]:

$$R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B^{+}_{\mathrm{dR}} \cong R\Gamma_{\mathrm{cris}}(X^{\mathrm{ad}}_{C}/B^{+}_{\mathrm{dR}}). \tag{1.3.3}$$

The identification (1.3.3) is important for ensuring that the semistable comparison (1.3.1) is compatible with the de Rham comparison proved in [Sch13a], and hence that it respects filtrations.

As for the question posed in § 1.1, even though it only involves the étale and the de Rham cohomologies, the resolution of its torsion-free case outlined in § 1.2 uses both (1.3.2) and (1.3.3) (so also the bulk of the material of this paper). This is because we need to ensure that the determination of $H^{i}_{\mathrm{dR}}(X/K)$ by $H^{i}_{\text{ét}}(X_{\overline{K}},\mathbb{Q}_p)$ via the de Rham comparison of p-adic Hodge theory is compatible with the determination of $H^{i}_{\log\,\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i}_{\log\,\mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K)$ by $H^{i}_{\text{ét}}(X_{\overline{K}},\mathbb{Z}_p)$ via A_{inf}-cohomology and Breuil–Kisin–Fargues modules. In fact, to show that the cohomology modules of $R\Gamma_{A_{\mathrm{inf}}}(\mathcal{X})$ are Breuil–Kisin–Fargues, we already use the absolute crystalline comparison (1.3.2).

1.4 The object $A\Omega_{\mathfrak{X}}$ and its base changes

Even though we have so far focused on schemes, the construction and the analysis of $R\Gamma_{A_{\mathrm{inf}}}(-)$ works for any p-adic formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ that is semistable in the sense described in § 1.5 (see (1.5.1)) and that, whenever needed, is assumed to be proper. Specifically, for such an $\mathfrak{X}$, in § 2.2 we use the (variant for the étale topology of the) definition of Bhatt–Morrow–Scholze from [BMS18] to build an object

$$A\Omega_{\mathfrak{X}} \in D^{\geqslant 0}(\mathfrak{X}_{\text{ét}},A_{\mathrm{inf}}) \quad \text{and to set} \quad R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) := R\Gamma(\mathfrak{X}_{\text{ét}},A\Omega_{\mathfrak{X}}).$$

As in the smooth case of [BMS18], the relation of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ to the p-adic étale cohomology of the adic generic fiber $\mathfrak{X}^{\mathrm{ad}}_{C}$ of $\mathfrak{X}$ follows from the results of [Sch13a] (see § 2). In turn, the

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relations to the logarithmic de Rham and crystalline cohomologies are the subjects of §§ 4 and 5, respectively, and rest on the following identifications established in Theorems 4.17 and 5.4:

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log} \quad \text{and} \quad A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \cong Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}), \tag{1.4.1}$$

where $u\colon (\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log\mathrm{cris}} \to \mathfrak{X}_{\text{ét}}$ is the forgetful map of topoi. The arguments for (1.4.1) build on the same general skeleton as in [BMS18] but differ, among other aspects, in how they handle the interaction of the Deligne–Berthelot–Ogus décalage functor $L\eta$ used in the definition of $A\Omega_{\mathfrak{X}}$ with the intervening base changes and with the almost isomorphisms supplied by the almost purity theorem. Namely, for this, the nonflatness over the singular points of $\mathfrak{X}$ of the explicit perfectoid proétale covers that we construct makes it difficult to directly adapt the arguments from [BMS18]. Instead, we take advantage of several general results about $L\eta$ from [Bha18]. Verifying their assumptions in our case amounts to the analysis in § 3 of continuous group cohomology modules built using the aforementioned perfectoid cover. The typical conclusion of this analysis is that these modules have no nonzero 'almost torsion' and that the element $\mu \in A_{\mathrm{inf}}$ kills their 'nonintegral parts'.

Further and more specific overviews of our arguments are given in the beginning of each section that follows. In the rest of this introduction, we fix the precise notational setup for the remainder of the paper (see § 1.5), discuss the logarithmic structure on $\mathfrak{X}$ that we later use without notational explication (see § 1.6), and review the relevant general notational conventions (see § 1.7).

1.5 The setup

In what follows, we fix the notational setup.

- We fix an algebraically closed field k of characteristic $p > 0$, let C be the completed algebraic closure of $W(k)[\frac{1}{p}]$, and let $\mathfrak{m} \subset \mathcal{O}_C$ be the maximal ideal in the valuation ring of C.
- For convenience, we fix an embedding $p^{\mathbb{Q}} \subset C$, that is, for every prime ℓ, we fix a system of compatible ℓ^n-power roots $p^{1/\ell^\infty} := (p^{1/\ell^n})_{n\geqslant 0}$ of p in $\mathcal{O}_C$.
- We fix a p-adic formal scheme $\mathfrak{X}$ over $\mathcal{O}_C$ that in the étale topology may be covered by open affines $\mathfrak{U}$ which admit an étale $\mathcal{O}_C$-morphism

$$\mathfrak{U} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}) \quad \text{with} \quad R^{\square} := \mathcal{O}_C\{t_0,\dots,t_r,t_{r+1}^{\pm 1},\dots,t_d^{\pm 1}\}/(t_0\cdots t_r - p^q) \tag{1.5.1}$$

  for some $d \geqslant 0$, some $0 \leqslant r \leqslant d$, and some $q \in \mathbb{Q}_{>0}$ (where d, r, and q may depend on $\mathfrak{U}$).

For example, C could be the completed algebraic closure of any discretely valued field K of mixed characteristic $(0,p)$ with a perfect residue field. In addition, no generality is gained by allowing p^q in (1.5.1) to be *any* nonunit $\pi \in \mathcal{O}_C \setminus \{0\}$. The role of the embedding $p^{\mathbb{Q}} \subset C$ is to simplify arguments with explicit charts for the log structure on $\mathfrak{X}$ (defined in § 1.6); this is particularly useful in § 5, especially in §§ 5.25–5.26. Our C is less general than in [BMS18], where any complete algebraically closed nonarchimedean extension of $\mathbb{Q}_p$ is typically allowed. One of the main reasons for this is that we want to be able to apply, especially in § 5, certain auxiliary results from [Bei13a] (in any event, relations $t_0\cdots t_r - \pi$ in which π has a nonrational valuation go beyond 'semistable reduction').

The existence of étale local *semistable coordinates* (1.5.1) implies that each $\mathfrak{X}_{\mathcal{O}_C/p^n}$ is flat and locally of finite presentation over $\mathcal{O}_C/p^n$ and $\mathfrak{X}^{\mathrm{sm}}_{\mathcal{O}_C/p^n}$ is dense in $\mathfrak{X}_{\mathcal{O}_C/p^n}$. By [SP, 04D1] and limit arguments, equation (1.5.1) is the formal p-adic completion of the $\overline{W(k)}$-base change of an étale $\mathcal{O}$-morphism

$$U \to \mathrm{Spec}\big(\mathcal{O}[t_0,\dots,t_r,t_{r+1}^{\pm 1},\dots,t_d^{\pm 1}]/(t_0\cdots t_r - p^q)\big) \tag{1.5.2}$$

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for some discrete valuation subring $\mathcal{O} \subset \overline{W(k)}$ that contains p^q. The references [SP, 04D1] and [GR03, 7.1.6(i)] also imply that R is $R^{\square}$-flat. In addition, if $R \otimes_{\mathcal{O}_C} k$ is not k-smooth, then R determines q.[2]

Any smooth p-adic formal $\mathcal{O}_C$-scheme $\mathfrak{X}$ meets the requirements above: indeed, then the cover $\{\mathfrak{U}\}$ exists already for the Zariski topology with $r = 0$ and $q = 1$ for all $\mathfrak{U}$, see [FK18, I.5.3.18]. Another key example is

$$\mathfrak{X} = \widehat{\mathcal{X}_{\mathcal{O}_C}} \tag{1.5.3}$$

for some discrete valuation subring $\mathcal{O} \subset \mathcal{O}_C$ with a perfect residue field and a uniformizer $\pi \in \mathcal{O}$ and a locally of finite type, flat $\mathcal{O}$-scheme $\mathcal{X}$ that is *semistable* in the sense that $\mathcal{X}_{\mathcal{O}/\pi}$ is a normal crossings divisor in $\mathcal{X}$ (as defined in [SP, 0BSF]), so that, in particular, $\mathcal{X}$ is regular at every point of $\mathcal{X}_{\mathcal{O}/\pi}$.[3] Moreover, if $\mathcal{X}$ is even *strictly semistable* in the sense that $\mathcal{X}_{\mathcal{O}/\pi}$ is even a strict normal crossings divisor in $\mathcal{X}$ (as defined in [SP, 0BI9]), then the étale maps (1.5.4) exist even Zariski locally on $\mathcal{X}$, and so the cover $\{\mathfrak{U}\}$ also exists already for the Zariski topology of $\mathfrak{X}$.

- We let $\mathfrak{X}_C^{\mathrm{ad}}$ denote the adic generic fiber of $\mathfrak{X}$. By (1.5.1) and [Hub96, 3.5.1], the adic space $\mathfrak{X}_C^{\mathrm{ad}}$ is smooth over C; by [Hub96, 1.3.18 ii)], if $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then $\mathfrak{X}_C^{\mathrm{ad}}$ is C-proper.
- We let $(\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{proét}}$ denote the proétale site of $\mathfrak{X}_C^{\mathrm{ad}}$ (reviewed in [BMS18, § 5.1] and defined in [Sch13a, 3.9] and [Sch16, (1)]) and let

$$\nu \colon (\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{proét}} \to \mathfrak{X}_{\mathrm{ét}} \tag{1.5.5}$$

be the morphism to the étale site of $\mathfrak{X}$ that sends any étale $\mathfrak{U} \to \mathfrak{X}$ to the constant pro-system associated to its adic generic fiber. By [SP, 00X6], this functor indeed defines a morphism of sites: by [Hub96, 3.5.1], it preserves coverings, commutes with fiber products, and respects final objects. Thus, ν induces a morphism of topoi (ν^{-1}, ν_*) (see [SP, 00XC]).

1.6 The logarithmic structure on $\mathfrak{X}$

Unless noted otherwise, we always equip

(1) the ring $\mathcal{O}_C$ (respectively, $\mathcal{O}_C/p^n$ or k) with the log structure $\mathcal{O}_C \setminus \{0\} \hookrightarrow \mathcal{O}_C$ (respectively, its pullback);
(2) the formal scheme $\mathfrak{X}$ (respectively, $\mathfrak{X}_{\mathcal{O}_C/p^n}$ or $\mathfrak{X}_k$) with the log structure given by the subsheaf associated to the subpresheaf[4] $\mathcal{O}_{\mathfrak{X},\text{ét}} \cap (\mathcal{O}_{\mathfrak{X},\text{ét}}[\frac{1}{p}])^{\times} \hookrightarrow \mathcal{O}_{\mathfrak{X},\text{ét}}$ (respectively, its pullback log structure).

[2] The following argument justifies this. Choose an $n \in \mathbb{Z}_{>q}$ and let A be the local ring of $\mathrm{Spec}(R/p^n)$ at some singular point. Without loss of generality, all the t_i with $0 \leqslant i \leqslant r$ are noninvertible in A, so, in particular, $r \geqslant 1$. The dth Fitting ideal $\mathrm{Fitt}_d(\Omega^1_{(R^{\square}/p^n)/(\mathcal{O}_C/p^n)}) \subset R^{\square}/p^n$ is generated by the r-fold partial products $t_0 \cdots \widehat{t_i} \cdots t_r$ with $0 \leqslant i \leqslant r$, so the same holds for $\mathrm{Fitt}_d(\Omega^1_{A/(\mathcal{O}_C/p^n)}) \subset A$ (see [SGA 7_{I}, VI, 5.1(a)]). Consequently, the quotient $(R^{\square}/p^n)/(\mathrm{Fitt}_d(\Omega^1_{(R^{\square}/p^n)/(\mathcal{O}_C/p^n)}))$ is faithfully flat over $\mathcal{O}_C/(p^q)$, and hence so is $A/(\mathrm{Fitt}_d(\Omega^1_{A/(\mathcal{O}_C/p^n)}))$. It follows that $(p^q) \subset \mathcal{O}_C$ is the preimage of $\mathrm{Fitt}_d(\Omega^1_{A/(\mathcal{O}_C/p^n)}) \subset A$, to the effect that R determines q.

[3] To justify that any $\mathfrak{X}$ as in (1.5.3) meets the requirements, we first note that étale locally on $\mathcal{X}$ there exists a regular sequence such that the product its $r+1$ first terms cuts out $\mathcal{X}_{\mathcal{O}/\pi}$. Thus, since any finite extension of $\mathcal{O}/\pi$ is separable, the miracle flatness theorem [EGA IV_2, 6.1.5] ensures that every $x \in \mathcal{X}_{\mathcal{O}/\pi}$ has an étale neighborhood $U \to \mathcal{X}$ that admits an étale $\mathcal{O}$-morphism $U \to \mathrm{Spec}(\mathcal{O}[t_0, \ldots, t_d]/(t_0 \cdots t_r - \pi))$ or, equivalently, an étale morphism

$$U \to \mathrm{Spec}(\mathcal{O}[t_0, \ldots, t_r, t_{r+1}^{\pm 1}, \ldots, t_d^{\pm 1}]/(t_0 \cdots t_r - \pi))). \tag{1.5.4}$$

[4] The subpresheaf and its associated subsheaf necessarily agree on every quasi-compact object $\mathfrak{U}$ of $\mathfrak{X}_{\text{ét}}$.

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Both (1) and (2) determine the same log structure on $\mathrm{Spf}(\mathcal{O}_C)$, so the map $\mathfrak{X} \to \mathrm{Spf}(\mathcal{O}_C)$ is that of log formal schemes. Moreover, étale locally on $\mathfrak{X}$, the log structure may be made explicit: in the presence of a coordinate morphism (1.5.1), Claims 1.6.1 and 1.6.3 below give an explicit chart for the log structure of $\mathfrak{U}$, namely, the chart (1.6.2) in which we replace $\mathcal{O}$ by $\mathcal{O}_C$, replace U by $\mathfrak{U}$, and set $\pi := p^q$. This chart shows, in particular, that $\mathfrak{U}$ and $\mathcal{O}_C$ may be endowed with fine log structures whose base changes along a 'change of log structure' self-map of $\mathcal{O}_C$ recover the log structures described in (1)–(2) (for example, the fine log structure on $\mathcal{O}_C$ could be that determined by the chart $\mathbb{N}_{\geqslant 0} \xrightarrow{a \mapsto (p^q)^a} \mathcal{O}_C$, in which case the 'change of log structure' self-map of $\mathcal{O}_C$ is the identity on the underlying scheme $\mathrm{Spec}(\mathcal{O}_C)$ and is determined on the log structures by the map of charts $\mathbb{N}_{\geqslant 0} \xrightarrow{a \mapsto (p^q)^a} \mathcal{O}_C \setminus \{0\}$). Since many common properties of maps of log schemes are stable under base change, in practice this means that we may often deal with the log structures in (1)–(2) as if they were fine and, in particular, we may cite [Kat89] for certain purposes.

By the preceding discussion, all the log structures above are quasi-coherent and integral. Moreover, by [Kat89, 3.7(2)], each $\mathfrak{X}_{\mathcal{O}_C/p^n}$ is log smooth over $\mathcal{O}_C/p^n$, so that, by [Kat89, 3.10], the $\mathcal{O}_{\mathfrak{X}}$-module $\Omega^1_{\mathfrak{X}/\mathcal{O}_C,\log}$ of logarithmic differentials is finite locally free. We set

$$\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log} := \textstyle\bigwedge^i \Omega^1_{\mathfrak{X}/\mathcal{O}_C,\log},$$

let $\Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log}$ denote the logarithmic de Rham complex, and set

$$R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) := R\Gamma(\mathfrak{X}_{\text{ét}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C,\log}).$$

Claim 1.6.1. *For a valuation subring $\mathcal{O} \subset \overline{W(k)}$ and an $\mathcal{O}$-scheme U that has an étale morphism*

$$U \to \mathrm{Spec}\big(\mathcal{O}[t_0,\dots,t_r,t_{r+1}^{\pm 1},\dots,t_d^{\pm 1}]/(t_0\cdots t_r - \pi)\big) \quad \text{for some nonunit} \quad \pi \in \mathcal{O}\setminus\{0\},$$

the log structure on U associated to $\mathcal{O}_{U,\text{ét}} \cap (\mathcal{O}_{U,\text{ét}}[\frac{1}{p}])^\times$ has the chart

$$\mathbb{N}_{\geqslant 0}^{r+1} \sqcup_{\mathbb{N}_{\geqslant 0}} (\mathcal{O}\setminus\{0\}) \to \Gamma(U,\mathcal{O}_U) \tag{1.6.2}$$

given by $(a_i)_{0\leqslant i\leqslant r} \mapsto \prod_{0\leqslant i\leqslant r} t_i^{a_i}$ on $\mathbb{N}_{\geqslant 0}^{r+1}$, the diagonal $\mathbb{N}_{\geqslant 0} \to \mathbb{N}_{\geqslant 0}^{r+1}$ and $\mathbb{N}_{\geqslant 0} \xrightarrow{a\mapsto \pi^a} (\mathcal{O}\setminus\{0\})$ on $\mathbb{N}_{\geqslant 0}$, and the structure map $(\mathcal{O}\setminus\{0\}) \to \Gamma(U,\mathcal{O}_U)$ on $\mathcal{O}\setminus\{0\}$.

Proof. Without loss of generality, U is affine, so, by a limit argument, we may assume that $\mathcal{O}$ is discretely valued. Then U, endowed with the log structure associated to (1.6.2), is logarithmically regular in the sense of [Kat94b, 2.1] (compare with [Bei12, § 4.1, proof of Lemma]). Therefore, since the locus of triviality of this log structure is $U[\frac{1}{p}]$, the claim follows from [Kat94b, 11.6]. □

Claim 1.6.3. *For $\mathcal{O}$ as in Claim 1.6.1, a flat $\mathcal{O}$-scheme U (respectively, and its formal p-adic completion $\mathfrak{U}$) endowed with the log structure associated to $\mathcal{O}_{U,\text{ét}} \cap (\mathcal{O}_{U,\text{ét}}[\frac{1}{p}])^\times$ (respectively, $\mathcal{O}_{\mathfrak{U},\text{ét}} \cap (\mathcal{O}_{\mathfrak{U},\text{ét}}[\frac{1}{p}])^\times$),*

$$\text{the formal } p\text{-adic completion morphism} \quad j\colon \mathfrak{U} \to U \quad \text{of log ringed étale sites is strict.} \tag{1.6.4}$$

Proof. For a geometric point $\overline{u}$ of $\mathfrak{U}$, due to [SP, 04D1], the stalk map $\mathcal{O}_{U,\overline{u}} \cong j^{-1}(\mathcal{O}_{U,\overline{u}}) \to \mathcal{O}_{\mathfrak{U},\overline{u}}$ induces an isomorphism $\mathcal{O}_{U,\overline{u}}/p^n \cong \mathcal{O}_{\mathfrak{U},\overline{u}}/p^n$ for every $n > 0$. We consider the stalk map

$$\mathcal{O}_{U,\overline{u}} \cap \big(\mathcal{O}_{U,\overline{u}}\big[\tfrac{1}{p}\big]\big)^\times \cong j^{-1}\big(\mathcal{O}_{U,\overline{u}} \cap \big(\mathcal{O}_{U,\overline{u}}\big[\tfrac{1}{p}\big]\big)^\times\big) \to \mathcal{O}_{\mathfrak{U},\overline{u}} \cap \big(\mathcal{O}_{\mathfrak{U},\overline{u}}\big[\tfrac{1}{p}\big]\big)^\times. \tag{1.6.5}$$

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Every element x of the target of (1.6.5) satisfies the equation $xy = p^n$ for some $n > 0$. We choose an $\widetilde{x} \in \mathcal{O}_{U,\overline{u}}$ congruent to x modulo p^{n+1}, so that $\widetilde{x}\widetilde{y} = p^n + p^{n+1}\widetilde{z}$ for some $\widetilde{y}, \widetilde{z} \in \mathcal{O}_{U,\overline{u}}$. Since $1+p\widetilde{z} \in \mathcal{O}_{U,\overline{u}}^{\times}$, we adjust $\widetilde{y}$ to get $\widetilde{x}\widetilde{y} = p^n$, which shows that $\widetilde{x} \in \mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\frac{1}{p}])^{\times}$ and $(p^n) \subset (\widetilde{x})$. Thus, the image of $\widetilde{x}$ in $\mathcal{O}_{\mathfrak{U},\overline{u}}$ and x generate the same ideal, and hence are unit multiples of each other. Conversely, if $\widetilde{x}_1, \widetilde{x}_2 \in \mathcal{O}_{U,\overline{u}} \cap (\mathcal{O}_{U,\overline{u}}[\frac{1}{p}])^{\times}$ are unit multiples of each other in $\mathcal{O}_{\mathfrak{U},\overline{u}}$, then, by reducing modulo p^n for a large enough n, we see that they generate the same ideal in $\mathcal{O}_{U,\overline{u}}$, so are unit multiples of each other already in $\mathcal{O}_{U,\overline{u}}$. In conclusion, the map (1.6.5) induces an isomorphism

$$\big(\mathcal{O}_{U,\overline{u}} \cap \big(\mathcal{O}_{U,\overline{u}}\big[\tfrac{1}{p}\big]\big)^{\times}\big)/\mathcal{O}_{U,\overline{u}}^{\times} \xrightarrow{\sim} \big(\mathcal{O}_{\mathfrak{U},\overline{u}} \cap \big(\mathcal{O}_{\mathfrak{U},\overline{u}}\big[\tfrac{1}{p}\big]\big)^{\times}\big)/\mathcal{O}_{\mathfrak{U},\overline{u}}^{\times},$$

to the effect that the map (1.6.4) is indeed strict, as claimed. □

1.7 Conventions and additional notation

For a field K, we let $\overline{K}$ be its algebraic closure (taken inside C if K is given as a subfield of C). If K has a valuation, we let $\mathcal{O}_K$ be its valuation subring and write $\overline{\mathcal{O}}_K$ for the integral closure of $\mathcal{O}_K$ in $\overline{K}$. In mixed characteristic, we normalize the valuations by requiring that $v(p) = 1$. We let $(-)^{\mathrm{sm}}$ denote the smooth locus of a (formal) scheme over an implicitly understood base. For power series rings, we use $\{-\}$ to indicate decaying coefficients. For a topological ring R, we let R° denote the subset of powerbounded elements.

We let $W(-)$ (respectively $W_n(-)$) denote p-typical Witt vectors (respectively, their length n truncation), and let $[-]$ denote Teichmüller representatives. We let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ at p, let μ_{p^n} be the group scheme of p^nth roots of unity, and let ζ_{p^n} denote a primitive p^nth root of unity. For brevity, we set $\mathbb{Z}_p(1) := \varprojlim(\mu_{p^n}(C))$. We let $\widehat{M}$ denote the (by default, p-adic) completion of a module M and, similarly, let $\widehat{\bigoplus}$ denote the completion of a direct sum. Unless specified otherwise, we endow a p-adically complete module with the inverse limit of the discrete topologies.

We use the definition of a perfectoid ring given in [BMS18, 3.5] (the compatibility with prior definitions is discussed in [BMS18, 3.20]). Explicitly, by [BMS18, 3.9 and 3.10], a p-torsion-free ring S is *perfectoid* if and only if S is p-adically complete and the divisor $(p) \subset S$ has a p-power root in the sense that there is a $\varpi \in S$ with $(\varpi^p) = (p)$ and $S/\varpi S \xrightarrow[x \mapsto x^p]{\sim} S/pS$. In particular, for such an S, any p-adically formally étale S-algebra S' that is p-adically complete is also perfectoid.

For a ring object R of a topos $\mathscr{T}$, we write $D(\mathscr{T}, R)$, or simply $D(R)$, for the derived category of R-modules. For an object M of a derived category, we denote its derived p-adic completion by

$$\widehat{M} := R\lim_n(M \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}), \quad \text{and also set} \quad * \widehat{\otimes}^{\mathbb{L}}_{\cdot} - := R\lim_n((* \otimes^{\mathbb{L}}_{\cdot} -) \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \tag{1.7.1}$$

(see [SP, 0940] for the definition of $R\lim$). For a morphism f of ringed topoi, we use the commutativity of the functor Rf_* with derived limits and derived completions, see [SP, 0A07 and 0944].

For a profinite group H and a continuous H-module M, we write $R\Gamma_{\mathrm{cont}}(H, M)$ for the continuous cochain complex. Whenever convenient, we also view $R\Gamma_{\mathrm{cont}}(H, -)$ as the derived global sections functor of the site of profinite H-sets (see [Sch13a, 3.7(iii)] and [Sch16, (1)]).

For commuting endomorphisms $f_1, \dots, f_n$ of an abelian group A, we recall the *Koszul complex*:

$$K_A(f_1, \dots, f_n) := A \otimes_{\mathbb{Z}[x_1,\dots,x_n]} \bigotimes_{i=1}^{n} \big(\mathbb{Z}[x_1, \dots, x_n] \xrightarrow{x_i} \mathbb{Z}[x_1, \dots, x_n]\big), \tag{1.7.2}$$

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where A is regarded as a $\mathbb{Z}[x_1, \dots, x_n]$-module by letting x_j act as f_j, the tensor products are over $\mathbb{Z}[x_1, \dots, x_n]$, and the factor complexes are concentrated in degrees 0 and 1.

For an ideal I of a ring R and an R-module complex $(M^\bullet, d^\bullet)$ with $M^j \cong 0$ for $j < 0$, the subcomplex

$$\eta_I(M^\bullet) \subset M^\bullet \quad \text{is defined by} \quad (\eta_I(M^\bullet))^j := \{m \in I^j M^j \mid d^j(m) \in I^{j+1} M^{j+1}\}. \tag{1.7.3}$$

We will mostly (but *a priori* not always, see Proposition 5.34) use $\eta_I(M^\bullet)$ as in [BMS18, 6.2], namely, when I is generated by a nonzero-divisor and the M^j have no nonzero I-torsion.

A *logarithmic divided power thickening* (or, for brevity, a *log PD thickening*) is an exact closed immersion of logarithmic (often abbreviated to log) schemes equipped with a divided power structure on the quasi-coherent sheaf of ideals that defines the underlying closed immersion of schemes.

2. The object $A\Omega_{\mathfrak{X}}$ and the p-adic étale cohomology of $\mathfrak{X}$

As in the case when $\mathfrak{X}$ is smooth treated in [BMS18], the eventual construction of the A_{inf}-cohomology modules of $\mathfrak{X}$ rests on the object $A\Omega_{\mathfrak{X}}$ that lives in a derived category of A_{inf}-module sheaves on $\mathfrak{X}$. In this short section, we review the definition of $A\Omega_{\mathfrak{X}}$ in § 2.2 and then, in the case when $\mathfrak{X}$ is proper, review the connection between $A\Omega_{\mathfrak{X}}$ and the integral p-adic étale cohomology of $\mathfrak{X}_C^{\mathrm{ad}}$ in Theorem 2.3. We begin by fixing the basic notation that concerns the ring A_{inf} of integral p-adic Hodge theory. The setup of §§ 2.1–2.2 will be used freely in the rest of the paper.

2.1 The ring A_{inf}

We denote the tilt of $\mathcal{O}_C$ by

$$\mathcal{O}_C^\flat := \varprojlim_{y \mapsto y^p} (\mathcal{O}_C/p), \quad \text{so that, by reduction mod } p, \quad \varprojlim_{y \mapsto y^p} \mathcal{O}_C \xrightarrow{\sim} \varprojlim_{y \mapsto y^p} (\mathcal{O}_C/p) = \mathcal{O}_C^\flat$$

as multiplicative monoids (see [Sch12, 3.4(i)]). We regard p^{1/p^∞} fixed in § 1.5 as an element of $\mathcal{O}_C^\flat$. Due to the fixed embedding $p^{\mathbb{Q}_{\geqslant 0}} \subset \mathcal{O}_C$, this element comes equipped with well-defined powers $(p^{1/p^\infty})^q \in \mathcal{O}_C^\flat$ for $q \in \mathbb{Q}_{\geqslant 0}$. For each $x \in \mathcal{O}_C^\flat$, we let $(\dots, x^{(1)}, x^{(0)})$ denote its preimage in $\varprojlim_{y \mapsto y^p} \mathcal{O}_C$. The map $x \mapsto \mathrm{val}_{\mathcal{O}_C}(x^{(0)})$ makes $\mathcal{O}_C^\flat$ a complete valuation ring of height 1 whose fraction field $C^\flat := \mathrm{Frac}(\mathcal{O}_C^\flat)$ is algebraically closed (see [Sch12, 3.4(iii), 3.7(ii)]). We let $\mathfrak{m}^\flat$ denote the maximal ideal of $\mathcal{O}_C^\flat$.

The basic period ring A_{inf} of Fontaine is defined by

$$A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat) \quad \text{and comes equipped with the Witt vector Frobenius} \quad \varphi \colon A_{\mathrm{inf}} \xrightarrow{\sim} A_{\mathrm{inf}}.$$

We equip the local domain A_{inf} with the product of the valuation topologies via the Witt coordinate bijection $W(\mathcal{O}_C^\flat) \cong \prod_{n=1}^\infty \mathcal{O}_C^\flat$. Then A_{inf} is complete and its topology agrees with the $(p, [x])$-adic topology for any nonzero nonunit $x \in \mathcal{O}_C^\flat$. We fix (once and for all) a compatible system $\epsilon = (\dots, \zeta_{p^2}, \zeta_p, 1)$ of p-power roots of unity in $\mathcal{O}_C$, so that $\epsilon \in \mathcal{O}_C^\flat$, and set

$$\mu := [\epsilon] - 1 \in A_{\mathrm{inf}}. \tag{2.1.1}$$

Since $(p, \mu) = (p, [\epsilon - 1])$, the topology of A_{inf} is (p, μ)-adic. By forming the limit of the sequences

$$0 \to W_n(\mathcal{O}_C^\flat) \xrightarrow{\mu} W_n(\mathcal{O}_C^\flat) \to W_n(\mathcal{O}_C^\flat)/\mu \to 0, \tag{2.1.2}$$

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we see that A_{inf}/μ is p-adically complete and that the ideal $(\mu) \subset A_{\mathrm{inf}}$ does not depend on the choice of ϵ (use the fact that the valuation of $\zeta_p - 1$ does not depend on ζ_p).

The assignment $[x] \mapsto x^{(0)}$ extends uniquely to a ring homomorphism

$$\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C, \quad \text{the } \textit{de Rham specialization} \text{ map of } A_{\mathrm{inf}}, \tag{2.1.3}$$

which is surjective, as indicated, and intertwines the Frobenius φ of A_{inf} with the absolute Frobenius of $\mathcal{O}_C/p$. Its kernel $\mathrm{Ker}(\theta) \subset A_{\mathrm{inf}}$ is principal and generated by the element

$$\xi := \textstyle\sum_{i=0}^{p-1} [\epsilon^{i/p}] \tag{2.1.4}$$

(see [BMS18, 3.16]). Analogues of the sequences (2.1.2) show that each A_{inf}/ξ^n is p-adically complete. In fact, the map θ identifies A_{inf}/ξ^n with the initial p-adically complete infinitesimal thickening of $\mathcal{O}_C$ of order $n-1$, see [SZ18, 3.13]. The composition

$$\theta \circ \varphi^{-1}\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C \quad \text{is the } \textit{Hodge–Tate specialization} \text{ map of } A_{\mathrm{inf}},$$

and its kernel is generated by the element $\varphi(\xi) = \sum_{i=0}^{p-1} [\epsilon^i]$.

Due to the nature of our C (see § 1.5), the ring $\mathcal{O}_C/p$ is a k-algebra, so A_{inf} is a $W(k)$-algebra.

2.2 The object $A\Omega_{\mathfrak{X}}$

The operations that define $\mathcal{O}_C^{\flat}$ and A_{inf} make sense on the proétale site $(\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}$: namely, as in [Sch13a, 4.1, 5.10, and 6.1], we have the integral completed structure sheaf

$$\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}} := \varprojlim_n (\mathcal{O}^+_{\mathfrak{X}_C^{\mathrm{ad}},\,\mathrm{pro\acute{e}t}}/p^n), \quad \text{its tilt} \quad \widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C^{\mathrm{ad}}} := \varprojlim_{y \mapsto y^p} (\mathcal{O}^+_{\mathfrak{X}_C^{\mathrm{ad}},\,\mathrm{pro\acute{e}t}}/p), \tag{2.2.1}$$

and the basic period sheaf

$$\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}} := W(\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C^{\mathrm{ad}}}).$$

For brevity, we often denote these sheaves simply by $\widehat{\mathcal{O}}^+$, $\widehat{\mathcal{O}}^{+,\flat}$, and $\mathbb{A}_{\mathrm{inf}}$. Affinoid perfectoids form a basis for $(\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}$ (see [Sch13a, 4.7]) and the construction of the map θ of (2.1.3) makes sense for any perfectoid $\mathcal{O}_C$-algebra (see [BMS18, § 3]). In particular, $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ comes equipped with the map

$$\theta_{\mathfrak{X}_C^{\mathrm{ad}}}\colon \mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}} \to \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}, \tag{2.2.2}$$

which, by construction, is compatible with the map $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_C$, intertwines the Witt vector Frobenius φ of $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ with the absolute Frobenius of $\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}/p$, and, by [Sch13a, 6.3 and 6.5], is surjective with $\mathrm{Ker}(\theta_{\mathfrak{X}_C^{\mathrm{ad}}}) = \xi \cdot \mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ (in addition, ξ is not a zero divisor in $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$).

The key object that we are going to study in this paper is

$$A\Omega_{\mathfrak{X}} := L\eta_{(\mu)}(R\nu_*(\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}})) \in D^{\geqslant 0}(\mathfrak{X}_{\text{ét}}, A_{\mathrm{inf}}), \tag{2.2.3}$$

where the décalage functor $L\eta$ of [BMS18, § 6] is formed with respect to the ideal (μ) of the constant sheaf A_{inf} of $\mathfrak{X}_{\text{ét}}$ (the definition of $L\eta_{(\mu)}$ builds on the formula (1.7.3) for $\eta_{(\mu)}$). The formula (2.2.3) may also be executed with the Zariski site $\mathfrak{X}_{\mathrm{Zar}}$ as the target of ν, and it then defines the object

$$A\Omega_{\mathfrak{X}_{\mathrm{Zar}}} \in D^{\geqslant 0}(\mathfrak{X}_{\mathrm{Zar}}, A_{\mathrm{inf}}), \tag{2.2.4}$$

which is the $A\Omega_{\mathfrak{X}}$ that was used in [BMS18]. We will only use $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$ in Corollary 4.21 (and in some results that lead to it) for comparison with $A\Omega_{\mathfrak{X}}$.

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Since $\varphi(\mu) = \varphi(\xi)\mu$, by [BMS18, 6.11], we have $L\eta_{(\varphi(\mu))} \cong L\eta_{(\varphi(\xi))} \circ L\eta_{(\mu)}$, so the Frobenius automorphism of $\mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}$ gives the Frobenius morphism

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \cong L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}}) \xrightarrow{\text{[BMS18, 6.10 and 3.17(ii)]}} A\Omega_{\mathfrak{X}} \quad \text{in} \quad D^{\geqslant 0}(\mathfrak{X}_{\text{ét}}, A_{\mathrm{inf}}), \tag{2.2.5}$$

which, by [BMS18, 6.14], induces an isomorphism

$$(A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}})\left[\tfrac{1}{\varphi(\xi)}\right] \xrightarrow{\sim} (A\Omega_{\mathfrak{X}})\left[\tfrac{1}{\varphi(\xi)}\right]. \tag{2.2.6}$$

In addition, by *loc. cit.*, we also have

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right] \cong (R\nu_*(\mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right], \tag{2.2.7}$$

so a result of Scholze [BMS18, 5.6] supplies the following relation to integral p-adic étale cohomology.

Theorem 2.3. *If $\mathfrak{X}$ is proper over $\mathcal{O}_C$, then there is an identification*

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right] \cong R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right]. \tag{2.3.1}$$

In broad strokes, the proof of Theorem 2.3 given in [BMS18, 5.6] goes as follows: one considers the map

$$R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}} \cong R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}} \to R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}) \tag{2.3.2}$$

induced by the inclusion $A_{\mathrm{inf}} \hookrightarrow \mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}$ and deduces from the almost purity theorem with, for instance, Lemma 3.17 below that the ideal

$$W(\mathfrak{m}^\flat) := \mathrm{Ker}(W(\mathcal{O}_C^\flat) \twoheadrightarrow W(k)) \quad \text{of} \quad A_{\mathrm{inf}} \tag{2.3.3}$$

kills the cohomology of its cone. Since μ lies in $W(\mathfrak{m}^\flat)$ and we have the identification (2.2.7), it follows that the map (2.3.2) induces the identification (2.3.1).

Remark 2.4. In practice, $\mathfrak{X}$ often arises as the formal p-adic completion of a proper, finitely presented $\mathcal{O}_C$-scheme $\mathcal{X}$. In this situation, $\mathfrak{X}_C^{\mathrm{ad}}$ agrees with the adic space associated to $\mathcal{X}_C$ (see [Con99, 5.3.1 4.], [Hub94, 4.6(i)], and [Hub96, 1.9.2 ii)]) and, by [Hub96, 3.7.2], we have

$$R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \cong R\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p).$$

3. The local analysis of $A\Omega_{\mathfrak{X}}$

Even though the definition of the object $A\Omega_{\mathfrak{X}}$ given in (2.2.3) is global, the key computations that will eventually relate it to the logarithmic de Rham and crystalline cohomologies are local and are presented in this section. Under the assumption that $\mathfrak{X}$ has a coordinate morphism as in (1.5.1) (or (3.1.1) below), their basic goal is to express the cohomology of the proétale sheaf $\mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}$, at least after applying $L\eta_{(\mu)}$, in terms of continuous group cohomology formed using an explicit perfectoid proétale cover $\mathfrak{X}_{C,\infty}^{\mathrm{ad}}$ of $\mathfrak{X}_C^{\mathrm{ad}}$ (see Theorem 3.20). The basic relation of this sort is supplied by the almost purity theorem, so the key point is to explicate the appearing group cohomology modules well enough in order to eliminate the 'almost' ambiguities inherent in

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this theorem with the help of Lemma 3.18 below that comes from [Bha18]. We first carry out this program for the simpler sheaf $\widehat{\mathcal{O}}^{+}_{\mathfrak{X}^{\mathrm{ad}}_C}$, and then build on this case to address $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}^{\mathrm{ad}}_C}$.

In comparison with the local analysis carried out in the smooth case in [BMS18], one complication is that the perfectoid cover of $\mathfrak{X}$ that gives rise to $\mathfrak{X}^{\mathrm{ad}}_{C,\,\infty}$ is not flat over the singular points of $\mathfrak{X}_k$. This makes it difficult to transfer various arguments with 'q-de Rham complexes' across the coordinate morphism (3.1.1). In fact, we avoid q-de Rham complexes altogether and instead phrase the intermediate steps of the local analysis purely in terms of continuous group cohomology modules.

3.1 The local setup

We assume throughout § 3 that $\mathfrak{X} = \mathrm{Spf}(R)$ and for some $d \geqslant 0$, some $0 \leqslant r \leqslant d$, and some $q \in \mathbb{Q}_{>0}$, there is an étale $\mathrm{Spf}(\mathcal{O}_C)$-morphism as in (1.5.1):

$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}) =: \mathfrak{X}^{\square} \quad \text{with} \quad R^{\square} := \mathcal{O}_C\{t_0, \ldots, t_r, t_{r+1}^{\pm 1}, \ldots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q). \tag{3.1.1}$$

Due to our assumptions from § 1.5, a general $\mathfrak{X}$ is of this form on a basis for its étale topology.

3.2 The perfectoid cover $\mathfrak{X}^{\mathrm{ad}}_{C,\,\infty}$

For each $m \geqslant 0$, we consider the $R^{\square}$-algebra

$$R^{\square}_m := \mathcal{O}_C\{t_0^{1/p^m}, \ldots, t_r^{1/p^m}, t_{r+1}^{\pm 1/p^m}, \ldots, t_d^{\pm 1/p^m}\}/(t_0^{1/p^m} \cdots t_r^{1/p^m} - p^{q/p^m}), \quad \text{and} \quad R^{\square}_{\infty} := (\varinjlim R^{\square}_m)\widehat{\ },$$

where, as always unless mentioned otherwise (see § 1.7), the completion is p-adic. Explicitly, we have the p-adically completed direct sum decomposition

$$R^{\square}_{\infty} \cong \widehat{\bigoplus}_{\substack{(a_0,\ldots,a_d) \in (\mathbb{Z}[\frac{1}{p}]_{\geqslant 0})^{\oplus(r+1)} \oplus (\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)}, \\ a_j = 0 \text{ for some } 0 \leqslant j \leqslant r}} \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}, \tag{3.2.1}$$

which shows that $R^{\square}_{\infty}$ is perfectoid (see § 1.7) and that, for each $m \geqslant 0$, the map $R^{\square}_m \to R^{\square}_{\infty}$ is an inclusion of an $R^{\square}_m$-module direct summand composed of those summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of (3.2.1) for which $p^m a_j \in \mathbb{Z}$ for all j.

The corresponding R-algebras are

$$R_m := R \otimes_{R^{\square}} R^{\square}_m \quad \text{and} \quad R_{\infty} := \big(\varinjlim R_m\big)\widehat{\ } \cong (R \otimes_{R^{\square}} R^{\square}_{\infty})\widehat{\ }.$$

Each R_m (respectively, R_{∞}) is a p-torsion-free p-adically formally étale $R^{\square}_m$-algebra (respectively, $R^{\square}_{\infty}$-algebra). In particular, R_{∞} is perfectoid (see § 1.7). By [GR03, 7.1.6(ii)], each R_m is p-adically complete.

The summands in (3.2.1) with $a_j \notin \mathbb{Z}$ for some $0 \leqslant j \leqslant d$ comprise an $R^{\square}$-submodule $M^{\square}_{\infty}$ of $R^{\square}_{\infty}$, and we set $M_{\infty} := R \mathbin{\widehat{\otimes}}_{R^{\square}} M^{\square}_{\infty}$. Thus, we have the $R^{\square}$-module (respectively, R-module) decomposition

$$R^{\square}_{\infty} \cong R^{\square} \oplus M^{\square}_{\infty} \quad (\text{respectively,} \quad R_{\infty} \cong R \oplus M_{\infty}). \tag{3.2.2}$$

The profinite group

$$\Delta := \Big\{(\epsilon_0, \ldots, \epsilon_d) \in \Big(\varprojlim_{m \geqslant 0} (\mu_{p^m}(\mathcal{O}_C))\Big)^{\oplus(d+1)} \;\Big|\; \epsilon_0 \cdots \epsilon_r = 1\Big\} \simeq \mathbb{Z}_p^{\oplus d}$$

acts $R^{\square}$-linearly on $R^{\square}_m$ by scaling each t_j^{1/p^m} by the μ_{p^m}-component of ϵ_j. The induced actions of Δ on $R^{\square}_{\infty}$ and R_{∞} are continuous, compatible, and preserve the decompositions (3.2.1) and

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(3.2.2). In terms of the element ϵ fixed in § 2.1, Δ is topologically freely generated by the following d elements:

$$\delta_i := (\epsilon^{-1}, 1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = 1, \dots, r \quad \text{where the 0th and } i\text{th entries are nonidentity;}$$
$$\delta_i := (1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = r+1, \dots, d \quad \text{where the } i\text{th entry is nonidentity.}$$

After inverting p, for each $m \geqslant 0$, we have

$$R^{\square}_m\left[\tfrac{1}{p}\right] \cong \bigoplus\nolimits_{a_1,\dots,a_d \in \{0, \frac{1}{p^m}, \dots, \frac{p^m-1}{p^m}\}} R^{\square}\left[\tfrac{1}{p}\right] \cdot t_1^{a_1} \cdots t_d^{a_d},$$

so $R^{\square}_m[\frac{1}{p}]$ is the $R^{\square}[\frac{1}{p}]$-algebra obtained by adjoining the $(p^m)^{\text{th}}$ roots of $t_1, \dots, t_d \in (R^{\square}[\frac{1}{p}])^{\times}$, and hence is finite étale over $R^{\square}[\frac{1}{p}]$. Therefore, $\varinjlim_m (R^{\square}_m[\frac{1}{p}])$ is a pro-(finite étale) Δ-cover of $R^{\square}[\frac{1}{p}]$. The explicit description (3.2.1) implies that $R^{\square}_m = (R^{\square}_m[\frac{1}{p}])^{\circ}$, so the pro-object

$$(\mathfrak{X}^{\square})^{\mathrm{ad}}_{C,\infty} := \varprojlim \operatorname{Spa}\left(R^{\square}_m\left[\tfrac{1}{p}\right], R^{\square}_m\right) \quad \text{which determines the perfectoid space} \quad \operatorname{Spa}\left(R^{\square}_{\infty}\left[\tfrac{1}{p}\right], R^{\square}_{\infty}\right),$$

is an affinoid perfectoid pro-(finite étale) Δ-cover of the adic generic fiber $(\mathfrak{X}^{\square})^{\mathrm{ad}}_C$ of $\operatorname{Spf}(R^{\square})$; in particular, $(\mathfrak{X}^{\square})^{\mathrm{ad}}_{C,\infty}$ is an affinoid perfectoid object of the proétale site $((\mathfrak{X}^{\square})^{\mathrm{ad}}_C)_{\mathrm{proét}}$. Consequently, the $\mathfrak{X}^{\mathrm{ad}}_C$-base change of $(\mathfrak{X}^{\square})^{\mathrm{ad}}_{C,\infty}$, namely, the tower

$$\mathfrak{X}^{\mathrm{ad}}_{C,\infty} := \varprojlim \operatorname{Spa}\left(R_m\left[\tfrac{1}{p}\right], R_m\right) \quad \text{which determines the perfectoid space} \quad \operatorname{Spa}\left(R_{\infty}\left[\tfrac{1}{p}\right], R_{\infty}\right),$$

is an affinoid perfectoid pro-(finite étale) Δ-cover of $\mathfrak{X}^{\mathrm{ad}}_C$, so, in particular, is an affinoid perfectoid object of $(\mathfrak{X}^{\mathrm{ad}}_C)_{\mathrm{proét}}$.

By [Sch13a, 4.10(iii)], the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ of the sheaf $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$ reviewed in (2.2.1) is R_{∞}.

3.3 The cohomology of $\widehat{\mathcal{O}}^+$ and continuous group cohomology

By [Sch13a, 3.5, 3.7(iii) and its proof, 6.6] (see also [Sch16]), the Čech complex of the sheaf $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$ with respect to the pro-(finite étale) affinoid perfectoid cover

$$\mathfrak{X}^{\mathrm{ad}}_{C,\infty} \twoheadrightarrow \mathfrak{X}^{\mathrm{ad}}_C$$

is identified with the continuous cochain complex $R\Gamma_{\mathrm{cont}}(\Delta, R_{\infty})$. In particular, by using [SP, 01GY], we obtain the edge map to the proétale cohomology of $\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}$:

$$e\colon R\Gamma_{\mathrm{cont}}(\Delta, R_{\infty}) \to R\Gamma_{\mathrm{proét}}(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+), \tag{3.3.1}$$

which on the level of cohomology is described by the Cartan–Leray spectral sequence (see [SP, 01GY] or [SGA 4_{II}, V.3.3]). By the almost purity theorem [Sch13a, 4.10 (v)], the maximal ideal $\mathfrak{m} \subset \mathcal{O}_C$ kills the cohomology groups of $\mathrm{Cone}(e)$.

We will show in Theorem 3.9 that $L\eta_{(\zeta_p-1)}(e)$ is an isomorphism, so that

$$L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{proét}}(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+))$$

is computed in terms of continuous group cohomology. For this, we will use the following lemma.

Lemma 3.4 [BMS18, 8.11(i)]. *An $\mathcal{O}_C$-module map $f\colon M \to M'$ with*

$$M[\mathfrak{m}] = \left(\tfrac{M}{(\zeta_p-1)M}\right)[\mathfrak{m}] = 0$$

and both $\operatorname{Ker} f$ *and* $\operatorname{Coker} f$ *killed by* $\mathfrak{m}$ *induces an isomorphism*

$$\tfrac{M}{M[\zeta_p-1]} \xrightarrow{\sim} \tfrac{M'}{M'[\zeta_p-1]}.$$

□

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In order to apply Lemma 3.4, we will check in Proposition 3.8 that the cohomology modules $H^i_{\mathrm{cont}}(\Delta, R_\infty)$ have no nonzero $\mathfrak{m}$-torsion. This will use the following general lemmas.

LEMMA 3.5. *For an inclusion $\mathfrak{o} \subset \mathfrak{O}$ of a discrete valuation ring into a nondiscrete valuation ring of rank 1, if N is an $\mathfrak{o}$-module and $\mathfrak{M} \subset \mathfrak{O}$ denotes the maximal ideal, then $(N \otimes_{\mathfrak{o}} \mathfrak{O})[\mathfrak{M}] = 0$.*

Proof. The $\mathfrak{o}$-flatness of $\mathfrak{O}$ reduces us to the case when N is finitely generated, so it suffices to observe that $(\mathfrak{O}/(a))[\mathfrak{M}] = 0$ whenever $a \in \mathfrak{O}$. □

LEMMA 3.6. *Fix an $i \in \mathbb{Z}_{\geqslant 0}$, let H be a profinite group, let $\{M_j\}_{j\in J}$ be p-adically complete, p-torsion-free, continuous H-modules, and suppose that either:*

(i) *the group $H^i_{\mathrm{cont}}(H, M_j)$ is p-torsion free for every j; or*
(ii) *some p^n kills $H^i_{\mathrm{cont}}(H, M_j)$ for every j.*

Then the following map is injective:

$$H^i_{\mathrm{cont}}\big(H, \widehat{\bigoplus}_{j\in J} M_j\big) \hookrightarrow \prod_{j\in J} H^i_{\mathrm{cont}}(H, M_j) \quad \text{where the completion is } p\text{-adic.}$$

In particular, in the case (i) (respectively, (ii)), $H^i_{\mathrm{cont}}(H, \widehat{\bigoplus}_{j\in J} M_j)$ is p-torsion free (respectively, killed by p^n).

Proof. Let c be a continuous $(\widehat{\bigoplus}_{j\in J} M_j)$-valued i-cocycle that represents an element of the kernel. For each j, let c_j be the 'jth coordinate' of c. We discard the j with $c_j = 0$ and, for each remaining j, we choose the maximal $n_j \in \mathbb{Z}_{\geqslant 0}$ such that c_j is $(p^{n_j} M_j)$-valued, so that the function $j \mapsto n_j$ is finite-to-one. Since each M_j is p-torsion free, each $p^{-n_j} c_j$ is an M_j-valued continuous i-cocycle.

In the case (i), the class of $p^{-n_j} c_j$ in $H^i_{\mathrm{cont}}(H, M_j)$ vanishes, so each c_j is the coboundary of a $(p^{n_j} M_j)$-valued continuous $(i-1)$-cochain b_j. In the case (ii), p^n kills $H^i_{\mathrm{cont}}(H, M_j)$, so c_j is the coboundary of a $(p^{n_j - n} M_j)$-valued continuous $(i-1)$-cochain b_j whenever $n_j \geqslant n$.

In both cases, the b_j exhibit c as a continuous coboundary. □

LEMMA 3.7 [BMS18, 7.3(ii)]. *Let H be a profinite group isomorphic to $\mathbb{Z}_p^{\oplus d}$ for some $d > 0$, and let $M \cong \varprojlim_{n\geqslant 1} M_n$ be a continuous H-module with each M_n a discrete, p^n-torsion, continuous H-module. For any $\gamma_1, \ldots, \gamma_d \in H$ that topologically freely generate H, there is a natural identification*

$$R\Gamma_{\mathrm{cont}}(H, M) \cong K_M(\gamma_1 - 1, \ldots, \gamma_d - 1), \quad \text{so also} \quad H^j_{\mathrm{cont}}(H, M) \cong H^j(K_M(\gamma_1 - 1, \ldots, \gamma_d - 1)),$$

in the derived category (see § 1.7 for the notation). □

PROPOSITION 3.8. *The element $\zeta_p - 1$ kills the $\mathcal{O}_C$-modules $H^i_{\mathrm{cont}}(\Delta, M_\infty)$. Moreover, for each $b \in \mathcal{O}_C$, the $\mathcal{O}_C$-modules R_∞/b and $H^i_{\mathrm{cont}}(\Delta, R_\infty/b)$ have no nonzero $\mathfrak{m}$-torsion.*

Proof. Let $S := \mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ be a summand of (3.2.1). By Lemma 3.7, the $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, S)$ is the ith cohomology of the $\mathcal{O}_C$-tensor product of d complexes of the form $\mathcal{O}_C \xrightarrow{\zeta - 1} \mathcal{O}_C$ for suitable p-power roots of unity ζ. Moreover, since the d complexes may be defined over some discrete valuation subring of $\mathcal{O}_C$, Lemma 3.5 ensures that

$$H^i_{\mathrm{cont}}(\Delta, S) \quad \text{has no nonzero } \mathfrak{m}\text{-torsion.} \tag{3.8.1}$$

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If S contributes to M_∞, that is, if $a_j \notin \mathbb{Z}$ for some j, then some ζ is not 1, and the corresponding factor complex is quasi-isomorphic to $\mathcal{O}_C/(\zeta - 1)$. Thus, in this case,

$$\zeta - 1 \quad \text{and hence also} \quad \zeta_p - 1, \quad \text{kills} \quad H^i_{\mathrm{cont}}(\Delta, S). \tag{3.8.2}$$

For $m > 0$, let $M_m^{\square}$ denote the p-adically completed direct sum of those summands $\mathcal{O}_C \cdot t_0^{a_0} \cdots t_d^{a_d}$ of (3.2.1) for which m is the smallest *nonnegative* integer with $p^m \cdot (a_0, \ldots, a_d) \in \mathbb{Z}^{\oplus(d+1)}$. Lemma 3.6 and (3.8.1)–(3.8.2) imply that the $\mathcal{O}_C$-module

$$H^i_{\mathrm{cont}}(\Delta, M_m^{\square}) \quad \text{has no nonzero } \mathfrak{m}\text{-torsion and is killed by } \zeta_p - 1. \tag{3.8.3}$$

Since R is $R^{\square}$-flat and $R \otimes_{R^{\square}} M_m^{\square}$ is p-adically complete (see §§ 1.5 and 3.2), Lemma 3.7 gives

$$H^i_{\mathrm{cont}}(\Delta, R \otimes_{R^{\square}} M_m^{\square}) \cong R \otimes_{R^{\square}} H^i_{\mathrm{cont}}(\Delta, M_m^{\square}). \tag{3.8.4}$$

Since $M_\infty \cong \widehat{\bigoplus}_m (R \otimes_{R^{\square}} M_m^{\square})$, equations (3.8.3)–(3.8.4) and Lemma 3.6 imply that $\zeta_p - 1$ kills $H^i_{\mathrm{cont}}(\Delta, M_\infty)$.

Since R_∞/b is p-adically complete and each of the summands of the decomposition

$$R_\infty/(b, p^n) \cong R/(b, p^n) \oplus \textstyle\bigoplus_{m>0} (R \otimes_{R^{\square}} M_m^{\square})/(b, p^n) \quad \text{for } n > 0$$

may be defined over a suitably large discrete valuation subring of $\mathcal{O}_C$, Lemma 3.5 ensures that R_∞/b has no nonzero $\mathfrak{m}$-torsion. In addition, the Δ-action on each summand may be defined over a possibly larger such subring, so, by Lemmas 3.5 and 3.7, in the case $b \neq 0$ each

$$H^i_{\mathrm{cont}}(\Delta, (R \otimes_{R^{\square}} M_m^{\square})/b) \quad \text{so also} \quad H^i_{\mathrm{cont}}(\Delta, M_\infty/b), \quad \text{has no nonzero } \mathfrak{m}\text{-torsion.}$$

This conclusion extends to the case $b = 0$ because the $(\zeta_p - 1)$-annihilation of $H^i_{\mathrm{cont}}(\Delta, M_\infty)$ supplies the injection $H^i_{\mathrm{cont}}(\Delta, M_\infty) \hookrightarrow H^i_{\mathrm{cont}}(\Delta, M_\infty/(\zeta_p - 1))$. It remains to observe that the $\mathcal{O}_C$-module $H^i_{\mathrm{cont}}(\Delta, R/b)$ also has no nonzero $\mathfrak{m}$-torsion: Δ acts trivially on R/b, so Lemma 3.7 ensures that $H^i_{\mathrm{cont}}(\Delta, R/b)$ is a direct sum of copies of R/b. □

Theorem 3.9. *The edge map e defined in (3.3.1) induces the isomorphism*

$$L\eta_{(\zeta_p - 1)}(e) \colon L\eta_{(\zeta_p - 1)}(R\Gamma_{\mathrm{cont}}(\Delta, R_\infty)) \xrightarrow{\sim} L\eta_{(\zeta_p - 1)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)).$$

Proof. Proposition 3.8 ensures that the $\mathcal{O}_C$-modules $H^i_{\mathrm{cont}}(\Delta, R_\infty)$ have no nonzero $\mathfrak{m}$-torsion and that

$$\frac{H^i_{\mathrm{cont}}(\Delta, R_\infty)}{H^i_{\mathrm{cont}}(\Delta, R_\infty)[\zeta_p - 1]} \cong \frac{H^i_{\mathrm{cont}}(\Delta, R)}{H^i_{\mathrm{cont}}(\Delta, R)[\zeta_p - 1]}.$$

Since Δ acts trivially on R, this last quotient is a finite direct sum of copies of R (see Lemma 3.7), so, by Proposition 3.8, it has no nonzero $\mathfrak{m}$-torsion. Consequently, since $\mathfrak{m}$ kills the kernel and the cokernel of each map

$$H^i(e) \colon H^i_{\mathrm{cont}}(\Delta, R_\infty) \to H^i(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)$$

(see § 3.3), Lemma 3.4 applies to these maps and gives the desired conclusion. □

Remark 3.10. Theorem 3.9 extends as follows: for any profinite group Δ' equipped with a continuous surjection $\Delta' \twoheadrightarrow \Delta$ and any pro-(finite étale) affinoid perfectoid Δ'-cover

$$\mathrm{Spa}\big(R'_\infty\big[\tfrac{1}{p}\big], R'_\infty\big) \to \mathrm{Spa}\big(R\big[\tfrac{1}{p}\big], R\big) \cong \mathfrak{X}_C^{\mathrm{ad}} \quad \text{that refines the } \Delta\text{-cover} \quad \mathfrak{X}_{C,\infty}^{\mathrm{ad}} \to \mathfrak{X}_C^{\mathrm{ad}} \quad \text{of § 3.2}$$

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compatibly with the surjection $\Delta' \twoheadrightarrow \Delta$, the edge map e' defined analogously to (3.3.1) induces the isomorphism

$$L\eta_{(\zeta_p-1)}(e')\colon L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{cont}}(\Delta', R'_\infty)) \xrightarrow{\sim} L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)).$$

Indeed, by the almost purity theorem [Sch13a, 4.10(v)], the ideal $\mathfrak{m}$ kills the cohomology of $\mathrm{Cone}(e')$ (in addition to that of $\mathrm{Cone}(e)$), so the octahedral axiom (see [BBD82, 1.1.7.1]) ensures that it also kills the cohomology of the cone of the map $R\Gamma_{\mathrm{cont}}(\Delta, R_\infty) \to R\Gamma_{\mathrm{cont}}(\Delta', R'_\infty)$; Lemma 3.4 then applies to this map and combines with Theorem 3.9 to give the claim.

The main goal of this section is an analogue of Theorem 3.9 for the sheaf $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ (see Theorem 3.20). To prepare for it, in §§ 3.11 and 3.14 we describe the values of the sheaves $\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C^{\mathrm{ad}}}$ and $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ on $\mathfrak{X}^{\mathrm{ad}}_{C,\,\infty}$.

3.11 The tilt $R^\flat_\infty$

Thanks to the explicit description (3.2.1) of the perfectoid ring $R^\square_\infty$, its tilt

$$(R^\square_\infty)^\flat := \varprojlim_{y\mapsto y^p}(R^\square_\infty/p)$$

is described explicitly by the identification

$$\begin{aligned}(R^\square_\infty)^\flat &\cong (\varinjlim_m (\mathcal{O}^\flat_C[x_0^{1/p^m}, \dots, x_r^{1/p^m}, x_{r+1}^{\pm 1/p^m}, \dots, x_d^{\pm 1/p^m}]/(x_0^{1/p^m}\cdots x_r^{1/p^m} - (p^{1/p^\infty})^{q/p^m})))^\wedge \\ &\cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d)\in(\mathbb{Z}[\frac{1}{p}]_{\geqslant 0})^{\oplus(r+1)}\oplus(\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)},\\ a_j=0 \text{ for some } 0\leqslant j\leqslant r}} \mathcal{O}^\flat_C\cdot x_0^{a_0}\cdots x_d^{a_d},\end{aligned}$$

where x_i^{1/p^m} corresponds to the p-power compatible sequence $(\dots, t_i^{1/p^{m+1}}, t_i^{1/p^m})$ of elements of $R^\square_\infty$, the completions are p^{1/p^∞}-adic, and the decomposition is as $\mathcal{O}^\flat_C$-modules. Thus,

$$\text{the tilt}\quad R^\flat_\infty := \varprojlim_{y\mapsto y^p}(R_\infty/p)\quad \text{of the perfectoid ring}\quad R_\infty$$

is identified with the p^{1/p^∞}-adic completion of any lift of the étale $R^\square_\infty/p$-algebra R_∞/p to an étale $(R^\square_\infty)^\flat$-algebra (such a lift exists, see [SP, 04D1]). By [Sch13a, 5.11(i)], the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\,\infty}$ of the sheaf $\widehat{\mathcal{O}}^{+,\flat}_{\mathfrak{X}_C^{\mathrm{ad}}}$ reviewed in (2.2.1) is the ring $R^\flat_\infty$.

By functoriality, the group Δ acts continuously and $\mathcal{O}^\flat_C$-linearly on $(R^\square_\infty)^\flat$ and $R^\flat_\infty$. Explicitly, Δ respects the completed direct sum decomposition and an $(\epsilon_0,\dots,\epsilon_d)\in\Delta$ scales $x_j^{a_j}$ by $\epsilon_j^{a_j}\in\mathcal{O}^\flat_C$.

Our analysis in § 3.14 of the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\,\infty}$ of the sheaf $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ will hinge on the following lemmas.

LEMMA 3.12. *Both $R^\flat_\infty/b$ and $H^i_{\mathrm{cont}}(\Delta, R^\flat_\infty/b)$ for each $b\in\mathcal{O}^\flat_C\setminus\{0\}$ have no nonzero $\mathfrak{m}^\flat$-torsion.*

Proof. We may assume that $b\in\mathfrak{m}^\flat$, so, by using Frobenius, that $b\,|\,p^{1/p^\infty}$ in $\mathcal{O}^\flat_C$. Then Proposition 3.8 and the Δ-isomorphism $R^\flat_\infty/b\cong R_\infty/b^\sharp$ for some $b^\sharp\in\mathcal{O}_C$ gives the claim. □

LEMMA 3.13. *For any affinoid perfectoid* $\mathrm{Spa}(R'_\infty[\frac{1}{p}], R'_\infty)$ *over* $\mathrm{Spa}(C,\mathcal{O}_C)$*, the ring*

$$\mathbb{A}_{\mathrm{inf}}(R'_\infty) := W((R'_\infty)^\flat)\quad (\textit{respectively,}\quad \mathbb{A}_{\mathrm{inf}}(R'_\infty)/\mu)$$

is (p,μ)-adically complete (respectively, p-adically complete). Moreover, for any $n,n'>0$, the sequence $(p^n,\mu^{n'})$ is $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-regular and the $A_{\mathrm{inf}}/(p^n,\mu^{n'})$-algebra $\mathbb{A}_{\mathrm{inf}}(R'_\infty)/(p^n,\mu^{n'})$ is flat.

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Proof. By its definition, the perfect $\mathcal{O}_C^\flat$-algebra $(R'_\infty)^\flat := \varprojlim_{y\mapsto y^p}(R'_\infty/p)$ has no nonzero p^{1/p^∞}-torsion (that is, it is $\mathcal{O}_C^\flat$-flat), so the regular sequence claim follows from [SP, 07DV]. The formal criterion of flatness [Bou, Ch. III, § 5.2, Theorem 1(i)⇔(iv)] then implies the $A_{\mathrm{inf}}/(p^n,\mu^{n'})$-flatness of $\mathbb{A}_{\mathrm{inf}}(R'_\infty)/(p^n,\mu^{n'})$ (even with $n'=0$). In addition, the short exact sequences (2.1.2) with $(R'_\infty)^\flat$ in place of $\mathcal{O}_C^\flat$ imply the p-adic completeness of $\mathbb{A}_{\mathrm{inf}}(R'_\infty)/\mu$.

Analogously to the case of A_{inf} discussed in § 2.1, we use the Witt coordinate bijection and the μ-adic topology on $(R'_\infty)^\flat$ to topologize $\mathbb{A}_{\mathrm{inf}}(R'_\infty)\cong\prod_{n=1}^\infty(R'_\infty)^\flat$ and we see that this topology agrees with the (p,μ)-adic topology. Thus, $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$ is (p,μ)-adically complete. □

3.14 The ring $\mathbb{A}_{\mathrm{inf}}(R_\infty)$

By [Sch13a, 6.5(i)], the value on $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$ of the sheaf $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}^{\mathrm{ad}}_C}$ is the ring

$$\mathbb{A}_{\mathrm{inf}}(R_\infty) := W(R_\infty^\flat).$$

By Lemma 3.13 and the formal criterion of flatness, $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ is (p,μ)-adically formally flat as an A_{inf}-algebra and (p,μ)-adically formally étale as an $\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)$-algebra. By using, in addition, Lemma 3.12, we see that each quotient

$$\mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n,\mu^{n'}),\quad\text{so also}\quad\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu,\quad\text{has no nonzero } W(\mathfrak{m}^\flat)\text{-torsion.}\tag{3.14.1}$$

In general, for a perfect $\mathbb{F}_p$-algebra A, the Witt ring $W(A)$ is the unique p-adically complete p-torsion-free $\mathbb{Z}_p$-algebra $\widetilde{A}$ equipped with an isomorphism $\widetilde{A}/p\simeq A$ (see [Bha18, 2.5]). For an $a\in A$, the Teichmüller $[a]\in\widetilde{A}$ is $\lim_{n\to\infty}(\widetilde{a}_n^{p^n})$ where $\widetilde{a}_n\in\widetilde{A}$ is any lift of a^{1/p^n} (see [Bha18, 2.4]). Therefore,

$$\begin{aligned}\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)&\cong\big(\varinjlim_m A_{\mathrm{inf}}[X_0^{1/p^m},\dots,X_r^{1/p^m},X_{r+1}^{\pm1/p^m},\dots,X_d^{\pm1/p^m}]/(\textstyle\prod_{i=0}^r X_i^{1/p^m}-[(p^{1/p^\infty})^{q/p^m}])\big)^{\widehat{}}\\&\cong\widehat{\bigoplus}_{\substack{(a_0,\dots,a_d)\in(\mathbb{Z}[\frac1p]_{\geqslant0})^{\oplus(r+1)}\oplus(\mathbb{Z}[\frac1p])^{\oplus(d-r)},\\ a_j=0\text{ for some }0\leqslant j\leqslant r}} A_{\mathrm{inf}}\cdot X_0^{a_0}\cdots X_d^{a_d},\end{aligned}$$

where the completions are (p,μ)-adic, the decomposition is as A_{inf}-modules, and, in terms of § 3.11, we have $X_i^{1/p^m}=[x_i^{1/p^m}]$. The summands for which $a_i\in\mathbb{Z}$ for all i comprise a subring

$$A(R^\square)\cong A_{\mathrm{inf}}\{X_0,\dots,X_r,X_{r+1}^{\pm1},\dots,X_d^{\pm1}\}/(X_0\cdots X_r-[(p^{1/p^\infty})^q])\quad\text{inside}\quad\mathbb{A}_{\mathrm{inf}}(R^\square_\infty),\tag{3.14.2}$$

where the convergence is (p,μ)-adic. The remaining summands, that is, those for which $a_i\notin\mathbb{Z}$ for some i, comprise an $A(R^\square)$-submodule $N^\square_\infty\subset\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)$.

On sections over $\mathfrak{X}^{\mathrm{ad}}_{C,\infty}$, the map θ from (2.2.2) is identified with the unique ring homomorphism

$$\theta\colon\mathbb{A}_{\mathrm{inf}}(R_\infty)\twoheadrightarrow R_\infty\quad\text{such that}\quad[x]\mapsto x^{(0)},$$

is surjective with the kernel generated by the regular element ξ (see [BMS18, 3.10, 3.11]), and intertwines the Witt vector Frobenius of $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ with the absolute Frobenius of R_∞/p. Thus,

$$\theta\colon A(R^\square)\twoheadrightarrow R^\square\quad\text{is described by}\quad X_i\mapsto t_i.\tag{3.14.3}$$

We use the surjection (3.14.3) to uniquely lift the étale $R^\square/p$-algebra R/p to a (p,μ)-adically complete, formally étale $A(R^\square)$-algebra $A(R)$. By construction, we have the identification

$$\mathbb{A}_{\mathrm{inf}}(R_\infty)\cong\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)\mathbin{\widehat{\otimes}}_{A(R^\square)}A(R),\tag{3.14.4}$$

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where the completion is (p,μ)-adic. Therefore, by setting $N_\infty := N_\infty^{\square} \widehat{\otimes}_{A(R^{\square})} A(R)$, we arrive at the decompositions of $\mathbb{A}_{\mathrm{inf}}(R_\infty^{\square})$ and $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ into 'integral' and 'nonintegral' parts:

$$\mathbb{A}_{\mathrm{inf}}(R_\infty^{\square}) \cong A(R^{\square}) \oplus N_\infty^{\square} \quad \text{and} \quad \mathbb{A}_{\mathrm{inf}}(R_\infty) \cong A(R) \oplus N_\infty. \tag{3.14.5}$$

Modulo $\operatorname{Ker}\theta$ (that is, modulo ξ), these decompositions reduce to the decompositions (3.2.2).

The Witt vector Frobenius of $\mathbb{A}_{\mathrm{inf}}(R_\infty^{\square})$ preserves $A(R^{\square})$; explicitly: it is semilinear with respect to the Frobenius of A_{inf} and raises each X_i^{1/p^m} to the pth power. By construction, $A(R)$ inherits a Frobenius ring endomorphism from $A(R^{\square})$, and the identification (3.14.4) is Frobenius-equivariant.

The natural Δ-action on $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ is continuous and commutes with the Frobenius. Explicitly, Δ respects the completed direct sum decomposition and an $(\epsilon_0,\dots,\epsilon_d) \in \Delta$ scales $X_j^{a_j}$ by $[\epsilon_j^{a_j}] \in A_{\mathrm{inf}}$. The Δ-action on $A(R^{\square})$ lifts uniquely to a necessarily Frobenius-equivariant Δ-action on $A(R)$. In particular, Δ acts trivially on $A(R)/\mu$. The identifications (3.14.4) and (3.14.5) are Δ-equivariant.

3.15 The cohomology of $\mathbb{A}_{\mathrm{inf}}$ and continuous group cohomology

Similarly to § 3.3, the Čech complex of the sheaf $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$ with respect to the pro-(finite étale) affinoid perfectoid cover $\mathfrak{X}_{C,\infty}^{\mathrm{ad}} \to \mathfrak{X}_C^{\mathrm{ad}}$ is identified with the continuous cochain complex $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))$. Thus, by using [SP, 01GY], we obtain the edge map to the proétale cohomology of $\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}$:

$$e\colon R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf}}). \tag{3.15.1}$$

By the almost purity theorem, more precisely, by [Sch13a, 6.5(ii)], the subset $[\mathfrak{m}^\flat] \subset A_{\mathrm{inf}}$ that consists of the Teichmüller lifts of the elements in the maximal ideal $\mathfrak{m}^\flat \subset \mathcal{O}_C^\flat$ kills all the cohomology groups of $\operatorname{Cone}(e)$. Since $\mu \in W(\mathfrak{m}^\flat)$ (see (2.3.3)), it will be useful to strengthen this annihilation as follows.

LEMMA 3.16. *The ideal $W(\mathfrak{m}^\flat) \subset A_{\mathrm{inf}}$ defined in (2.3.3) kills each $H^i(\operatorname{Cone}(e))$.*

Proof. We argue similarly to [BMS18, proof of Theorem 5.6]. Both the source and the target of e are derived p-adically complete (see § 1.7), so, by [BS15, 3.4.4 and 3.4.14], each $H^i(\operatorname{Cone}(e))$ is also derived p-adically complete. Thus, the desired conclusion follows from the following lemma. □

LEMMA 3.17. *If $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$ kills a derived p-adically complete A_{inf}-module H, then so does $W(\mathfrak{m}^\flat)$.*

Proof. By the derived p-adic completeness, any free A_{inf}-module resolution $F^\bullet$ of H satisfies

$$H \cong \operatorname{Coker}\left(\varprojlim_n (F^{-1}/p^n) \to \varprojlim_n (F^0/p^n)\right).$$

Moreover, for every $n \geqslant 1$ the ideals $[\mathfrak{m}^\flat]\cdot W_n(\mathcal{O}_C^\flat)$ and $W_n(\mathfrak{m}^\flat) := \operatorname{Ker}(W_n(\mathcal{O}_C^\flat) \to W_n(k))$ of $W_n(\mathcal{O}_C^\flat)$ agree. Thus, the $([\mathfrak{m}^\flat]A_{\mathrm{inf}})$-annihilation of H implies that $W_n(\mathfrak{m}^\flat)$ kills both

$$H/p^n \cong H^0(F^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n) \quad \text{and} \quad \operatorname{Tor}_1^{A_{\mathrm{inf}}}(H, A_{\mathrm{inf}}/p^n) \cong H^{-1}(F^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n).$$

Thus, since $[\mathfrak{m}^\flat]^2 = [\mathfrak{m}^\flat]$ and F_0/p^n has no nonzero m-torsion for every nonzero $m \in [\mathfrak{m}^\flat]$, any element $x \in W_{n+1}(\mathfrak{m}^\flat)\cdot(F_0/p^{n+1})$ may be lifted to $W_{n+1}(\mathfrak{m}^\flat)\cdot(F_{-1}/p^{n+1})$, compatibly with a specified lift of its image $\overline{x} \in W_n(\mathfrak{m}^\flat)\cdot(F_0/p^n)$ to $W_n(\mathfrak{m}^\flat)\cdot(F_{-1}/p^n)$. In particular, $W(\mathfrak{m}^\flat)\cdot(\varprojlim_n (F^0/p^n))$ lies in the image of $\varprojlim_n (F^{-1}/p^n)$, that is, $W(\mathfrak{m}^\flat)$ kills H, as desired. □

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We will show in Theorem 3.20 that $L\eta_{(\mu)}(e)$ is an isomorphism, so that continuous group cohomology computes $L\eta_{(\mu)}(R\Gamma_{\text{proét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{A}_{\text{inf}}))$. For this, we will use the following lemma.

Lemma 3.18. *If $B \xrightarrow{b} B'$ is a morphism in $D(A_{\text{inf}})$ such that each $H^i(B \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu)$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion and $W(\mathfrak{m}^\flat)$ kills each $H^i(\mathrm{Cone}(b))$, then $L\eta_{(\mu)}(b)$ is an isomorphism.*

Proof. Since $L\eta$ is not a triangulated functor, the fact that $L\eta_{(\mu)}(\mathrm{Cone}(b)) \cong 0$ does not *a priori* suffice. Instead, the ideal $(W(\mathfrak{m}^\flat))^2$ kills the cohomology of $\mathrm{Cone}(b) \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu$, so the sequences

$$0 \to H^i(B \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu) \to H^i(B' \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu) \to H^i(\mathrm{Cone}(b) \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu) \to 0$$

are short exact. By the Bockstein construction (see [BMS18, 6.12]), as i varies, they comprise a short exact sequence whose terms are complexes that compute $L\eta_{(\mu)}(B) \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu$, etc. Thus, the vanishing of $L\eta_{(\mu)}(\mathrm{Cone}(b))$ implies that $(L\eta_{(\mu)}(b)) \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu$ is an isomorphism. It follows that $\mathrm{Cone}(L\eta_{(\mu)}(b)) \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}/\mu \cong 0$, so μ acts invertibly on the cohomology of $\mathrm{Cone}(L\eta_{(\mu)}(b))$. But then, as we see after applying $- \otimes^{\mathbb{L}}_{A_{\text{inf}}} A_{\text{inf}}[\frac{1}{\mu}]$, this cohomology vanishes. □

We now verify that the edge map e defined in (3.15.1) also meets the first assumption of Lemma 3.18.

Proposition 3.19. *For each $i \in \mathbb{Z}$, the A_{inf}-module $H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/\mu)$ is p-torsion free and p-adically complete; moreover, the following natural maps are isomorphisms:*

$$H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/\mu) \otimes_{A_{\text{inf}}} A_{\text{inf}}/p^n \xrightarrow{\sim} H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/(\mu, p^n)) \quad \text{for} \quad n > 0 \tag{3.19.1}$$

and

$$H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/\mu) \xrightarrow{\sim} \varprojlim_n (H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/(\mu, p^n))). \tag{3.19.2}$$

In addition, $H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/(\mu, p^n))$ and $H^i_{\text{cont}}(\Delta, \mathbb{A}_{\text{inf}}(R_\infty)/\mu)$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion.

Proof. Since $A(R)/\mu$ is p-adically complete and has a trivial Δ-action (see Lemma 3.13 and § 3.14), Lemma 3.7 implies that $H^i_{\text{cont}}(\Delta, A(R)/\mu)$ is a direct sum of copies of $A(R)/\mu$, and likewise for $H^i_{\text{cont}}(\Delta, A(R)/(\mu, p^n))$. Consequently, since, by (3.14.1), the rings $A(R)/(\mu, p^n)$ and $A(R)/\mu$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion, the analogues of all the claims with $A(R)$ in place of $\mathbb{A}_{\text{inf}}(R_\infty)$ follow. Thus, due to (3.14.5), we only need to establish these analogues with N_∞ in place of $\mathbb{A}_{\text{inf}}(R_\infty)$.

To prepare for treating N_∞, we start by building on the ideas of [Bha18, proof of Lemma 4.6] to analyze a single summand $S := A_{\text{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d}$ that, as in § 3.14, contributes to $N_\infty^{\square}$. We set

$$b_j := a_j - a_0 \quad \text{for} \quad 1 \leqslant j \leqslant r \quad \text{and} \quad b_j := a_j \quad \text{for} \quad r+1 \leqslant j \leqslant d, \tag{3.19.3}$$

and let $m \in \mathbb{Z}_{>0}$ be the minimal such that $p^m b_j \in \mathbb{Z}$ for all j. Lemma 3.7 applied with the topological generators $\delta_1, \ldots, \delta_d$ of Δ defined in § 3.2 gives an A_{inf}-isomorphism

$$H^i_{\text{cont}}(\Delta, S/\mu) \simeq H^i(C^\bullet),$$

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where $C^\bullet$ is the (A_{inf}/μ)-tensor product of the d complexes

$$[A_{\mathrm{inf}}/\mu \xrightarrow{[\epsilon^{b_j}]-1} A_{\mathrm{inf}}/\mu] \cong A_{\mathrm{inf}}/([\epsilon^{b_j}]-1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu. \tag{3.19.4}$$

By reordering the b_j, we may assume that for all j we have $b_j/b_1 \in \mathbb{Z}_{(p)}$, so that $b_1 \notin \mathbb{Z}$ and both $[\epsilon^{b_1}]-1 \,|\, [\epsilon^{b_j}]-1$ and $[\epsilon^{b_1}]-1 \,|\, \mu$. Then the object (3.19.4) with $j=1$ is given by the complex $[A_{\mathrm{inf}}/([\epsilon^{b_1}]-1) \xrightarrow{0} A_{\mathrm{inf}}/([\epsilon^{b_1}]-1)]$ and, by using the left-hand sides of (3.19.4) for the factors with $j \neq 1$, we see that $C^\bullet$ is quasi-isomorphic to a direct sum of shifts of

$$A_{\mathrm{inf}}/([\epsilon^{b_1}]-1) \cong A_{\mathrm{inf}}/\varphi^{-m}(\mu).$$

Thus, for $i \in \mathbb{Z}$,

$$H^i_{\mathrm{cont}}(\Delta, S/\mu) \simeq \textstyle\bigoplus_I A_{\mathrm{inf}}/\varphi^{-m}(\mu) \quad \text{for some set } I, \quad \text{and hence} \quad H^i_{\mathrm{cont}}(\Delta, S/\mu)[p] = 0. \tag{3.19.5}$$

By Lemma 3.7 and [SP, 061Z, 0662], this implies that

$$H^i_{\mathrm{cont}}(\Delta, S/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, S/(\mu, p^n)). \tag{3.19.6}$$

We now analyze $N^\square_\infty$. Since $\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu$ is p-adically complete, § 3.14 gives the Δ-decomposition

$$\mathbb{A}_{\mathrm{inf}}(R^\square_\infty)/\mu \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d)\in(\mathbb{Z}[\frac{1}{p}]_{\geqslant 0})^{\oplus(r+1)}\oplus(\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)},\\ a_j=0 \text{ for some } 0\leqslant j\leqslant r}} A_{\mathrm{inf}}/\mu \cdot X_0^{a_0}\cdots X_d^{a_d}$$

in which the completion is p-adic. Lemma 3.6 (i) then combines with (3.19.5) to prove that

$$H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/\mu)[p] = 0 \quad \text{for each} \quad i \in \mathbb{Z}.$$

Analogously to (3.19.6), this, in turn, implies that

$$H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/(\mu, p^n)). \tag{3.19.7}$$

Finally, we analyze N_∞. The identification

$$N_\infty/(\mu, p^n) \cong N^\square_\infty/(\mu, p^n) \otimes_{A(R^\square)} A(R)$$

is Δ-equivariant and $A(R)/(\mu, p^n)$ is $(A(R^\square)/(\mu, p^n))$-flat, so Lemma 3.7 gives the identifications

$$H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n)) \cong H^i_{\mathrm{cont}}(\Delta, N^\square_\infty/(\mu, p^n)) \otimes_{A(R^\square)} A(R) \quad \text{for} \quad n \geqslant 1, \tag{3.19.8}$$

which are compatible as n varies. Consequently, for $n > 1$, the sequences

$$0 \to H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))[p] \to H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n)) \to H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^{n-1})) \to 0 \tag{3.19.9}$$

are short exact because, by (3.19.5) and (3.19.7), so are their analogues with $N^\square_\infty$ in place of N_∞. By taking the inverse limit of these sequences for varying n and using [SP, 0D6K], we obtain

$$H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu) \xrightarrow{\sim} \varprojlim_n (H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))), \tag{3.19.10}$$

which is the sought analogue of (3.19.2). The p-torsion-freeness of $H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu)$ follows from (3.19.9)–(3.19.10) and, as in (3.19.6), it implies that

$$H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n \xrightarrow{\sim} H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n)).$$

It remains to show that each $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion.

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The surjectivity aspect of the short exact sequences (3.19.9) implies that the sequences

$$0 \to N_\infty/(\mu, p) \xrightarrow{p^{n-1}} N_\infty/(\mu, p^n) \to N_\infty/(\mu, p^{n-1}) \to 0$$

remain short exact after applying $H^i_{\mathrm{cont}}(\Delta, -)$. Thus, $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p^n))$ is a successive extension of copies of $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p))$. Consequently, it has no nonzero $W(\mathfrak{m}^\flat)$-torsion because, by Lemma 3.12, neither does $H^i_{\mathrm{cont}}(\Delta, N_\infty/(\mu, p))$ (note that $N_\infty/(\mu, p)$ is a direct summand of $\mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p) \cong R^\flat_\infty/\mu$). □

Theorem 3.20. *The edge map e defined in (3.15.1) induces the isomorphism*

$$L\eta_{(\mu)}(e)\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}^{\mathrm{ad}}_C})).$$

Proof. By the projection formula [SP, 0944],

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu), \tag{3.20.1}$$

so Proposition 3.19 implies that the cohomology modules of $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion. Thus, the claim follows from Lemmas 3.16 and 3.18. □

Remark 3.21. Analogously to Remark 3.10, Theorem 3.20 extends as follows: for any pro-(finite étale) affinoid perfectoid Δ'-cover

$$\mathrm{Spa}\big(R'_\infty\big[\tfrac{1}{p}\big], R'_\infty\big) \to \mathrm{Spa}\big(R\big[\tfrac{1}{p}\big], R\big) \cong \mathfrak{X}^{\mathrm{ad}}_C \quad \text{that refines} \quad \mathfrak{X}^{\mathrm{ad}}_{C,\infty} \to \mathfrak{X}^{\mathrm{ad}}_C$$

subject to the same conditions as in Remark 3.10, the edge map e' defined analogously to (3.15.1) induces the isomorphism

$$L\eta_{(\mu)}(e')\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}_{\mathrm{inf}}(R'_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf},\, \mathfrak{X}^{\mathrm{ad}}_C})).$$

Indeed, as in Remark 3.10, by the almost purity theorem and the octahedral axiom, $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$ kills the cohomology modules of the cone of the map

$$e_0\colon R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \to R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}_{\mathrm{inf}}(R'_\infty))$$

and, by [BS15, 3.4.4 and 3.4.14], these modules are derived p-adically complete; thus, by Lemma 3.17, even $W(\mathfrak{m}^\flat)$ kills them, to the effect that Lemma 3.18 applies to the map e_0 and proves the claim.

As a final goal of § 3, we wish to show in Theorem 3.34 that even the maps $L\eta_{(\mu)}(e \mathbin{\widehat{\otimes}}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$ are isomorphisms for A_{inf}-algebras $A^{(m)}_{\mathrm{cris}}$ reviewed in § 3.26 below. This extension of Theorem 3.20 will be important for relating $A\Omega_{\mathfrak{X}}$ to logarithmic crystalline cohomology in § 5. Our analysis of $L\eta_{(\mu)}(e \mathbin{\widehat{\otimes}}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$ will use the following further consequences of the proof of Proposition 3.19.

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3.22 The decomposition of N_∞

For $m \geqslant 0$, let $N_m^{\square}$ be the (p,μ)-adically completed direct sum of those summands $A_{\mathrm{inf}} \cdot X_0^{a_0} \cdots X_d^{a_d}$ that contribute to $\mathbb{A}_{\mathrm{inf}}(R_\infty^{\square})$ in § 3.14 for which m is the smallest nonnegative integer such that $p^m a_j \in \mathbb{Z}$ for all j (equivalently, in the notation of (3.19.3), such that $p^m b_j \in \mathbb{Z}$ for all j). For varying $m > 0$, the $A(R^{\square})$-modules $N_m^{\square}$ and the $A(R)$-modules $N_m := N_m^{\square} \widehat{\otimes}_{A(R^{\square})} A(R)$ comprise the (p,μ)-adically completed direct sum decompositions

$$N_\infty^{\square} \cong \widehat{\bigoplus}_{m>0} N_m^{\square} \quad \text{and} \quad N_\infty \cong \widehat{\bigoplus}_{m>0} N_m. \tag{3.22.1}$$

For a fixed i, Lemma 3.7 and (3.19.5)–(3.19.6) imply that

$$H^i_{\mathrm{cont}}(\Delta, N_m^{\square}/(\mu,p^n)) \simeq \bigoplus_{I'} A_{\mathrm{inf}}/(\varphi^{-m}(\mu),p^n) \quad \text{for some set } I' \text{ and every } n > 0. \tag{3.22.2}$$

COROLLARY 3.23. *For all i and $n, m \geqslant 0$,*

$$H^i_{\mathrm{cont}}(\Delta, N_m/(\mu,p^n)) \quad \textit{is killed by } \varphi^{-m}(\mu) \quad \textit{and} \quad \textit{is a flat } A_{\mathrm{inf}}/(\varphi^{-m}(\mu),p^n)\textit{-module.}$$

Proof. If $R = R^{\square}$, then (3.22.2) suffices. In addition, by Lazard's theorem, $A(R)/(\mu,p^n)$ is a filtered direct limit of finite free $A(R^{\square})/(\mu,p^n)$-modules. Thus, the general case of the claim follows by using (3.19.8) and its analogue for N_0 and $N_0^{\square}$. □

We wish to supplement Proposition 3.19 with Proposition 3.25 that analyzes the cohomology of N_∞ without reducing modulo μ. Its proof will use the following base change result for $L\eta$.

LEMMA 3.24 [Bha18, 5.16]. *For a ring A, elements $f, g \in A$ with g a nonzero-divisor, and a $K \in D(A)$, if the modules $H^i(K \otimes^{\mathbb{L}}_A A/f)$ have no nonzero g-torsion, then the natural map*

$$L\eta_{(f)}(K) \otimes^{\mathbb{L}}_A A/g \to L\eta_{(\overline{f})}(K \otimes^{\mathbb{L}}_A A/g) \quad \textit{where } \overline{f} \textit{ denotes the image of } f \textit{ in } A/g,$$

is an isomorphism. □

PROPOSITION 3.25. *The element μ kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty)$.*

Proof. Let $\delta_1, \ldots, \delta_d$ be the free generators of Δ fixed in § 3.2. By Lemma 3.7, we need to prove that

$$L\eta_{(\mu)}(K_{N_\infty}(\delta_1 - 1, \ldots, \delta_d - 1)) \cong 0. \tag{3.25.1}$$

The key point, with which we start, is to prove the vanishing (3.25.1) modulo $\varphi(\xi)$. The isomorphism

$$K_{N_\infty}(\delta_1 - 1, \ldots, \delta_d - 1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu \cong K_{N_\infty/\mu}(\delta_1 - 1, \ldots, \delta_d - 1),$$

Lemma 3.7, and Proposition 3.19 show that the cohomology of $K_{N_\infty}(\delta_1 - 1, \ldots, \delta_d - 1) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ is p-torsion free. Therefore, Lemma 3.24 supplies the identification

$$L\eta_{(\mu)}(K_{N_\infty}(\delta_1 - 1, \ldots, \delta_d - 1)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\varphi(\xi) \cong L\eta_{(\zeta_p - 1)}(K_{N_\infty/\varphi(\xi)}(\delta_1 - 1, \ldots, \delta_d - 1)). \tag{3.25.2}$$

The inverse Frobenius φ^{-1} maps $N_\infty^{\square}$ isomorphically onto a direct summand of $N_\infty^{\square}$, so it maps N_∞ isomorphically onto a direct summand of N_∞. Thus, φ^{-1} maps $N_\infty/\varphi(\xi)$ isomorphically onto a direct summand of $N_\infty/\xi \cong M_\infty$ (see (3.14.5)). In particular, by Lemma 3.7 and Proposition 3.8, $\zeta_p - 1$ kills the cohomology of $K_{N_\infty/\varphi(\xi)}(\delta_1 - 1, \ldots, \delta_d - 1)$, so both sides of (3.25.2) are acyclic.

Since N_∞ is (p,μ)-adically complete, it is also $\varphi(\xi)$-adically complete (see [SP, 090T]). Thus, $K_{N_\infty}(\delta_1 - 1, \ldots, \delta_d - 1)$ is derived $\varphi(\xi)$-adically complete, and [BMS18, 6.19] implies the same for $L\eta_{(\mu)}(K_{N_\infty}(\delta_1 - 1, \ldots, \delta_d - 1))$. The established acyclicity of the left-hand side of (3.25.2) therefore implies the desired vanishing (3.25.1). □

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3.26 The A_{inf}-algebras $A_{\mathrm{cris}}^{(m)}$

For $m \in \mathbb{Z}_{\geqslant 1}$, we let $A_{\mathrm{cris}}^{(m)}$ be the p-adic completion of the A_{inf}-subalgebra $A_{\mathrm{cris}}^{0,(m)}$ of $A_{\mathrm{inf}}[\frac{1}{p}]$ generated by the elements $\xi^s/s!$ with $s \leqslant m$. In particular, $A_{\mathrm{cris}}^{(m)} \cong A_{\mathrm{inf}}$ for $m < p$. In contrast, if $m \geqslant p$, then, since $\mu^p/p! \in A_{\mathrm{cris}}^{(m)}$, the p-adic and (p,μ)-adic topologies of $A_{\mathrm{cris}}^{(m)}$ agree. By its definition, $A_{\mathrm{cris}}^{(m)}$ is p-torsion free; in fact, although we will not use this, Proposition 5.36 below implies that $A_{\mathrm{cris}}^{(m)}$ is even a domain. The map θ of (2.1.3) extends to $A_{\mathrm{cris}}^{(m)}$:

$$\theta\colon A_{\mathrm{cris}}^{(m)} \twoheadrightarrow \mathcal{O}_C. \tag{3.26.1}$$

Due to the 'finite-type nature' of the A_{inf}-algebra $A_{\mathrm{cris}}^{(m)}$, more precisely, due to [BMS18, 12.8(ii)], the systems of ideals

$$(p^n A_{\mathrm{cris}}^{(m)})_{n\geqslant 1} \quad \text{and} \quad (\{x \in A_{\mathrm{cris}}^{(m)} \mid \mu x \in p^n A_{\mathrm{cris}}^{(m)}\})_{n\geqslant 1} \quad \text{of } A_{\mathrm{cris}}^{(m)} \text{ are intertwined.}$$

Equivalently,

$$\text{for every} \quad n \geqslant 1, \quad \text{the map} \quad (A_{\mathrm{cris}}^{(m)}/p^{n'})[\mu] \to A_{\mathrm{cris}}^{(m)}/p^n \quad \text{vanishes for large} \quad n' > n. \tag{3.26.2}$$

Therefore, by taking the inverse limit over n of the sequences

$$0 \to (A_{\mathrm{cris}}^{(m)}/p^n)[\mu] \to A_{\mathrm{cris}}^{(m)}/p^n \xrightarrow{\mu} A_{\mathrm{cris}}^{(m)}/p^n \to A_{\mathrm{cris}}^{(m)}/(\mu, p^n) \to 0, \tag{3.26.3}$$

we conclude that

$$A_{\mathrm{cris}}^{(m)} \quad \text{is } \mu\text{-torsion free} \quad \text{and} \quad A_{\mathrm{cris}}^{(m)}/\mu \quad \text{is } p\text{-adically complete.} \tag{3.26.4}$$

The Frobenius automorphism of A_{inf} preserves the subring $A_{\mathrm{cris}}^{0,(m)} \subset A_{\mathrm{inf}}[\frac{1}{p}]$: indeed, for $m \geqslant p$, since $\xi = \sum_{i=0}^{p-1}[\epsilon^{i/p}]$ and $\xi^p \in pA_{\mathrm{cris}}^{0,(m)}$, we have $\varphi(\xi) = \sum_{i=0}^{p-1}[\epsilon^i]$ and $\varphi(\xi) \in pA_{\mathrm{cris}}^{0,(m)}$. Thus, the Frobenius induces a ring endomorphism

$$\varphi\colon A_{\mathrm{cris}}^{(m)} \to A_{\mathrm{cris}}^{(m)},$$

which, via the map θ, intertwines the absolute Frobenius of $\mathcal{O}_C/p$ (compare with (2.1.3)).

3.27 The $A(R)$-algebras $A_{\mathrm{cris}}^{(m)}(R)$

The 'relative version' of $A_{\mathrm{cris}}^{(m)}$ (respectively, a 'highly ramified cover' of this relative version) is the $A(R)$-algebra (respectively, $\mathbb{A}_{\mathrm{inf}}(R_\infty)$-algebra)

$$A_{\mathrm{cris}}^{(m)}(R) := A(R)\mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)} \quad (\text{respectively,} \quad \mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty) := \mathbb{A}_{\mathrm{inf}}(R_\infty)\mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}),$$

where the completion is (p,μ)-adic (equivalently, p-adic if $m \geqslant p$). In the case $m < p$, due to Lemma 3.13 and § 3.26, we have $A_{\mathrm{cris}}^{(m)}(R) \cong A(R)$ and $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R_\infty)$.

Due to the decomposition (3.14.5), the subring $A_{\mathrm{cris}}^{(m)}(R) \subset \mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty)$ is an $A_{\mathrm{cris}}^{(m)}(R)$-module direct summand. Explicitly, the decomposition of $\mathbb{A}_{\mathrm{inf}}(R_\infty^\square)$ described in § 3.14 gives the decomposition

$$\mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty^\square) \cong \widehat{\bigoplus}_{\substack{(a_0,\dots,a_d)\in(\mathbb{Z}[\frac{1}{p}]_{\geqslant 0})^{\oplus(r+1)}\oplus(\mathbb{Z}[\frac{1}{p}])^{\oplus(d-r)},\\ a_j=0 \text{ for some } 0\leqslant j\leqslant r}} A_{\mathrm{cris}}^{(m)} \cdot X_0^{a_0}\cdots X_d^{a_d}, \tag{3.27.1}$$

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where the completion is (p,μ)-adic (equivalently, p-adic if $m \geqslant p$), and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ is (p,μ)-adically formally étale over $\mathbb{A}^{(m)}_{\mathrm{cris}}(R^{\square}_\infty)$ (see § 3.14). In particular, (3.26.2) holds with $A^{(m)}_{\mathrm{cris}}$ replaced by $\mathbb{A}^{(m)}_{\mathrm{cris}}(R^{\square}_\infty)$, and hence also by $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$. Consequently, the generalization of (3.26.4) holds too:

$$\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty) \quad \text{is } \mu\text{-torsion free} \quad \text{and} \quad \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu \quad \text{is } p\text{-adically complete.} \tag{3.27.2}$$

In addition, by (3.27.1) and the formal étaleness, each $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ is p-torsion free. By §§ 3.14 and 3.26, the rings $A^{(m)}_{\mathrm{cris}}(R)$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ come equipped with compatible $A^{(m)}_{\mathrm{cris}}$-semilinear Frobenius endomorphisms that are compatible as m varies.

The group Δ acts continuously, Frobenius-equivariantly, and $A^{(m)}_{\mathrm{cris}}$-linearly on $A^{(m)}_{\mathrm{cris}}(R)$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$. For each $\delta \in \Delta$, the A_{inf}-module endomorphism $\frac{\delta-1}{\mu}$ of $A(R)$ induces an $A^{(m)}_{\mathrm{cris}}$-module endomorphism $\frac{\delta-1}{\mu}$ of $A^{(m)}_{\mathrm{cris}}(R)$ that satisfies $\delta = 1 + \mu \cdot \frac{\delta-1}{\mu}$. In particular, Δ acts trivially on $A^{(m)}_{\mathrm{cris}}(R)/\mu$.

3.28 The $A^{(m)}_{\mathrm{cris}}$-base change of the edge map

Since $A^{(m)}_{\mathrm{cris}} \cong A_{\mathrm{inf}}$ for $m < p$, for the sake of analyzing the map $e \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}}$, where e is as in (3.15.1), we suppose that $m \geqslant p$. Then, for each $n > 0$, we have $A^{(m)}_{\mathrm{cris}}/p^n \cong A^{(m)}_{\mathrm{cris}}/(p^n, \mu^{n'})$ for every large enough $n' > 0$ (see § 3.26). Consequently, since $(p^n, \mu^{n'})$ is an $\mathbb{A}_{\mathrm{inf}}(R_\infty)$-regular sequence with $\mathbb{A}_{\mathrm{inf}}(R_\infty)/(p^n, \mu^{n'})$ flat over $A_{\mathrm{inf}}/(p^n, \mu^{n'})$ (see Lemma 3.13), the projection formula [SP, 0944] and Lemma 3.7 imply that

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}} \cong R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)).$$

Consequently, the edge map e defined in (3.15.1) gives rise to the map

$$e \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}} \colon R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)) \to R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}}. \tag{3.28.1}$$

Since $[\mathfrak{m}^\flat]$ kills each $H^i(\mathrm{Cone}(e))$ (see § 3.15) and $[\mathfrak{m}^\flat]^2 = [\mathfrak{m}^\flat]$, by using a free A_{inf}-module resolution of $A^{(m)}_{\mathrm{cris}}/p^n$ and the definition [SP, 064M], we see that $[\mathfrak{m}^\flat]$ also kills each $H^i(\mathrm{Cone}(e) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n)$. Consequently, by [SP, 0D6K], the ideal $[\mathfrak{m}^\flat]A_{\mathrm{inf}}$ kills each $H^i(\mathrm{Cone}(e) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}})$, to the effect that, by Lemma 3.17 (and [BS15, 3.4.4 and 3.4.14]), so does $W(\mathfrak{m}^\flat)$. In conclusion,

$$W(\mathfrak{m}^\flat) \quad \text{kills the cohomology modules of} \quad \mathrm{Cone}(e \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}}) \simeq \mathrm{Cone}(e) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}}. \tag{3.28.2}$$

By applying Lemma 3.18, we will show in Theorem 3.34 that $L\eta_{(\mu)}(e \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}})$ is an isomorphism. Thus, we need to know that the A_{inf}-modules $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu)$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion (compare with Proposition 3.19 for $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$). The following result is a step in that direction.

PROPOSITION 3.29. *Each $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n)$ and also $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu$ have no nonzero $W(\mathfrak{m}^\flat)$-torsion.*

Proof. By the p-adic completeness of $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu$ (see (3.27.2)), we may focus on the rings $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n)$. The argument for the latter is similar to that of [BMS18, 12.8(iii)] and uses

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approximation by Noetherian rings. Namely, by the $(p, \varphi^{-1}(\mu))$-adic completeness of A_{inf}, the assignment

$$T \mapsto [\epsilon]^{1/p} - 1 \quad \text{defines a } \mathbb{Z}_p\text{-algebra morphism} \quad \mathbb{Z}_p[\![T]\!] \to A_{\mathrm{inf}}. \tag{3.29.1}$$

By [BMS18, 4.31], this makes A_{inf} a faithfully flat $\mathbb{Z}_p[\![T]\!]$-algebra. Thus, letting M be the mod $((T+1)^p-1, p^n)$ reduction of the $\mathbb{Z}_p[\![T]\!]$-subalgebra of $\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ generated by the $\frac{1}{s!}(\sum_{i=0}^{p-1}(T+1)^i)^s$ with $s \leqslant m$, we have the identification

$$\mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty)/(\mu, p^n) \cong M \otimes_{\mathbb{Z}_p[\![T]\!]/((T+1)^p-1,\, p^n)} \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n).$$

The $(\mathbb{Z}_p[\![T]\!]/((T+1)^p - 1, p^n))$-flatness of $\mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n)$ ensures that the $\varphi^{-1}(\mu)$-torsion submodule of $\mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty)/(\mu, p^n)$ is the base change of the T-torsion submodule $M[T] \subset M$. Consequently, since $\varphi^{-1}(\mu) \in W(\mathfrak{m}^\flat)$, the consideration of the p-adic filtration of $M[T]$ reduces us to proving that

$$\mathbb{F}_p \otimes_{\mathbb{Z}_p[\![T]\!]/((T+1)^p-1,\, p^n)} \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n) \cong R_\infty^\flat/\varphi^{-1}(\mu) \quad \text{has no nonzero } \mathfrak{m}^\flat\text{-torsion},$$

which follows from Lemma 3.12. □

To relate $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty)/\mu)$ to $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$ in Proposition 3.33, we will use the following general result about exactness properties of p-adically completed tensor products. For concreteness, we state it for A_{inf} and its algebra $A_{\mathrm{cris}}^{(m)}$, but the proof is not specific to these choices.

Lemma 3.30. *For a fixed $m \geqslant p$, consider the following condition on an A_{inf}-module L:*

$$\text{for } j > 0, \quad \{\mathrm{Tor}_j^{A_{\mathrm{inf}}}(L, A_{\mathrm{cris}}^{(m)}/p^n)\}_{n>0} \quad \text{is Mittag–Leffler with vanishing eventual images,} \quad (\star)$$

which means that for every j, n, the map $\mathrm{Tor}_j^{A_{\mathrm{inf}}}(L, A_{\mathrm{cris}}^{(m)}/p^{n'}) \to \mathrm{Tor}_j^{A_{\mathrm{inf}}}(L, A_{\mathrm{cris}}^{(m)}/p^n)$ vanishes for some $n' > n$. For a bounded complex

$$M^\bullet = \ldots \to M^i \xrightarrow{d^i} M^{i+1} \to \ldots$$

of A_{inf}-modules, if each M^i and each $H^i(M^\bullet)$ satisfy $(\star)$, then, for every i, we have

$$H^i(M^\bullet \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}) \cong \varprojlim_n (H^i(M^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n)) \cong H^i(M^\bullet) \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}. \tag{3.30.1}$$

Proof. For an inverse system $\{0 \to I'_n \to I_n \to I''_n \to 0\}_{n>0}$ of short exact sequences of abelian groups, $\{I_n\}_{n>0}$ is Mittag–Leffler with vanishing eventual images if and only if so are both $\{I'_n\}_{n>0}$ and $\{I''_n\}_{n>0}$. Therefore, the short exact sequences

$$0 \to \mathrm{Ker}(d^i) \to M^i \to \mathrm{Im}(d^i) \to 0 \quad \text{and} \quad 0 \to \mathrm{Im}(d^{i-1}) \to \mathrm{Ker}(d^i) \to H^i(M^\bullet) \to 0 \tag{3.30.2}$$

imply, by descending induction on i, that each $\mathrm{Ker}(d^i)$ and each $\mathrm{Im}(d^i)$ satisfy $(\star)$. Consequently, these sequences remain short exact after applying $- \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}$, to the effect that the flanking terms of (3.30.1) get identified. By construction, this identification is compatible with the canonical maps to $\varprojlim_n (H^i(M^\bullet \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}/p^n))$, so it remains to establish the second identification in (3.30.1).

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By [SP, 0662 and 0130], the spectral sequences associated to a double complex give the following spectral sequences that converge to $H^{i+j}(M^\bullet \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n)$:

$$ {}^{(n)}E_2^{ij} = H^i(H^j(M^\bullet) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n) \quad \text{and} \quad {}^{(n)\prime}E_1^{ij} = H^j(M^i \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n), $$

where the differential on the ${}^{(n)\prime}E_1$-page is $H^j(d^i \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n)$. As n varies, both families of spectral sequences form inverse systems. Moreover, by assumption, the systems $\{{}^{(n)}E_2^{ij}\}_{n>0}$ with $i \neq 0$ and $\{{}^{(n)\prime}E_1^{ij}\}_{n>0}$ with $j \neq 0$ are Mittag–Leffler with vanishing eventual images. This persists to the subsequent pages: namely, by the first sentence of the proof, to the systems $\{{}^{(n)}E_s^{ij}\}_{n>0}$ with $i \neq 0$ and $\{{}^{(n)\prime}E_s^{ij}\}_{n>0}$ with $j \neq 0$ for any $s \leqslant \infty$. Consequently, the edge maps

$$ H^i(M^\bullet) \otimes A^{(m)}_{\mathrm{cris}}/p^n \to H^i(M^\bullet \otimes^{\mathbb{L}} A^{(m)}_{\mathrm{cris}}/p^n) \quad \text{and} \quad H^i(M^\bullet \otimes^{\mathbb{L}} A^{(m)}_{\mathrm{cris}}/p^n) \to H^i(M^\bullet \otimes A^{(m)}_{\mathrm{cris}}/p^n) $$

become isomorphisms after applying the functor $\varprojlim_n$. It remains to note that then so does their composition, which is the canonical map $H^i(M^\bullet) \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n \to H^i(M^\bullet \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n)$. □

To make Lemma 3.30 practical, we now establish its condition (⋆) in several key cases.

LEMMA 3.31. *For a fixed $m \geqslant p$, the condition (⋆) holds in any of the following cases:*

(i) *for any $n, n' > 0$, the sequence $(p^n, \mu^{n'})$ is L-regular and $L/(p^n, \mu^{n'})$ is $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-flat;*
(ii) *the module L has no nonzero p-torsion and each L/p^n is a filtered direct limit of direct sums of A_{inf}-modules of the form $A_{\mathrm{inf}}/(\varphi^{-s}(\mu), p^n)$ for variable $s \geqslant 0$.*

Thus, (⋆) holds for $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ and $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$, and for each $H^i_{\mathrm{cont}}(\Delta, N_\infty)$ and $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$.

Proof. If (i) holds, then, by the regular sequence aspect, $L \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(p^n, \mu^{n'}) \cong L/(p^n, \mu^{n'})$, so, by the flatness aspect, $L \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n$ is concentrated in degree 0. Thus, in the case (i), the inverse systems in (⋆) vanish termwise.

If (ii) holds, then each $L \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/p^n$ is concentrated in degree 0, so

$$ \{\mathrm{Tor}_j^{A_{\mathrm{inf}}}(L, A^{(m)}_{\mathrm{cris}}/p^n)\}_{n>0} \cong \{\mathrm{Tor}_j^{A_{\mathrm{inf}}/p^n}(L/p^n, A^{(m)}_{\mathrm{cris}}/p^n)\}_{n>0} \tag{3.31.1} $$

for every $j \geqslant 0$. In addition, since $\varphi^{-s}(\mu) \mid \mu$ for $s \geqslant 0$ and each A_{inf}/p^n has no nonzero μ-torsion, the assumption on L/p^n in (ii) ensures that the right-hand side of (3.31.1) vanishes termwise for $j > 1$. In contrast, for $j = 1$ and every $n > 0$, there is an $n' > n$ such that the transition map between positions n' and n in the right-hand side system of (3.31.1) vanishes: this follows from the identification

$$ \mathrm{Tor}_1^{A_{\mathrm{inf}}/p^{n'}}(A_{\mathrm{inf}}/(\varphi^{-s}(\mu), p^{n'}), A^{(m)}_{\mathrm{cris}}/p^{n'}) \cong (A^{(m)}_{\mathrm{cris}}/p^{n'})[\varphi^{-s}(\mu)] $$

and (3.26.2). Consequently, (ii) implies (⋆), as claimed.

By Lemma 3.13, (i) holds for $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ and then, by Lazard's theorem, (ii) holds for $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$. Likewise, Proposition 3.19, Corollary 3.23, and Lazard's theorem imply that (ii) holds for each $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$. By Lemma 3.7, $H^i_{\mathrm{cont}}(\Delta, N_\infty)$ vanishes for large i and, by Proposition 3.25, we have the short exact sequences

$$ 0 \to H^i_{\mathrm{cont}}(\Delta, N_\infty) \to H^i_{\mathrm{cont}}(\Delta, N_\infty/\mu) \to H^{i+1}_{\mathrm{cont}}(\Delta, N_\infty) \to 0. $$

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Therefore, due to the first sentence of the proof of Lemma 3.30, descending induction on i shows that $(\star)$ for $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu)$ implies $(\star)$ for $H^i_{\mathrm{cont}}(\Delta, N_\infty)$. □

Thanks to Lemma 3.31, we may draw the following concrete consequences from Lemma 3.30.

Proposition 3.32. *For every $m \geqslant p$ and $i \in \mathbb{Z}$, we have the identifications*

$$\begin{aligned} H^i_{\mathrm{cont}}(\Delta, N_\infty \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}) &\cong \varprojlim_n (H^i_{\mathrm{cont}}(\Delta, N_\infty \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n)) \\ &\cong H^i_{\mathrm{cont}}(\Delta, N_\infty) \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}. \end{aligned} \tag{3.32.1}$$

In particular, μ kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$.

Proof. By Lemma 3.7, the Koszul complex $M^\bullet$ of N_∞ with respect to $\delta_1, \dots, \delta_d$ satisfies

$$H^i(M^\bullet) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty) \quad \text{and} \quad H^i(M^\bullet \mathbin{\widehat{\otimes}} A^{(m)}_{\mathrm{cris}}) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}),$$

as well as

$$H^i(M^\bullet \otimes A^{(m)}_{\mathrm{cris}}/p^n) \cong H^i_{\mathrm{cont}}(\Delta, N_\infty \otimes A^{(m)}_{\mathrm{cris}}/p^n)$$

for every $n > 0$. Moreover, by Lemma 3.31, each M^i and each $H^i(M^\bullet)$ satisfy $(\star)$. Thus, (3.32.1) is a special case of (3.30.1). By Proposition 3.25, μ kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty)$, so, by (3.32.1), it also kills every $H^i_{\mathrm{cont}}(\Delta, N_\infty \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$. □

Proposition 3.33. *For every $m \geqslant p$ and $i \in \mathbb{Z}$, we have the identifications*

$$\begin{aligned} H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu) &\cong \varprojlim_n (H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/(\mu, p^n))) \\ &\cong H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu) \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}. \end{aligned}$$

Moreover, the A_{inf}-module $H^i_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)/\mu)$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion.

Proof. Similarly to the proof of Proposition 3.32, Lemma 3.30 applies to the Koszul complex of $\mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu$ and, due to (3.27.2), gives the identifications. Thus, it suffices to show that each

$$H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/\mu) \otimes_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}/p^n \overset{(3.19.1)}{\cong} H^i_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/p^n} A^{(m)}_{\mathrm{cris}}/p^n$$

has no nonzero $W(\mathfrak{m}^\flat)$-torsion. Since Δ acts trivially on $A(R)/(\mu, p^n)$, Lemma 3.7 and Proposition 3.29 imply that each $H^i_{\mathrm{cont}}(\Delta, A(R)/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/p^n} A^{(m)}_{\mathrm{cris}}/p^n$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion. Consequently, due to the decomposition (3.22.1), it suffices to show that for $j > 0$, the module

$$H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/p^n} A^{(m)}_{\mathrm{cris}}/p^n \overset{3.23}{\cong} H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \otimes_{A_{\mathrm{inf}}/(\varphi^{-j}(\mu), p^n)} A^{(m)}_{\mathrm{cris}}/(\varphi^{-j}(\mu), p^n)$$

has no nonzero $W(\mathfrak{m}^\flat)$-torsion. For this, similarly to the proof of Proposition 3.29, we will approximate by Noetherian rings. More precisely, similarly to (3.29.1), the assignment

$$T \mapsto [\epsilon]^{1/p^j} - 1 \quad \text{defines a } \mathbb{Z}_p\text{-algebra morphism} \quad \mathbb{Z}_p[\![T]\!] \to A_{\mathrm{inf}},$$

for which A_{inf} is $\mathbb{Z}[\![T]\!]$-flat. In terms of this morphism, the A_{inf}-algebra $A^{(m)}_{\mathrm{cris}}/(\varphi^{-j}(\mu), p^n)$ is the $A_{\mathrm{inf}}/(\varphi^{-j}(\mu), p^n)$-base change of the mod (T, p^n) reduction M of the $\mathbb{Z}_p[\![T]\!]$-subalgebra of

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$\mathbb{Z}_p[\![T]\!][\frac{1}{p}]$ generated by the elements $\frac{1}{s!}(\sum_{i=0}^{p-1}(T+1)^{p^{j-1}\cdot i})^s$ with $s \leqslant m$. Consequently, we need to show that

$$H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \otimes_{\mathbb{Z}_p[\![T]\!]/(T,\, p^n)} M$$

has no nonzero $W(\mathfrak{m}^\flat)$-torsion. By Corollary 3.23, the module

$$H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n)) \quad \text{is} \quad \mathbb{Z}_p[\![T]\!]/(T,\, p^n)\text{-flat.}$$

Thus, by p-adically filtering M, we reduce to showing that $H^i_{\mathrm{cont}}(\Delta, N_j/(\mu, p^n))/p$ has no nonzero $W(\mathfrak{m}^\flat)$-torsion. This, in turn, follows from Proposition 3.19 and Lemma 3.12. □

With Proposition 3.33 in hand, we are ready for the promised claim about $L\eta_{(\mu)}(e \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$.

THEOREM 3.34. *For each $m \geqslant p$, the map $e \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}$ from (3.28.1) induces the isomorphism*

$$L\eta_{(\mu)}(e \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf}}) \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}).$$

Proof. By (3.28.2), the ideal $W(\mathfrak{m}^\flat) \subset A_{\mathrm{inf}}$ kills the cohomology of $\mathrm{Cone}(e \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})$. By Proposition 3.33 (and the projection formula [SP, 0944] with (3.27.2)), the cohomology modules of

$$R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$$

have no nonzero $W(\mathfrak{m}^\flat)$-torsion. Thus, Lemma 3.18 applies and gives the desired conclusion. □

Remark 3.35. Analogously to Remark 3.21, we may extend Theorem 3.34 to any affinoid perfectoid Δ'-cover that refines $\mathfrak{X}^{\mathrm{ad}}_{C,\infty} \to \mathfrak{X}^{\mathrm{ad}}_C$ and is subject to the same conditions as in Remark 3.10: more precisely, with the notation used there, we have

$$L\eta_{(\mu)}(e' \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}})\colon L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty))) \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf}}) \,\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}),$$

where $\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty) := \mathbb{A}_{\mathrm{inf}}(R'_\infty) \,\widehat{\otimes}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}$. Indeed, as there (see also § 3.28), the ideal $W(\mathfrak{m}^\flat)$ kills the cohomology of the cone of the map $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)) \to R\Gamma_{\mathrm{cont}}(\Delta', \mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty))$, so Lemma 3.18 applies to this map and gives the claim.

4. The de Rham specialization of $A\Omega_{\mathfrak{X}}$

With the local analysis of § 3 at our disposal, we turn to relating $A\Omega_{\mathfrak{X}}$ to the logarithmic de Rham complex of $\mathfrak{X}$ in Theorem 4.17. The key steps for this are the identification and the analysis of the Hodge–Tate specialization of $A\Omega_{\mathfrak{X}}$ in Theorems 4.2 and 4.11. These steps were also used in the smooth case in [BMS18, §§ 8 and 9] but, due to the difficulties mentioned in the beginning of § 3, we carry them out differently. Namely, we rely on the analysis of group cohomology presented in § 3 and, in the identification step, we use Lemma 3.24 (which comes from [Bha18]). Nevertheless, similarly to [BMS18, § 9.2], we will take advantage of the following formalism of presheaves.

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4.1 The presheaf version $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$

In addition to the étale site $\mathfrak{X}_{\text{ét}}$, we consider the site $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$ whose objects are those connected affine opens of $\mathfrak{X}_{\text{ét}}$ that have an étale coordinate map (1.5.1) and coverings are the isomorphisms. Thus, the topology of $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$ is the coarsest possible and any presheaf is already a sheaf. Since the objects of $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$ form a basis of $\mathfrak{X}_{\text{ét}}$, there is a morphism of topoi

$$(\phi^{-1}, \phi_*)\colon \mathfrak{X}_{\text{ét}} \to \mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$$

for which ϕ_* is given by restricting sheaves on $\mathfrak{X}_{\text{ét}}$ to $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$ and ϕ^{-1} is given by sheafifying. In particular, since any sheaf is the sheafification of its associated presheaf, $\phi^{-1} \circ \phi_* \cong \mathrm{id}$. We let

$$\nu^{\mathrm{psh}} := \phi \circ \nu\colon (\mathfrak{X}_C^{\mathrm{ad}})_{\text{proét}} \to \mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$$

be the indicated composition of morphisms of topoi (with ν defined in (1.5.5)) and set

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} := L\eta_{(\mu)}(R\nu_*^{\mathrm{psh}}(\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}})) \in D^{\geqslant 0}(\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}, A_{\mathrm{inf}}). \tag{4.1.1}$$

Since $L\eta$ commutes with pullback under flat morphisms of ringed topoi (see [BMS18, 6.14]),

$$\phi^{-1}(A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \cong A\Omega_{\mathfrak{X}}. \tag{4.1.2}$$

Moreover, $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ may be described explicitly: for every object $\mathfrak{U}$ of $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$, we have

$$R\Gamma(\mathfrak{U}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \cong L\eta_{(\mu)}(R\Gamma((\mathfrak{U}_C^{\mathrm{ad}})_{\text{proét}}, \mathbb{A}_{\mathrm{inf},\,\mathfrak{U}_C^{\mathrm{ad}}})). \tag{4.1.3}$$

In particular, since, by [BMS18, 6.19], the functor $L\eta$ preserves derived completeness when used in the context of a *replete* topos (such as that of sets), we see from (4.1.3) that $A\Omega_{\mathfrak{X}}^{\mathrm{psh}}$ is derived ξ-adically (and also $\varphi(\xi)$-adically) complete (compare with Corollary 4.6 below).

Armed with the formalism of § 4.1, we now identify the Hodge–Tate specialization of $A\Omega_{\mathfrak{X}}$.

Theorem 4.2. *We have the identification*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\,\theta\circ\varphi^{-1}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p-1)}(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}})), \tag{4.2.1}$$

where in the target $L\eta$ is with respect to the ideal sheaf $(\zeta_p - 1)\mathcal{O}_{\mathfrak{X},\,\text{ét}} \subset \mathcal{O}_{\mathfrak{X},\,\text{ét}}$. If the coordinate morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$, then (4.2.1) also holds for $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$ (defined in (2.2.4)).

Proof. The kernel of $\theta_{\mathfrak{X}_C^{\mathrm{ad}}} \circ \varphi^{-1}\colon \mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}} \twoheadrightarrow \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}$ is generated by the nonzero-divisor $\varphi(\xi)$ (see § 2.2), so the projection formula [SP, 0944] provides the identification

$$R\nu_*(\mathbb{A}_{\mathrm{inf},\,\mathfrak{X}_C^{\mathrm{ad}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\,\theta\circ\varphi^{-1}} \mathcal{O}_C \cong R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}).$$

Since $(\theta \circ \varphi^{-1})(\mu) = \zeta_p - 1$, this induces the map (4.2.1) and, likewise, also its presheaf version

$$A\Omega_{\mathfrak{X}}^{\mathrm{psh}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\,\theta\circ\varphi^{-1}} \mathcal{O}_C \to L\eta_{(\zeta_p-1)}(R\phi_*(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}))). \tag{4.2.2}$$

Due to (4.1.2), the map ϕ^{-1} brings (4.2.2) to (4.2.1), so we seek to show that (4.2.2) is an isomorphism.

For every object $\mathfrak{U} \cong \mathrm{Spf}(R)$ of $\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}$ equipped with an étale morphism as in (1.5.1), the discussion and the notation of § 3 apply. In particular, Proposition 3.19 and (3.20.1) ensure

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that the cohomology of $R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty)) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\mu$ is p-torsion free. Thus, since we have $\varphi(\xi) \equiv p \bmod (\mu)$ (see § 2.1), Lemma 3.24 implies that

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_\infty))) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p-1)}(R\Gamma_{\mathrm{cont}}(\Delta, R_\infty)).$$

Since the edge maps (3.3.1) and (3.15.1) are compatible, Theorems 3.9 and 3.20 then imply that

$$L\eta_{(\mu)}(R\Gamma((\mathfrak{U}^{\mathrm{ad}}_C)_{\mathrm{pro\acute{e}t}}, \mathbb{A}_{\mathrm{inf}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\, \theta\circ\varphi^{-1}} \mathcal{O}_C \xrightarrow{\sim} L\eta_{(\zeta_p-1)}(R\Gamma((\mathfrak{U}^{\mathrm{ad}}_C)_{\mathrm{pro\acute{e}t}}, \widehat{\mathcal{O}}^+)).$$

Consequently, (4.2.2) is an isomorphism on every $\mathfrak{U}$, as desired. □

4.3 The object $\widetilde{\Omega}_{\mathfrak{X}}$

To proceed further, we need to analyze the right-hand side of (4.2.1), namely,

$$\widetilde{\Omega}_{\mathfrak{X}} := L\eta_{(\zeta_p-1)}(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C})) \in D^{\geqslant 0}(\mathcal{O}_{\mathfrak{X},\,\text{ét}}), \tag{4.3.1}$$

where, as in Theorem 4.2, the functor $L\eta$ is formed with respect to the ideal sheaf $(\zeta_p - 1)\mathcal{O}_{\mathfrak{X},\,\text{ét}}$.

PROPOSITION 4.4. *For $i \geqslant 0$, the $\mathcal{O}_{\mathfrak{X},\,\text{ét}}$-module $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is locally free of rank $\binom{\dim_x(\mathfrak{X}_k)}{i}$ at a variable closed point x of $\mathfrak{X}_k$ (in particular, each $H^i(\widetilde{\Omega}_{\mathfrak{X}})/p^n$ is a quasi-coherent $\mathcal{O}_{\mathfrak{X},\,\text{ét}}/p^n$-module). Moreover,*

$$\nu^\sharp\colon \mathcal{O}_{\mathfrak{X},\,\text{ét}} \xrightarrow{\sim} \nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}^{\mathrm{ad}}_C}) \quad \textit{so that} \quad H^0(\widetilde{\Omega}_{\mathfrak{X}}) \cong \mathcal{O}_{\mathfrak{X},\,\text{ét}}. \tag{4.4.1}$$

Proof. The claims are étale local (see [SP, 058S]), so we assume that $\mathfrak{X} = \mathrm{Spf}(R)$, that $\mathfrak{X}$ is connected, and that there is an étale $\mathrm{Spf}(\mathcal{O}_C)$-morphism as in (1.5.1):

$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^\square) =: \mathfrak{X}^\square \quad \text{with} \quad R^\square := \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q), \tag{4.4.2}$$

so that the discussion and the notation of § 3 apply. In particular, since R is $R^\square$-flat (see § 1.5) and Δ acts trivially on $R^\square$ and R, Lemma 3.7 and Proposition 3.8 imply that

$$R^{\oplus\binom{d}{i}} \cong H^i_{\mathrm{cont}}(\Delta, R^\square) \otimes_{R^\square} R \cong \frac{H^i_{\mathrm{cont}}(\Delta, R^\square_\infty)}{H^i_{\mathrm{cont}}(\Delta, R^\square_\infty)[\zeta_p - 1]} \otimes_{R^\square} R \xrightarrow{\sim} \frac{H^i_{\mathrm{cont}}(\Delta, R_\infty)}{H^i_{\mathrm{cont}}(\Delta, R_\infty)[\zeta_p - 1]}. \tag{4.4.3}$$

Thus, since the edge maps e of (3.3.1) are compatible for R and $R^\square$, Theorem 3.9 shows that

$$\frac{H^i((\mathfrak{X}^\square)^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i((\mathfrak{X}^\square)^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]} \otimes_{R^\square} R \xrightarrow{\sim} \frac{H^i(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i(\mathfrak{X}^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]} \tag{4.4.4}$$

is an isomorphism of free R-modules of rank $\binom{d}{i}$. Consequently,

$$\frac{H^i((\mathfrak{X}^\square)^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)}{H^i((\mathfrak{X}^\square)^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]} \otimes_{R^\square} \mathcal{O}_{\mathrm{Spf}(R),\,\text{ét}} \xrightarrow{\sim} \frac{R^i\nu_*(\widehat{\mathcal{O}}^+)}{(R^i\nu_*(\widehat{\mathcal{O}}^+))[\zeta_p - 1]} \cong H^i(\widetilde{\Omega}_{\mathfrak{X}}), \tag{4.4.5}$$

to the effect that $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ is free of rank $\binom{d}{i}$, as desired. For (4.4.1), by § 3.3, we need to show that $R \xrightarrow{\sim} (R_\infty)^\Delta$. This map is an inclusion of a direct summand whose complementary summand M^Δ_∞ is both p-torsion free and, by Proposition 3.8, killed by $\zeta_p - 1$, so the claim follows. □

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Remark 4.5. The proof of Proposition 4.4, specifically, (4.4.4) and (4.4.5), shows that if $\mathfrak{X}$ is affine, connected, and admits a coordinate map as in (1.5.1), then the presheaf assigning $H^i(\mathfrak{X}'^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)/H^i(\mathfrak{X}'^{\mathrm{ad}}_C, \widehat{\mathcal{O}}^+)[\zeta_p - 1]$ to a variable $\mathfrak{X}$-étale affine $\mathfrak{X}'$ is already a sheaf. In particular, if the coordinate maps (1.5.1) exist Zariski locally on $\mathfrak{X}$ (for instance, if $\mathfrak{X}$ is $\mathcal{O}_C$-smooth or arises as in (1.5.3) from a strictly semistable $\mathcal{X}$), then the sheaves $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ may be computed using the Zariski topology: more precisely, then the object $\widetilde{\Omega}_{\mathfrak{X}_{\mathrm{Zar}}}$ defined by the formula (4.3.1) using the Zariski topology of $\mathfrak{X}$ satisfies

$$H^i(\widetilde{\Omega}_{\mathfrak{X}_{\mathrm{Zar}}}) \xrightarrow{\sim} (H^i(\widetilde{\Omega}_{\mathfrak{X}}))|_{\mathfrak{X}_{\mathrm{Zar}}} \quad \text{for every } i. \tag{4.5.1}$$

Corollary 4.6. *The object $A\Omega_{\mathfrak{X}}$ is derived ξ-adically complete and*

$$A\Omega^{\mathrm{psh}}_{\mathfrak{X}} \xrightarrow{\sim} R\phi_*(A\Omega_{\mathfrak{X}}) \overset{(4.1.2)}{\cong} R\phi_*(\phi^{-1}(A\Omega^{\mathrm{psh}}_{\mathfrak{X}})). \tag{4.6.1}$$

Proof. For the derived ξ-adic completeness, since $\phi^{-1} \circ R\phi_* \cong \mathrm{id}$, it suffices to show that the map

$$A\Omega_{\mathfrak{X}} \to R\lim_n (A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/\xi^n)$$

becomes an isomorphism after applying $R\phi_*$. Thus, since $A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$ is derived ξ-adically complete (see § 4.1), it suffices to establish the adjunction isomorphism (4.6.1). For this, by the definition of $\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}$ given in § 4.1, we may assume that $\mathfrak{X}$ is affine, connected, and admits an étale morphism (1.5.1). In addition, since $A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$ is derived $\varphi(\xi)$-adically complete, the $\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}$-analogue of [BMS18, 9.15] reduces us to proving that

$$A\Omega^{\mathrm{psh}}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\varphi(\xi)^n) \xrightarrow{\sim} R\phi_*(\phi^{-1}(A\Omega^{\mathrm{psh}}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\varphi(\xi)^n))).$$

By the five lemmas, we may assume that $n = 1$ and, by the proof of Theorem 4.2,

$$A\Omega^{\mathrm{psh}}_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\varphi(\xi)) \cong L\eta_{(\zeta_p - 1)}(R\phi_*(R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}'^{\mathrm{ad}}_C}))) =: \Omega^{\mathrm{psh}}_{\mathfrak{X}}.$$

It remains to recall from Remark 4.5 that the cohomology presheaves of $\Omega^{\mathrm{psh}}_{\mathfrak{X}}$ are sheaves. □

Our next task is to identify the vector bundles $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ with the twists of the bundles given by logarithmic differentials (see Theorem 4.11). For this, in Proposition 4.8, we first express $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ as $\bigwedge^i H^1(\widetilde{\Omega}_{\mathfrak{X}})$, and then, in (4.10.2), construct a map that relates $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ to Kähler differentials.

4.7 The cup product maps

By the same arguments as in [SP, 068G], there are product maps

$$R^j\nu_*(\widehat{\mathcal{O}}^+) \otimes_{\mathcal{O}_{\mathfrak{X},\text{ét}}} R^{j'}\nu_*(\widehat{\mathcal{O}}^+) \xrightarrow{-\cup-} H^{j+j'}(R\nu_*(\widehat{\mathcal{O}}^+) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\text{ét}}} R\nu_*(\widehat{\mathcal{O}}^+))$$

that satisfy $x \cup y = (-1)^{jj'} y \cup x$ (see [SP, 0BYI]). By [SP, 0B6C], there is a cup product map

$$R\nu_*(\widehat{\mathcal{O}}^+) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\text{ét}}} R\nu_*(\widehat{\mathcal{O}}^+) \to R\nu_*(\widehat{\mathcal{O}}^+).$$

These maps combine to give the 'cup product map' (where the tensor product is over $\mathcal{O}_{\mathfrak{X},\text{ét}}$)

$$\bigotimes_{s=1}^{i} R^1\nu_*(\widehat{\mathcal{O}}^+) \to R^i\nu_*(\widehat{\mathcal{O}}^+) \quad \text{for each} \quad i > 0. \tag{4.7.1}$$

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PROPOSITION 4.8. *For each $i > 0$, the map (4.7.1) induces the isomorphism*

$$\bigwedge^i \left(\frac{R^1\nu_*(\widehat{\mathcal{O}}^+)}{R^1\nu_*(\widehat{\mathcal{O}}^+)[\zeta_p - 1]} \right) \cong \bigwedge^i H^1(\widetilde{\Omega}_{\mathfrak{X}}) \xrightarrow{\sim} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \cong \frac{R^i\nu_*(\widehat{\mathcal{O}}^+)}{R^i\nu_*(\widehat{\mathcal{O}}^+)[\zeta_p - 1]}. \tag{4.8.1}$$

Proof. By Proposition 4.4, each $H^i(\widetilde{\Omega}_{\mathfrak{X}})$ has no nontrivial 2-torsion, so the antisymmetry of the map (4.7.1) in each pair of variables indeed induces the $\mathcal{O}_{\mathfrak{X},\text{ét}}$-module map (4.8.1). For the isomorphism claim, we may work étale locally, so we put ourselves in the situation (4.4.2). The edge maps

$$e\colon H^i_{\mathrm{cont}}(\Delta, R_\infty) \to H^i(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)$$

of (3.3.1) are compatible with cup products: to check this, one identifies $H^i(\mathfrak{X}_C^{\mathrm{ad}}, \widehat{\mathcal{O}}^+)$ with the direct limit of the ith Čech cohomology groups of $\widehat{\mathcal{O}}^+$ with respect to a variable proétale hypercovering of $\mathfrak{X}_C^{\mathrm{ad}}$ (see [SP, 01H0]) and uses the hypercovering construction of the cup product (see [SP, 01FP]). Due to Theorem 3.9 and (4.4.3), it then remains to argue that via the cup product the identification

$$H^1_{\mathrm{cont}}(\Delta, R) \overset{3.7}{\cong} R^d \quad \text{induces} \quad H^i_{\mathrm{cont}}(\Delta, R) \overset{3.7}{\cong} \bigwedge^i (R^d),$$

which follows from [BMS18, 7.3 and 7.5]. □

To relate $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ to Kähler differentials, we now review the needed material on cotangent complexes.

4.9 The completed cotangent complex $\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p}$

Affinoid perfectoids form a basis of $(\mathfrak{X}_C^{\mathrm{ad}})_{\mathrm{pro\acute{e}t}}$ (see [Sch13a, 4.7]). Therefore, [BMS18, 3.14] ensures that for the sheaf of rings $\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}$, the cotangent complex $\mathbb{L}_{\widehat{\mathcal{O}}^+/\mathcal{O}_C}$, whose terms are $\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}$-flat and which gives an object of $D^{\leqslant 0}(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}})$, satisfies

$$\mathbb{L}_{\widehat{\mathcal{O}}^+/\mathcal{O}_C} \otimes^{\mathbb{L}}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \cong 0, \quad \text{and, hence, also} \quad \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathcal{O}_C} \cong 0.$$

Consequently, the derived p-adic completion turns the canonical morphism

$$\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}} \to \mathbb{L}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p} \quad \text{into an isomorphism} \quad (\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}})\widehat{\ } \xrightarrow{\sim} \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p}$$

in the derived category. By [GR03, 6.5.12(ii)], the complex $\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p}$ is quasi-isomorphic to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ placed in degree 0. The p-divisibility of $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$ then ensures that for every $n > 0$ we have

$$\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes^{\mathbb{L}}_{\mathcal{O}_C} (\widehat{\mathcal{O}}^+/p^n\widehat{\mathcal{O}}^+) \cong (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n] \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+)[1] \overset{\text{[Sch13a, 4.2(iii)]}}{\cong} (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n] \otimes_{\mathcal{O}_C} (\mathcal{O}^+/p^n\mathcal{O}^+))[1],$$

where $\mathcal{O}^+$ abbreviates the integral structure sheaf $\mathcal{O}^+_{\mathfrak{X}_C^{\mathrm{ad}}}$. Moreover, by [Fon82, Theorem 1′(ii)],[5]

$$\mathcal{O}_C\{1\} := \varprojlim_{n,\, y \mapsto py} (\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}[p^n]) \quad \text{is a free } \mathcal{O}_C\text{-module of rank 1.}$$

In conclusion, letting $\{1\}$ abbreviate the $\mathcal{O}_C$-tensor product with $\mathcal{O}_C\{1\}$, we obtain an isomorphism

$$(\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}})\widehat{\ } \cong (\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}\{1\})[1], \quad \text{so also} \quad \widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p} \cong (\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}\{1\})[1], \quad \text{in} \quad D(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}). \tag{4.9.1}$$

[5] For passage from $\Omega^1_{\overline{\mathbb{Z}}_p/\mathbb{Z}_p}$ of [Fon82] to $\Omega^1_{\mathcal{O}_C/\mathbb{Z}_p}$, one may use [GR03, 6.5.20(i)] to conclude that $\Omega^1_{\mathcal{O}_C/\overline{\mathbb{Z}}_p}[p] = 0$.

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4.10 The relation between $\widetilde{\Omega}_{\mathfrak{X}}$ and Kähler differentials

The functoriality of the cotangent complex supplies the pullback morphism

$$\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\text{ét}}/\mathbb{Z}_p} \to R\nu_*(\widehat{\mathbb{L}}_{\widehat{\mathcal{O}}^+/\mathbb{Z}_p}) \overset{(4.9.1)}{\cong} (R\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}\{1\}))[1]. \tag{4.10.1}$$

To explicate its source, we note that, as in § 4.9, the explicit description of $\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p}$ gives

$$(\mathbb{L}_{\mathcal{O}_C/\mathbb{Z}_p} \otimes_{\mathcal{O}_C} \mathcal{O}_{\mathfrak{X},\text{ét}})\widehat{\ } \cong (\mathcal{O}_{\mathfrak{X},\text{ét}}\{1\})[1], \quad \text{so} \quad H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\text{ét}}/\mathbb{Z}_p}) \cong H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\text{ét}}/\mathcal{O}_C}).$$

Moreover, the short exact sequence [SP, 0D6K] leads to the identification

$$H^0(\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X},\text{ét}}/\mathcal{O}_C}) \cong \Omega^1_{\mathfrak{X}/\mathcal{O}_C}$$

(the $R^1\lim$ term vanishes due to the description [Ill71, III.3.2.7]: each $\mathfrak{X}_{\mathcal{O}_C/p^n}$ is a local complete intersection over $\mathcal{O}_C/p^n$ and, as may be seen using (1.5.1), no nonzero local section of a vector bundle on $\mathfrak{X}_{\mathcal{O}_C/p^n}$ vanishes on $\mathfrak{X}^{\mathrm{sm}}_{\mathcal{O}_C/p^n}$). By [Ill71, III.3.1.2], over $\mathfrak{X}^{\mathrm{sm}}$, this identification gives a quasi-isomorphism between

$$\widehat{\mathbb{L}}_{\mathcal{O}_{\mathfrak{X}^{\mathrm{sm}},\text{ét}}/\mathcal{O}_C} \quad \text{and} \quad \Omega^1_{\mathfrak{X}^{\mathrm{sm}}/\mathcal{O}_C} \quad \text{placed in degree 0.}$$

Consequently, by applying $H^0(-)$ to the map (4.10.1) and twisting by $\mathcal{O}_C\{-1\}$ we obtain the first map in the following composition of $\mathcal{O}_{\mathfrak{X},\text{ét}}$-module morphisms:

$$\Omega^1_{\mathfrak{X}/\mathcal{O}_C}\{-1\} \to R^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}) \twoheadrightarrow \frac{R^1\nu_*(\widehat{\mathcal{O}}^+)}{(R^1\nu_*(\widehat{\mathcal{O}}^+))[\zeta_p-1]} \cong H^1(\widetilde{\Omega}_{\mathfrak{X}}). \tag{4.10.2}$$

By [BMS18, 8.15 and its proof], the restriction of this composition to $\mathfrak{X}^{\mathrm{sm}}$ is an isomorphism onto $((\zeta_p-1)\cdot H^1(\widetilde{\Omega}_{\mathfrak{X}}))|_{\mathfrak{X}^{\mathrm{sm}}}$. Moreover, by Proposition 4.4, the $\mathcal{O}_{\mathfrak{X},\text{ét}}$-module $H^1(\widetilde{\Omega}_{\mathfrak{X}})$ is a vector bundle, so it has no nonzero (ζ_p-1)-torsion and $(H^1(\widetilde{\Omega}_{\mathfrak{X}}))/(\zeta_p-1)$ has no nonzero local sections that vanish on $\mathfrak{X}^{\mathrm{sm}}_{\mathcal{O}_C/(\zeta_p-1)}$. In conclusion, we may divide the composition (4.10.2) by ζ_p-1 to obtain a map

$$\Omega^1_{\mathfrak{X}/\mathcal{O}_C}\{-1\} \to H^1(\widetilde{\Omega}_{\mathfrak{X}}) \quad \text{that is an isomorphism over } \mathfrak{X}^{\mathrm{sm}}. \tag{4.10.3}$$

Theorem 4.11. *The restriction of the map (4.10.3) to $\mathfrak{X}^{\mathrm{sm}}$ extends uniquely to an $\mathcal{O}_{\mathfrak{X},\text{ét}}$-isomorphism*

$$\Omega^1_{\mathfrak{X}/\mathcal{O}_C,\log}\{-1\} \cong H^1(\widetilde{\Omega}_{\mathfrak{X}}), \tag{4.11.1}$$

which, by (4.4.1) and Proposition 4.8, induces an $\mathcal{O}_{\mathfrak{X},\text{ét}}$-module identification

$$\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}\{-i\} \cong H^i(\widetilde{\Omega}_{\mathfrak{X}}) \quad \textit{for every } i \geqslant 0. \tag{4.11.2}$$

The proof of Theorem 4.11 will use the formal GAGA and Grothendieck existence theorems. The Noetherian cases of these theorems proved in [EGA III$_1$, § 5] have been extended to suitable non-Noetherian settings by K. Fujiwara and F. Kato (with important inputs by O. Gabber). The following theorem summarizes the relevant to our aims special case of this extension.

Theorem 4.12 (Fujiwara–Kato). *For a valuation ring V of height 1, a nonzero nonunit $a \in V$ such that V is a-adically complete, and a proper, finitely presented V-scheme Y, the functor*

$$\mathcal{F} \mapsto (\mathcal{F}/a^n\mathcal{F})_{n>0} \tag{4.12.1}$$

is an equivalence from the category of finitely presented $\mathcal{O}_Y$-modules $\mathcal{F}$ to that of sequences $(\mathcal{F}_n)_{n>0}$ of finitely presented $\mathcal{O}_{Y_{V/a^n}}$-modules $\mathcal{F}_n$ equipped with isomorphisms $\mathcal{F}_{n+1}|_{Y_{V/a^n}} \simeq \mathcal{F}_n$.

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Proof. The claim is a special case of [FK18, I.10.1.2]. In order to explain why [FK18, I.10.1.2] implies our assertion, we first reinterpret our source and target categories.

By a result of Gabber [FK18, 0.9.2.7], the ring V is 'a-adically topologically universally adhesive', so, by [FK18, 0.8.5.25(2)], it is also 'topologically universally coherent with respect to (a)'. In particular, by [FK18, 0.8.5.24], every finitely presented V-algebra is a coherent ring and, hence, by [FK18, 0.5.1.2], the $\mathcal{O}_Y$-module $\mathcal{O}_Y$ is coherent (in the sense of [FK18, 0.4.1.4(2)] or [EGA I, 0.5.3.1]). In particular, by [FK18, 0.4.1.8], an $\mathcal{O}_Y$-module $\mathcal{F}$ is finitely presented if and only if $\mathcal{F}$ is coherent, and likewise for $\mathcal{O}_{Y_{V/a^n}}$-modules for $n > 0$.

By [FK18, 0.8.4.2 and 0.8.5.19(3)], the formal a-adic completion $\widehat{Y}$ of Y is covered by open affines whose coordinate rings are 'topologically universally adhesive' and, hence, by [FK18, 0.8.5.18], also 'topologically universally Noetherian outside (a)'. In particular, by [FK18, I.2.1.1(1) and I.2.1.7], the topological ring V is 'topologically universally rigid-Noetherian' and the formal scheme $\widehat{Y}$ is 'universally rigid-Noetherian'. In addition, by [FK18, 0.8.4.5], the formal scheme $\widehat{Y}$ is locally of finite presentation over $\mathrm{Spf}(V)$. Thus, [FK18, I.7.2.2] applied with $A = V$ and [FK18, I.7.2.1] imply that $\widehat{Y}$ is 'universally cohesive'. Then, by [FK18, I.7.2.4 and I.3.4.1], the functor $(\mathcal{F}_n) \mapsto \varprojlim \mathcal{F}_n$ is an equivalence from the target category of (4.12.1) to the category of coherent $\mathcal{O}_{\widehat{Y}}$-modules.

In conclusion, our claim is that the quasi-coherent pullback i^* along the morphism $i\colon \widehat{Y} \to Y$ of locally ringed spaces induces an equivalence between the category of coherent $\mathcal{O}_Y$-modules and that of coherent $\mathcal{O}_{\widehat{Y}}$-modules. This is a special case of [FK18, I.10.1.2] (see also [FK18, I.§ 9.1]). □

Remarks.

4.13. In Theorem 4.12, if each $\mathcal{F}_n$ is locally free, then the $\mathcal{O}_Y$-module $\mathcal{F}$ that algebraizes the sequence $(\mathcal{F}_n)_{n>0}$ is also locally free. Indeed, it is enough to argue that the stalks of $\mathcal{F}$ at the points of $Y_{V/a}$ are flat, so, since i is flat by [FK18, I.1.4.7(2), 0.8.5.8(2), 0.8.5.17], it suffices to note that the $\mathcal{O}_{\widehat{Y}}$-module $i^*\mathcal{F} \cong \varprojlim \mathcal{F}_n$ is locally free because the Nakayama lemma ensures that $\mathcal{F}_{n+1}$ is locally trivialized by any lifts of local sections that trivialize $\mathcal{F}_n$.

4.14. Remark 4.13. and the proof of Theorem 4.12 also show that i is flat and that the functor $(\mathcal{F}_n) \mapsto \varprojlim \mathcal{F}_n$ is an equivalence to the category of finitely presented $\mathcal{O}_{\widehat{Y}}$-modules.

4.15 Proof of Theorem 4.11

As we observed in § 4.10, no nonzero local section of a vector bundle on $\mathfrak{X}$ vanishes on $\mathfrak{X}^{\mathrm{sm}}$. Thus, the desired isomorphism (4.11.1) is unique if it exists. Consequently, we may assume that $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q))$ with r, d, and q as in (1.5.1). In this case, $\mathfrak{X}$ is an open subscheme of the formal p-adic completion of some *proper*, flat $\overline{W(k)}$-scheme $\mathcal{X}$ that Zariski locally has étale 'coordinate morphisms' as in (1.5.2) with $\mathcal{O}$ there replaced by $\overline{W(k)}$. Thus, finally, we may drop the previous assumptions and assume instead that $\mathfrak{X} = \widehat{\mathcal{X}}$ with $\mathcal{X}$ as above. We equip $\mathcal{X}$ with the log structure $\mathcal{O}_{\mathcal{X}} \cap (\mathcal{O}_{\mathcal{X}}[\frac{1}{p}])^{\times}$, so that $\mathcal{X}$ is log smooth over $\overline{W(k)}$ (see § 1.6, especially, Claim 1.6.1) and the map $\mathfrak{X} \to \mathcal{X}$ of log ringed étale sites is strict (see Claim 1.6.3). By Theorem 4.12, the map (4.10.3) algebraizes to an $\mathcal{O}_{\mathcal{X}}$-module map

$$f\colon \Omega^1_{\mathcal{X}/\mathcal{O}_C}\{-1\} \to \mathcal{H}.$$

By Proposition 4.4 and Remark 4.13., the $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{H}$ is locally free. By (4.10.3) and the Nakayama lemma, f is surjective at every point of $\mathcal{X}_k^{\mathrm{sm}}$.

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Claim 4.15.1. *There is an isomorphism* $\mathcal{H}_C \simeq \Omega^1_{\mathcal{X}_C/C}$.

Proof. By the adic GAGA (see [Sch13a, 9.1(i)]), it suffices to find an analogous isomorphism after pullback to $(\mathcal{X}_C)^{\mathrm{ad}} \cong \mathfrak{X}_C^{\mathrm{ad}}$. On the one hand, such a pullback of $\mathcal{H}_C$ is isomorphic to $(R^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}))[\frac{1}{p}]$. On the other hand, [Sch13b, 3.23–3.24 and their proofs] supply an isomorphism between $(R^1\nu_*(\widehat{\mathcal{O}}^+_{\mathfrak{X}_C^{\mathrm{ad}}}))[\frac{1}{p}]$ and the pullback of $\Omega^1_{\mathcal{X}_C/C}$ to $(\mathcal{X}_C)^{\mathrm{ad}}$. □

Claim 4.15.1 ensures that f_C is a generically surjective morphism between isomorphic vector bundles on $\mathcal{X}_C$. Since $\mathcal{X}_C$ is proper and smooth, every global section of the structure sheaf of each connected component of $\mathcal{X}_C$ is constant, so $\det(f_C)$ is an isomorphism, and hence f_C is also an isomorphism. In conclusion, $f|_{\mathcal{X}^{\mathrm{sm}}}$ is a surjection between vector bundles of the same rank, so

$$f|_{\mathcal{X}^{\mathrm{sm}}} \colon \Omega^1_{\mathcal{X}^{\mathrm{sm}}/\mathcal{O}_C}\{-1\} \xrightarrow{\sim} \mathcal{H}|_{\mathcal{X}^{\mathrm{sm}}}. \tag{4.15.2}$$

Since $\mathcal{X} \setminus \mathcal{X}^{\mathrm{sm}}$ is of codimension $\geqslant 2$ in $\mathcal{X}$, limit arguments and [EGA IV$_2$, 5.10.5] ensure that $\mathcal{H}$ is the unique vector bundle extension of $\mathcal{H}|_{\mathcal{X}^{\mathrm{sm}}}$ to $\mathcal{X}$. The isomorphism (4.15.2) then leads to an isomorphism $\Omega^1_{\mathcal{X}/\mathcal{O}_C,\log}\{-1\} \simeq \mathcal{H}$ whose formal p-adic completion gives the desired (4.11.1). □

Remark 4.16. If the coordinate morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$, then, by (4.5.1), the identifications of Theorem 4.11 hold already for the Zariski topology; more precisely, then

$$H^i(\widetilde{\Omega}_{\mathfrak{X}_{\mathrm{Zar}}}) \cong \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}\{-i\} \quad \text{as } \mathcal{O}_{\mathfrak{X}_{\mathrm{Zar}}}\text{-modules for every } i \geqslant 0.$$

We are ready to relate the de Rham specialization of $A\Omega_{\mathfrak{X}}$ to differential forms by combining the results above with the argument from the proof of [BMS18, 14.1].

Theorem 4.17. *There is an identification*

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}. \tag{4.17.1}$$

If the coordinate morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$*, then (4.17.1) also holds for* $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$.

Proof. Since $\varphi(\mu) = \varphi(\xi)\mu$ (see § 2.1), [BMS18, 6.11] gives the second identification in

$$A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C \cong (L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C.$$

By [BMS18, 6.12], since $A_{\mathrm{inf}}/(\varphi(\xi)) \cong \mathcal{O}_C$ via $\theta\circ\varphi^{-1}$, the object $(L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}})) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C$ is identified with the complex whose ith degree term is

$$H^i(A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta\circ\varphi^{-1}} \mathcal{O}_C) \otimes_{\mathcal{O}_C} \left(\tfrac{\mathrm{Ker}(\theta\circ\varphi^{-1})}{(\mathrm{Ker}(\theta\circ\varphi^{-1}))^2}\right)^{\otimes i} \overset{(4.2.1)}{\cong} H^i(\widetilde{\Omega}_{\mathfrak{X}}) \otimes_{\mathcal{O}_C} \left(\tfrac{\mathrm{Ker}(\theta\circ\varphi^{-1})}{(\mathrm{Ker}(\theta\circ\varphi^{-1}))^2}\right)^{\otimes i}$$

and the differentials are given by Bockstein homomorphisms.

Since $\mathcal{O}_C^\flat$ is perfect, $\widehat{\mathbb{L}}_{A_{\mathrm{inf}}/\mathbb{Z}_p} \cong 0$. Moreover, (4.9.1) applied with $\mathfrak{X} = \mathrm{Spf}(\mathcal{O}_C)$ implies that $\widehat{\mathbb{L}}_{\mathcal{O}_C/\mathbb{Z}_p} \cong (\mathcal{O}_C\{1\})[1]$. Thus, $\widehat{\mathbb{L}}_{\mathcal{O}_C/A_{\mathrm{inf}}} \cong (\mathcal{O}_C\{1\})[1]$, where $\mathcal{O}_C$ is an A_{inf}-algebra via $\theta\circ\varphi^{-1}$. In particular, due to [Ill71, III.3.2.4(iii)], we have $\mathrm{Ker}(\theta\circ\varphi^{-1})/(\mathrm{Ker}(\theta\circ\varphi^{-1}))^2 \cong \mathcal{O}_C\{1\}$.

In conclusion, by (4.11.2) and the preceding discussion, $A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C$ is identified with the complex whose ith degree term is $\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}$ and the differentials are certain Bockstein homomorphisms. Each $\Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}$ is a vector bundle, so the agreement of the Bockstein differentials with those of $\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}$ may be checked over $\mathfrak{X}^{\mathrm{sm}}$ (compare with the argument for (4.10.3)), where it follows from [BMS18, 14.1(ii)] (or [Bha18, proof of Proposition 7.9]).

Due to Remark 4.16, the proof for $A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}$ is the same. □

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COROLLARY 4.18. *The de Rham specialization of $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ may be identified as follows:*

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C). \tag{4.18.1}$$

Proof. The claim follows from Theorem 4.17 and the projection formula [SP, 0944]. □

Remark 4.19. In the case when $\mathfrak{X} \cong \widehat{\mathcal{X}}$ for a proper, flat $\overline{W(k)}$-scheme $\mathcal{X}$ that étale locally has étale coordinate morphisms (1.5.2) with $\mathcal{O}$ there replaced by $\overline{W(k)}$, we have the further identification

$$R\Gamma(\mathcal{X}_{\text{ét}}, \Omega^{\bullet}_{\mathcal{X}/\overline{W(k)},\log}) \otimes^{\mathbb{L}}_{\overline{W(k)}} \mathcal{O}_C \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\text{ét}}, \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}) = R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C),$$

where $\mathcal{X}$ is endowed with the log structure $\mathcal{O}_{\mathcal{X},\text{ét}} \cap (\mathcal{O}_{\mathcal{X},\text{ét}}[\frac{1}{p}])^{\times}$ (whose pullback to $\mathfrak{X}$ is the log structure $\mathcal{O}_{\mathfrak{X},\text{ét}} \cap (\mathcal{O}_{\mathfrak{X},\text{ét}}[\frac{1}{p}])^{\times}$ of $\mathfrak{X}$, see Claim 1.6.3) and $\overline{W(k)}$ is endowed with the log structure associated to $\overline{W(k)} \setminus \{0\} \hookrightarrow \overline{W(k)}$. Indeed, the pullback map between the E_1-spectral sequences

$$H^j(\mathcal{X}_{\mathcal{O}_C}, \Omega^i_{\mathcal{X}_{\mathcal{O}_C}/\mathcal{O}_C,\log}) \Rightarrow H^{i+j}(R\Gamma(\mathcal{X}_{\text{ét}}, \Omega^{\bullet}_{\mathcal{X}_{\mathcal{O}_C}/\mathcal{O}_C,\log})),$$
$$H^j(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}) \Rightarrow H^{i+j}(R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C))$$

is an isomorphism because, by the Grothendieck finiteness and comparison theorems [EGA III_1, 3.2.1 and 4.1.7] (combined with limit arguments, which use Claim 1.6.1 and the fact that $\mathcal{X}$ is necessarily finitely presented, see [SP, 053E]; alternatively, directly by [FK18, I.9.2.1]),

$$H^j(\mathcal{X}_{\mathcal{O}_C}, \Omega^i_{\mathcal{X}_{\mathcal{O}_C}/\mathcal{O}_C,\log}) \xrightarrow{\sim} H^j(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathcal{O}_C,\log}) \quad \text{for all } i, j.$$

COROLLARY 4.20. *If $\mathfrak{X}$ is proper over $\mathcal{O}_C$, then $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ is a perfect object of $D^{\geqslant 0}(A_{\mathrm{inf}})$; in other words, then $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ is quasi-isomorphic to a bounded complex of finite free A_{inf}-modules.*

Proof. By the Grothendieck finiteness theorem [Ull95, 5.3] and the spectral sequence as in Remark 4.19, the $\mathcal{O}_C$-modules $H^j(R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C))$ are finitely presented and, hence, also perfect (see [SP, 0ASP]). Thus, by Corollary 4.18 and [SP, 066U], the object

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}/(\xi)$$

of $D^{\geqslant 0}(\mathcal{O}_C)$ is perfect. Moreover, by Corollary 4.6, the object $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ is derived ξ-adically complete. Therefore, by [SP, 09AW], it is perfect as well, as desired. □

We close the section by comparing $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ with its analogue defined using the Zariski topology.

COROLLARY 4.21. *If the coordinate morphisms (1.5.1) exist Zariski locally on $\mathfrak{X}$, then the cohomology $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ may be computed using the Zariski topology of $\mathfrak{X}$; more precisely, then*

$$R\Gamma(\mathfrak{X}_{\mathrm{Zar}}, A\Omega_{\mathfrak{X}_{\mathrm{Zar}}}) \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}). \tag{4.21.1}$$

Proof. By Theorem 4.17 and Corollary 4.18, the reduction of (4.21.1) modulo ξ is identified with

$$R\Gamma(\mathfrak{X}_{\mathrm{Zar}}, \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}) \xrightarrow{\sim} R\Gamma(\mathfrak{X}_{\text{ét}}, \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}),$$

and, hence, is an isomorphism as indicated. Thus, since, by Corollary 4.6 (and its Zariski analogue), $R\Gamma(\mathfrak{X}_{\mathrm{Zar}}, A\Omega_{\mathfrak{X}_{\mathrm{Zar}}})$ and $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ are derived ξ-adic complete, (4.21.1) is an isomorphism. □

Example 4.22. By § 1.5, Corollary 4.21 applies to any $\mathcal{O}_C$-smooth $\mathfrak{X}$ and, more generally, to any $\mathfrak{X}$ that Zariski locally arises from a strictly semistable scheme defined over a discrete valuation ring.

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5. The absolute crystalline comparison isomorphism

In Theorem 4.17, we identified the $\mathcal{O}_C$-base change along θ of the object $A\Omega_{\mathfrak{X}}$ with $\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}$. The goal of this section is to similarly identify the A_{cris}-base change of $A\Omega_{\mathfrak{X}}$ with an object that computes the logarithmic crystalline (that is, Hyodo–Kato) cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$ over A_{cris} (see Theorem 5.4). This is more general because, on the one hand, θ factors through the morphism $A_{\mathrm{inf}} \to A_{\mathrm{cris}}$, while, on the other hand, $\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}$ computes the log crystalline cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathcal{O}_C$. In fact, even the map $A_{\mathrm{inf}} \twoheadrightarrow A_{\mathrm{inf}}/\mu$ factors through $A_{\mathrm{inf}} \to A_{\mathrm{cris}}$, so the identification of the A_{cris}-base change of $A\Omega_{\mathfrak{X}}$ will capture the entire $\mu = 0$ locus of A_{inf} (in contrast, the comparison with the p-adic étale cohomology captured the $\mu \neq 0$ locus, see Theorem 2.3).

In comparison with the case when $\mathfrak{X}$ is smooth treated in [BMS18, § 12], controlling the interaction of the functor $L\eta_{(\mu)}$ with the relevant base changes seems more subtle. To overcome this, we resort to the analysis of continuous group cohomology carried out in § 3. Another major complication is the presence of log structures. Specifically, not knowing the existence of logarithmic divided power envelopes of certain nonexact logarithmic closed immersions in mixed characteristic, we are forced to devise slightly indirect arguments when analyzing the relevant divided power envelopes. For this, we rely on the results and arguments from [Kat89] and [Bei13b];[6] the latter reference is especially useful for us because some log structures that we use are not coherent (only quasi-coherent).

5.1 The ring A_{cris}

Using the generator ξ of the kernel of $\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C$, we let A^0_{cris} be the A_{inf}-subalgebra of $A_{\mathrm{inf}}[\frac{1}{p}]$ generated by the divided powers $\frac{\xi^n}{n!}$ for $n \geqslant 1$. The induced map $\theta\colon A^0_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C$ identifies A^0_{cris} with the divided power envelope of $\theta\colon A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_C/p$ over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ equipped with the unique divided powers on $p\mathbb{Z}_p$, see [Tsu99, A2.8]. Since $\theta(\mu) = 0$, we have $\mu^p \in pA^0_{\mathrm{cris}}$, so the p-adic topology of A^0_{cris} agrees with the (p,μ)-adic topology. We set

$$A_{\mathrm{cris}} := (A^0_{\mathrm{cris}})^{\wedge} \quad \text{where the completion is } p\text{-adic (equivalently, } (p,\mu)\text{-adic).}$$

The induced map $\theta\colon A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C$ identifies A_{cris} with the initial p-adically complete divided power thickening of $\mathcal{O}_C$ over $\mathbb{Z}_p$ (see [Tsu99, A1.3 and A1.5]). By Proposition 5.36 below (or by [Tsu99, A2.13] and [Bri06, 2.33]), the map $A^0_{\mathrm{cris}} \to A_{\mathrm{cris}}$ is an injection into an integral domain.

Analogously to § 3.26, the ring A_{cris} comes equipped with the Frobenius endomorphism φ that intertwines the absolute Frobenius endomorphism of $\mathcal{O}_C/p$ via the map θ. The identification

$$A_{\mathrm{cris}} \cong (\varinjlim_m A^{(m)}_{\mathrm{cris}})^{\wedge} \quad \text{which results from the evident} \quad A^0_{\mathrm{cris}} \cong \varinjlim_m A^{0,(m)}_{\mathrm{cris}}, \tag{5.1.1}$$

is Frobenius equivariant and compatible with the maps θ.

5.2 The log structure on A_{cris}

For each $n > 0$, the ring A_{cris}/p^n is a divided power thickening of $\mathcal{O}_C/p$ over $\mathbb{Z}/p^n$. Therefore, by [Bei13b, § 1.17, Lemma], every quasi-coherent, integral log structure $\mathcal{N}$ on $\mathcal{O}_C/p$ for which $\mathcal{N}/(\mathcal{O}_C/p)^{\times}$ is uniquely p-divisible lifts uniquely to a quasi-coherent, integral log structure on A_{cris}/p^n. Thus, letting $\mathcal{N}$ be the default log structure § 1.6 (1) on $\mathcal{O}_C/p$, for which

$$\mathcal{N}/(\mathcal{O}_C/p)^{\times} \cong \mathbb{Q}_{\geqslant 0},$$

[6] We are citing the post-publication arXiv version of the article, which slightly differs from the published version.

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we obtain compatible, quasi-coherent, integral log structures on the rings A_{cris}/p^n, to the effect that each A_{cris}/p^n becomes a log PD thickening of $\mathcal{O}_C/p$. Explicitly, these log structures are the pullbacks of the log structure on A_{cris} associated to the prelog structure

$$\mathcal{O}_C^{\flat} \setminus \{0\} \to A_{\mathrm{cris}}, \quad x \mapsto [x]. \tag{5.2.1}$$

In what follows, we always equip:

- each A_{cris}/p^n, as well as A_{cris}, with the log structure described above;
- each $\mathbb{Z}/p^n\mathbb{Z}$ with the standard divided powers on $p\mathbb{Z}/p^n\mathbb{Z}$ and the trivial log structure.

For every divided power thickening $\widetilde{Z}$ over $\mathbb{Z}/p^n\mathbb{Z}$ of an $\mathcal{O}_C/p$-scheme Z, the morphism $Z \xrightarrow{z} \mathrm{Spec}(\mathcal{O}_C/p)$ extends uniquely to a PD morphism $\widetilde{Z} \xrightarrow{\widetilde{z}} \mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ (see the proof of [Tsu99, A1.5]). If, in addition, $\widetilde{Z}$ is equipped with a quasi-coherent, integral log structure for which z is enhanced to a morphism $z^\sharp$ of log schemes, then, by [Bei13b, § 1.17, Exercise], the morphism $z^\sharp$ extends uniquely to a PD morphism $\widetilde{z}^\sharp\colon \widetilde{Z} \to \mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ of log schemes.

5.3 The absolute crystalline cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$

We let

$$(\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p)_{\log\,\mathrm{cris}}$$

be the log crystalline site of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathbb{Z}_p$ defined as in [Bei13b, § 1.12]: the objects are the étale $\mathfrak{X}_{\mathcal{O}_C/p}$-schemes Z equipped with a divided power thickening $\widetilde{Z}$ over some $\mathbb{Z}/p^n\mathbb{Z}$ such that $\widetilde{Z}$ is, in turn, equipped with a quasi-coherent, integral log structure whose pullback to Z is identified with the pullback of the log structure of $\mathfrak{X}_{\mathcal{O}_C/p}$ (which is defined in § 1.6 (2)); the coverings are the jointly surjective étale log PD morphisms. The universal property of A_{cris} reviewed in the last paragraph of § 5.2 gives the following identification of sites:

$$(\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p)_{\log\,\mathrm{cris}} \cong (\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log\,\mathrm{cris}},$$

where $(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log\,\mathrm{cris}}$ is the log crystalline site of $\mathfrak{X}_{\mathcal{O}_C/p}$ over A_{cris} defined analogously to the site $(\mathfrak{X}_{\mathcal{O}_C/p}/\mathbb{Z}_p)_{\log\,\mathrm{cris}}$ reviewed above (simply replace $\mathbb{Z}/p^n\mathbb{Z}$ by A_{cris}/p^n). The *absolute logarithmic crystalline cohomology* of $\mathfrak{X}_{\mathcal{O}_C/p}$ is the cohomology of the structure sheaf:

$$R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) := R\Gamma((\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log\,\mathrm{cris}}, \mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}).$$

We consider the morphism of topoi

$$u\colon (\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})_{\log\,\mathrm{cris}} \to (\mathfrak{X}_{\mathcal{O}_C/p})_{\text{ét}} \cong \mathfrak{X}_{\text{ét}}$$

that 'forgets the thickenings $\widetilde{Z}$' (see [Bei13b, § 1.5]), and we use it to obtain the identification

$$R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \cong R\Gamma(\mathfrak{X}_{\text{ét}}, Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})).$$

By the functoriality discussed in [Bei13b, § 1.5, Corollary], the absolute Frobenius of $\mathfrak{X}_{\mathcal{O}_C/p}$ (which is the multiplication by p on log structures) and the Frobenius of A_{cris} induce the A_{cris}-semilinear Frobenius endomorphisms of $Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})$ and $R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$.

The main goal of this section is the following identification of the A_{cris}-base change of $A\Omega_{\mathfrak{X}}$.

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Theorem 5.4. *There is a Frobenius-equivariant identification*

$$A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \cong Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}), \tag{5.4.1}$$

where the Frobenii result from those discussed in §§ 2.1, 2.2, 5.1, and 5.3 and, consistently with the notation (1.7.1), we have $A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} = R\lim_n (A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}/p^n)$.

We will first prove a version of Theorem 5.4 in the presence of fixed semistable coordinates. We will then complete the proof by using 'all possible coordinates' to globalize the argument. This overall strategy is similar to that used in [BMS18, § 12] in the smooth case.

5.5 The local setup

For the local argument, we assume until § 5.17 that $\mathfrak{X} = \mathrm{Spf}(R)$, that $\mathfrak{X}$ is connected, and that for some $0 \leqslant r \leqslant d$ and $q \in \mathbb{Q}_{>0}$ there is an étale $\mathcal{O}_C$-morphism

$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}) \quad \text{with} \quad R^{\square} = \mathcal{O}_C\{t_0, \dots, t_r, t_{r+1}^{\pm 1}, \dots, t_d^{\pm 1}\}/(t_0 \cdots t_r - p^q). \tag{5.5.1}$$

We use the rings $R^{\square}_{\infty}$ and R_{∞}, the group Δ, and its generators δ_i introduced in § 3.2, the rings $\mathbb{A}_{\mathrm{inf}}(R^{\square}_{\infty})$, $\mathbb{A}_{\mathrm{inf}}(R_{\infty})$, $A(R^{\square})$, and $A(R)$ and the modules $N^{\square}_{\infty}$ and N_{∞} introduced in § 3.14, the rings $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\infty})$ and $A^{(m)}_{\mathrm{cris}}(R)$ introduced in § 3.27, and the object $A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$ introduced in § 4.1.

Roughly speaking, with the coordinates above, we will access the right-hand side of (5.4.1) through the logarithmic de Rham complex of an explicit log smooth lift $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ over $\mathrm{Spf}(A_{\mathrm{cris}})$ of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathrm{Spec}(\mathcal{O}_C/p)$ (see Proposition 5.13). This complex may be made explicit by expressing its differentials in terms of the Δ-action on $A_{\mathrm{cris}}(R)$ (see Lemma 5.15). In contrast, results from § 3, namely, Theorem 3.20 and (3.25.1), make the left-hand side of (5.4.1) explicit. Once both sides are explicit, one identifies them and establishes (the presheaf version of) the local case of Theorem 5.4.

However, this relatively short local proof, whose detailed version in the good reduction case is given in [BMS18, 12.5], is ill-suited for globalization. This is because it appears difficult to extend the implicit exchange of the order of the functors $L\eta_{(\mu)}$ and $- \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ in this argument to general perfectoid covers that appear in the 'all possible coordinates' technique. For instance, one may attempt to use the almost purity theorem and Lemma 3.18 to reduce such commutativity to the 'base case' of R_{∞}, but this requires understanding the $W(\mathfrak{m}^{\flat})$-torsion in the groups

$$H^i_{\mathrm{cont}}(\Delta, (\mathbb{A}_{\mathrm{inf}}(R_{\infty}) \widehat{\otimes}_{A_{\mathrm{inf}}} A_{\mathrm{cris}})/\mu)$$

that seem difficult to access due to pathologies of the ring A_{cris}/μ.

Similarly to [BMS18, § 12.2], to overcome this difficulty we will use the rings $A^{(m)}_{\mathrm{cris}}$ reviewed in § 3.26 that retain better finite-type properties over A_{inf} than A_{cris}. In particular, we use the work of § 3 to commute the functors $L\eta_{(\mu)}$ and $- \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}$ in the following proposition.

Proposition 5.6. *In the local setting of § 5.5, for every $m \geqslant p$, we have*

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf}})) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}} \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{pro\acute{e}t}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{A}_{\mathrm{inf}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}). \tag{5.6.1}$$

Proof. The map (5.6.1) exists because its target is derived p-adically complete (see [BMS18, 6.19]). Moreover, by Theorems 3.20 and 3.34, it suffices to prove that

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}_{\mathrm{inf}}(R_{\infty}))) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}} \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\infty}))).$$

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By Propositions 3.25 and 3.32, the 'nonintegral' part N_∞ does not contribute, so it suffices to show:

$$L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, A(R))) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}} \xrightarrow{\sim} L\eta_{(\mu)}(R\Gamma_{\mathrm{cont}}(\Delta, A^{(m)}_{\mathrm{cris}}(R))). \tag{5.6.2}$$

In turn, (5.6.2) follows from the triviality of the Δ-action on $A(R)/\mu$ and $A^{(m)}_{\mathrm{cris}}(R)/\mu$ (see §§ 3.14 and 3.27): namely, due to Lemma 3.7 and this triviality, the left-hand (respectively, right-hand) side of (5.6.2) becomes

$$K_{A(R)}\big(\tfrac{\delta_1-1}{\mu}, \dots, \tfrac{\delta_d-1}{\mu}\big) \mathbin{\widehat{\otimes}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}} \quad \big(\text{respectively,} \quad K_{A^{(m)}_{\mathrm{cris}}(R)}\big(\tfrac{\delta_1-1}{\mu}, \dots, \tfrac{\delta_d-1}{\mu}\big)\big),$$

where the completed tensor product is nonderived (that is, termwise) because each p^n, $\mu^{n'}$ is an $A(R)$-regular sequence with $A(R)/(p^n, \mu^{n'})$ flat over $A_{\mathrm{inf}}/(p^n, \mu^{n'})$ (see Lemma 3.13); the two Koszul complexes may then be identified termwise (see § 3.27). □

Continuing to work in the local setting, we now express the (presheaf version of the) left-hand side of (5.4.1) in the form that will be convenient for the 'all possible coordinates' technique.

Corollary 5.7. *In the local setting of § 5.5, there is a Frobenius-equivariant identification*

$$R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}} \cong \Big(\varinjlim\nolimits_m (\eta_{(\mu)}(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)}(\delta_1 - 1, \dots, \delta_d - 1)))\Big)^{\widehat{}} \tag{5.7.1}$$

(see (4.1.1) for $A\Omega^{\mathrm{psh}}_{\mathfrak{X}}$) where, on the right-hand side, the direct limit and the p-adic completion are termwise.

Proof. The Δ-equivariant Frobenii of the rings $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ are compatible as m varies (see § 3.27), so, due to the divisibility $\mu \mid \varphi(\mu)$, they induce the Frobenius on the right-hand side of (5.7.1). Proposition 5.6 and Theorem 3.34 give the Frobenius-equivariant identification

$$R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}} \cong \eta_{(\mu)}(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)}(\delta_1 - 1, \dots, \delta_d - 1)),$$

so it remains to pass to the direct limit and to form the p-adic completion. □

We turn to the right-hand side of (5.4.1) and begin by constructing a log smooth lift $A_{\mathrm{cris}}(R)$ of R/p.

5.8 The ring $A_{\mathrm{cris}}(R)$

The 'relative version' of A_{cris} (respectively, a 'highly ramified cover' of this relative version) is the $A(R)$-algebra (respectively, $\mathbb{A}_{\mathrm{inf}}(R_\infty)$-algebra)

$$A_{\mathrm{cris}}(R) := A(R) \mathbin{\widehat{\otimes}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}} \quad \big(\text{respectively,} \quad \mathbb{A}_{\mathrm{cris}}(R_\infty) := \mathbb{A}_{\mathrm{inf}}(R_\infty) \mathbin{\widehat{\otimes}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}}\big),$$

where the completion is p-adic (equivalently, (p, μ)-adic, see § 5.1). Due to the decomposition (3.14.5), the subring $A_{\mathrm{cris}}(R) \subset \mathbb{A}_{\mathrm{cris}}(R_\infty)$ is an $A_{\mathrm{cris}}(R)$-module direct summand. The maps θ from §§ 3.14 and 5.1 induce compatible surjections

$$\theta\colon A_{\mathrm{cris}}(R) \twoheadrightarrow R \quad \text{and} \quad \theta\colon \mathbb{A}_{\mathrm{cris}}(R_\infty) \twoheadrightarrow R_\infty.$$

We let $\mathbb{A}^0_{\mathrm{cris}}(R_\infty)$ be the $\mathbb{A}_{\mathrm{inf}}(R_\infty)$-subalgebra of $\mathbb{A}_{\mathrm{inf}}(R_\infty)[\frac{1}{p}]$ generated by the elements $\xi^n/n!$ for $n \geqslant 1$. By [Tsu99, proof of A2.8], letting $\mathbb{A}_{\mathrm{inf}}(R_\infty)[\frac{T^n}{n!}]_{n\geqslant 1}$ denote the divided power polynomial algebra over $\mathbb{A}_{\mathrm{inf}}(R_\infty)$ in one variable, we have

$$\mathbb{A}^0_{\mathrm{cris}}(R_\infty) \cong \big(\mathbb{A}_{\mathrm{inf}}(R_\infty)\big[\tfrac{T^n}{n!}\big]_{n\geqslant 1}\big)/(T - \xi) \quad \text{so also} \quad \mathbb{A}^0_{\mathrm{cris}}(R_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R_\infty) \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}.$$

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Consequently, since ξ generates $\mathrm{Ker}(\theta) \subset \mathbb{A}_{\mathrm{inf}}(R_\infty)$, the ring $\mathbb{A}^0_{\mathrm{cris}}(R_\infty)$ is identified with the divided power envelope of $(\mathbb{A}_{\mathrm{inf}}(R_\infty), \mathrm{Ker}(\theta)+p\mathbb{A}_{\mathrm{inf}}(R_\infty))$ over $(\mathbb{Z}_p, p\mathbb{Z}_p)$. By the previous display,

$$\mathbb{A}_{\mathrm{cris}}(R_\infty) \cong (\mathbb{A}^0_{\mathrm{cris}}(R_\infty))^\frown.$$

By § 3.14, the ring $\mathbb{A}_{\mathrm{cris}}(R_\infty)$ (respectively, $A_{\mathrm{cris}}(R)$) is p-adically formally étale as an $\mathbb{A}_{\mathrm{cris}}(R^\square_\infty)$-algebra (respectively, $A_{\mathrm{cris}}(R^\square)$-algebra) and p-adically formally flat as an A_{cris}-algebra. In particular, $\mathbb{A}_{\mathrm{cris}}(R_\infty)$ inherits p-torsion freeness from A_{cris}. Moreover, even though we will not use this, $\mathbb{A}_{\mathrm{cris}}(R_\infty)$ is also μ-torsion free, as follows from Proposition 5.36 below (contrast this with an argument for (3.27.2)).

The rings $A_{\mathrm{cris}}(R)$ and $\mathbb{A}_{\mathrm{cris}}(R_\infty)$ come equipped with A_{cris}-semilinear Frobenius endomorphisms that are compatible with their counterparts for $A^{(m)}_{\mathrm{cris}}(R)$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_\infty)$ discussed in § 3.27. The profinite group Δ acts continuously, Frobenius-equivariantly, and A_{cris}-linearly on $A_{\mathrm{cris}}(R)$ and $\mathbb{A}_{\mathrm{cris}}(R_\infty)$. As in § 3.27, the induced Δ-action on $A_{\mathrm{cris}}(R)/\mu$ is trivial.

We will endow $A_{\mathrm{cris}}(R)$ with a log structure, which will, in fact, come from $A(R)$.

5.9 The log structure on $A(R)$

Provisionally, we consider the (fine) log structures on A_{inf} and $A(R)$ associated to the prelog structures

$$\mathbb{N}_{\geqslant 0} \xrightarrow{a \mapsto [(p^{1/p^\infty})^q]^a} A_{\mathrm{inf}} \quad \text{and} \quad \mathbb{N}^{r+1}_{\geqslant 0} \xrightarrow{(a_i) \mapsto \prod X_i^{a_i}} A(R).$$

Then, under the diagonal map $\mathbb{N}_{\geqslant 0} \to \mathbb{N}^{r+1}_{\geqslant 0}$, the ring $A(R)$ is a (p, μ)-adically formally log smooth A_{inf}-algebra (see (3.14.2) and [Kat89, 3.5–3.6]). To eliminate the dependence on q, we always, unless noted otherwise, equip A_{inf} with the log structure associated to the prelog structure

$$\mathcal{O}^\flat_C \setminus \{0\} \to A_{\mathrm{inf}}, \quad x \mapsto [x]. \tag{5.9.1}$$

Likewise, we always equip $A(R)$ with the log structure that is the base change of the fine log structure on $A(R)$ described above along the 'change of log structure' self-map of A_{inf} determined by $\mathbb{N}_{\geqslant 0} \xrightarrow{a \mapsto ((p^{1/p^\infty})^q)^a} \mathcal{O}^\flat_C \setminus \{0\}$. Explicitly, this log structure is associated to the prelog structure

$$\mathbb{N}^{r+1}_{\geqslant 0} \bigsqcup_{\mathbb{N}_{\geqslant 0}} (\mathcal{O}^\flat_C \setminus \{0\}) \to A(R) \tag{5.9.2}$$

that embeds $\mathbb{N}_{\geqslant 0}$ diagonally into $\mathbb{N}^{r+1}_{\geqslant 0}$, sends an $a \in \mathbb{N}_{\geqslant 0}$ to $((p^{1/p^\infty})^q)^a$, and sends the ith standard basis vector of $\mathbb{N}^{r+1}_{\geqslant 0}$ (respectively, an $x \in \mathcal{O}^\flat_C \setminus \{0\}$) to X_i (respectively, to $[x]$).

These latter 'default' log structures on A_{inf} and $A(R)$ are quasi-coherent and integral and, by base change, with them $A(R)$ is (p, μ)-adically formally log smooth over A_{inf}. In fact, via the map θ, the ring $A(R)$ over A_{inf} becomes a (p, μ)-adically formally log smooth thickening of R/p over $\mathcal{O}_C/p$ (where R/p is endowed with the log structure discussed in § 1.6).

The Frobenii of A_{inf} and $A(R)$ extend to the log structures by letting them act as multiplication by p on $\mathbb{N}^{r+1}_{\geqslant 0}$ and $\mathbb{N}_{\geqslant 0}$ and as the pth power map on $\mathcal{O}^\flat_C \setminus \{0\}$. Consequently, the Frobenius of the log A_{inf}-algebra $A(R)$ lifts the absolute Frobenius of the log $\mathcal{O}_C/p$-algebra R/p.

The Frobenius-equivariant Δ-action on the A_{inf}-algebra $A(R)$ (see § 3.14) extends to a Frobenius-equivariant Δ-action on the log A_{inf}-scheme $\mathrm{Spec}(A(R))$: indeed, a $\delta \in \Delta$ sends each X_i with $0 \leqslant i \leqslant r$ to $u_{\delta,i} \cdot X_i$ for some Teichmüller unit $u_{\delta,i} \in A(R)^\times$ (see § 3.14) and the prelog structures

$$\mathbb{N}^{r+1}_{\geqslant 0} \xrightarrow{(a_i) \mapsto \prod X_i^{a_i}} A(R) \quad \text{and} \quad \mathbb{N}^{r+1}_{\geqslant 0} \xrightarrow{(a_i) \mapsto \prod (u_{\delta,i} \cdot X_i)^{a_i}} A(R)$$

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determine the same log structure on $\mathrm{Spec}(A(R))$, namely, the one associated to the prelog structure

$$\mathbb{Z}^{r+1} \times \mathbb{N}^{r+1}_{\geqslant 0} \xrightarrow{((z_i),(a_i)) \mapsto \prod u_{\delta,i}^{z_i} \cdot \prod X_i^{a_i}} A(R).$$

5.10 The logarithmic de Rham complex

With a slight abuse of notation, we let

$$\Omega^{\bullet}_{A(R)/A_{\mathrm{inf}},\log}$$

be the (global section complex of the) logarithmic de Rham complex of $\mathrm{Spf}(A(R))$ over $\mathrm{Spf}(A_{\mathrm{inf}})$. More precisely, $\Omega^{\bullet}_{A(R)/A_{\mathrm{inf}},\log}$ is the (termwise) inverse limit over $n, n' > 0$ of the logarithmic de Rham complexes of $A(R)/(p^n, \mu^{n'})$ over $A_{\mathrm{inf}}/(p^n, \mu^{n'})$ (described, for instance, in [Ogu18, V.2.1.1]). Due to the formal log smoothness of $A(R)$ over A_{inf}, each $\Omega^{i}_{A(R)/A_{\mathrm{inf}},\log}$ is a free $A(R)$-module: indeed, the logarithmic differentials

$$d\log(X_1), \ldots, d\log(X_d)$$

form an $A(R)$-basis of $\Omega^{1}_{A(R)/A_{\mathrm{inf}},\log}$. We let

$$\tfrac{\partial}{\partial \log(X_i)} \colon A(R) \to A(R) \quad \text{for} \quad i = 1, \ldots, d \tag{5.10.1}$$

denote the dual basis of log A_{inf}-derivations (we do not notationally explicate the accompanying homomorphisms from the log structure to $A(R)$). These satisfy the following explicit formulas that are derived using the relation $d\log(X_0) + \cdots + d\log(X_r) = 0$:

$$\tfrac{\partial}{\partial \log(X_i)}(X_j) = \begin{cases} 0 & \text{if } 0 < j \neq i, \\ X_i & \text{if } j = i, \end{cases} \quad \text{and} \quad \tfrac{\partial}{\partial \log(X_i)}(X_0) = \begin{cases} -X_0 & \text{if } 0 < i \leqslant r, \\ 0 & \text{if } r < i. \end{cases} \tag{5.10.2}$$

The $\frac{\partial}{\partial \log(X_i)}$ also define an isomorphism $\Omega^{1}_{A(R)/A_{\mathrm{inf}},\log} \cong A(R)^{\oplus d}$, which extends to an isomorphism

$$\Omega^{\bullet}_{A(R)/A_{\mathrm{inf}},\log} \cong K_{A(R)}\big(\tfrac{\partial}{\partial \log(X_1)}, \ldots, \tfrac{\partial}{\partial \log(X_d)}\big) \tag{5.10.3}$$

that may be considered canonical because its construction only uses data determined by the local coordinate map (5.5.1). The Frobenius of the log A_{inf}-algebra $A(R)$ multiplies each $d\log(X_i)$ by p, so its effect on the right-hand side of (5.10.3) is given in each degree j by p^j times the Frobenius of $A(R)$.

5.11 The log structure on $A_{\mathrm{cris}}(R)$

Unless specified otherwise, we equip the $A(R)$-algebras $A_{\mathrm{cris}}(R)$ and $A_{\mathrm{cris}}(R)/p^n$ for $n > 0$ with the pullback of the log structure on $A(R)$ determined by (5.9.2). Thus, since the log structures (5.9.1) on A_{inf} and (5.2.1) on A_{cris} agree, $A_{\mathrm{cris}}(R)$ is p-adically formally log smooth over A_{cris}. Letting the completion be p-adic, we set

$$\Omega^{\bullet}_{A_{\mathrm{cris}}(R)/A_{\mathrm{cris}},\log} := \Omega^{\bullet}_{A(R)/A_{\mathrm{inf}},\log} \mathbin{\widehat{\otimes}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}},$$

which is the (global sections of the) logarithmic de Rham complex of $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ over $\mathrm{Spf}(A_{\mathrm{cris}})$.

We use the p-adic completeness of $A_{\mathrm{cris}}(R)$ and its p-adic formal flatness over A_{cris} (see § 5.8) to extend the divided power structure of A_{cris} to $A_{\mathrm{cris}}(R)$ (see [SP, 07H1]). In effect, $A_{\mathrm{cris}}(R)$ over

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A_{cris} becomes a p-adically formally log smooth log PD thickening of R/p over $\mathcal{O}_C/p$ (compare with § 5.9).

Through results of [Bei13b], the following lemma will be key for relating the right-hand side of (5.4.1) to the logarithmic de Rham cohomology of $\mathrm{Spf}(A_{\mathrm{cris}}(R))$ over $\mathrm{Spf}(A_{\mathrm{cris}})$ in Proposition 5.13.

Lemma 5.12. *For each $n \geqslant 1$, the log smooth log PD thickening $A_{\mathrm{cris}}(R)/p^n$ over A_{cris}/p^n of R/p over $\mathcal{O}_C/p$ is PD smooth in the sense of [Bei13b, § 1.4] (see the proof for the definition).*

Proof. The PD smoothness is the claim that for every log PD thickening $U \hookrightarrow \widetilde{U}$ over the log PD scheme A_{cris}/p^n such that U is affine and the log structure of $\widetilde{U}$ (and, hence, also of U) is integral and quasi-coherent, the indicated diagonal log PD morphism exists in every commutative diagram

$$\begin{array}{ccccc} U & \longrightarrow & \mathrm{Spec}(R/p) & \hookrightarrow & \mathrm{Spec}(A_{\mathrm{cris}}(R)/p^n) \\ \downarrow & \overset{\text{log PD}}{\nearrow} & & & \downarrow \text{log PD} \\ \widetilde{U} & & \xrightarrow[\text{log PD}]{} & & \mathrm{Spec}(A_{\mathrm{cris}}/p^n) \end{array}$$

of log schemes and log (or log PD where indicated) scheme morphisms over A_{cris}/p^n (see [Bei13a, § 1.4]).

This sought property of $A_{\mathrm{cris}}(R)/p^n$ is invariant under base change that changes the log structure on A_{cris}/p^n, so we may assume that A_{cris}/p^n and $A_{\mathrm{cris}}(R)/p^n$ are instead equipped with the pullbacks of the 'provisional' fine log structures defined in § 5.9. Moreover, since the PD structure of $A_{\mathrm{cris}}(R)/p^n$ is extended from A_{cris}/p^n, the log PD thickening

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(A_{\mathrm{cris}}(R)/p^n)$$

over A_{cris}/p^n is its own log PD-envelope over A_{cris}/p^n (in the sense of [Bei13b, § 1.3]). Thus, the log smoothness of $A_{\mathrm{cris}}(R)/p^n$ over A_{cris}/p^n and [Bei13b, § 1.4, Remarks (ii)] give the claimed PD smoothness. □

Proposition 5.13. *In the local setting of § 5.5, letting $\frac{\partial}{\partial \log(X_i)} \colon A_{\mathrm{cris}}^{(m)}(R) \to A_{\mathrm{cris}}^{(m)}(R)$ denote the $A_{\mathrm{cris}}^{(m)}$-derivations induced from (5.10.1) by base change, we have Frobenius-equivariant identifications*

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong \Omega^{\bullet}_{A_{\mathrm{cris}}(R)/A_{\mathrm{cris}},\log} \overset{(5.10.3)}{\cong} \left(\varinjlim_{m \geqslant p} \left(K_{A_{\mathrm{cris}}^{(m)}(R)}\left(\frac{\partial}{\partial \log(X_1)}, \ldots, \frac{\partial}{\partial \log(X_d)}\right)\right)\right)^{\widehat{}}$$

(the Frobenius action on the last term is analogous to that described after (5.10.3)).

Proof. By Lemma 5.12, each $A_{\mathrm{cris}}(R)/p^n$ over A_{cris}/p^n is a PD smooth thickening of R/p over $\mathcal{O}_C/p$, so [Bei13b, (1.8.1)] gives the Frobenius-equivariant identification[7]

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong R\Gamma_{\text{ét}}(\mathrm{Spf}(A_{\mathrm{cris}}(R)), \Omega^{\bullet}_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\,\mathrm{Spf}(A_{\mathrm{cris}}),\log}).$$

[7] Equation (1.8.1) of [Bei13a] uses the logarithmic PD de Rham complex, that is, the quotient of $\Omega^{\bullet}_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\,\mathrm{Spf}(A_{\mathrm{cris}}),\log}$ by the PD relations $d(u^{[m]}) = u^{[m-1]}du$, see [Bei13b, § 1.7]. In our situation, there is no difference: since the PD structure of $A_{\mathrm{cris}}(R)/p^n$ is extended from the base A_{cris}/p^n, the PD relations hold already in $\Omega^{\bullet}_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\,\mathrm{Spf}(A_{\mathrm{cris}}),\log}$.

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Since the sheaves $\Omega^i_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}$ are locally free and, in particular, quasi-coherent, they are acyclic for $\Gamma_{\text{ét}}(\mathrm{Spf}(A_{\mathrm{cris}}(R)),-)$ (see [FK18, I.1.1.23(2)]), so we have

$$R\Gamma_{\text{ét}}(\mathrm{Spf}(A_{\mathrm{cris}}(R)),\Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log})\cong\Gamma_{\text{ét}}(\mathrm{Spf}(A_{\mathrm{cris}}(R)),\Omega^\bullet_{\mathrm{Spf}(A_{\mathrm{cris}}(R))/\mathrm{Spf}(A_{\mathrm{cris}}),\log}).$$

It remains to observe that the latter complex is identified with $\Omega^\bullet_{A_{\mathrm{cris}}(R)/A_{\mathrm{cris}},\log}$. □

Having expressed the presheaf versions of both sides of (5.4.1) in the desired forms in Corollary 5.7 and Proposition 5.13, we now seek to exhibit an isomorphism between them in Proposition 5.16.

5.14 The element $\log([\epsilon])$

Fix an $m\geqslant p^2$. By the proof of [BMS18, 12.2],[8] each $\mu^n/(n+1)!\in A^{(m)}_{\mathrm{cris}}[\frac{1}{p}]$ with $n\geqslant 1$ lies in $A^{(m)}_{\mathrm{cris}}$, is p-adically topologically nilpotent there, and p-adically tends to 0 as $n\to\infty$. Consequently, recalling that $\mu=[\epsilon]-1$, we may define

$$\log([\epsilon]):=\mu-\tfrac{\mu^2}{2}+\tfrac{\mu^3}{3}-\ldots\quad\text{in}\quad A^{(m)}_{\mathrm{cris}},$$

so that the Frobenius maps $\log([\epsilon])$ to $p\cdot\log([\epsilon])$. By [BMS18, 12.2],[9] the elements $\log([\epsilon])$ and μ are unit multiples of each other in $A^{(m)}_{\mathrm{cris}}$, so $\frac{(\log([\epsilon]))^n}{\mu\cdot n!}$ lies in $A^{(m)}_{\mathrm{cris}}$, is topologically nilpotent if $n>1$, and p-adically tends to 0 in $A^{(m)}_{\mathrm{cris}}$ as $n\to\infty$.

The following lemma describes the Δ-action on $A^{(m)}_{\mathrm{cris}}(R)$ in terms of the derivations $\frac{\partial}{\partial\log(X_i)}$ induced on $A^{(m)}_{\mathrm{cris}}(R)$ by base change from the derivations (5.10.1).

LEMMA 5.15. *For $m\geqslant p^2$, a $\delta_i\in\Delta$ with $i=1,\ldots,d$ (see § 3.2) acts on $A^{(m)}_{\mathrm{cris}}(R)$ as the series*

$$\exp\big(\log([\epsilon])\cdot\tfrac{\partial}{\partial\log(X_i)}\big):=\textstyle\sum_{n\geqslant 0}\frac{(\log([\epsilon]))^n}{n!}\big(\frac{\partial}{\partial\log(X_i)}\big)^n.\tag{5.15.1}$$

In particular, for m and i as above, we have the following description of the 'q-derivative' $\frac{\delta_i-1}{\mu}$:

$$\tfrac{\delta_i-1}{\mu}=\tfrac{\partial}{\partial\log(X_i)}\cdot\big(\textstyle\sum_{n\geqslant 1}\frac{(\log([\epsilon]))^n}{\mu\cdot n!}\big(\frac{\partial}{\partial\log(X_i)}\big)^{n-1}\big)\quad\text{as maps}\quad A^{(m)}_{\mathrm{cris}}(R)\to A^{(m)}_{\mathrm{cris}}(R),\tag{5.15.2}$$

where the parenthetical factor defines an $A^{(m)}_{\mathrm{cris}}$-linear additive automorphism of $A^{(m)}_{\mathrm{cris}}(R)$.

Proof. The argument is similar to that of [BMS18, 12.4]. First, $\frac{(\log([\epsilon]))^n}{n!}$ tends to 0 in the p-adic topology of $A^{(m)}_{\mathrm{cris}}$ (see § 5.14), so the series (5.15.1) does define an $A^{(m)}_{\mathrm{cris}}$-linear additive endomorphism of $A^{(m)}_{\mathrm{cris}}(R)$. This endomorphism is also multiplicative because, by the Leibniz rule,

$$\tfrac{(\log([\epsilon]))^n}{n!}\big(\tfrac{\partial}{\partial\log(X_i)}\big)^n(ab)=\textstyle\sum_{j=0}^n\Big(\frac{(\log([\epsilon]))^j}{j!}\big(\frac{\partial}{\partial\log(X_i)}\big)^j(a)\cdot\frac{(\log([\epsilon]))^{n-j}}{(n-j)!}\big(\frac{\partial}{\partial\log(X_i)}\big)^{n-j}(b)\Big).$$

[8] The argument is as follows. Since $p,\mu\xi$ is an A_{inf}-regular sequence, $\mu^p-\mu\xi^p\in p\mu\xi A_{\mathrm{inf}}$, so $\mu^{p-1}/p=\xi^p/p+\xi a$ with $a\in A_{\mathrm{inf}}$. Thus, since $(p^2)!/p^p\in p\mathbb{Z}$, we have $(\mu^{p-1}/p)^p\in pA^{(m)}_{\mathrm{cris}}$, so μ^{p-1}/p is topologically nilpotent in $A^{(m)}_{\mathrm{cris}}$. In effect, since $\frac{1}{(n+1)!}p^{\lfloor n/(p-1)\rfloor}\in\mathbb{Z}_p$, the elements $\mu^n/(n+1)!$ tend to 0 in the p-adic topology of $A^{(m)}_{\mathrm{cris}}$ and are topologically nilpotent.

[9] The argument is as follows. By the previous footnote, $\sum_{n\geqslant p}\frac{(-1)^n\mu^n}{n+1}$ lies in $pA^{(m)}_{\mathrm{cris}}$. Thus, since each $\frac{\mu^n}{n+1}$ with $0<n<p$ is topologically nilpotent in $A^{(m)}_{\mathrm{cris}}$, so is $\sum_{n\geqslant 1}\frac{(-1)^n\mu^n}{n+1}$. In conclusion, $\frac{\log([\epsilon])}{\mu}$ is a unit in $A^{(m)}_{\mathrm{cris}}$.

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Therefore, in the case $R = R^{\square}$, the desired equality

$$\delta_i = \exp\bigl(\log([\epsilon]) \cdot \tfrac{\partial}{\partial \log(X_i)}\bigr) \quad \text{of endomorphisms} \quad A_{\mathrm{cris}}^{(m)}(R^{\square}) \to A_{\mathrm{cris}}^{(m)}(R^{\square}) \tag{5.15.3}$$

follows by noting that both of its sides agree on every X_j: indeed, due to the formulas (5.10.2), they send X_i to $[\epsilon]X_i$, fix each X_j with $0 < j \neq i$, and send X_0 to $[\epsilon^{-1}]X_0$ if $i \leqslant r$ and to X_0 if $r < i$.

In the general case, since μ, and hence also ξ, divides each $\frac{(\log([\epsilon]))^n}{n!}$ with $n \geqslant 1$ (see § 5.14), both sides of the equality (5.15.3) induce the trivial action modulo (p, ξ) (see § 3.27). Therefore, due to the formal étaleness of $A_{\mathrm{cris}}^{(m)}(R)$ over $A_{\mathrm{cris}}^{(m)}(R^{\square})$ and the settled $R = R^{\square}$ case, the sides agree.

Since $A_{\mathrm{cris}}^{(m)}(R)$ is μ-torsion free (see (3.27.2)) and $\mu \,|\, \frac{(\log([\epsilon]))^n}{n!}$ in $A_{\mathrm{cris}}^{(m)}$, the equality (5.15.2) follows from (5.15.3). Since $\frac{(\log([\epsilon]))^n}{\mu \cdot n!}$ is a unit for $n = 1$, is topologically nilpotent for $n > 1$ (see § 5.14), and p-adically tends to 0 as $n \to \infty$, the parenthetical factor of (5.15.2) is indeed an automorphism. □

We are ready to settle the (presheaf version of the) local case of Theorem 5.4.

Proposition 5.16. *In the local setting of § 5.5, for $m \geqslant p^2$ and $i = 1, \dots, d$, the morphism*

$$\Bigl(A_{\mathrm{cris}}^{(m)}(R) \xrightarrow{\frac{\partial}{\partial \log(X_i)}} A_{\mathrm{cris}}^{(m)}(R)\Bigr) \xrightarrow{\bigl(\mathrm{id},\, \sum_{n \geqslant 1} \frac{(\log([\epsilon]))^n}{n!} \bigl(\frac{\partial}{\partial \log(X_i)}\bigr)^{n-1}\bigr)} \Bigl(A_{\mathrm{cris}}^{(m)}(R) \xrightarrow{\delta_i - 1} A_{\mathrm{cris}}^{(m)}(R)\Bigr) \tag{5.16.1}$$

of complexes in degrees 0 and 1 is Frobenius equivariant, granted that the usual Frobenius action on the copy of $A_{\mathrm{cris}}^{(m)}(R)$ in degree 1 of the source is multiplied by p (compare with the description after (5.10.3)). For $m \geqslant p^2$, these morphisms induce a Frobenius-equivariant quasi-isomorphism

$$K_{A_{\mathrm{cris}}^{(m)}(R)}\Bigl(\tfrac{\partial}{\partial \log(X_1)}, \dots, \tfrac{\partial}{\partial \log(X_d)}\Bigr) \xrightarrow{\sim} \eta_{(\mu)}\Bigl(K_{\mathbb{A}_{\mathrm{cris}}^{(m)}(R_\infty)}(\delta_1 - 1, \dots, \delta_d - 1)\Bigr), \tag{5.16.2}$$

which, as m varies, induces the Frobenius-equivariant identification (a local version of (5.4.1)):

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong R\Gamma(\mathfrak{X}_{\text{ét}}^{\mathrm{psh}}, A\Omega_{\mathfrak{X}}^{\mathrm{psh}}) \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}. \tag{5.16.3}$$

Proof. The Frobenius-equivariance of (5.16.1) follows from the equations

$$\tfrac{\partial}{\partial \log(X_i)} \circ \varphi = p \cdot \bigl(\varphi \circ \tfrac{\partial}{\partial \log(X_i)}\bigr) \quad \text{and} \quad \varphi(\log([\epsilon])) = p \cdot \log([\epsilon])$$

(see §§ 5.10 and 5.14). Since Δ acts trivially on $A_{\mathrm{cris}}^{(m)}(R)/\mu$ (see § 3.27), the subcomplex

$$\eta_{(\mu)}\Bigl(K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)\Bigr) \subset K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)$$

is obtained by letting its jth term be the submodule of the jth term of

$$K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)$$

composed of the μ^j-multiples (see (1.7.2) and (1.7.3)); since $\mu \,|\, \varphi(\mu)$, this subcomplex is Frobenius-stable. Thus, Lemma 5.15 implies that the morphisms (5.16.1) induce an isomorphism

$$K_{A_{\mathrm{cris}}^{(m)}(R)}\Bigl(\tfrac{\partial}{\partial \log(X_1)}, \dots, \tfrac{\partial}{\partial \log(X_d)}\Bigr) \xrightarrow{\sim} \eta_{(\mu)}\Bigl(K_{A_{\mathrm{cris}}^{(m)}(R)}(\delta_1 - 1, \dots, \delta_d - 1)\Bigr). \tag{5.16.4}$$

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Proposition 3.32 (with Lemma 3.7) implies that the natural inclusion of the target of (5.16.4) into the target of (5.16.2) is a quasi-isomorphism, and (5.16.2) follows. The maps (5.16.2) are compatible as m varies, so, by passing to their limit over m, forming the termwise p-adic completions, and applying Corollary 5.7 and Proposition 5.13, we obtain the desired identification (5.16.3). □

Proposition 5.16 concludes the 'single coordinate patch' part of the proof of Theorem 5.4, so we turn to the 'all possible coordinates' technique that will globalize the argument. For this, the key steps are, for a small enough affine $\mathfrak{X}$, to build in § 5.21 a *functorial* in $\mathfrak{X}$ explicit complex that computes the presheaf version of the left-hand side of (5.4.1), to then build in § 5.32 such a complex for the right-hand side of (5.4.1), and, finally, to build in § 5.38 and Proposition 5.39 a natural quasi-isomorphism between these complexes. Each of these steps will use our work in the setting of § 5.5 discussed so far.

5.17 More general coordinates

Continuing to work locally, we now assume until the final part of the proof of Theorem 5.4 given in § 5.40 that $\mathfrak{X} = \mathrm{Spf}(R)$ is affine and nonempty, that every two irreducible components of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$ meet (so that $\mathfrak{X}$ is connected), and that we have:

- a finite set Σ that indexes the coordinates of the formal $\mathcal{O}_C$-torus
$$R^{\square}_{\Sigma} := \mathcal{O}_C\{t_\sigma^{\pm 1} \,|\, \sigma \in \Sigma\};$$
- a nonempty finite set Λ and, for each $\lambda \in \Lambda$, an $\mathcal{O}_C$-algebra
$$R^{\square}_{\lambda} := \mathcal{O}_C\{t_{\lambda,0}, \dots, t_{\lambda,r_\lambda}, t^{\pm 1}_{\lambda,r_\lambda+1}, \dots, t^{\pm 1}_{\lambda,d}\}/(t_{\lambda,0} \cdots t_{\lambda,r_\lambda} - p^{q_\lambda}) \quad \text{with} \quad q_\lambda \in \mathbb{Q}_{>0};$$
- a closed immersion
$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}_{\Sigma}) \times \textstyle\prod_{\lambda\in\Lambda} \mathrm{Spf}(R^{\square}_{\lambda}), \tag{5.17.1}$$
where the products are formed over $\mathrm{Spf}(\mathcal{O}_C)$, subject to the requirements that already
$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}_{\Sigma}) \quad \text{is a closed immersion} \tag{5.17.2}$$
and, for each $\lambda \in \Lambda$, the induced map
$$\mathfrak{X} = \mathrm{Spf}(R) \to \mathrm{Spf}(R^{\square}_{\lambda}) \quad \text{is étale.} \tag{5.17.3}$$

By (5.17.3), for each $\lambda \in \Lambda$, the irreducible components of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$ are *a priori* identified with the connected components of $\bigsqcup_i \mathrm{Spec}((R \otimes_{\mathcal{O}_C} k)/(t_{\lambda,i}))$. Thus, our assumption on $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$ implies that each irreducible component of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$ is cut out by a unique $t_{\lambda,i}$ with $0 \leqslant i \leqslant r_\lambda$.

By § 1.5, if $R \otimes_{\mathcal{O}_C} k$ is not k-smooth, then R determines q_λ, which therefore does not depend on λ. On the other hand, if $R \otimes_{\mathcal{O}_C} k$ is k-smooth, then q_λ may depend on λ. This, together with the possibility that $r_\lambda > 0$, complicates matters in the 'simpler' smooth case but is crucial to allow in order for the eventual 'all possible coordinates' constructions to be functorial in R.

For any $\mathfrak{X}$, the data above exist on a basis for $\mathfrak{X}_{\text{ét}}$: indeed, a coordinate map (5.17.3) exists étale locally on $\mathfrak{X}$ (see § 1.5), and then R is the p-adic completion of a finite-type $\mathcal{O}_C$-algebra, so the Zariski topology of $\mathrm{Spf}(R)$ has a basis whose elements embed into some (variable) $\widehat{\mathbb{G}}_m^{\Sigma}$.

Each (5.17.3) is an instance of the local setting of § 5.5, so the discussion between § 5.5 and the present section applies to it. Another instance is the identity map $\mathrm{Spf}(R^{\square}_{\Sigma}) \xrightarrow{=} \mathrm{Spf}(R^{\square}_{\Sigma})$ (with $r = 0$ and $d = \#\Sigma$), so the indicated discussion also applies to the ring $R^{\square}_{\Sigma}$ in place of $R^{\square}$.

Our first aim in this setup is to reexpress the (presheaf version of the) left-hand side of (5.4.1) in § 5.21.

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5.18 The perfectoid $R_{\Sigma,\Lambda,\infty}$
For each $\lambda \in \Lambda$, we set

$$\Delta_\lambda := \left\{ (\epsilon_0, \dots, \epsilon_d) \in \left(\varprojlim_{m \geqslant 0} (\mu_{p^m}(\mathcal{O}_C))\right)^{\oplus(d+1)} \;\middle|\; \epsilon_0 \cdots \epsilon_{r_\lambda} = 1 \right\} \simeq \mathbb{Z}_p^{\oplus d}$$

and let

$$\mathrm{Spa}\big(R_{\lambda,\infty}\big[\tfrac{1}{p}\big], R_{\lambda,\infty}\big) \to \mathrm{Spa}\big(R\big[\tfrac{1}{p}\big], R\big) \quad \text{and} \quad \mathrm{Spa}\big(R^\square_{\lambda,\infty}\big[\tfrac{1}{p}\big], R^\square_{\lambda,\infty}\big) \to \mathrm{Spa}\big(R^\square_\lambda\big[\tfrac{1}{p}\big], R^\square_\lambda\big)$$

be the affinoid perfectoid pro-(finite étale) Δ_λ-covers defined as in § 3.2 using the coordinate map $\mathrm{Spf}(R) \xrightarrow{(5.17.3)} \mathrm{Spf}(R^\square_\lambda)$. Similarly, we set

$$\Delta_\Sigma := \left(\varprojlim_{m \geqslant 0} (\mu_{p^m}(\mathcal{O}_C))\right)^{\Sigma} \simeq \mathbb{Z}_p^{\Sigma}$$

and let

$$\mathrm{Spa}\big(R^\square_{\Sigma,\infty}\big[\tfrac{1}{p}\big], R^\square_{\Sigma,\infty}\big) \to \mathrm{Spa}\big(R^\square_\Sigma\big[\tfrac{1}{p}\big], R^\square_\Sigma\big)$$

be the affinoid perfectoid pro-(finite étale) Δ_Σ-cover defined as in § 3.2 using the coordinate map $\mathrm{Spf}(R^\square_\Sigma) \xrightarrow{=} \mathrm{Spf}(R^\square_\Sigma)$, so that, explicitly,

$$R^\square_{\Sigma,\infty} \cong \left(\varinjlim_{m \geqslant 0} (\mathcal{O}_C\{t_\sigma^{\pm 1/p^m} \mid \sigma \in \Sigma\})\right)^\wedge.$$

By forming products over $\mathrm{Spa}(\mathcal{O}_C[\frac{1}{p}], \mathcal{O}_C)$ and setting

$$\Delta_{\Sigma,\Lambda} := \Delta_\Sigma \times \textstyle\prod_{\lambda \in \Lambda} \Delta_\lambda,$$

we obtain the affinoid perfectoid pro-(finite étale) $\Delta_{\Sigma,\Lambda}$-cover

$$\mathrm{Spa}\big(R^\square_{\Sigma,\infty}\big[\tfrac{1}{p}\big], R^\square_{\Sigma,\infty}\big) \times \textstyle\prod_{\lambda \in \Lambda} \mathrm{Spa}\big(R^\square_{\lambda,\infty}\big[\tfrac{1}{p}\big], R^\square_{\lambda,\infty}\big) \to \mathrm{Spa}\big(R^\square_\Sigma\big[\tfrac{1}{p}\big], R^\square_\Sigma\big) \times \textstyle\prod_{\lambda \in \Lambda} \mathrm{Spa}\big(R^\square_\lambda\big[\tfrac{1}{p}\big], R^\square_\lambda\big),$$

which we abbreviate as

$$\mathrm{Spa}\big(R^\square_{\Sigma,\Lambda,\infty}\big[\tfrac{1}{p}\big], R^\square_{\Sigma,\Lambda,\infty}\big) \to \mathrm{Spa}\big(R^\square_{\Sigma,\Lambda}\big[\tfrac{1}{p}\big], R^\square_{\Sigma,\Lambda}\big).$$

Its base change along the generic fiber of (5.17.1) is the pro-(finite étale) $\Delta_{\Sigma,\Lambda}$-cover

$$\mathrm{Spa}\big(R_{\Sigma,\Lambda,\infty}\big[\tfrac{1}{p}\big], R_{\Sigma,\Lambda,\infty}\big) \to \mathrm{Spa}\big(R\big[\tfrac{1}{p}\big], R\big), \tag{5.18.1}$$

which contains each $\mathrm{Spa}(R_{\lambda,\infty}[\frac{1}{p}], R_{\lambda,\infty}) \to \mathrm{Spa}(R[\frac{1}{p}], R)$ as a subcover. Thus, by the almost purity theorem [Sch12, 7.9(iii)], the $\mathcal{O}_C$-algebra $R_{\Sigma,\Lambda,\infty}$ defined by (5.18.1) is perfectoid (the notions of 'perfectoid' used here and in [Sch12] agree by [BMS18, 3.20]).

The topological generators for Δ_Σ and Δ_λ fixed in § 3.2 are

$$\delta_\sigma := (1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } \sigma \in \Sigma, \quad \text{where the } \sigma\text{th entry is nonidentity,}$$

and

$$\delta_{\lambda,i} := (\epsilon^{-1}, 1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = 1, \dots, r_\lambda, \quad \text{where the 0th and } i\text{th entries are nonidentity;}$$
$$\delta_{\lambda,i} := (1, \dots, 1, \epsilon, 1, \dots, 1) \quad \text{for } i = r_\lambda + 1, \dots, d, \quad \text{where the } i\text{th entry is nonidentity.}$$

Jointly, the δ_σ and $\delta_{\lambda,i}$ topologically freely generate $\Delta_{\Sigma,\Lambda}$.

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5.19 The rings $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$
Similarly to § 3.14, we set

$$\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) := W(R^{\flat}_{\Sigma,\Lambda,\infty}).$$

By Lemma 3.13, for $n, n' > 0$, the sequence $(p^n, \mu^{n'})$ is $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$-regular,

$$\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/(p^n, \mu^{n'})$$

is $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-flat, and $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/\mu$ is p-adically complete. As in § 3.14, we have the surjection

$$\theta\colon \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) \twoheadrightarrow R_{\Sigma,\Lambda,\infty} \tag{5.19.1}$$

that intertwines the Witt vector Frobenius of $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ with the absolute Frobenius of $R_{\Sigma,\Lambda,\infty}/p$ and whose kernel is generated by the regular element ξ. To fix further notation, we let

$$\begin{aligned} A(R^{\square}_{\Sigma}) &\cong A_{\mathrm{inf}}\{X_{\sigma}^{\pm 1} \mid \sigma \in \Sigma\}, \\ A(R^{\square}_{\lambda}) &\cong A_{\mathrm{inf}}\{X_{\lambda,0}, \dots, X_{\lambda,r_\lambda}, X^{\pm 1}_{\lambda,r_\lambda+1}, \dots, X^{\pm 1}_{\lambda,d}\}/(X_{\lambda,0}\cdots X_{\lambda,r_\lambda} - [(p^{1/p^\infty})^{q_\lambda}]) \end{aligned} \tag{5.19.2}$$

be the isomorphisms (3.14.2) for $R^{\square}_{\Sigma}$ and $R^{\square}_{\lambda}$. Similarly to § 3.27, for an $m \in \mathbb{Z}_{\geqslant 1}$, we set

$$\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) := \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A^{(m)}_{\mathrm{cris}}, \tag{5.19.3}$$

where the completion is (p,μ)-adic (equivalently, p-adic if $m \geqslant p$). Since $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is (p,μ)-adically formally flat over A_{inf}, the ring $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ inherits p-torsion freeness from $A^{(m)}_{\mathrm{cris}}$. By also using the short exact sequences (3.26.3) and the vanishing (3.26.2), we see that

$$\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \quad \text{is } \mu\text{-torsion free} \quad \text{and} \quad \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})/\mu \quad \text{is } p\text{-adically complete.}$$

As in § 3.27, the rings $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ come equipped with $A^{(m)}_{\mathrm{cris}}$-semilinear Frobenius endomorphisms that are compatible as m varies. The maps (3.26.1) and (5.19.1) give rise to the surjection

$$\theta\colon \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \twoheadrightarrow R_{\Sigma,\Lambda,\infty}. \tag{5.19.4}$$

The actions of the profinite group $\Delta_{\Sigma,\Lambda}$ on $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ are compatible, continuous, and Frobenius-equivariant.

The following consequence of Remark 3.35 will help us build a desired functorial complex in § 5.21.

PROPOSITION 5.20. *In the local setting of § 5.17, for every $m \geqslant p$, the analogue for $R_{\Sigma,\Lambda,\infty}$ of the edge map (3.15.1) induces the Frobenius-equivariant identification*

$$\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda, 1\leqslant i\leqslant d})\right) \xrightarrow{\sim} R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}}. \tag{5.20.1}$$

In particular, we have the following Frobenius-equivariant identification in the derived category:

$$\left(\varinjlim_m \left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda, 1\leqslant i\leqslant d})\right)\right)\right)^{\wedge} \xrightarrow{\sim} R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}},$$

where the direct limit and the p-adic completion of the complexes in the source are formed termwise.

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Proof. Proposition 5.6 gives the Frobenius-equivariant identification

$$R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}} \cong L\eta_{(\mu)}(R\Gamma_{\text{proét}}(\mathfrak{X}^{\mathrm{ad}}_{C}, \mathbb{A}_{\mathrm{inf}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A^{(m)}_{\mathrm{cris}}).$$

Therefore, since the pro-(finite étale) affinoid perfectoid $\Delta_{\Sigma,\Lambda}$-cover $\mathrm{Spa}(R_{\Sigma,\Lambda,\infty}[\frac{1}{p}], R_{\Sigma,\Lambda,\infty})$ of $\mathrm{Spa}(R[\frac{1}{p}], R)$ contains $\mathrm{Spa}(R_{\lambda,\infty}[\frac{1}{p}], R_{\lambda,\infty})$ as a subcover (see § 5.18), Remark 3.35 applies and (with Lemma 3.7) gives (5.20.1). The remaining assertion follows: each $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ is p-torsion free, so the termwise p-adic completion of the source there agrees with the derived p-adic completion. □

5.21 A functorial complex that computes $R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}}$

For a fixed R, we form the filtered direct limit over the closed immersions (5.17.1) for varying Σ and Λ to build the complex

$$\varinjlim_{\Sigma,\Lambda}\Big(\Big(\varinjlim_{m\geqslant p}\Big(\eta_{(\mu)}\Big(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\sigma-1)_{\sigma\in\Sigma},(\delta_{\lambda,i}-1)_{\lambda\in\Lambda,1\leqslant i\leqslant d})\Big)\Big)\Big)^{\widehat{}}\Big), \tag{5.21.1}$$

where the direct limits and the p-adic completion are termwise. By its construction, this complex comes equipped with an A_{cris}-semilinear Frobenius endomorphism. The isomorphisms of Proposition 5.20 are compatible with enlarging Σ and Λ, so they show that in the derived category the complex (5.21.1) is canonically and Frobenius-equivariantly identified with

$$R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}}.$$

Moreover, if R' is a p-adically formally étale R-algebra equipped with data as in § 5.17 for some sets Σ' and Λ', then the term indexed by Σ, Λ (and by the closed immersion (5.17.1)) of the direct limit (5.21.1) maps to the term indexed by $\Sigma\cup\Sigma'$, $\Lambda\cup\Lambda'$ (and by a closed immersion of $\mathrm{Spf}(R')$) of the analogous direct limit for R', compatibly with the transition maps in (5.21.1) and with the Frobenius. Thus, the complex (5.21.1) equipped with its Frobenius is functorial in R, and so is its identification with $R\Gamma(\mathfrak{X}^{\mathrm{psh}}_{\text{ét}}, A\Omega^{\mathrm{psh}}_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}}$.

Our next aim is to similarly reexpress the (presheaf version of the) right-hand side of (5.4.1) in § 5.32.

5.22 The completed log PD envelope $D_{\Sigma,\Lambda}$

By § 5.9, the maps $\theta\colon A(R^{\square}_{\lambda}) \twoheadrightarrow R^{\square}_{\lambda}$ of (3.14.3) are compatible with log structures. Thus, they give rise to a Frobenius-equivariant closed immersion

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(A(R^{\square}_{\Sigma})) \times \textstyle\prod_{\lambda\in\Lambda} \mathrm{Spf}(A(R^{\square}_{\lambda})) =: \mathrm{Spf}(A^{\square}_{\Sigma,\Lambda}) \tag{5.22.1}$$

of (p,μ)-adic formal log schemes, where the products are over the (p,μ)-adic formal log scheme $\mathrm{Spf}(A_{\mathrm{inf}})$. By [Kat89, 4.1, 4.4], for $n, n' > 0$, the quasi-coherent log structure of

$$\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda}/(p^n,\mu^{n'}))$$

and the log scheme map

$$\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda}/(p^n,\mu^{n'})) \to \mathrm{Spec}(A_{\mathrm{inf}}/(p^n,\mu^{n'}))$$

are integral.

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For $n, n' > 0$, by [Bei13b, 1.3, Theorem], the $A_{\mathrm{inf}}/(p^n, \mu^{n'})$-base change of the closed immersion (5.22.1) has a log PD envelope

$$\mathrm{Spec}(D_{\Sigma,\Lambda,n,n'}) \quad \text{over} \quad (\mathbb{Z}/p^n\mathbb{Z}, p\mathbb{Z}/p^n\mathbb{Z}),$$

which, in particular, is a nil thickening of $\mathrm{Spec}(R/p)$, so is affine as indicated (see [SP, 01ZT]). In fact, $D_{\Sigma,\Lambda,n,n'}$ is supplied already by [Kat89, 5.4] because the closed immersion (5.22.1) is the base change of a similar closed immersion of *fine* formal log schemes along a 'change of log structure' self-map of A_{inf} (see § 5.9).[10]

If n' is large enough relative to n, so that $\mu^{n'} \in p^n A_{\mathrm{cris}}$, then, by §§ 5.1–5.2, $\mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ is identified with the log PD envelope of the exact log closed immersion

$$\mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spec}(A_{\mathrm{inf}}/(p^n, \mu^{n'})) \quad \text{over} \quad (\mathbb{Z}/p^n\mathbb{Z}, p\mathbb{Z}/p^n\mathbb{Z}).$$

Thus, for such n, n', the envelope $\mathrm{Spec}(D_{\Sigma,\Lambda,n,n'})$ comes equipped with a canonical log PD morphism to $\mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ that identifies it with the log PD envelope of

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}/p^n) \quad \text{over} \quad \mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spec}(A_{\mathrm{cris}}/p^n).$$

Thus, letting $D_{\Sigma,\Lambda,n}$ be this log PD envelope, that is, the common $D_{\Sigma,\Lambda,n,n'}$ for large n', we have $D_{\Sigma,\Lambda,n}/p^{n-1} \cong D_{\Sigma,\Lambda,n-1}$ for $n > 1$ and obtain a p-adic formal log $\mathrm{Spf}(A_{\mathrm{cris}})$-scheme $\mathrm{Spf}(D_{\Sigma,\Lambda})$ that fits into a factorization

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(D_{\Sigma,\Lambda}) \to \mathrm{Spf}(A_{\mathrm{cris}}(R^{\square}_{\Sigma})) \times \textstyle\prod_{\lambda\in\Lambda} \mathrm{Spf}(A_{\mathrm{cris}}(R^{\square}_{\lambda})) =: \mathrm{Spf}(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}), \quad (5.22.2)$$

where the products are formed over the p-adic formal log scheme $\mathrm{Spf}(A_{\mathrm{cris}})$ and we have

$$A^{\square}_{\Sigma,\Lambda,\mathrm{cris}} \cong A^{\square}_{\Sigma,\Lambda} \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}.$$

By functoriality, $\mathrm{Spf}(D_{\Sigma,\Lambda})$ comes equipped with an A_{cris}-semilinear Frobenius. In addition, since, for each $n > 0$, the ideal defining the exact closed immersion $\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(R/p^n)$ inherits divided powers from $\mathbb{Z}/p^n$, the universal property of $D_{\Sigma,\Lambda}$ supplies the factorization

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(R) \hookrightarrow \mathrm{Spf}(D_{\Sigma,\Lambda}) \quad \text{over} \quad \mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spf}(\mathcal{O}_C) \hookrightarrow \mathrm{Spf}(A_{\mathrm{cris}}). \quad (5.22.3)$$

The profinite group $\Delta_{\Sigma,\Lambda}$ acts continuously and Frobenius-equivariantly on $A^{\square}_{\Sigma,\Lambda}$ over A_{inf} (see § 3.14) and, due to the last paragraph of § 5.9, this action extends to a $\Delta_{\Sigma,\Lambda}$-action on the (p,μ)-adic formal log scheme $\mathrm{Spf}(A^{\square}_{\Sigma,\Lambda})$. Thus, since the closed immersion (5.22.1) is $\Delta_{\Sigma,\Lambda}$-equivariant, $\Delta_{\Sigma,\Lambda}$ acts A_{cris}-linearly and Frobenius-equivariantly on each $D_{\Sigma,\Lambda,n}$ and also on $D_{\Sigma,\Lambda}$.

The main practical deficiency of $D_{\Sigma,\Lambda}$ is its inexplicit nature, for instance, we do not know whether $D_{\Sigma,\Lambda}$ is p-torsion free. In contrast, its utility for us manifests itself through the following proposition.

PROPOSITION 5.23. *In the local setting of § 5.17, the complex (where the inverse limit is termwise)*

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} := \varprojlim_{n>0} (\Omega^{\bullet}_{(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n)/(A_{\mathrm{cris}}/p^n),\log} \otimes_{A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n} D_{\Sigma,\Lambda,n})$$

[10] The two references characterize the log PD envelope differently, but they give the same $\mathrm{Spec}(D_{\Sigma,\Lambda,n,n'})$, in essence because the image of any monoid morphism $M \to M'$ with M finitely generated is finitely generated.

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is canonically and Frobenius-equivariantly identified in the derived category as follows:

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong \Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}}. \tag{5.23.1}$$

Under this identification, the map

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \to R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \quad \textit{is} \quad \Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \xrightarrow{(5.22.3)} \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}. \tag{5.23.2}$$

In particular, we have a Frobenius-equivariant identification

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong K_{D_{\Sigma,\Lambda}}\left(\left(\frac{\partial}{\partial\log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\frac{\partial}{\partial\log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\leqslant i\leqslant d}\right) \tag{5.23.3}$$

where the $\frac{\partial}{\partial\log(X_\sigma)}$ *(respectively,* $\frac{\partial}{\partial\log(X_{\lambda,i})}$*) are as in (5.10.1) with* $R^{\square}_{\Sigma}$ *(respectively,* $R^{\square}_{\lambda}$*) in place of* R *and the Frobenius acts in degree* j *on the right-hand side as* p^j *times the Frobenius of* $D_{\Sigma,\Lambda}$ *(compare with § 5.10).*

Proof. By § 5.11, each $A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n$ is a log smooth thickening of R/p over A_{cris}/p^n. Therefore, by [Bei13a, 1.4, Remarks (ii)] (and the second paragraph of § 5.22), the log PD thickening $D_{\Sigma,\Lambda,n}$ of R/p is PD smooth over A_{cris}/p^n (see the proof of Lemma 5.12). Thus, as in the proof of Proposition 5.13 above, [Bei13a, (1.8.1)] ensures that the logarithmic PD de Rham complex $\Omega^{\bullet}_{D_{\Sigma,\Lambda,n}/(A_{\mathrm{cris}}/p^n),\log,\mathrm{PD}}$ Frobenius-equivariantly computes $R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/(A_{\mathrm{cris}}/p^n)})$. By [Bei13a, 1.7, Exercises, (i)],

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda,n}/(A_{\mathrm{cris}}/p^n),\log,\mathrm{PD}} \cong \Omega^{\bullet}_{(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n)/(A_{\mathrm{cris}}/p^n),\log} \otimes_{A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n} D_{\Sigma,\Lambda,n}, \tag{5.23.4}$$

so (5.23.1) follows. Then, since each R/p^n is a log smooth log PD thickening of R/p over $\mathcal{O}_C/p^n$, analogous reasoning applies to

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/\mathcal{O}_C}) \overset{[\text{Bei13a, (1.8.1)}]}{\cong} R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$$

(compare with the proof of Proposition 5.13) and gives (5.23.2).

Finally, the identification (5.23.3) results from (5.23.1) and the Frobenius-equivariant identifications

$$\Omega^{\bullet}_{(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n)/(A_{\mathrm{cris}}/p^n),\log} \cong K_{A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n}\left(\left(\frac{\partial}{\partial\log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\frac{\partial}{\partial\log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\leqslant i\leqslant d}\right)$$

supplied by (5.10.3). □

Remark 5.24. By [Bei13a, (1.11.1)], the first map in (5.23.2) induces the identification

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} \mathcal{O}_C/p \cong R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes^{\mathbb{L}}_{\mathcal{O}_C} \mathcal{O}_C/p \tag{5.24.1}$$

in the derived category, so the same holds for the second map:

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} \mathcal{O}_C/p \cong \Omega^{\bullet}_{(R/p)/(\mathcal{O}_C/p),\log}.$$

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To make the identification (5.23.3) analogous to the identification in Proposition 5.20, we will express $D_{\Sigma,\Lambda}$ as a completed direct limit of rings $D^{(m)}_{\Sigma,\Lambda}$ that 'are generated by divided powers of degree at most m', see (5.30.1). For this, we will build on the ideas from the proof of [Kat89, 4.10(1)] to identify $D_{\Sigma,\Lambda}$ with the p-adic completion of the (nonlog) divided power envelope of an *exact* closed immersion in Lemma 5.29.[11] This will also make $D_{\Sigma,\Lambda}$ more explicit and easier to analyze.

5.25 A chart for $A^{\square}_{\Sigma,\Lambda}$

To express $D_{\Sigma,\Lambda}$ as the p-adic completion of a usual (nonlog) divided power envelope, in §§ 5.25–5.27 we build a chart for the (fine version) of the log closed immersion

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda}). \tag{5.25.1}$$

For this, we fix the unique $q \in \mathbb{Q}_{>0}$ for which

$$\mathbb{Z}\cdot q = \Sigma_{\lambda\in\Lambda}\,\mathbb{Z}\cdot q_\lambda \quad \text{inside} \quad \mathbb{Q},$$

so that $q_\lambda/q \in \mathbb{Z}_{>0}$ for every λ (and even $q_\lambda = q$ in the case when $R\otimes_{\mathcal{O}_C} k$ is not k-smooth, see § 5.17). We endow $\mathcal{O}_C/p$ and A_{inf} with the compatible via θ fine log structures determined by

$$\mathbb{N}_{\geqslant 0} \to \mathcal{O}_C/p \quad \text{with} \quad 1\mapsto p^q \quad \text{and} \quad \mathbb{N}_{\geqslant 0} \to A_{\mathrm{inf}} \quad \text{with} \quad 1\mapsto [(p^{1/p^\infty})^q].$$

For each $\lambda\in\Lambda$, we consider the submonoid

$$Q_\lambda \subset \tfrac{q}{q_\lambda}\textstyle\prod_{0\leqslant i\leqslant r_\lambda}\mathbb{N}_{\geqslant 0} \quad \text{generated by} \quad \prod_{0\leqslant i\leqslant r_\lambda}\mathbb{N}_{\geqslant 0} \quad \text{and the diagonal} \quad \big(\tfrac{q}{q_\lambda},\dots,\tfrac{q}{q_\lambda}\big),$$

so that the chart

$$Q_\lambda \to A(R^{\square}_\lambda) \quad \text{given by} \quad \textstyle\prod_{0\leqslant i\leqslant r_\lambda}\mathbb{N}_{\geqslant 0} \xrightarrow{(n_i)\mapsto \prod X_{\lambda,i}^{n_i}} A(R^{\square}_\lambda) \quad \text{and} \quad \big(\tfrac{q}{q_\lambda},\dots,\tfrac{q}{q_\lambda}\big)\mapsto [(p^{1/p^\infty})^q]$$

makes $\mathrm{Spec}(A(R^{\square}_\lambda))$ a fine log $\mathrm{Spec}(A_{\mathrm{inf}})$-scheme. We let

$$Q := \big(\textstyle\prod_{\lambda\in\Lambda} Q_\lambda\big)/\left(\big(\tfrac{q}{q_{\lambda_1}},\dots,\tfrac{q}{q_{\lambda_1}}\big) = \big(\tfrac{q}{q_{\lambda_2}},\dots,\tfrac{q}{q_{\lambda_2}}\big)\right)_{\lambda_1\neq\lambda_2}$$

be the quotient monoid obtained by identifying the diagonal elements $(q/q_\lambda,\dots,q/q_\lambda)$, so that the map

$$Q \to A^{\square}_{\Sigma,\Lambda} \quad \text{that results from the charts} \quad Q_\lambda \to A(R^{\square}_\lambda)$$

is a chart for the target $\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda})$ of a fine version of the log closed immersion (5.25.1). In terms of this chart, the Frobenius of $A^{\square}_{\Sigma,\Lambda}$ multiplies each element of Q by p (see § 5.9).

5.26 A convenient chart in the smooth case

We consider the case when $R\otimes_{\mathcal{O}_C} k$ is k-smooth, so that for each $\lambda\in\Lambda$ there is a unique $0\leqslant i_\lambda\leqslant r_\lambda$ with $t_{\lambda,i_\lambda}\notin R^\times$, and build the monoids

$$P_{\lambda_0} := \big(\mathbb{N}_{\geqslant 0}\times\textstyle\prod_{0\leqslant i\leqslant r_{\lambda_0},\, i\neq i_{\lambda_0}}\mathbb{Z}\big)\times\prod_{\lambda\neq\lambda_0}\big(\big(\prod_{0\leqslant i\leqslant r_\lambda}\mathbb{Z}\big)/\mathbb{Z}\big) \quad \text{for} \quad \lambda_0\in\Lambda, \tag{5.26.1}$$

[11] The arguments below would become more direct if we could 'uncomplete' $D_{\Sigma,\Lambda}$ by constructing the log PD envelope of the (possibly nonexact) log closed immersion $\mathrm{Spec}(R/p)\hookrightarrow\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda})$. Neither [Kat89, 5.4] nor [Bei13a, 1.3, Theorem] gives this hypothetical envelope because p is not nilpotent in $A^{\square}_{\Sigma,\Lambda}$.

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where each $\mathbb{Z}$ by which we quotient is embedded diagonally. For each (λ, i) with $0 \leqslant i \leqslant r_\lambda$,

$$t_{\lambda,i} = (p^q)^{n_{\lambda,i}} \cdot v_{\lambda,i} \quad \text{in} \quad R \quad \text{for unique} \quad n_{\lambda,i} \in \mathbb{Z}_{\geqslant 0} \quad \text{and} \quad v_{\lambda,i} \in R^\times; \tag{5.26.2}$$

explicitly, $n_{\lambda,i_\lambda} = \frac{q_\lambda}{q}$ and $n_{\lambda,i} = 0$ for $i \neq i_\lambda$. In particular, $\prod_{0\leqslant i\leqslant r_\lambda} v_{\lambda,i} = 1$ for each λ. The map

$$P_{\lambda_0} \to R/p \quad \text{given by} \quad \mathbb{N}_{\geqslant 0} \ni 1 \mapsto p^q, \quad \mathbb{Z}_{(\lambda,i)} \ni 1 \mapsto v_{\lambda,i},$$

where the subscript (λ, i) indicates the factor $\mathbb{Z}$ of (5.26.1) being considered, is a chart for the source $\mathrm{Spec}(R/p)$ of a fine version of the log closed immersion (5.25.1). In terms of this chart, the Frobenius of R/p multiplies each element of P_{λ_0} by p.

Due to (5.26.2), knowing the indices i_λ, we may evidently express the image of every generator of Q under $Q \to A^{\square}_{\Sigma,\Lambda} \twoheadrightarrow R/p$ in terms of the images of elements of P_{λ_0} without knowing the 'values' of these images. Thus, the log closed immersion (5.25.1) has a natural Frobenius-equivariant chart

$$Q \to P_{\lambda_0} = \big(\mathbb{N}_{\geqslant 0} \times \textstyle\prod_{0\leqslant i\leqslant r_{\lambda_0}, i\neq i_{\lambda_0}} \mathbb{Z}\big) \times \prod_{\lambda\neq\lambda_0} \big((\prod_{0\leqslant i\leqslant r_\lambda} \mathbb{Z})/\mathbb{Z}\big)$$

that, for instance, sends $1 \in (\mathbb{N}_{\geqslant 0})_{(\lambda_0, i_{\lambda_0})}$ to the element $(q_{\lambda_0}/q, -1, \ldots, -1)$ of

$$\mathbb{N}_{\geqslant 0} \times \textstyle\prod_{0\leqslant i\leqslant r_{\lambda_0}, i\neq i_{\lambda_0}} \mathbb{Z},$$

each $(q/q_\lambda, \ldots, q/q_\lambda)$ to $1 \in \mathbb{N}_{\geqslant 0}$, each $1 \in (\mathbb{N}_{\geqslant 0})_{(\lambda,i)}$ with $i \neq i_\lambda$ to $1 \in \mathbb{Z}_{(\lambda,i)}$, etc.

More precisely, the resulting $A^{\square}_{\Sigma,\Lambda}$-algebra

$$A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$$

comes equipped with an $A^{\square}_{\Sigma,\Lambda}$-semilinear Frobenius and is initial among the $A^{\square}_{\Sigma,\Lambda}$-algebras B equipped with a unit $V_{\lambda,i} \in B^\times$ for each (λ, i) with $0 \leqslant i \leqslant r_\lambda$ subject to the relations

$$X_{\lambda,i} = [((p^{1/p^\infty})^q)^{n_{\lambda,i}}] \cdot V_{\lambda,i}, \qquad \textstyle\prod_{0\leqslant i\leqslant r_\lambda} V_{\lambda,i} = 1. \tag{5.26.3}$$

In particular,

$$R \quad \text{is naturally an} \quad (A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])\text{-algebra} \quad (\text{with } V_{\lambda,i} = v_{\lambda,i}). \tag{5.26.4}$$

A fine version of the log closed immersion (5.25.1) factors Frobenius-equivariantly as follows:

$$\mathrm{Spec}(R/p) \xhookrightarrow{j_{\lambda_0}} \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) \xrightarrow{q_{\lambda_0}} \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda}), \tag{5.26.5}$$

where $\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$ is equipped with the log structure determined by P_{λ_0}. By construction, j_{λ_0} is an exact closed immersion and, by [Kat89, 3.5], the projection q_{λ_0} is log étale.

The relations (5.26.3) do not depend on the choice of λ_0, so neither does the factorization (5.26.5). More precisely, for another $\lambda_0' \in \Lambda$, we have a natural isomorphism over Q of charts for R/p:

$$P_{\lambda_0} \xrightarrow{\sim} P_{\lambda_0'}, \tag{5.26.6}$$

which gives rise to the vertical Frobenius-equivariant isomorphism in the following commutative diagram.

$$\begin{array}{ccccc} & \xhookrightarrow{j_{\lambda_0}} & \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) & \xrightarrow{q_{\lambda_0}} & \\ \mathrm{Spec}(R/p) & & \wr\uparrow & & \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda}). \\ & \xhookrightarrow{j_{\lambda_0'}} & \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0'}]) & \xrightarrow{q_{\lambda_0'}} & \end{array} \tag{5.26.7}$$

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5.27 A convenient chart in the nonsmooth case

We now consider the case when $R \otimes_{\mathcal{O}_C} k$ is not k-smooth, so that $q_\lambda = q$ and $Q_\lambda \cong \prod_{0 \leqslant i \leqslant r_\lambda} \mathbb{N}_{\geqslant 0}$ for every $\lambda \in \Lambda$. Letting $\Delta_\lambda \subset Q_\lambda$ be the diagonal copy of $\mathbb{N}_{\geqslant 0}$, we can then describe the chart Q for a fine version of $\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda})$ as follows:

$$Q \cong \left(\prod_{\lambda \in \Lambda} \left(\prod_{0 \leqslant i \leqslant r_\lambda} \mathbb{N}_{\geqslant 0}\right)\right) \Big/ (\Delta_{\lambda_1} = \Delta_{\lambda_2})_{\lambda_1 \neq \lambda_2}.$$

By § 5.17, each $t_{\lambda,i} \notin R^\times$ cuts out a unique irreducible component $\overline{\{y_{\lambda,i}\}}$ of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$. Its generic point $y_{\lambda,i}$ determines the ideal $(t_{\lambda,i}) \subset R$: indeed, $(p^q) \subset (t_{\lambda,i})$ in R and the ideal $(t_{\lambda,i})/(p^q) \subset R/(p^q)$ is the kernel of the localization map $R/(p^q) \to (R/(p^q))_{y_{\lambda,i}}$, as may be seen over $R^{\square}_\lambda$. Conversely, for each generic point y of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$ and $\lambda \in \Lambda$, a unique $t_{\lambda,i_\lambda(y)}$ with $0 \leqslant i_\lambda(y) \leqslant r_\lambda$ cuts out $\overline{\{y\}}$ (see § 5.17). Consequently, for each y and $\lambda, \lambda_0 \in \Lambda$,

$$t_{\lambda,i_\lambda(y)} = u_{\lambda,\lambda_0,y} \cdot t_{\lambda_0,i_{\lambda_0}(y)} \quad \text{in} \quad R \quad \text{for a unique} \quad u_{\lambda,\lambda_0,y} \in R^\times. \tag{5.27.1}$$

Letting $\mathcal{Y}$ denote the set of the generic points of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$, for $\lambda_0 \in \Lambda$ we build the monoid

$$P_{\lambda_0} := \left(\left(\prod_{\mathcal{Y}} \mathbb{N}_{\geqslant 0} \times \prod_{\{0 \leqslant i \leqslant r_{\lambda_0}\} \setminus i_{\lambda_0}(\mathcal{Y})} \mathbb{Z}\right) \times \prod_{\lambda \neq \lambda_0} \left(\prod_{0 \leqslant i \leqslant r_\lambda} \mathbb{Z}\right)\right) \Big/ (\Delta_\lambda = \Delta_{\lambda_0})_{\lambda \neq \lambda_0}, \tag{5.27.2}$$

where the quotient means that for every $\lambda \neq \lambda_0$ we are identifying every diagonal element of $\prod_{0 \leqslant i \leqslant r_\lambda} \mathbb{Z}$ with the corresponding diagonal element of $\prod_{\{0 \leqslant i \leqslant r_{\lambda_0}\} \setminus i_{\lambda_0}(\mathcal{Y})} \mathbb{Z}$ (interpreted to be 0 if the indexing set is empty). The assignment (as in § 5.26, subscripts indicate factors in (5.27.2))

$$(\mathbb{N}_{\geqslant 0})_y \ni 1 \mapsto t_{\lambda_0,i_{\lambda_0}(y)}, \quad \mathbb{Z}_{(\lambda,i)} \ni 1 \mapsto t_{\lambda,i} \quad \text{if} \quad i \notin i_\lambda(\mathcal{Y}), \quad \mathbb{Z}_{(\lambda,i)} \ni 1 \mapsto u_{\lambda,\lambda_0,y} \quad \text{if} \quad i = i_\lambda(y)$$

determines a chart

$$P_{\lambda_0} \to R/p$$

for the source $\mathrm{Spec}(R/p)$ of a fine version of the log closed immersion (5.25.1). In terms of this chart, the Frobenius of R/p multiplies each element of P_{λ_0} by p.

Due to the relation (5.27.1), the images in R/p of the generators of Q are evidently expressible in terms of the images of the elements of P_{λ_0} (without knowing the 'values' of these images), so, as in the smooth case, there is a natural Frobenius-equivariant chart for a fine version of (5.25.1):

$$Q \to P_{\lambda_0}$$

that, for instance, for $\lambda \neq \lambda_0$ and $y \in \mathcal{Y}$, sends $1 \in (\mathbb{N}_{\geqslant 0})_{(\lambda,i_\lambda(y))}$ to $(1,1) \in (\mathbb{N}_{\geqslant 0})_y \times \mathbb{Z}_{(\lambda,i_\lambda(y))}$.

The resulting $A^{\square}_{\Sigma,\Lambda}$-algebra

$$A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$$

comes equipped with an $A^{\square}_{\Sigma,\Lambda}$-semilinear Frobenius endomorphism and is initial among the $A^{\square}_{\Sigma,\Lambda}$-algebras B for which $X_{\lambda,i} \in B^\times$ when $i \notin i_\lambda(\mathcal{Y})$ and that are equipped with, for each $y \in \mathcal{Y}$ and $\lambda \in \Lambda$, a unit $U_{\lambda,\lambda_0,y} \in B^\times$ subject to the relations

$$\begin{aligned} X_{\lambda,i_\lambda(y)} &= U_{\lambda,\lambda_0,y} \cdot X_{\lambda_0,i_{\lambda_0}(y)}, \quad U_{\lambda_0,\lambda_0,y} = 1, \quad \text{and} \\ \prod_{y \in \mathcal{Y}} U_{\lambda,\lambda_0,y} &= \left(\prod_{\{0 \leqslant i \leqslant r_{\lambda_0}\} \setminus i_{\lambda_0}(\mathcal{Y})} X_{\lambda_0,i}\right) \Big/ \left(\prod_{\{0 \leqslant i \leqslant r_\lambda\} \setminus i_\lambda(\mathcal{Y})} X_{\lambda,i}\right) \quad \text{for} \quad \lambda \in \Lambda. \end{aligned} \tag{5.27.3}$$

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For a $\lambda_0' \in \Lambda$, we may set $U_{\lambda,\lambda_0',y} = U_{\lambda,\lambda_0,y} \cdot U_{\lambda_0',\lambda_0,y}^{-1}$ to express the $U_{\lambda,\lambda_0',y}$ in terms of the $U_{\lambda,\lambda_0,y}$, so, up to a canonical $A^{\square}_{\Sigma,\Lambda}$-isomorphism, $A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ does not depend on λ_0. Moreover,

$$R \quad \text{is naturally an} \quad (A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])\text{-algebra} \quad (\text{with } U_{\lambda,\lambda_0,y} = u_{\lambda,\lambda_0,y}). \tag{5.27.4}$$

As in the smooth case, we equip $\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$ with the log structure determined by P_{λ_0}, so a fine version of the log closed immersion (5.25.1) factors Frobenius-equivariantly as follows:

$$\mathrm{Spec}(R/p) \overset{j_{\lambda_0}}{\hookrightarrow} \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) \xrightarrow{q_{\lambda_0}} \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda}), \tag{5.27.5}$$

where j_{λ_0} is an exact closed immersion and, by [Kat89, 3.5], the projection q_{λ_0} is log étale. As in § 5.26, we have natural isomorphisms $P_{\lambda_0} \simeq P_{\lambda_0'}$ over Q and the compatibility diagram (5.26.7).

We now use the charts $Q \to P_{\lambda_0}$ to build a (nonlog) PD envelope whose p-adic completion is $D_{\Sigma,\Lambda}$.

5.28 The divided power envelope of j_{λ_0}

For $\lambda_0 \in \Lambda$, we let $D_{j_{\lambda_0}}$ be the divided power envelope over $(\mathbb{Z}_p, p\mathbb{Z}_p)$ of the closed immersion j_{λ_0} defined in (5.26.5) and (5.27.5). The universal property of A^0_{cris} (see § 5.1) identifies $D_{j_{\lambda_0}}$ with the divided power envelope of the closed immersion

$$j_{\lambda_0,\mathrm{cris}}\colon \mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}((A^{\square}_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) \quad \text{over} \quad \mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spec}(A^0_{\mathrm{cris}})$$

(compare with § 5.22). Since j_{λ_0} underlies an *exact* closed immersion of log schemes, we may, in addition, identify $D_{j_{\lambda_0}}$ endowed with the log structure determined by P_{λ_0} with the log PD envelope of j_{λ_0} over $\mathbb{Z}_p$, or of $j_{\lambda_0,\mathrm{cris}}$ over A^0_{cris} (compare with [Kat89, 5.5.1]). For $\lambda_0' \in \Lambda$, the vertical isomorphism in (5.26.7) induces an isomorphism

$$D_{j_{\lambda_0}} \cong D_{j_{\lambda_0'}}. \tag{5.28.1}$$

By functoriality, $D_{j_{\lambda_0}}$ comes equipped with an A^0_{cris}-semilinear Frobenius endomorphism, and the isomorphisms (5.28.1) are Frobenius equivariant. Due to (5.26.4) and (5.27.4), there is a map

$$D_{j_{\lambda_0}} \to R \quad \text{that lifts} \quad D_{j_{\lambda_0}} \twoheadrightarrow R/p; \tag{5.28.2}$$

its formation is compatible with the isomorphisms (5.28.1).

By the universal property of $A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ (see (5.26.3) and (5.27.3)), the continuous $\Delta_{\Sigma,\Lambda}$-action on $A^{\square}_{\Sigma,\Lambda}$ extends to a (p,μ)-adically continuous $\Delta_{\Sigma,\Lambda}$-action on $A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$, so it induces an A^0_{cris}-linear $\Delta_{\Sigma,\Lambda}$-action on $D_{j_{\lambda_0}}$. As an $(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$-algebra, $D_{j_{\lambda_0}}$ is generated by the divided powers of the elements of the ideal of j_{λ_0}, so this action is p-adically continuous.

Lemma 5.29. *For $\lambda_0 \in \Lambda$, the map q_{λ_0} induces Frobenius- and $\Delta_{\Sigma,\Lambda}$-equivariant isomorphisms*

$$D_{\Sigma,\Lambda,n} \cong D_{j_{\lambda_0}}/p^n \quad \text{for} \quad n > 0 \quad (\text{respectively}, \quad D_{\Sigma,\Lambda} \cong \widehat{D_{j_{\lambda_0}}}) \tag{5.29.1}$$

that are A_{cris}-linear and compatible with divided powers, maps to R/p^n (respectively, R; see (5.22.3) and (5.28.2)), and the isomorphisms (5.28.1). In particular, $D_{\Sigma,\Lambda}$ is p-adically complete,

$$D_{\Sigma,\Lambda}/p^n \xrightarrow{\sim} D_{\Sigma,\Lambda,n} \quad \text{for} \quad n > 0$$

and the $\Delta_{\Sigma,\Lambda}$-action on $D_{\Sigma,\Lambda}$ is p-adically continuous.

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Proof. We may identify $D_{\Sigma,\Lambda,n}$ with the log PD envelope of

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n) \quad \text{over} \quad \mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spec}(A_{\mathrm{cris}}/p^n)$$

defined using fine log structures (see § 5.22). On the other hand, we may identify $D_{j_{\lambda_0}}/p^n$ with the (log) divided power envelope of $j_{\lambda_0,\mathrm{cris}} \otimes_{A^0_{\mathrm{cris}}} A^0_{\mathrm{cris}}/p^n$, that is, of

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spec}((A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]), \quad \text{over} \quad \mathrm{Spec}(\mathcal{O}_C/p) \hookrightarrow \mathrm{Spec}(A_{\mathrm{cris}}/p^n)$$

(see § 5.28 and [SP, 07HB]). Consider a commutative square

$$\begin{array}{ccc} T_0 & \longrightarrow & \mathrm{Spec}((A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]) \\ \downarrow & \overset{?}{\nearrow} & \downarrow{\scriptstyle q_{\lambda_0} \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}/p^n} \\ T & \longrightarrow & \mathrm{Spec}(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n) \end{array} \tag{5.29.2}$$

of log schemes over A_{cris}/p^n in which $T_0 \hookrightarrow T$ is a log PD thickening such that the log structure $\mathcal{N}_T$ of T (and, hence, also $\mathcal{N}_{T_0}$ of T_0) is integral and quasi-coherent and the log structures of $\mathrm{Spec}((A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$ and $\mathrm{Spec}(A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}/p^n)$ are determined by the charts P_{λ_0} and Q, respectively (see §§ 5.25–5.27). By [Bei13a, 1.1 Exercises (iii)], for any $t, t' \in \Gamma(T, \mathcal{N}_T)$ and $u_0 \in \mathcal{O}^{\times}_{T_0}$ with $t|_{T_0} = u_0 \cdot t'|_{T_0}$, there exists a unique lift $u \in \mathcal{O}^{\times}_T$ of u_0 such that $t = ut'$. Thus, by the construction of P_{λ_0} and the universal property described by (5.26.3) and (5.27.3), there is a unique log morphism indicated by the dashed arrow in (5.29.2) that makes the diagram commute. Consequently, q_{λ_0} induces an isomorphism between the log PD envelopes:

$$D_{j_{\lambda_0}}/p^n \cong D_{\Sigma,\Lambda,n}, \quad \text{and, by letting } n \text{ vary, also} \quad \widehat{D_{j_{\lambda_0}}} \cong D_{\Sigma,\Lambda}.$$

Functoriality implies the claimed compatibilities, and (5.29.1) implies the 'in particular' assertion. □

We now use Lemma 5.29 to define the rings $D^{(m)}_{\Sigma,\Lambda}$ that are analogous to the rings $A^{(m)}_{\mathrm{cris}}(R)$ of § 3.27.

5.30 The rings $D^{(m)}_{\Sigma,\Lambda}$

For $\lambda_0 \in \Lambda$, the divided powers of the elements of the ideal of j_{λ_0} generate $D_{j_{\lambda_0}}$ as an algebra over $A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$. In turn, for a fixed $m \in \mathbb{Z}_{\geqslant 1}$, the divided powers of degree at most m generate a Frobenius-stable $(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$-subalgebra

$$D^{(m)}_{j_{\lambda_0}} \subset D_{j_{\lambda_0}}, \qquad \text{and} \qquad D_{j_{\lambda_0}} = \bigcup_{m \geqslant 1} D^{(m)}_{j_{\lambda_0}}.$$

Since $D_{j_{\lambda_0}}$ is naturally and Frobenius-semilinearly an A^0_{cris}-algebra (see § 5.28), $D^{(m)}_{j_{\lambda_0}}$ is naturally and Frobenius-semilinearly an algebra over the subring $A^{0,(m)}_{\mathrm{cris}} \subset A^0_{\mathrm{cris}}$ defined in § 3.26. By Lemma 5.29,

$$\text{the image } D^0_{\Sigma,\Lambda} \text{ of } D_{j_{\lambda_0}} \text{ in } D_{\Sigma,\Lambda} \overset{(5.29.1)}{\cong} \widehat{D_{j_{\lambda_0}}} \text{ is Frobenius-stable and independent of } \lambda_0,$$

and the same holds for the image $D^{0,(m)}_{\Sigma,\Lambda} \subset D^0_{\Sigma,\Lambda}$ of $D^{(m)}_{j_{\lambda_0}}$ in $D_{\Sigma,\Lambda}$. For $m \geqslant p$, the p-adic completion

$$D^{(m)}_{\Sigma,\Lambda} := (D^{0,(m)}_{\Sigma,\Lambda})^{\wedge} \quad \text{is naturally an algebra over} \quad A^{(m)}_{\mathrm{cris}}$$

and comes equipped with an $A^{(m)}_{\mathrm{cris}}$-semilinear Frobenius.

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By Lemma 5.29, the composition $D_{j_{\lambda_0}} \twoheadrightarrow D^0_{\Sigma,\Lambda} \hookrightarrow D_{\Sigma,\Lambda}$ induces an isomorphism modulo p^n and, hence, so do both maps that comprise it. Thus, since $D^0_{\Sigma,\Lambda} = \bigcup_{m \geqslant p} D^{0,(m)}_{\Sigma,\Lambda}$, we have

$$D_{\Sigma,\Lambda} \cong (D^0_{\Sigma,\Lambda})^\wedge \cong \left(\varinjlim\nolimits_{m \geqslant p} D^{(m)}_{\Sigma,\Lambda}\right)^\wedge \quad \text{over} \quad A_{\mathrm{cris}} \tag{5.30.1}$$

compatibly with the Frobenii. In what follows, $D^0_{\Sigma,\Lambda}$ plays the role of the ring that underlies the hypothetical log PD envelope of the log closed immersion (5.25.1) (see footnote 11).

By Lemma 5.29, the $\Delta_{\Sigma,\Lambda}$-action on $D_{\Sigma,\Lambda}$ respects the subrings $D^{0,(m)}_{\Sigma,\Lambda} \subset D_{\Sigma,\Lambda}$. The induced $A^{(m)}_{\mathrm{cris}}$-linear $\Delta_{\Sigma,\Lambda}$-action on $D^{(m)}_{\Sigma,\Lambda}$ is p-adically continuous and compatible as m varies. The identifications in (5.30.1) are $\Delta_{\Sigma,\Lambda}$-equivariant.

5.31 The derivations $\frac{\partial}{\partial \log(X_\tau)}$

For brevity, let τ denote either the index 'σ' for some $\sigma \in \Sigma$ or the index 'λ, i' for some $\lambda \in \Lambda$ and $i = 1, \dots, d$. The log derivations $\frac{\partial}{\partial \log(X_\tau)}$ defined in (5.10.1) with $R^\square_\Sigma$ or $R^\square_\lambda$ in place of R give rise to the log A_{inf}-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon A^\square_{\Sigma,\Lambda} \to A^\square_{\Sigma,\Lambda} \tag{5.31.1}$$

(as in § 5.10, we do not explicate the accompanying homomorphisms from the log structure). These, in turn, induce the divided power A_{cris}-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon D_{\Sigma,\Lambda} \to D_{\Sigma,\Lambda} \tag{5.31.2}$$

(compare with Proposition 5.23 and its proof, especially (5.23.4)), where a *divided power A_{cris}-derivation* ∂ is required to satisfy $\partial(x^{[m]}) = x^{[m-1]}\partial(x)$ for divided powers $x^{[m]}$ with $m \geqslant 1$, in addition to the A_{cris}-linearity and the Leibniz rule.

Since q_{λ_0} is log étale (see §§ 5.26–5.27), the derivations (5.31.1) uniquely extend to log A_{inf}-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon A^\square_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \to A^\square_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \quad \text{for every} \quad \lambda_0 \in \Lambda. \tag{5.31.3}$$

These, in turn, induce divided power A^0_{cris}-derivations (see [SP, 07HW])

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon D_{j_{\lambda_0}} \to D_{j_{\lambda_0}}. \tag{5.31.4}$$

By construction, the derivations (5.31.2) and (5.31.4) are compatible, so they induce $A^{(m)}_{\mathrm{cris}}$-derivations

$$\tfrac{\partial}{\partial \log(X_\tau)} \colon D^{(m)}_{\Sigma,\Lambda} \to D^{(m)}_{\Sigma,\Lambda} \quad \text{for} \quad m \geqslant p$$

that are compatible as m varies and recover (5.31.2) under the identification

$$D_{\Sigma,\Lambda} \cong (\varinjlim D^{(m)}_{\Sigma,\Lambda})^\wedge.$$

Consequently, we may reexpress the identification (5.23.3) as the Frobenius-equivariant identification

$$R\Gamma_{\log \mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \cong \left(\varinjlim\nolimits_{m \geqslant p} \left(K_{D^{(m)}_{\Sigma,\Lambda}}\left(\left(\tfrac{\partial}{\partial \log(X_\tau)}\right)_\tau\right)\right)\right)^\wedge, \tag{5.31.5}$$

where in degree j of $K_{D^{(m)}_{\Sigma,\Lambda}}((\frac{\partial}{\partial \log(X_\tau)})_\tau)$ the Frobenius acts as p^j times the Frobenius of $D^{(m)}_{\Sigma,\Lambda}$.

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5.32 A functorial complex that computes $R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$

For a fixed R, the formation of the rings $D_{\Sigma,\Lambda}$, $D_{j_{\lambda_0}}$, $D^0_{\Sigma,\Lambda}$, and $D^{(m)}_{\Sigma,\Lambda}$ and the morphisms j_{λ_0} and q_{λ_0} is compatible with enlarging Σ and Λ, and the same holds for the identification (5.31.5). Thus, we may form the filtered direct limit over all the closed immersions (5.17.1) for varying Σ and Λ to build the complex

$$\varinjlim_{\Sigma,\Lambda}\Big(\Big(\varinjlim_{m\geqslant p}\Big(K_{D^{(m)}_{\Sigma,\Lambda}}\Big(\Big(\frac{\partial}{\partial\log(X_\sigma)}\Big)_{\sigma\in\Sigma},\Big(\frac{\partial}{\partial\log(X_{\lambda,i})}\Big)_{\lambda\in\Lambda,\,1\leqslant i\leqslant d}\Big)\Big)\Big)^{\widehat{}}\Big),\tag{5.32.1}$$

where the direct limits and the p-adic completion are termwise. By construction, this complex comes equipped with an A_{cris}-semilinear Frobenius endomorphism (see the end of § 5.31) and, by (5.31.5), in the derived category it is canonically and Frobenius-equivariantly identified with

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}).$$

Moreover, if R' is a p-adically formally étale R-algebra equipped with data as in § 5.17 for some sets Σ' and Λ', then the term indexed by Σ, Λ (and by the closed immersion (5.17.1)) of the direct limit (5.32.1) maps to[12] the term indexed by $\Sigma\cup\Sigma'$, $\Lambda\cup\Lambda'$ (and by a closed immersion of $\mathrm{Spf}(R')$) of the analogous direct limit for R', compatibly with the transition maps in (5.32.1) and with the Frobenius. Thus, the complex (5.32.1) equipped with its Frobenius is functorial in R, and so is its identification with $R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})$.

Since the formation of the maps (5.23.2) is compatible with enlarging Σ and Λ, and then also with replacing R by R', the map $R\Gamma_{\log\mathrm{cris}}(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}})\to R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is identified with a map

$$\varinjlim_{\Sigma,\Lambda}\Big(\Big(\varinjlim_{m\geqslant p}\Big(K_{D^{(m)}_{\Sigma,\Lambda}}\Big(\Big(\frac{\partial}{\partial\log(X_\sigma)}\Big)_{\sigma\in\Sigma},\Big(\frac{\partial}{\partial\log(X_{\lambda,i})}\Big)_{\lambda\in\Lambda,\,1\leqslant i\leqslant d}\Big)\Big)\Big)^{\widehat{}}\Big)\to\Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}$$

whose formation is compatible with replacing R by R'.

Having constructed the functorial complexes (5.21.1) and (5.32.1), we seek to exhibit a natural map between them and to prove that it is an isomorphism. These tasks, which will be completed in § 5.38 and Proposition 5.39, are the last stepping stones to the proof of Theorem 5.4 given in § 5.40. We begin with the following variants of Lemma 5.15 and Proposition 5.16.

LEMMA 5.33. *For $m\geqslant p^2$, the element $\delta_\tau\in\Delta_{\Sigma,\Lambda}$, where the index τ is either 'σ' for some $\sigma\in\Sigma$ or 'λ,i' for some $\lambda\in\Lambda$ and $i=1,\dots,d$ (see § 5.18), acts on $D^{(m)}_{\Sigma,\Lambda}$ as the endomorphism*

$$\textstyle\sum_{n\geqslant 0}\frac{(\log([\epsilon]))^n}{n!}\big(\frac{\partial}{\partial\log(X_\tau)}\big)^n,\tag{5.33.1}$$

where $\frac{(\log([\epsilon]))^n}{n!}$ lies in $A^{(m)}_{\mathrm{cris}}$ and p-adically tends to 0 (see § 5.14).

Proof. Analogously to the proof of Lemma 5.15, the series (5.33.1) *a priori* defines an A_{cris}-algebra endomorphism of $D_{\Sigma,\Lambda}$. Moreover, by Lemma 5.15, the action of δ_τ on the ring $A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}$ defined in (5.22.2) is given by (5.33.1). Thus, due to the universal properties (5.26.3) and (5.27.3), the same holds for the action of δ_τ on

$$A^{\square}_{\Sigma,\Lambda,\mathrm{cris}}\mathbin{\widehat{\otimes}}_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}]\cong((A^{\square}_{\Sigma,\Lambda}\otimes_{A_{\mathrm{inf}}}A^0_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}])^{\widehat{}}$$

[12] One uses the universal properties described in (5.26.3) and (5.27.3) and keeps in mind the case when $R\otimes_{\mathcal{O}_C}k$ is not k-smooth but $R'\otimes_{\mathcal{O}_C}k$ is.

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(see (5.31.3)), where the completion is p-adic. Then, by the universal property of $D_{j_{\lambda_0}}$ (see § 5.28) and (5.29.1), the element δ_τ acts on $D_{\Sigma,\Lambda}$, and hence also on $D^{0,(m)}_{\Sigma,\Lambda}$ and $D^{(m)}_{\Sigma,\Lambda}$, by the series (5.33.1). □

Proposition 5.34. *In the local setting of § 5.17, for $m \geqslant p^2$, the additive morphisms*

$$\Big(D^{(m)}_{\Sigma,\Lambda} \xrightarrow{\frac{\partial}{\partial \log(X_\tau)}} D^{(m)}_{\Sigma,\Lambda}\Big) \xrightarrow{\Big(\mathrm{id},\, \sum_{n\geqslant 1} \frac{(\log([\epsilon]))^n}{n!}\Big(\frac{\partial}{\partial \log(X_\tau)}\Big)^{n-1}\Big)} \Big(D^{(m)}_{\Sigma,\Lambda} \xrightarrow{\delta_\tau - 1} D^{(m)}_{\Sigma,\Lambda}\Big) \tag{5.34.1}$$

of complexes in degree 0 and 1, where the index τ ranges over 'σ' for $\sigma \in \Sigma$ and 'λ, i' for $\lambda \in \Lambda$ and $i = 1, \ldots, d$, define a Frobenius-equivariant morphism (whose target is defined as in (1.7.3))

$$K_{D^{(m)}_{\Sigma,\Lambda}}\Big(\big(\tfrac{\partial}{\partial \log(X_\sigma)}\big)_{\sigma\in\Sigma}, \big(\tfrac{\partial}{\partial \log(X_{\lambda,i})}\big)_{\lambda\in\Lambda,\, 1\leqslant i\leqslant d}\Big) \to \eta_{(\mu)}\big(K_{D^{(m)}_{\Sigma,\Lambda}}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda,\, 1\leqslant i\leqslant d})\big),$$

where in each degree j of the source the Frobenius acts as p^j times the Frobenius of $D^{(m)}_{\Sigma,\Lambda}$.

Proof. By Lemma 5.33, the morphism (5.34.1) is well defined. Moreover, the image of its degree 1 component lies in $\mu \cdot D^{(m)}_{\Sigma,\Lambda}$ because, by § 5.14, the element $\frac{(\log([\epsilon]))^n}{\mu \cdot n!}$ lies in $A^{(m)}_{\mathrm{cris}}$ and p-adically tends to 0. The rest of the claim then follows from the definitions (1.7.2) and (1.7.3), granted that one argues the Frobenius-equivariance as in the proof of Proposition 5.16. □

Proposition 5.34 reduces the task of exhibiting a natural map from the complex (5.32.1) to the complex (5.21.1) to exhibiting a natural $\Delta_{\Sigma,\Lambda}$-equivariant ring morphism

$$D^{(m)}_{\Sigma,\Lambda} \to \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}).$$

To build the latter, we will realize $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ inside the following ring $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$.

5.35 The ring $\mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$
For an affinoid perfectoid $\mathrm{Spa}(R'_\infty[\frac{1}{p}], R'_\infty)$ over $\mathrm{Spa}(C, \mathcal{O}_C)$ (such as that with $R'_\infty = R_{\Sigma,\Lambda,\infty}$), we consider the $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-subalgebra

$$\mathbb{A}^0_{\mathrm{cris}}(R'_\infty) \subset \mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{1}{p}\big]$$

generated by the elements $\xi^n/n!$ for $n \geqslant 1$. Analogously to § 5.8, by [Tsu99, proof of A2.8], we have

$$\mathbb{A}^0_{\mathrm{cris}}(R'_\infty) \cong \big(\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{T^n}{n!}\big]_{n\geqslant 1}\big)/(T - \xi), \quad \text{so} \quad \mathbb{A}^0_{\mathrm{cris}}(R'_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R'_\infty) \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}.$$

Thus, analogously to § 5.8, the ring $\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)$ is identified with the divided power envelope of $(\mathbb{A}_{\mathrm{inf}}(R'_\infty), \mathrm{Ker}(\theta) + p \cdot \mathbb{A}_{\mathrm{inf}}(R'_\infty))$ over $(\mathbb{Z}_p, p\mathbb{Z}_p)$, and

$$\mathbb{A}_{\mathrm{cris}}(R'_\infty) := \mathbb{A}_{\mathrm{inf}}(R'_\infty) \mathbin{\widehat{\otimes}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} \quad \text{is identified with} \quad (\mathbb{A}^0_{\mathrm{cris}}(R'_\infty))^\frown.$$

For an $m \in \mathbb{Z}_{\geqslant 1}$, we let $\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty) \subset \mathbb{A}^0_{\mathrm{cris}}(R'_\infty)$ be the $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-subalgebra generated by the elements $\xi^n/n!$ with $n \leqslant m$ (compare with § 3.26). For a fixed m, the subalgebra

$$\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{T^n}{n!}\big]_{m\geqslant n\geqslant 1} \subset \mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{T^n}{n!}\big]_{n\geqslant 1} \subset \big(\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{1}{p}\big]\big)[T]$$

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is described by explicit lower bounds on the 'p-adic valuations' of the coefficients of T^N for $N \geqslant 1$. Thus, since the sequence (p, ξ) is $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-regular (compare with Lemma 3.13), the quotient of $\mathbb{A}_{\mathrm{inf}}(R'_\infty)[\frac{T^n}{n!}]_{n\geqslant 1}$ by $\mathbb{A}_{\mathrm{inf}}(R'_\infty)[\frac{T^n}{n!}]_{m\geqslant n\geqslant 1}$ has no nonzero $(T-\xi)$-torsion. Consequently,

$$\mathbb{A}_{\mathrm{cris}}^{0,(m)}(R'_\infty) \cong (\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{T^n}{n!}\big]_{m\geqslant n\geqslant 1})/(T-\xi), \tag{5.35.1}$$

to the effect that

$$\mathbb{A}_{\mathrm{cris}}^{0,(m)}(R'_\infty) \cong \mathbb{A}_{\mathrm{inf}}(R'_\infty) \otimes_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{0,(m)}.$$

Thus, by letting the completion be p-adic if $m \geqslant p$ and (p,μ)-adic if $m < p$, we obtain the following identification with the $A_{\mathrm{cris}}^{(m)}$-algebra $\mathbb{A}_{\mathrm{cris}}^{(m)}(R'_\infty)$ defined as in (5.19.3):

$$(\mathbb{A}_{\mathrm{cris}}^{0,(m)}(R'_\infty))^\wedge \cong \mathbb{A}_{\mathrm{cris}}^{(m)}(R'_\infty) := \mathbb{A}_{\mathrm{inf}}(R'_\infty)\,\widehat{\otimes}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}^{(m)}. \tag{5.35.2}$$

PROPOSITION 5.36. *For an affinoid perfectoid* $\mathrm{Spa}(R'_\infty[\frac{1}{p}], R'_\infty)$ *over* $\mathrm{Spa}(C, \mathcal{O}_C)$ *and an* $m \geqslant 1$, *the following ring homomorphisms are injective:*

$$\mathbb{A}_{\mathrm{cris}}^{0,(m)}(R'_\infty) \hookrightarrow \mathbb{A}_{\mathrm{cris}}^{(m)}(R'_\infty) \hookrightarrow \mathbb{A}_{\mathrm{cris}}(R'_\infty) \hookrightarrow \mathbb{B}_{\mathrm{dR}}^{+}(R'_\infty) := \big(\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{1}{p}\big]\big)^\wedge \tag{5.36.1}$$

where the completion is ξ*-adic and the definition of the last map is explained in the proof. In particular, the* A_{inf}*-algebras in (5.36.1) have no nonzero* μ*-torsion.*

Proof. The assertion about the μ-torsion follows from the rest because μ/ξ is a unit in $\mathbb{B}_{\mathrm{dR}}^{+}(R'_\infty)$ (see (2.1.1)–(2.1.4)) and $\mathbb{B}_{\mathrm{dR}}^{+}(R'_\infty)$ inherits ξ-torsion freeness from $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$.

The sequence (p,ξ) is $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-regular and $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$ is ξ-adically separated (see [SP, 090T]), so the ring $\mathbb{A}_{\mathrm{inf}}(R'_\infty)[\frac{1}{p}]$ is also ξ-adically separated. Thus, we obtain the injection

$$\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{1}{p}\big] \hookrightarrow \mathbb{B}_{\mathrm{dR}}^{+}(R'_\infty), \quad \text{and hence also} \quad \mathbb{A}_{\mathrm{cris}}^{0,(m)}(R'_\infty) \hookrightarrow \mathbb{B}_{\mathrm{dR}}^{+}(R'_\infty),$$

which, in particular, allows us to assume that $m \geqslant p$. For varying $n \geqslant 0$, the $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-submodules

$$\mathrm{Fil}_n^0 \subset \mathbb{A}_{\mathrm{cris}}^0(R'_\infty) \quad \text{generated by the} \quad \tfrac{\xi^{n'}}{n'!} \quad \text{for} \quad n' \geqslant n$$

form a decreasing filtration of $\mathbb{A}_{\mathrm{cris}}^0(R'_\infty)$ by ideals. By [Tsu99, A2.9(2)],[13] each

$$\mathbb{A}_{\mathrm{cris}}^0(R'_\infty)/\,\mathrm{Fil}_n^0 \quad \text{is } p\text{-torsion free and } p\text{-adically complete.} \tag{5.36.2}$$

Thus, the p-adic completions $\mathrm{Fil}_n := (\mathrm{Fil}_n^0)^\wedge$ form a decreasing filtration of $\mathbb{A}_{\mathrm{cris}}(R'_\infty)$ by ideals with

$$\mathbb{A}_{\mathrm{cris}}(R'_\infty)/\,\mathrm{Fil}_n \cong \mathbb{A}_{\mathrm{cris}}^0(R'_\infty)/\,\mathrm{Fil}_n^0\,. \tag{5.36.3}$$

The p-torsion freeness also supplies a decreasing filtration modulo p:

$$\mathrm{Fil}_n^0\,/p\,\mathrm{Fil}_n^0 \hookrightarrow \mathbb{A}_{\mathrm{cris}}^0(R'_\infty)/p\mathbb{A}_{\mathrm{cris}}^0(R'_\infty).$$

The isomorphism $\mathbb{A}_{\mathrm{cris}}^0(R'_\infty) \cong (\mathbb{A}_{\mathrm{inf}}(R'_\infty)[T^n/n!]_{n\geqslant 1})/(T-\xi)$ gives the explicit description

$$\mathbb{A}_{\mathrm{cris}}^0(R'_\infty)/p\mathbb{A}_{\mathrm{cris}}^0(R'_\infty) \cong (R'^{\flat}_\infty/\xi^p)[Y_1, Y_2, \ldots]/(Y_1^p, Y_2^p, \ldots) \quad \text{with} \quad Y_j = \frac{T^{p^j}}{(p^j)!} \tag{5.36.4}$$

[13] Note that [Tsu99, A2.9(2)] is written in a different setting, but its proof continues to work if A there is replaced by our R'_∞.

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(compare with [BC09, 9.4.1(3)]), so the filtration $\{\mathrm{Fil}^0_n/p\,\mathrm{Fil}^0_n\}_{n\geqslant 0}$ is separated. Thus, since

$$\mathrm{Fil}^0_n/p\,\mathrm{Fil}^0_n \cong \mathrm{Fil}_n/p\,\mathrm{Fil}_n \quad \text{compatibly with} \quad \mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/p\mathbb{A}^0_{\mathrm{cris}}(R'_\infty) \cong \mathbb{A}_{\mathrm{cris}}(R'_\infty)/p\mathbb{A}_{\mathrm{cris}}(R'_\infty),$$

the p-adic separatedness of $\mathbb{A}_{\mathrm{cris}}(R'_\infty)$ and (5.36.2) force the filtration $\{\mathrm{Fil}_n\}_{n\geqslant 0}$ to be separated too:

$$\mathbb{A}_{\mathrm{cris}}(R'_\infty) \hookrightarrow \varprojlim_n (\mathbb{A}_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}_n) \overset{(5.36.3)}{\cong} \varprojlim_n (\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^0_n).$$

Moreover, we have the injection $\varprojlim_n (\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^0_n) \hookrightarrow \mathbb{B}^+_{\mathrm{dR}}(R'_\infty)$ that results from the injections

$$\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^0_n \hookrightarrow (\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^0_n)\big[\tfrac{1}{p}\big] \cong \big(\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{1}{p}\big]\big)/\xi^n \cong \mathbb{B}^+_{\mathrm{dR}}(R'_\infty)/\xi^n.$$

Thus, we obtain the desired natural injection $\mathbb{A}_{\mathrm{cris}}(R'_\infty) \hookrightarrow \mathbb{B}^+_{\mathrm{dR}}(R'_\infty)$ of $\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)$-algebras.

We turn to the remaining injection $\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty) \hookrightarrow \mathbb{A}_{\mathrm{cris}}(R'_\infty)$. For $n \geqslant 0$, we consider the ideal

$$\mathrm{Fil}^{0,(m)}_n := \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty) \bigcap \mathrm{Fil}^0_n = \mathrm{Ker}\big(\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty) \to \big(\mathbb{A}_{\mathrm{inf}}(R'_\infty)\big[\tfrac{1}{p}\big]\big)/\xi^n\big) \subset \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty),$$

so that $\{\mathrm{Fil}^{0,(m)}_n\}_{n\geqslant 0}$ forms a filtration of $\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)$. Explicitly, as an $\mathbb{A}_{\mathrm{inf}}(R'_\infty)$-module, $\mathrm{Fil}^{0,(m)}_n$ is generated by the products $\xi^{n_1}/n_1!\cdots\xi^{n_s}/n_s!$ with $n_1+\cdots+n_s \geqslant n$ and $0 \leqslant n_i \leqslant m$. By (5.36.2), each

$$\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^{0,(m)}_n \quad \text{is } p\text{-torsion free,} \tag{5.36.5}$$

so we again get the induced filtration modulo p:

$$\mathrm{Fil}^{0,(m)}_n/p\,\mathrm{Fil}^{0,(m)}_n \hookrightarrow \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/p\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty).$$

As in the case of $\{\mathrm{Fil}^0_n/p\,\mathrm{Fil}^0_n\}_{n\geqslant 0}$, the analogous to (5.36.4) description of

$$\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/p\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)$$

supplied by the isomorphism (5.35.1) shows that the filtration $\{\mathrm{Fil}^{0,(m)}_n/p\,\mathrm{Fil}^{0,(m)}_n\}_{n\geqslant 0}$ is separated.

For each $n > 0$, there is a $j_n > 0$ such that p^{j_n} kills

$$\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/(\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty) + \mathrm{Fil}^0_n)$$

(for instance, $j_n := \mathrm{ord}_p(n!)$ has this property). Consequently, p^{j_n} kills the kernel of the map

$$\frac{\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^{0,(m)}_n}{p^j\cdot(\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^{0,(m)}_n)} \to \frac{\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^0_n}{p^j\cdot(\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^0_n)} \quad \text{for each} \quad j > 0,$$

so, for $j > j_n$, every element of this kernel is a multiple of p^{j-j_n}. The short exact sequences

$$0 \to \frac{\mathrm{Fil}^{0,(m)}_n}{p^j\cdot\mathrm{Fil}^{0,(m)}_n} \xrightarrow{(5.36.5)} \frac{\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)}{p^j\cdot\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)} \to \frac{\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^{0,(m)}_n}{p^j\cdot(\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/\mathrm{Fil}^{0,(m)}_n)} \to 0$$

then show that for each $n > 0$, every element of the kernel

$$\mathrm{Ker}(\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty) \to \mathbb{A}_{\mathrm{cris}}(R'_\infty)) \cong \mathrm{Ker}\left(\varprojlim_{j>0}(\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/p^j) \to \varprojlim_{j>0}(\mathbb{A}^0_{\mathrm{cris}}(R'_\infty)/p^j)\right) \tag{5.36.6}$$

lies in $\varprojlim_{j>0}(\mathrm{Fil}^{0,(m)}_n/p^j\,\mathrm{Fil}^{0,(m)}_n)$. In particular, this kernel maps to $\mathrm{Fil}^{0,(m)}_n/p \subset \mathbb{A}^{0,(m)}_{\mathrm{cris}}(R'_\infty)/p$ for each $n > 0$. However, by the previous paragraph, $\bigcap_{n>0}(\mathrm{Fil}^{0,(m)}_n/p) = 0$, so the kernel (5.36.6) lies in $p\cdot\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty)$. Since $\mathbb{A}_{\mathrm{cris}}(R'_\infty)$ has no nonzero p-torsion and $\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty)$ is p-adically separated, this implies that the map $\mathbb{A}^{(m)}_{\mathrm{cris}}(R'_\infty) \to \mathbb{A}_{\mathrm{cris}}(R'_\infty)$ is injective, as desired. □

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The following lemma is the final step to building the desired map $D^{(m)}_{\Sigma,\Lambda} \to \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ in § 5.38.

LEMMA 5.37. *For $\lambda_0 \in \Lambda$, there is a $\Delta_{\Sigma,\Lambda}$-equivariant divided power morphism*

$$D_{j_{\lambda_0}} \to \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \tag{5.37.1}$$

that is compatible with the isomorphisms $D_{j_{\lambda_0}} \cong D_{j_{\lambda'_0}}$ of (5.28.1) and is Frobenius-equivariant.

Proof. By construction, $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is an $A^{\square}_{\Sigma,\Lambda}$-algebra. Moreover, since $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is ξ-adically complete with $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/\xi \cong R_{\Sigma,\Lambda,\infty}$ (see § 5.19), if $t_{\lambda,i}$ is a unit in $R \subset R_{\Sigma,\Lambda,\infty}$, then $X_{\lambda,i}$ is a unit in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ (see (3.14.3)). Thus, if $R \otimes_{\mathcal{O}_C} k$ is k-smooth, then equations (5.26.3) have a unique solution in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$, to the effect that, in this case, $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is naturally and $\Delta_{\Sigma,\Lambda}$-equivariantly an $(A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$-algebra, compatibly with the 'change of λ_0' isomorphisms of (5.26.7), the maps (5.26.4) and (5.19.1) to R and $R_{\Sigma,\Lambda,\infty}$, and the Frobenii.

If $R \otimes_{\mathcal{O}_C} k$ is not k-smooth, then, in the notation of § 5.27, for each $m \geqslant 0$ and a generic point $y \in \mathcal{Y}$ of $\mathrm{Spec}(R \otimes_{\mathcal{O}_C} k)$, the element $t^{1/p^m}_{\lambda_0, i_{\lambda_0}(y)}$ is not a zero divisor in $R_{\Sigma,\Lambda,\infty}$ and is a unit in $R_{\Sigma,\Lambda,\infty}[\frac{1}{p}]$. Thus, since $R_{\Sigma,\Lambda,\infty}$ is integrally closed in $R_{\Sigma,\Lambda,\infty}[\frac{1}{p}]$, we conclude from (5.27.1) that

$$t^{1/p^m}_{\lambda, i_\lambda(y)} / t^{1/p^m}_{\lambda_0, i_{\lambda_0}(y)} \in R^{\times}_{\Sigma,\Lambda,\infty} \quad \text{for every} \quad \lambda_0, \lambda \in \Lambda,\ m \geqslant 0.$$

In other words,

$$t^{1/p^m}_{\lambda, i_\lambda(y)} = u^{(m)}_{\lambda,\lambda_0,y} \cdot t^{1/p^m}_{\lambda_0, i_{\lambda_0}(y)} \quad \text{in} \quad R_{\Sigma,\Lambda,\infty} \quad \text{for a unique} \quad u^{(m)}_{\lambda,\lambda_0,y} \in R^{\times}_{\Sigma,\Lambda,\infty}.$$

By the uniqueness, $(u^{(m+1)}_{\lambda,\lambda_0,y})^p = u^{(m)}_{\lambda,\lambda_0,y}$, so $u^{\flat}_{\lambda,\lambda_0,y} := (u^{(m)}_{\lambda,\lambda_0,y})_{m\geqslant 0} \in (R^{\flat}_{\Sigma,\Lambda,\infty})^{\times}$ satisfies

$$X_{\lambda, i_\lambda(y)} = [u^{\flat}_{\lambda,\lambda_0,y}] \cdot X_{\lambda_0, i_{\lambda_0}(y)} \quad \text{in} \quad \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$$

(see §§ 3.11 and 3.14). Thus, since each $X_{\lambda,i}$ is a nonzero-divisor in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$, the $[u^{\flat}_{\lambda,\lambda_0,y}]$ solve the equations (5.27.3) in $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$, compatibly with the solution in $R \subset R_{\Sigma,\Lambda,\infty}$ of (5.27.4). Thus, in the nonsmooth case as well, $\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})$ is $\Delta_{\Sigma,\Lambda}$-equivariantly an algebra over $A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$, compatibly with the change of λ_0, the maps (5.27.4) and (5.19.1) to R and $R_{\Sigma,\Lambda,\infty}$, and the Frobenii.

In conclusion, in all the cases we obtain a commutative square which is compatible with the change of λ_0:

$$\begin{array}{ccc} A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] & \xrightarrow{j_{\lambda_0}} & R \\ \downarrow & & \downarrow \\ \mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty}) & \xrightarrow{\theta} & R_{\Sigma,\Lambda,\infty} \end{array} \quad \text{so also} \quad \begin{array}{ccc} A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] & \xrightarrow{j_{\lambda_0}} & R \\ \downarrow & & \downarrow \\ \mathbb{A}^0_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) & \xrightarrow{\theta} & R_{\Sigma,\Lambda,\infty}. \end{array}$$

The universal property of $D_{j_{\lambda_0}}$ then supplies the desired divided power morphism (5.37.1). □

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5.38 The comparison map

The p-adic completion of the morphism (5.37.1) is the morphism

$$D_{\Sigma,\Lambda} \to \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \tag{5.38.1}$$

(see Lemma 5.29), which does not depend on λ_0. By construction, it makes the following diagram commute.

$$\begin{array}{ccc} D_{\Sigma,\Lambda} & \xrightarrow{(5.22.3)} & R \\ \downarrow{\scriptstyle (5.38.1)} & & \downarrow \\ \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) & \xrightarrow{\theta} & R_{\Sigma,\Lambda,\infty} \end{array} \tag{5.38.2}$$

Its restriction to $D^{0,(m)}_{\Sigma,\Lambda}$ of § 5.30 lands in the subring $\mathbb{A}^{0,(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \overset{5.36}{\subset} \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$, so, by passing to p-adic completions and using (5.35.2), we obtain the compatible morphisms

$$D^{(m)}_{\Sigma,\Lambda} \to \mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \quad \text{for} \quad m \geqslant p \tag{5.38.3}$$

that recover (5.38.1) via the identifications (5.30.1) and (5.1.1). By construction and Lemma 5.37, the morphisms (5.38.3) are $\Delta_{\Sigma,\Lambda}$-equivariant and Frobenius-equivariant, so they give rise to the Frobenius-equivariant morphisms of complexes

$$K_{D^{(m)}_{\Sigma,\Lambda}}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda, 1\leqslant i\leqslant d}) \to K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\sigma - 1)_{\sigma\in\Sigma}, (\delta_{\lambda,i} - 1)_{\lambda\in\Lambda, 1\leqslant i\leqslant d}).$$

After applying the functor $\eta_{(\mu)}$, these morphisms compose with those constructed in Proposition 5.34 to give rise to the desired Frobenius-equivariant comparison map of complexes:

$$\left(\varinjlim_{m\geqslant p}\left(K_{D^{(m)}_{\Sigma,\Lambda}}\left(\left(\frac{\partial}{\partial \log(X_\tau)}\right)_\tau\right)\right)\right)^{\widehat{}} \to \left(\varinjlim_{m\geqslant p}\left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau - 1)_\tau)\right)\right)\right)^{\widehat{}}, \tag{5.38.4}$$

where the direct limits and the p-adic completions are formed termwise and, for brevity, we let τ range over the indices 'σ' for $\sigma \in \Sigma$ and '(λ, i)' for $\lambda \in \Lambda$ and $i = 1, \ldots, d$.

The source (respectively, the target) of the map (5.38.4) is a term of the direct limit (5.32.1) (respectively, (5.21.1)) and the formation of this map is compatible with enlarging Σ and Λ, that is, with the transition maps of the direct limits (5.32.1) and (5.21.1). Moreover, if R' is a p-adically formally étale R-algebra equipped with data as in (5.17.1) for some sets Σ' and Λ', then the map (5.38.4) and its analogue for R' and the sets $\Sigma \cup \Sigma'$, $\Lambda \cup \Lambda'$ (and the induced closed immersion (5.17.1)) are compatible with the maps between their sources (respectively, targets) discussed in §§ 5.21 and 5.32.

In conclusion, by taking the filtered direct limit of the maps (5.38.4) over all the closed immersions (5.17.1) for varying Σ and Λ (but a fixed R), we obtain a comparison map from the complex (5.32.1) to the complex (5.21.1), and the formation of this map is compatible with replacing R by a formally étale R-algebra R'. The following proposition implies that this map is a quasi-isomorphism.

Proposition 5.39. *The Frobenius-equivariant comparison map (5.38.4) is a quasi-isomorphism.*

Proof. The proof is similar to that of [BMS18, 12.9], and the idea is to reduce to the case of a single coordinate morphism settled in Proposition 5.16. More precisely, for $m \geqslant p$, let

$$\mathrm{Spec}(R/p) \hookrightarrow \mathrm{Spf}(D^{(m)}_{\Sigma,\Lambda})$$

be the closed immersion induced by its analogue for $D_{\Sigma,\Lambda}$, that is, by the first map in (5.22.2). For each $\lambda_0 \in \Lambda$, the ideal of $A^{\square}_{\Sigma,\Lambda} \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}]$ that cuts out R/p (see (5.27.5)) is finitely generated.

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Consequently, for each $m \geqslant p$, the ideal of $D^{(m)}_{\Sigma,\Lambda}$ that cuts out R/p is also finitely generated and, hence, due to divided powers, it is also p-adically topologically nilpotent. Thus, fixing a $\lambda \in \Lambda$ and, for $m \geqslant p$, letting $A^{(m)}_{\mathrm{cris}}(R)_\lambda$ denote the ring $A^{(m)}_{\mathrm{cris}}(R)$ of § 3.27 constructed using the coordinate morphism $R^{\square}_\lambda \to R$, we may use the p-adic formal étaleness of $A^{(m)}_{\mathrm{cris}}(R^{\square}_\lambda) \to A^{(m)}_{\mathrm{cris}}(R)_\lambda$ (see § 3.14) to obtain the unique indicated lifts in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Spec}(R_{\Sigma,\Lambda,\infty}/p) & \longrightarrow & \mathrm{Spec}(R/p) & \longrightarrow & \mathrm{Spf}(A^{(m)}_{\mathrm{cris}}(R)_\lambda) \\
\downarrow{\scriptstyle \theta} & & \downarrow & & \downarrow \\
\mathrm{Spf}(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})) & \xrightarrow[(5.38.3)]{} & \mathrm{Spf}(D^{(m)}_{\Sigma,\Lambda}) & \longrightarrow & \mathrm{Spf}(A^{(m)}_{\mathrm{cris}}(R^{\square}_\lambda))
\end{array}$$

in which the bottom right horizontal map results from the fact that, by construction, each $D^{(m)}_{\Sigma,\Lambda}$ is both an $A(R^{\square}_\lambda)$-algebra and an $A^{(m)}_{\mathrm{cris}}$-algebra. By the uniqueness of such lifts, the resulting maps

$$A^{(m)}_{\mathrm{cris}}(R)_\lambda \to D^{(m)}_{\Sigma,\Lambda} \tag{5.39.1}$$

are compatible as m varies, $\Delta_{\Sigma,\Lambda}$-equivariant, where $\Delta_{\Sigma,\Lambda}$ acts on $A^{(m)}_{\mathrm{cris}}(R)_\lambda$ through the projection $\Delta_{\Sigma,\Lambda} \twoheadrightarrow \Delta_\lambda$, and are compatible with the maps from its source and target to $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$. By construction, the maps (5.39.1) are also compatible with the derivations $\frac{\partial}{\partial \log(X_{\lambda,i})}$ for $i = 1,\dots,d$ discussed in §§ 5.10 and 5.31. Consequently, the diagram

$$\begin{array}{ccc}
K_{A^{(m)}_{\mathrm{cris}}(R)_\lambda}\left(\frac{\partial}{\partial \log(X_{\lambda,1})},\dots,\frac{\partial}{\partial \log(X_{\lambda,d})}\right) & \xrightarrow{(5.16.2)} & \eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\lambda,\infty})}(\delta_{\lambda,1}-1,\dots,\delta_{\lambda,d}-1)\right) \\
\downarrow{\scriptstyle (5.39.1)} & & \downarrow \\
K_{D^{(m)}_{\Sigma,\Lambda}}\left(\left(\frac{\partial}{\partial \log(X_\tau)}\right)_\tau\right) & \xrightarrow{5.34 \text{ and } (5.38.3)} & \eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)
\end{array} \tag{5.39.2}$$

commutes, where the index τ ranges over 'σ' for $\sigma \in \Sigma$ and '(λ', i)' for $\lambda' \in \Lambda$ and $i = 1,\dots,d$. By Proposition 5.16, for $m \geqslant p^2$, the top horizontal map in (5.39.2) is a quasi-isomorphism and, by Lemma 3.7 and Remark 3.35, so is the right vertical map. By Proposition 5.13 and (5.31.5), the left vertical map in (5.39.2) becomes a quasi-isomorphism after applying $\varinjlim_{m \geqslant p}$ and forming the termwise p-adic completion. These operations turn the bottom horizontal map in (5.39.2) into the comparison map (5.38.4), so we conclude that the latter is also a quasi-isomorphism, as desired. □

5.40 Proof of Theorem 5.4

By § 5.38 and Proposition 5.39, the functorial in R complexes (5.21.1) and (5.32.1) define canonically and Frobenius-equivariantly quasi-isomorphic complexes of presheaves on a basis for the topology of $\mathfrak{X}_{\text{ét}}$. Their associated complexes of sheaves on $\mathfrak{X}_{\text{ét}}$ are then also canonically and Frobenius-equivariantly quasi-isomorphic. By §§ 5.21 and 5.32, these complexes of sheaves Frobenius-compatibly represent $A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}$ and $Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p/A_{\mathrm{cris}}}})$, respectively, so that, in conclusion, Proposition 5.39 supplies a Frobenius-equivariant isomorphism

$$Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p/A_{\mathrm{cris}}}}) \xrightarrow{\sim} A\Omega_{\mathfrak{X}} \widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}, \tag{5.40.1}$$

which gives the desired identification (5.4.1). □

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We have obtained two ways to identify the de Rham specialization of $A\Omega_{\mathfrak{X}}$: we may either use (4.17.1) or combine (5.40.1) with the fact that the logarithmic crystalline cohomology of $\mathfrak{X}_{\mathcal{O}_C/p}$ over $\mathcal{O}_C$ is computed by $\Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}$. For use in § 8, we now check that these two identifications agree.

Proposition 5.41. *The following diagram commutes:*

$$\begin{array}{ccc} Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) & \xrightarrow[\sim]{(5.40.1)} & A\Omega_{\mathfrak{X}} \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}} \\ & \searrow \quad \swarrow_{(4.17.1)} & \\ & \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log} & \end{array} \tag{5.41.1}$$

where the unlabeled map results from the identification

$$Ru_*(\mathcal{O}_{\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{cris}},\theta}} \mathcal{O}_C \cong \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}$$

of [Bei13a, (1.8.1) and (1.11.1)] (compare with Remark 5.24).

Proof. The overall argument is similar to that given in the smooth case in [BMS18, proof of 14.1].

The claim is local, so we assume the setup of § 5.17. Then $d\log(X_\sigma)$ and $d\log(X_{\lambda,i})$ generate the differential graded algebra $\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}}$ over $D_{\Sigma,\Lambda}$ (see Proposition 5.23), so, since the terms of $\Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}$ are p-torsion free and each t_σ and $t_{\lambda,i}$ is a unit in $R[\frac{1}{p}]$, there is at most one map of differential graded algebras

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \to \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log} \quad \text{with} \quad D_{\Sigma,\Lambda} \xrightarrow{(5.22.3)} R \quad \text{in degree} \quad 0. \tag{5.41.2}$$

By Proposition 5.23, the unlabeled map of (5.41.1) is identified with this unique map, so we need to show that so is the composition in (5.41.1).

We recall from the proof of Theorem 4.17 that the right diagonal map in (5.41.1) is defined by composing the Frobenius of $A\Omega_{\mathfrak{X}}$, the reduction modulo $\varphi(\xi)$, and the canonical identification supplied by [BMS18, 6.12] of $(L\eta_{(\varphi(\xi))}(A\Omega_{\mathfrak{X}}))/\varphi(\xi)$ with the complex[14] $H^{\bullet}(A\Omega_{\mathfrak{X}}/\varphi(\xi))$ that is *a posteriori* identified with $\Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}$ and whose differentials are *a priori* given by Bockstein homomorphisms (defined in [BMS18, 6.12] using $A\Omega_{\mathfrak{X}}/(\varphi(\xi))^2$). Letting τ range over the usual indices (see § 5.31), we may also apply this construction to the complex

$$\eta_{(\mu)}(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau - 1)_\tau)).$$

Frobenius maps it isomorphically to $\eta_{(\varphi(\mu))}(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau - 1)_\tau))$, for which the reduction modulo $\varphi(\xi)$ map is

$$\eta_{(\varphi(\mu))}(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau - 1)_\tau)) \to H^{\bullet}((\eta_{(\mu)}(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau - 1)_\tau)))/\varphi(\xi)).$$

Moreover, due to the isomorphism (4.2.2) and Remarks 3.10 and 3.21,

$$(\eta_{(\mu)}(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau - 1)_\tau)))/\varphi(\xi) \xrightarrow{\sim} \eta_{(\zeta_p - 1)}(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau - 1)_\tau)).$$

[14] For notational simplicity, we suppress the twists inherent in the construction $H^{\bullet}(-)$ of [BMS18, 6.12].

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The cited remarks and Theorem 4.11 (with Remark 4.5) imply that this describes the de Rham specialization map $A\Omega_{\mathfrak{X}} \to \Omega^{\bullet}_{\mathfrak{X}/\mathcal{O}_C,\log}$ in terms of the complex $\eta_{(\mu)}(K_{\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau))$.

We now describe the right diagonal map of (5.41.1) in terms of $\eta_{(\mu)}(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau))$, which is a variable term that comprises the target of the comparison map (5.38.4). Namely, we first let $\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))$ be the analogue of the ring $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ built using the element $\varphi(\xi)$ instead of ξ, so that the Frobenius gives the isomorphism $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) \xrightarrow{\sim} \varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))$.[15] Then the Frobenius maps the complex $\eta_{(\mu)}(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau))$ isomorphically to the complex $\eta_{(\varphi(\mu))}(K_{\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))}((\delta_\tau-1)_\tau))$, for which the reduction modulo $\varphi(\xi)$ map is

$$\eta_{(\varphi(\mu))}(K_{\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))}((\delta_\tau-1)_\tau)) \to H^{\bullet}(\eta_{(\mu)}(K_{\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))}((\delta_\tau-1)_\tau))/\varphi(\xi)). \quad (5.41.3)$$

Via a morphism induced by $\theta\circ\varphi^{-1}\colon \varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})) \twoheadrightarrow R_{\Sigma,\Lambda,\infty}$, the target of (5.41.3) maps to

$$H^{\bullet}(\eta_{(\zeta_p-1)}(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau))) \overset{3.10 \text{ and } 4.11}{\cong} \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}$$

because, since each $H^i(\eta_{(\zeta_p-1)}(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau)))$ is p-torsion free, the agreement of the Bockstein differentials may be checked after inverting p by using the fact that $(\mathbb{A}_{\mathrm{inf}}(R_{\Sigma,\Lambda,\infty})/\varphi(\xi)^2)[\frac{1}{p}]$ is an algebra over $\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))$ via a map that lifts $\theta\circ\varphi^{-1}$. The resulting composition

$$\eta_{(\mu)}(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)) \to H^{\bullet}(\eta_{(\zeta_p-1)}(K_{R_{\Sigma,\Lambda,\infty}}((\delta_\tau-1)_\tau))) \cong \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log} \quad (5.41.4)$$

is the promised description of the right diagonal map of (5.41.1). In addition, by construction and [BMS18, 6.13], this composition is a morphism of differential graded algebras[16] that in degree 0 is given by the map θ of (5.19.4).

On the other hand, the comparison map

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \cong K_{D_{\Sigma,\Lambda}}\left(\left(\frac{\partial}{\partial\log(X_\tau)}\right)_\tau\right) \xrightarrow{(5.38.4)} \left(\varinjlim_{m\geqslant p}\left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)\right)\right)^{\widehat{}}$$

would be a map of differential graded algebras if in the formula

$$\log([\epsilon])\cdot\textstyle\sum_{n\geqslant 0}\left(\frac{(\log([\epsilon]))^n}{(n+1)!}\left(\frac{\partial}{\partial\log(X_\tau)}\right)^n\right)$$

that describes the morphism (5.34.1) in degree 1 we could disregard the terms with $n\geqslant 1$. However, $\log([\epsilon])$ and μ are unit multiples of each other (see § 5.14) and $\theta(\mu^n/(n+1)!)=0$ in $\mathcal{O}_C$ for $n\geqslant 1$, so we can indeed ignore these terms if we are only interested in the composition

$$\Omega^{\bullet}_{D_{\Sigma,\Lambda}/A_{\mathrm{cris}},\log,\mathrm{PD}} \to \left(\varinjlim_{m\geqslant p}\left(\eta_{(\mu)}\left(K_{\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})}((\delta_\tau-1)_\tau)\right)\right)\right)^{\widehat{}} \xrightarrow{5.41.4} \Omega^{\bullet}_{\mathrm{Spf}(R)/\mathcal{O}_C,\log}$$

that describes the composition in (5.41.1). In conclusion, this composition is a morphism of differential graded algebras that, due to (5.38.2), in degree 0 is the map $D_{\Sigma,\Lambda}\twoheadrightarrow R$ from (5.22.3). Therefore, as desired, it is the unique morphism (5.41.2). □

We now use Theorem 5.4 to analyze the crystalline specialization of $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$.

[15] Composing with the map $\varphi(\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}))\to\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$ recovers the Frobenius of $\mathbb{A}^{(m)}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty})$.

[16] The differential graded algebra structure on the Koszul complex $K_*((\delta_\tau-1)_\tau)$ that computes continuous group cohomology is described in [BMS18, 7.5] and its proof.

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5.42 The crystalline specialization map

The Witt vector functoriality gives the surjection

$$A_{\mathrm{inf}} \twoheadrightarrow W(k), \quad \text{the } \textit{crystalline specialization} \text{ map of } A_{\mathrm{inf}}.$$

Since ξ maps to p in $W(k)$, this surjection factors through A_{cris} as follows: $A_{\mathrm{inf}} \hookrightarrow A_{\mathrm{cris}} \twoheadrightarrow W(k)$. We equip $W(k)$ with the pullback of the log structure (5.2.1) on A_{cris}. Explicitly, the resulting log structure on $W(k)$ is associated to the prelog structure $\mathbb{Q}_{\geqslant 0} \xrightarrow{0 \neq q \mapsto 0,\, 0 \mapsto 1} W(k)$.

Corollary 5.43. *For quasi-compact and quasi-separated $\mathfrak{X}$, we have Frobenius-equivariant identifications*

$$\begin{aligned} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} A_{\mathrm{cris}} &\cong R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}), \\ R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{inf}}}} W(k) &\cong R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_k/W(k)). \end{aligned} \tag{5.43.1}$$

For $\mathcal{O}_C$-proper $\mathfrak{X}$, we have Frobenius-equivariant identifications

$$\begin{aligned} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}} &\cong R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}), \\ R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) &\cong R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_k/W(k)), \end{aligned} \tag{5.43.2}$$

and the cohomology modules of $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}[\frac{1}{p}]$ are finite free over $A_{\mathrm{cris}}[\frac{1}{p}]$.

Proof. By [BMS18, 4.9(i)], a finitely presented A_{inf}/p^n-module is perfect as an A_{inf}-module. Consequently, any A_{inf}/p^n-module M is a filtered direct limit of perfect A_{inf}-modules, so, by [SP, 0739],

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}} \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} M) \cong R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} M.$$

This applies to $M = A_{\mathrm{cris}}/p^n$, so the first identification in (5.43.1) follows from Theorem 5.4.

For each finite subextension of $C/(W(k)[\frac{1}{p}])$, we consider its ring of integers $\mathcal{O} \subset \mathcal{O}_C$ equipped with the (fine) log structure associated to $\mathcal{O} \cap (\mathcal{O}[\frac{1}{p}])^{\times} \hookrightarrow \mathcal{O}$. By using étale local semistable coordinates (1.5.1) and Claims 1.6.1 and 1.6.3, we employ limit arguments to find such an $\mathcal{O}$ and a quasi-compact and quasi-separated, fine, log smooth log scheme $\mathcal{X}$ over $\mathcal{O}/p$ that descends $\mathfrak{X}_{\mathcal{O}_C/p}$ and is of Cartier type (see [Kat89, 4.8]). Then the base change theorem [Bei13a, (1.11.1)] applies[17] and shows that

$$R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \mathbin{\widehat{\otimes}^{\mathbb{L}}_{A_{\mathrm{cris}}}} W(k) \cong R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_k/W(k)), \tag{5.43.3}$$

so the second identification in (5.43.1) follows from the first.

If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then, by Corollary 4.20, the object $R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}})$ is quasi-isomorphic to a bounded complex of finite free A_{inf}-modules, so the identifications in (5.43.2) follow from those in (5.43.1). Moreover, then $\mathcal{X}$ is $\mathcal{O}$-proper and [Bei13a, 1.18, Theorem] proves that the cohomology groups of

$$R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} A_{\mathrm{cris}}\left[\tfrac{1}{p}\right], \quad \text{and hence also of} \quad R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}\left[\tfrac{1}{p}\right],$$

are finite free $A_{\mathrm{cris}}[\frac{1}{p}]$-modules. □

[17] In [Bei13a, (1.11.1)], the map f of fine log schemes is quasi-compact and *separated*. One may relax this to quasi-compact and quasi-separated: once Y there is affine, the iterated intersections of opens in an affine cover of Z are quasi-compact and separated over Y, so the Čech technique (compare with [SP, 08BN]) reduces to the original assumptions.

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Remarks.

5.44. In the preceding proof, the special fiber $\mathcal{X}_k$ is a descent of $\mathfrak{X}_k$ to a fine, log smooth log scheme of Cartier type over k equipped with the log structure associated to

$$\mathbb{N}_{\geqslant 0} \xrightarrow{1\mapsto 0,\,0\mapsto 1} k$$

(the base change map is a 'change of log structure' self-map of k determined by the map $\mathbb{N}_{\geqslant 0} \to \mathbb{Q}_{\geqslant 0}$ that sends 1 to the valuation of a uniformizer of $\mathcal{O}$). Given such a descent, the base change theorem [Bei13a, (1.11.1)] gives the further Frobenius-equivariant identification

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_k/W(k)) \cong R\Gamma_{\log\mathrm{cris}}(\mathcal{X}_k/W(k)), \tag{5.44.1}$$

where the $W(k)$ on the left (respectively, right) is equipped with the log structure associated to $\mathbb{Q}_{\geqslant 0} \xrightarrow{0\neq q\mapsto 0,\,0\mapsto 1} W(k)$ (respectively, $\mathbb{N}_{\geqslant 0} \xrightarrow{1\mapsto 0,\,0\mapsto 1} W(k)$). Likewise, if $\mathfrak{X}_k$ is k-smooth, then [Bei13a, (1.11.1)] gives the Frobenius-equivariant identification

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_k/W(k)) \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}_k/W(k)). \tag{5.44.2}$$

5.45. The identification (5.43.3) expresses $R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_k/W(k))$ in terms of $R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$. Further results from [Bei13a] imply a converse for proper $\mathfrak{X}$ after extending coefficients to B^+_{st}, see (9.2.2) below (when $\mathfrak{X}_k$ is k-smooth, $A_{\mathrm{cris}}[\frac{1}{p}]$ in place of B^+_{st} suffices, see [BMS18, 13.21]).

6. The comparison to the B^+_{dR}-cohomology

The main goal of this section is to prove in Theorem 6.6 that for $\mathcal{O}_C$-proper $\mathfrak{X}$, we have

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}} \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}), \tag{6.0.1}$$

where the definition of $R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}})$, the 'crystalline cohomology of $\mathfrak{X}^{\mathrm{ad}}_C$ over B^+_{dR}', was given in [BMS18, § 13] (see § 6.2 for a brief review). This definition is purely in terms of $\mathfrak{X}^{\mathrm{ad}}_C$ and was engineered in [BMS18, § 13] to be compatible with $R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ when $\mathfrak{X}$ is smooth. Thus, for (6.0.1), we only need to check that a slightly more general definition that uses the étale topology and more general embeddings than those furnished by annuli leads to the same cohomology (see §§ 6.2–6.3). For this, we adapt the arguments of [BMS18, § 13]; in fact, our C is $(\overline{W(k)}[\frac{1}{p}])\widehat{\ }$ (see § 1.5), so we may simplify the descent to a discretely valued base aspects of these arguments by taking advantage of a result of Huber on the local structure of étale maps of adic spaces (see § 6.3).

6.1 The ring B^+_{dR}

Since ξ is not a zero divisor in $A_{\mathrm{inf}}[\frac{1}{p}]$ and generates $\mathrm{Ker}(\theta[\frac{1}{p}])$, the $(\mathrm{Ker}(\theta[\frac{1}{p}]))$-adic completion of $A_{\mathrm{inf}}[\frac{1}{p}]$ is a complete discrete valuation ring B^+_{dR} with ξ as a uniformizer and C as the residue field. By Proposition 5.36, both A_{inf} and A_{cris} are subalgebras of B^+_{dR}. By the 'glueing of flatness' [RG71, II.1.4.2.1], the ring B^+_{dR} is flat as an A_{inf}-algebra. We set

$$B_{\mathrm{dR}} := \mathrm{Frac}(B^+_{\mathrm{dR}}).$$

Our A_{inf} is a $W(k)$-algebra (see § 2.1), so, by the infinitesimal lifting, B^+_{dR} is a $\overline{W(k)}[\frac{1}{p}]$-algebra.

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6.2 The B_{dR}^+-cohomology using the étale topology

In [BMS18, § 13], Bhatt–Morrow–Scholze used the analytic topology of a smooth adic C-space X to define the 'B_{dR}^+-cohomology' of X,

$$R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{dR}}^+) \in D^{\geqslant 0}(B_{\mathrm{dR}}^+).$$

We now review their construction and show that it may also be carried out in the étale topology.

By [Hub96, 1.6.10, 2.2.8], the analytic (respectively, étale) topology of X has a basis of affinoids $\mathrm{Spa}(A, A^\circ)$ each of which admits a map

$$\mathrm{Spa}(A, A^\circ) \to \mathbb{T}_C^d := \mathrm{Spa}(C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle, \mathcal{O}_C\langle T_1^{\pm 1}, \dots, T_d^{\pm 1}\rangle) \quad \text{for some} \quad d \geqslant 0 \tag{6.2.1}$$

that is a composition of a rational embedding, a finite étale map, and a rational embedding. By localizing further, we refine the basis to consist of those $\mathrm{Spa}(A, A^\circ)$ as above for which there is a finite subset $\Psi \subset (A^\circ)^\times$ such that the following map is surjective:

$$C\langle (X_u^{\pm 1})_{u\in\Psi}\rangle \xrightarrow{X_u \mapsto u} A. \tag{6.2.2}$$

Then, by endowing each A_{inf}/ξ^n with the p-adic topology, each $(A_{\mathrm{inf}}/\xi^n)[\frac{1}{p}]$ with the unique ring topology for which A_{inf}/ξ^n is an open subring, setting

$$B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}\rangle := \varprojlim_{n>0}((B_{\mathrm{dR}}^+/\xi^n)\langle (X_u^{\pm 1})_{u\in\Psi}\rangle), \tag{6.2.3}$$

and composing the projection onto the $n = 1$ term with (6.2.2), we obtain the surjection

$$s\colon B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}\rangle \twoheadrightarrow A \quad \text{and set} \quad D_\Psi(A) := \varprojlim_{n>0}((B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}\rangle)/(\mathrm{Ker}\, s)^n).$$

By the Leibniz rule, any $(B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}\rangle)$-valued derivation of $B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}\rangle$ induces a $D_\Psi(A)$-valued derivation of $D_\Psi(A)$. Thus, the commuting derivations $\frac{\partial}{\partial \log(X_u)} := X_u \cdot \frac{\partial}{\partial X_u}$ give rise to the Koszul complex

$$\Omega^\bullet_{D_\Psi(A)/B_{\mathrm{dR}}^+} := K_{D_\Psi(A)}\left(\left(\frac{\partial}{\partial \log(X_u)}\right)_{u\in\Psi}\right)$$

that is functorial in enlarging Ψ. The resulting complex

$$\Omega^\bullet_{A/B_{\mathrm{dR}}^+} := \varinjlim_{\Psi}(\Omega^\bullet_{D_\Psi(A)/B_{\mathrm{dR}}^+}) \tag{6.2.4}$$

is contravariantly functorial in $\mathrm{Spa}(A, A^\circ)$. Consequently, by varying $\mathrm{Spa}(A, A^\circ)$, we obtain a complex of presheaves on the basis described above for the analytic (respectively, étale) topology of X. The hypercohomology of the associated complex of sheaves is, by definition, the B_{dR}^+-cohomology of X:

$$R\Gamma_{\mathrm{cris}}(X/B_{\mathrm{dR}}^+) \quad \text{(respectively, its variant for the étale topology} \quad R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B_{\mathrm{dR}}^+)).$$

By [BMS18, 13.12(ii), 13.13], if $\mathrm{Spa}(A, A^\circ)$ is fixed and Ψ is sufficiently large, then $D_\Psi(A)$ is ξ-torsion free and ξ-adically complete, $\Omega^\bullet_{D_\Psi(A)/B_{\mathrm{dR}}^+}$ maps quasi-isomorphically to $\Omega^\bullet_{A/B_{\mathrm{dR}}^+}$, and, in the derived category, we have a canonical identification

$$\Omega^\bullet_{D_\Psi(A)/B_{\mathrm{dR}}^+}/\xi \cong \Omega^{\bullet,\mathrm{cont}}_{A/C}, \quad \text{so also} \quad \Omega^\bullet_{A/B_{\mathrm{dR}}^+}/\xi \cong \Omega^{\bullet,\mathrm{cont}}_{A/C}.$$

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Consequently, by [BMS18, 9.15] (which we also used for proving Corollary 4.6), our definition of $R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}})$ agrees with that of [BMS18, § 13] (where one skips the sheafification step),

$$R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}}) \quad \text{and} \quad R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B^+_{\mathrm{dR}}) \quad \text{are derived } \xi\text{-adically complete,} \tag{6.2.5}$$

and their (derived) reductions modulo ξ are canonically and compatibly identified with the de Rham cohomology objects $R\Gamma(X, \Omega^{\bullet,\mathrm{cont}}_{X/C})$ and $R\Gamma(X_{\text{ét}}, \Omega^{\bullet,\mathrm{cont}}_{X/C})$, respectively, for instance:

$$R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}}) \otimes^{\mathbb{L}}_{B^+_{\mathrm{dR}}} C \cong R\Gamma(X, \Omega^{\bullet,\mathrm{cont}}_{X/C}) =: R\Gamma_{\mathrm{dR}}(X/C). \tag{6.2.6}$$

Thus, since, by the Hodge-to-de Rham spectral sequence and [Sch13a, 9.2(ii)], the formation of the de Rham cohomology is insensitive to passage to the étale topology, we have

$$R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}}) \xrightarrow{\sim} R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B^+_{\mathrm{dR}}) \tag{6.2.7}$$

via pullback. In addition, if X is proper over C and there is a complete, discretely valued subfield $K \subset C$ with a perfect residue field and a proper, smooth adic space X_0 over K equipped with an isomorphism $X \cong X_0 \widehat{\otimes}_K C$, then, by [BMS18, 13.20], there is a canonical identification

$$R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}}) \cong R\Gamma_{\mathrm{dR}}(X_0/K) \otimes_K B^+_{\mathrm{dR}}, \quad \text{where} \quad R\Gamma_{\mathrm{dR}}(X_0/K) := R\Gamma(X_0, \Omega^{\bullet,\mathrm{cont}}_{X_0/K}). \tag{6.2.8}$$

In this situation, by the proof of [BMS18, 13.20], the reduction modulo ξ of the identification (6.2.8) recovers the identification (6.2.6) under the base change identification

$$R\Gamma_{\mathrm{dR}}(X/C) \cong R\Gamma_{\mathrm{dR}}(X_0/K) \otimes^{\mathbb{L}}_K C.$$

6.3 The B^+_{dR}-cohomology using more general embeddings

To relate $R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B^+_{\mathrm{dR}})$ to the absolute crystalline cohomology of § 5.3, we now mildly generalize the construction of the former.

The étale topology of X has a basis of affinoids $\mathrm{Spa}(A, A^\circ)$ each of which admits an étale map

$$\begin{aligned}\mathrm{Spa}(A, A^\circ) \to \mathrm{Spa}(C\langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle/(T_0 \cdots T_r - p^q),\\ \mathcal{O}_C\langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle/(T_0 \cdots T_r - p^q))\end{aligned} \tag{6.3.1}$$

for some $d \geqslant r \geqslant 0$ and $q \in \mathbb{Q}_{>0}$ (we have seen in § 6.2 that even those with $r = 0$ would suffice). By [Hub96, 1.7.3(iii)][18] and limit arguments, there is a finite subextension $W(k)[\frac{1}{p}] \subset K \subset C$ with the ring of integers $\mathcal{O}$ containing p^q and a finite-type $(\mathcal{O}[T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]/(T_0 \cdots T_r - p^q))$-algebra A_0 that is étale after inverting p, flat over $\mathcal{O}$, normal, and such that the morphism (6.3.1) is the C-base change of an étale $\mathrm{Spa}(K, \mathcal{O})$-morphism

$$\begin{aligned}\mathrm{Spa}\big(\widehat{A_0}\big[\tfrac{1}{p}\big], \widehat{A_0}\big) \to \mathrm{Spa}(K\langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle/(T_0 \cdots T_r - p^q),\\ \mathcal{O}\langle T_0, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}\rangle/(T_0 \cdots T_r - p^q)).\end{aligned} \tag{6.3.2}$$

[18] Noncomplete A are allowed in [Hub96, 1.7.3(iii)], so we choose

$$A^+ := \overline{W(k)}[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_d^{\pm 1}]/(T_0 \cdots T_r - p^q) \quad \text{and} \quad A^\triangleright := A^+\big[\tfrac{1}{p}\big].$$

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By also using the reduced fiber theorem [SP, 09IL], we enlarge K to ensure that, in addition,

$$A_0\widehat{\otimes}_{\mathcal{O}}\overline{W(k)} \cong A^\circ. \tag{6.3.3}$$

The connected components of $\mathrm{Spec}(A_0)$ on which p is a unit do not contribute to $\widehat{A_0}$, so we lose no generality by assuming that $\mathrm{Spec}(A_0)$ has no such components.

By working locally on $\mathrm{Spa}(\widehat{A_0}[\frac{1}{p}], \widehat{A_0})$, we refine the basis above to consist of those $\mathrm{Spa}(A, A^\circ)$ for which, in addition, there are finite subsets $\Psi_0 \subset (\widehat{A_0})^\times$ and $\Xi_0 \subset \widehat{A_0} \cap (\widehat{A_0}[\frac{1}{p}])^\times$ such that the map

$$K\langle (x_u^{\pm 1})_{u\in\Psi_0}, (x_a)_{a\in\Xi_0}\rangle \xrightarrow{x_u\mapsto u,\, x_a\mapsto a} \widehat{A_0}\big[\tfrac{1}{p}\big] \tag{6.3.4}$$

is surjective. Then there are finite subsets $\Psi \subset (A^\circ)^\times$ and $\Xi \subset A^\circ \cap A^\times$ such that the map

$$C\langle (X_u^{\pm 1})_{u\in\Psi}, (X_a)_{a\in\Xi}\rangle \xrightarrow{X_u\mapsto u,\, X_a\mapsto a} A \tag{6.3.5}$$

is also surjective. Defining the ring $B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi},\, (X_a)_{a\in\Xi}\rangle$ analogously to (6.2.3), so that the map (6.3.5) gives rise to the surjection

$$s\colon B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi},\, (X_a)_{a\in\Xi}\rangle \twoheadrightarrow A,$$

we set

$$D_{\Psi,\Xi,n}(A) := (B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi},\, (X_a)_{a\in\Xi}\rangle)/(\mathrm{Ker}\, s)^n \quad \text{and} \quad D_{\Psi,\Xi}(A) := \varprojlim_{n>0} D_{\Psi,\Xi,n}(A).$$

By [BMS18, 13.4(ii)], each $D_{\Psi,\Xi,n}(A)$ is a complete, strongly Noetherian Tate ring (in the sense of [Hub93, § 1]), with the image of $(A_{\mathrm{inf}}/\xi^n)\langle (X_u^{\pm 1})_{u\in\Sigma}, (X_a)_{a\in\Xi}\rangle$ endowed with its p-adic topology as a ring of definition. By construction, $D_{\Psi,\Xi,n}(A)$ is a nilpotent thickening of $D_{\Psi,\Xi,1}(A) \cong A$.

The ring B_{dR}^+ is a K-algebra (see § 6.1), so we let $(B_{\mathrm{dR}}^+/\xi^n)_0 \subset B_{\mathrm{dR}}^+/\xi^n$ for $n > 0$ be the (module-finite) A_{inf}/ξ^n-subalgebra generated by the image of $\mathcal{O}$. The proof of [BMS18, 13.11] shows (with R_A there replaced by our $\widehat{A_0}[\frac{1}{p}]$)[19] that the B_{dR}^+-algebra

$$B_{\mathrm{dR}}^+\widehat{\otimes}_K\big(A_0\big[\tfrac{1}{p}\big]\big) := \varprojlim_{n>0}\big(((B_{\mathrm{dR}}^+/\xi^n)_0\widehat{\otimes}_{\mathcal{O}}A_0)\big[\tfrac{1}{p}\big]\big) \tag{6.3.6}$$

has no nonzero ξ-torsion and is ξ-adically complete with

$$\begin{aligned}\big(B_{\mathrm{dR}}^+\widehat{\otimes}_K\big(A_0\big[\tfrac{1}{p}\big]\big)\big)/\xi &\cong A \quad \text{and, more generally,}\\ \big(B_{\mathrm{dR}}^+\widehat{\otimes}_K\big(A_0\big[\tfrac{1}{p}\big]\big)\big)/\xi^n &\cong ((B_{\mathrm{dR}}^+/\xi^n)_0\widehat{\otimes}_{\mathcal{O}}A_0)\big[\tfrac{1}{p}\big].\end{aligned}$$

The argument of footnote 19 shows that the map $(B_{\mathrm{dR}}^+/\xi^{n+1})_0\widehat{\otimes}_{\mathcal{O}}A_0 \to (B_{\mathrm{dR}}^+/\xi^n)_0\widehat{\otimes}_{\mathcal{O}}A_0$ is surjective for $n > 0$ (with the kernel of square zero, as may be seen after inverting p), so the subring

$$\varprojlim_{n>0}((B_{\mathrm{dR}}^+/\xi^n)_0\widehat{\otimes}_{\mathcal{O}}A_0) \subset B_{\mathrm{dR}}^+\widehat{\otimes}_K\big(A_0\big[\tfrac{1}{p}\big]\big) \quad \text{surjects onto} \quad \mathcal{O}_C\widehat{\otimes}_{\mathcal{O}}A_0 \overset{(6.3.3)}{\cong} A^\circ. \tag{6.3.7}$$

Moreover, we have the following analogue of [BMS18, 13.12(ii)] whose proof will be given in § 6.4.

[19] In fact, in our case the argument is simpler, and we sketch it here. Since $\mathrm{Spec}(A_0)$ has no connected components on which p is a unit, by [RG71, I.3.3.5] and [SP, 0593], the ring A_0 is free as an $\mathcal{O}$-module. Thus, the nth term of the inverse limit in (6.3.6) is a p-adically completed direct sum of copies of $(A_{\mathrm{inf}}/\xi^n)[\frac{1}{p}]$. This makes the multiplication by ξ^m map on this nth term explicit and the desired claims follow by passing to the inverse limit over n.

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LEMMA 6.3.8. *If* $\mathrm{Spa}(A, A^\circ)$ *is an element of the refined basis for* $X_{\text{ét}}$ *described above,* $\Psi \subset (A^\circ)^\times$ *(respectively,* $\Xi \subset A^\circ \cap A^\times$*) contains the images of the* T_i *for* $r+1 \leqslant i \leqslant d$ *(respectively,* $1 \leqslant i \leqslant r$*) under a coordinate morphism as in (6.3.1), and* Ψ *and* Ξ *are large enough (see the proof for the precise meaning), then*

$$D_{\Psi,\Xi}(A) \cong \big(B^+_{\mathrm{dR}} \widehat{\otimes}_K \big(A_0\big[\tfrac{1}{p}\big]\big)\big)\llbracket (X_a - \widetilde{a})_{a\in(\Psi\cup\Xi)\setminus\{T_1,\dots,T_d\}}\rrbracket. \tag{6.3.9}$$

where $\widetilde{a} \in \varprojlim_{n>0}((B^+_{\mathrm{dR}}/\xi^n)_0 \widehat{\otimes}_{\mathcal{O}} A_0) \subset B^+_{\mathrm{dR}} \widehat{\otimes}_K (A_0[\frac{1}{p}])$ *is a fixed lift of* a *(see (6.3.7)). In particular, for large* Ψ *and* Ξ*, the* B^+_{dR}*-algebra* $D_{\Psi,\Xi}(A)$ *has no nonzero* ξ*-torsion and is* ξ*-adically complete.*

Similarly to § 6.2, for any Ψ and Ξ as in (6.3.5), the derivations $\frac{\partial}{\partial \log(X_a)} := X_a \cdot \frac{\partial}{\partial X_a}$ with $a \in \Psi \cup \Xi$ extend to $D_{\Psi,\Xi}(A)$, and we may define the Koszul complex

$$\Omega^\bullet_{D_{\Psi,\Xi}(A)/B^+_{\mathrm{dR}}} := K_{D_{\Psi,\Xi}(A)}\left(\left(\frac{\partial}{\partial \log(X_u)}\right)_{u\in\Psi}, \left(\frac{\partial}{\partial \log(X_a)}\right)_{a\in\Xi}\right)$$

that is functorial in replacing Ψ and Ξ by larger Ψ' and Ξ'. Since $a \in A^\times$ for $a \in \Psi \cup \Xi$, the proof of [BMS18, 13.13] shows that for Ψ and Ξ to which Lemma 6.3.8 applies,

$$\Omega^\bullet_{D_{\Psi,\Xi}(A)/B^+_{\mathrm{dR}}}/\xi \cong \Omega^{\bullet,\mathrm{cont}}_{A/C} \quad \text{in the derived category,} \tag{6.3.10}$$

compatibly with enlarging Ψ and Ξ. In particular, due to the derived ξ-adic completeness supplied by Lemma 6.3.8, for such large enough Ψ and Ξ, the map

$$\Omega^\bullet_{D_{\Psi,\Xi}(A)/B^+_{\mathrm{dR}}} \to \Omega^\bullet_{D_{\Psi',\Xi'}(A)/B^+_{\mathrm{dR}}} \quad \text{is a quasi-isomorphism.}$$

Thus, if the element $\mathrm{Spa}(A, A^\circ)$ of the refined basis above also belongs to the basis considered in § 6.2, that is, if it has an étale coordinate map as in (6.2.1) and a surjection (6.2.2), then we obtain the functorial in $\mathrm{Spa}(A, A^\circ)$ quasi-isomorphism with the complex $\Omega^\bullet_{A/B^+_{\mathrm{dR}}}$ of (6.2.4):

$$\Omega^\bullet_{A/B^+_{\mathrm{dR}}} \xrightarrow{\sim} \varinjlim_{\Psi,\Xi}(\Omega^\bullet_{D_{\Psi,\Xi}(A)/B^+_{\mathrm{dR}}}). \tag{6.3.11}$$

Such $\mathrm{Spa}(A, A^\circ)$ still form a basis for $X_{\text{ét}}$ (see the parenthetical remark after (6.3.1)), so we conclude that the hypercohomology of the sheafification of the complex of presheaves furnished by the target of (6.3.11) is identified with $R\Gamma_{\mathrm{cris}}(X_{\text{ét}}/B^+_{\mathrm{dR}})$. In conclusion, we may summarize informally:

$$\text{the complexes} \quad \Omega^\bullet_{D_{\Psi,\Xi}(A)/B^+_{\mathrm{dR}}} \quad \text{also compute the } B^+_{\mathrm{dR}}\text{-cohomology} \quad R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}}) \tag{6.3.12}$$

and the maps (6.3.10) recover the identification (6.2.6).

6.4 Proof of Lemma 6.3.8

We adapt the proof of [BMS18, 13.12(ii)] as follows. In addition to the coordinate morphism (6.3.1) and its descent (6.3.2) used in the statement, we fix subsets

$$\Psi_0 \subset (\widehat{A_0})^\times \quad \text{and} \quad \Xi_0 \subset \widehat{A_0} \cap (\widehat{A_0}\big[\tfrac{1}{p}\big])^\times$$

such that, as in (6.3.4), the map

$$s_0 \colon K\langle (x_u^{\pm 1})_{u\in\Psi_0}, (x_a)_{a\in\Xi_0}\rangle \xrightarrow{x_u\mapsto u,\, x_a\mapsto a} \widehat{A_0}\big[\tfrac{1}{p}\big]$$

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is surjective and Ψ_0 (respectively, Ξ_0) contains the images of the T_i for $r+1 \leqslant i \leqslant d$ (respectively, $1 \leqslant i \leqslant r$) under the map (6.3.2). We require that Ψ (respectively, Ξ) contains the image of this Ψ_0 (respectively, Ξ_0) in A°; this is the meaning of 'large enough' in the statement. We set

$$D_{0,n} := K\langle (x_u^{\pm 1})_{u\in\Psi_0}, (x_a)_{a\in\Xi_0}\rangle/(\operatorname{Ker} s_0)^n \quad \text{for} \quad n>0 \quad \text{and} \quad D_0 := \varprojlim_{n>0} D_{0,n}.$$

The continuous map $K \to B_{\mathrm{dR}}^+$ (see § 6.1) gives a continuous map compatible with s_0 and s:

$$K\langle (x_u^{\pm 1})_{u\in\Psi_0}, (x_a)_{a\in\Xi_0}\rangle \xrightarrow{x_u\mapsto X_u,\, x_a\mapsto X_a} B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}, (X_a)_{a\in\Xi}\rangle, \quad \text{so also} \quad D_{0,n} \to D_{\Psi,\Xi,n}(A).$$

By the $K[T_1,\dots,T_r,T_{r+1}^{\pm 1},\dots,T_d^{\pm 1}]$-étaleness of $A_0[\frac{1}{p}]$, the map

$$A_0\big[\tfrac{1}{p}\big] \to \widehat{A_0}\big[\tfrac{1}{p}\big] \quad \text{lifts to a map} \quad A_0\big[\tfrac{1}{p}\big] \to D_0 \quad \text{with} \quad T_i \mapsto x_{T_i}. \tag{6.4.1}$$

By [GR03, 7.3.15], for each $n>0$, the subring $D_{0,n}^\circ \subset D_{0,n}$ of powerbounded elements is the preimage of its counterpart $(\widehat{A_0}[\frac{1}{p}])^\circ \subset \widehat{A_0}[\frac{1}{p}]$. Thus, the lift (6.4.1) maps A_0 to $D_{0,n}^\circ$, so also to some ring of definition of $D_{0,n}$. By composing with the map $D_{0,n} \to D_{\Psi,\Xi,n}(A)$, we obtain the map $A_0 \to D_{\Psi,\Xi,n}(A)$ whose image lies in some ring of definition, so, as n varies, also the map

$$B_{\mathrm{dR}}^+ \widehat{\otimes}_K \big(A_0\big[\tfrac{1}{p}\big]\big) \to D_{\Psi,\Xi}(A) \quad \text{that is compatible with the maps to } A.$$

This gives rise to the continuous map y in the diagram

$$\begin{array}{ccccc} & \xrightarrow{X_{T_i}\mapsto T_i} & \big(B_{\mathrm{dR}}^+ \widehat{\otimes}_K \big(A_0\big[\tfrac{1}{p}\big]\big)\big)[\![(X_a-\widetilde{a})_{a\in(\Psi\cup\Xi)\setminus\{T_1,\dots,T_d\}}]\!] & & \\ B_{\mathrm{dR}}^+\langle (X_u^{\pm 1})_{u\in\Psi}, (X_a)_{a\in\Xi}\rangle & & \quad y \big(\quad \big) z & \xrightarrow{X_a\mapsto a} & A \\ & \searrow & D_{\Psi,\Xi}(A) & \nearrow & \end{array}$$

whose maps '$X_{T_i} \mapsto T_i$' and z are defined as follows.

- To define the map '$X_{T_i} \mapsto T_i$', one first forms the inverse limit over N of the maps

$$(A_{\mathrm{inf}}/\xi^n)\langle (X_u^{\pm 1})_{u\in\Psi}, (X_a)_{a\in\Xi}\rangle \xrightarrow{X_{T_i}\mapsto T_i} ((B_{\mathrm{dR}}^+/\xi^n)_0 \otimes_{\mathcal{O}} A_0/p^N)[\![(X_a-\widetilde{a})_{a\in(\Psi\cup\Xi)\setminus\{T_1,\dots,T_d\}}]\!]$$

  defined by using the fact that each $\widetilde{u}$ with $u\in\Psi$ is a unit in $(B_{\mathrm{dR}}^+/\xi^n)_0 \widehat{\otimes}_{\mathcal{O}} A_0$ (see the sentence around (6.3.7)) and the identity $X_u^{-1} = \widetilde{u}^{-1}(1-\widetilde{u}^{-1}(X_u-\widetilde{u})+\widetilde{u}^{-2}(X_u-\widetilde{u})^2-\cdots)$. Then one inverts p and forms the inverse limit over n.
- The continuous map z is defined by combining the top part of the diagram, the ξ-adic completeness of $B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}])$ (see § 6.3), and the definition of $D_{\Psi,\Xi}(A)$.

By construction, the diagram commutes, so $y\circ z = \mathrm{id}$. By $K[T_1,\dots,T_r,T_{r+1}^{\pm 1},\dots,T_d^{\pm 1}]$-étaleness of $A_0[\frac{1}{p}]$, the $K[T_1,\dots,T_r,T_{r+1}^{\pm 1},\dots,T_d^{\pm 1}]$-algebra endomorphism $z\circ y$ of the prothickening $(B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}]))[\![(X_a-\widetilde{a})_{a\in(\Psi\cup\Xi)\setminus\{T_1,\dots,T_d\}}]\!]$ of A is the identity on $A_0[\frac{1}{p}]$, so also on $(B_{\mathrm{dR}}^+ \widehat{\otimes}_K (A_0[\frac{1}{p}]))$. It also fixes every X_a, so it must be the identity. Thus, z is the desired isomorphism (6.3.9). □

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6.5 The map from the absolute crystalline cohomology

Returning to the $\mathfrak{X}$ of § 1.5, our next goal is to use the preceding discussion to exhibit a map

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+) \quad \text{over} \quad A_{\mathrm{cris}} \xrightarrow{\S 6.1} B_{\mathrm{dR}}^+. \tag{6.5.1}$$

For this, we use the basis of $\mathfrak{X}_{\text{ét}}$ consisting of the affine opens $\mathrm{Spf}(R)$ as in the 'all possible coordinates' setting of § 5.17 and adopt the subsequent notation of §§ 5.17–5.40. To relate to § 6.3, we set

$$A := R\big[\tfrac{1}{p}\big], \quad \Psi := \{t_\sigma\}_{\sigma\in\Sigma} \cup \bigcup_{\lambda\in\Lambda}\{t_{\lambda,r_\lambda+1},\dots,t_{\lambda,d}\}, \quad \text{and} \quad \Xi := \bigcup_{\lambda\in\Lambda}\{t_{\lambda,1},\dots,t_{\lambda,r_\lambda}\} \tag{6.5.2}$$

(so that $A^\circ \cong R$ and $t_{\lambda,0}$ are omitted). We may descend the étale map (5.17.3) for $\lambda \in \Lambda$ to the ring of integers of a finite subextension $W(k)[\frac{1}{p}] \subset K \subset C$ (see (1.5.2)) and then obtain the descended coordinate map (6.3.2) on the generic fiber. In addition, by enlarging K and using the closed immersion (5.17.2), we may ensure that the descent has a closed immersion (6.3.4) (with $\Xi_0 = \emptyset$); we then enlarge Ψ by adjoining the image in $(A^\circ)^\times$ of the resulting Ψ_0. Thus, the above choices of A, the enlarged Ψ, and Ξ satisfy the assumptions of § 6.3: specifically, $\mathrm{Spa}(A, A^\circ)$ is an element of the (refined) basis of $(\mathfrak{X}_C^{\mathrm{ad}})_{\text{ét}}$ considered there, Ψ (respectively, Ξ) contains $t_{\lambda,i}$ for $r_\lambda + 1 \leqslant i \leqslant d$ (respectively, $1 \leqslant i \leqslant r_\lambda$), and Lemma 6.3.8 applies to (the enlarged) Ψ and Ξ. In conclusion, with these choices, the entire § 6.3 applies.

By using descents and [GR03, 7.3.15] as in the proof of Lemma 6.3.8, we see that the elements $[(p^{1/p^\infty})^{q_\lambda}]/(X_{t_{\lambda,1}}\cdots X_{t_{\lambda,r_\lambda}})$ of $D_{\Psi,\Xi,n}(A)$ lie in $(D_{\Psi,\Xi,n}(A))^\circ$. Thus, each $(D_{\Psi,\Xi,n}(A))^\circ$, so also $D_{\Psi,\Xi}(A)$, is naturally an algebra over the ring $A^\square_{\Sigma,\Lambda}$ defined in (5.22.1). In fact, since each $D_{\Psi,\Xi,n}(A)$ is a $\mathbb{Q}$-algebra in which ξ^m vanishes for $m \geqslant n$ and each X_a is a unit in $D_{\Psi,\Xi}(A)$, the universal relations (5.26.3) and (5.27.3) imply that each $(D_{\Psi,\Xi,n}(A))^\circ$, so also $D_{\Psi,\Xi}(A)$, is naturally an algebra over

$$(A^\square_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \quad \text{for} \quad \lambda_0 \in \Lambda,$$

compatibly with the maps

$$(A^\square_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \xrightarrow{(5.26.4)\text{ and }(5.27.4)} R \quad \text{and} \quad (D_{\Psi,\Xi,n}(A))^\circ \to A^\circ \cong R \tag{6.5.3}$$

and the 'change of λ_0' isomorphisms (5.26.7). The resulting algebra structure map factors through some (necessarily p-adically complete) ring of definition $(D_{\Psi,\Xi,n}(A))_0$:

$$(A^\square_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \to (D_{\Psi,\Xi,n}(A))_0 \hookrightarrow (D_{\Psi,\Xi,n}(A))^\circ \hookrightarrow D_{\Psi,\Xi,n}(A),$$

so the map $(D_{\Psi,\Xi,n}(A))_0 \to R$ is surjective. In addition, by [SP, 07GM], the kernel of the map $(D_{\Psi,\Xi,n}(A))^\circ \twoheadrightarrow R/p$ has a unique divided power structure, so we obtain a map

$$D_{j_{\lambda_0}} \to (D_{\Psi,\Xi,n}(A))^\circ \tag{6.5.4}$$

from the divided power envelope $D_{j_{\lambda_0}}$ defined in § 5.28. For a fixed n and modulo the $\xi^m/m!$ with $m \geqslant n$, the kernel of the map $(A^\square_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \twoheadrightarrow R/p$ is finitely generated and, due to the surjectivity of $(A^\square_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}] \twoheadrightarrow R$, the generating set may be arranged to consist of p and a finite set of elements that vanish already in R. Thus, since $D_{j_{\lambda_0}}$ is generated as an $((A^\square_{\Sigma,\Lambda} \otimes_{A_{\mathrm{inf}}} A^0_{\mathrm{cris}}) \otimes_{\mathbb{Z}[Q]} \mathbb{Z}[P_{\lambda_0}])$-algebra by the divided powers of the elements of this kernel, by enlarging $(D_{\Psi,\Xi,n}(A))_0$ we may factor the map (6.5.4) as follows:

$$D_{j_{\lambda_0}} \to (D_{\Psi,\Xi,n}(A))_0 \hookrightarrow (D_{\Psi,\Xi,n}(A))^\circ \hookrightarrow D_{\Psi,\Xi,n}(A).$$

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Consequently, we obtain continuous maps that are independent of λ_0 and compatible as n varies:

$$\widehat{D_{j_{\lambda_0}}} \overset{(5.29.1)}{\cong} D_{\Sigma,\Lambda} \to (D_{\Psi,\Xi,n}(A))^\circ \hookrightarrow D_{\Psi,\Xi,n}(A), \quad \text{so also} \quad D_{\Sigma,\Lambda} \to D_{\Psi,\Xi}(A). \tag{6.5.5}$$

Via the last map, the $D_{j_{\lambda_0}}$-valued derivations $\frac{\partial}{\partial \log(X_\sigma)}$ for $\sigma \in \Sigma$ and $\frac{\partial}{\partial \log(X_{\lambda,i})}$ for $\lambda \in \Lambda$ and $1 \leqslant i \leqslant d$ of $D_{j_{\lambda_0}}$ are compatible with the corresponding $D_{\Psi,\Xi}(A)$-valued derivations of $D_{\Psi,\Xi}(A)$ (see (6.5.2) and §§ 5.31 and 6.3). Thus, due to the density of $D_{j_{\lambda_0}}$ in $D_{\Sigma,\Lambda}$, the same compatibility holds for the map $D_{\Sigma,\Lambda} \to D_{\Psi,\Xi}(A)$, to the effect that we obtain a map of complexes

$$K_{D_{\Sigma,\Lambda}}\left(\left(\frac{\partial}{\partial \log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\frac{\partial}{\partial \log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\leqslant i\leqslant d}\right) \to K_{D_{\Psi,\Xi}(A)}\left(\left(\frac{\partial}{\partial \log(X_a)}\right)_{a\in\Psi\cup\Xi}\right).$$

Its formation commutes with enlarging Σ and Λ (and, respectively, Ψ and Ξ), so we obtain the map

$$\varinjlim_{\Sigma,\Lambda}\left(K_{D_{\Sigma,\Lambda}}\left(\left(\frac{\partial}{\partial \log(X_\sigma)}\right)_{\sigma\in\Sigma}, \left(\frac{\partial}{\partial \log(X_{\lambda,i})}\right)_{\lambda\in\Lambda,\,1\leqslant i\leqslant d}\right)\right) \to \varinjlim_{\Psi,\Xi}\left(\Omega^\bullet_{D_{\Psi,\Xi}(R[\frac{1}{p}])/B^+_{\mathrm{dR}}}\right). \tag{6.5.6}$$

The formation of this map is compatible with replacing R by a p-adically formally étale R-algebra R' equipped with data as in § 5.17. The resulting map of complexes of presheaves gives rise to the map of complexes of sheaves on $\mathfrak{X}_{\text{ét}}$ from the complex whose $R\Gamma(\mathfrak{X}_{\text{ét}}, -)$ is identified with $R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ (see (5.23.3) and § 5.32) to the pushforward of the complex whose $R\Gamma((\mathfrak{X}^{\mathrm{ad}}_C)_{\text{ét}}, -)$ is identified with $R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}})$ (see § 6.3 and (6.3.12)). Thus, by applying $R\Gamma(\mathfrak{X}_{\text{ét}}, -)$, we obtain the desired map (6.5.1):

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \to R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}).$$

In addition, by its construction and Lemma 5.29, the map $D_{\Sigma,\Lambda} \to D_{\Psi,\Xi}(R[\frac{1}{p}])$ of (6.5.5) is compatible with the maps to $R[\frac{1}{p}]$ (see (6.5.3)). Thus, [BMS18, 13.13] used to obtain (6.3.10) implies that the map (6.5.6) is compatible with the maps in the derived category to $\Omega^{\bullet,\mathrm{cont}}_{R[\frac{1}{p}]/C}$ described in the last display of § 5.32 and (6.3.10). In conclusion, the map (6.5.1) fits into the following commutative square.

$$\begin{array}{ccc} R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) & \xrightarrow{(6.5.1)} & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}) \\ \downarrow{\scriptstyle (5.23.2)} & & \downarrow{\scriptstyle (6.2.6)} \\ R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) & \longrightarrow & R\Gamma_{\mathrm{dR}}(\mathfrak{X}^{\mathrm{ad}}_C/C) \end{array} \tag{6.5.7}$$

Having constructed the map (6.5.1), we are ready for the following extension of [BMS18, 13.23].

Theorem 6.6. *If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then the map (6.5.1) induces the identification*

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B^+_{\mathrm{dR}} \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}) \tag{6.6.1}$$

and the cohomology modules of $R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}})$ are finite free over B^+_{dR}. In particular, then

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}} \cong R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}), \tag{6.6.2}$$

compatibly with the identifications modulo ξ with $R\Gamma_{\mathrm{dR}}(\mathfrak{X}^{\mathrm{ad}}_C/C)$ given by (4.18.1) and (6.2.6).

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Proof. By Corollaries 4.20 and 5.43, the object $R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})$ of $D(A_{\mathrm{cris}})$ is perfect and its cohomology modules become finite free after inverting p. Therefore, due to (5.24.1) and the derived p-adic completeness, we have the identification

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}})\otimes^{\mathbb{L}}_{A_{\mathrm{cris}}}\mathcal{O}_C\xrightarrow[\sim]{(5.23.2)}R\Gamma_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C).$$

Consequently, both sides of (6.6.1) are derived ξ-adically complete (see (6.2.5)) and, due to (6.2.6) and the commutativity of the diagram (6.5.7), the map (6.5.1) identifies their reductions modulo ξ. In conclusion, (6.5.1) induces the desired identification (6.6.1) and the B^+_{dR}-freeness claim follows from the first sentence of the proof. The combination of (5.43.2) and (6.6.1) gives (6.6.2) and the asserted compatibility follows from Proposition 5.41 and the commutativity of (6.5.7). □

6.7 The B^+_{dR}-cohomology and the étale cohomology

For any proper, smooth adic space X over C, in [BMS18, 13.1] Bhatt–Morrow–Scholze constructed the functorial in X identification

$$R\Gamma_{\mathrm{cris}}(X/B^+_{\mathrm{dR}})\otimes_{B^+_{\mathrm{dR}}}B_{\mathrm{dR}}\cong R\Gamma_{\text{ét}}(X,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}B_{\mathrm{dR}}. \tag{6.7.1}$$

Due to the identification (6.2.8), when $X\cong X_0\widehat{\otimes}_K C$ for a proper, smooth adic space X_0 defined over a complete, discretely valued subfield $K\subset C$ that has a perfect residue field, the inverse of (6.7.1) supplies the functorial in X_0 de Rham comparison isomorphism

$$R\Gamma_{\text{ét}}(X_0\widehat{\otimes}_K C,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}B_{\mathrm{dR}}\cong R\Gamma_{\mathrm{dR}}(X_0/K)\otimes_K B_{\mathrm{dR}}. \tag{6.7.2}$$

If $C\cong\widehat{\overline{K}}$, then, by transport of structure, the identification (6.7.2) is $\mathrm{Gal}(\overline{K}/K)$-equivariant (with $\mathrm{Gal}(\overline{K}/K)$ acting trivially on $R\Gamma_{\mathrm{dR}}(X_0/K)$) and, by [BMS18, 13.1], it recovers the isomorphism constructed in [Sch13a, 8.4]. In particular, in this case, (6.7.2) is compatible with filtrations, where B_{dR} is filtered by its discrete valuation and $R\Gamma_{\mathrm{dR}}(X_0/K)$ (respectively, $R\Gamma_{\text{ét}}(X_0\widehat{\otimes}_K C,\mathbb{Z}_p)$) is equipped with the Hodge (respectively, trivial) filtration.

For proper $\mathfrak{X}$, we now have two ways to identify

$$R\Gamma(\mathfrak{X}_{\text{ét}},A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}B_{\mathrm{dR}}\quad\text{with}\quad R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C,\mathbb{Z}_p)\otimes^{\mathbb{L}}_{\mathbb{Z}_p}B_{\mathrm{dR}}\colon$$

we can either base change (2.3.1) to B_{dR} or combine (6.6.2) and (6.7.1). We now prove that the two ways give the same identification; this will be important in the proof of Theorem 8.7.

PROPOSITION 6.8. *If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then the map $R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}})\to R\Gamma(\mathfrak{X}^{\mathrm{ad}}_C,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}B^+_{\mathrm{dR}}$ of [BMS18, proof of 13.1] that underlies the identification (6.7.1) for $X=\mathfrak{X}^{\mathrm{ad}}_C$ makes the diagram*

$$\begin{array}{ccccc}
R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) & & \xrightarrow{\quad(6.5.1)\quad} & & R\Gamma_{\mathrm{cris}}(\mathfrak{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}) \\
\wr\downarrow{\scriptstyle(5.40.1)} & & & & \downarrow \\
R\Gamma(\mathfrak{X}_{\text{ét}},A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}A_{\mathrm{cris}} & \xrightarrow{\textit{[BMS18, 6.10]}} & R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C,\mathbb{A}_{\mathrm{inf},\mathfrak{X}^{\mathrm{ad}}_C})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}B^+_{\mathrm{dR}} & \xleftarrow[\sim]{(2.3.2)} & R\Gamma(\mathfrak{X}^{\mathrm{ad}}_C,\mathbb{Z}_p)\otimes^{\mathbb{L}}_{\mathbb{Z}_p}B^+_{\mathrm{dR}}
\end{array}$$

commute; in particular, the identification of $R\Gamma(\mathfrak{X}_{\text{ét}},A\Omega_{\mathfrak{X}})\otimes^{\mathbb{L}}_{A_{\mathrm{inf}}}B_{\mathrm{dR}}$ with $R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C,\mathbb{Z}_p)\otimes^{\mathbb{L}}_{\mathbb{Z}_p}B_{\mathrm{dR}}$ that results from (2.3.1) (and is encoded by the bottom part of the above diagram) agrees with the identification that results from (6.6.2) and (6.7.1) (and is encoded by the top part of the diagram).

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Proof. Since $\varphi^{-1}(\mu)$ lies in $W(\mathfrak{m}^\flat)$ and is a unit in B_{dR}^+, the discussion after Theorem 2.3 implies that the map labeled '(2.3.2)' in the diagram is an isomorphism. In particular, due to [Sch13a, 5.1], the object $R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+$ of $D(B_{\mathrm{dR}}^+)$ is perfect. We will now review the definition given in [BMS18, proof of 13.1] of the composition f of the right vertical map with this map '(2.3.2)'.

Let $\mathrm{Spa}(A, A^\circ)$ be an element of the basis for the analytic topology of $\mathfrak{X}_C^{\mathrm{ad}}$ discussed in § 6.2. For a large enough set Ψ as in § 6.2, we consider the surjection $C\langle (X_u^{\pm 1})_{u\in\Psi}\rangle \xrightarrow{X_u \mapsto u} A$ from (6.2.2), as well as the perfectoid $(\prod_\Psi \mathbb{Z}_p(1))$-cover $C\langle (X_u^{\pm 1/p^\infty})_{u\in\Psi}\rangle$ of $C\langle (X_u^{\pm 1})_{u\in\Psi}\rangle$. Granted that Ψ contains the images of the T_i under some étale coordinate map (6.2.1), the base change of this cover to $\mathrm{Spa}(A, A^\circ)$ is a perfectoid $(\prod_\Psi \mathbb{Z}_p(1))$-cover

$$\mathrm{Spa}(A_{\Psi,\infty}, A^+_{\Psi,\infty}) \to \mathrm{Spa}(A, A^\circ). \tag{6.8.1}$$

Each $u \in \Psi$ has a canonical system u^{1/p^∞} of p-power roots in $A^+_{\Psi,\infty}$, which gives the unit $[u^{1/p^\infty}]$ in the B_{dR}^+-algebra $\mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty})$ (see Proposition 5.36). Since $\mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty})$ may be viewed as a pro-(infinitesimal thickening) of $A_{\Psi,\infty}$, the map $X_u \mapsto [u^{1/p^\infty}]$ extends to a continuous B_{dR}^+-morphism

$$D_\Psi(A) \to \mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty}) \quad \text{over} \quad A \to A_{\Psi,\infty}. \tag{6.8.2}$$

By construction, for each $u \in \Psi$, this morphism intertwines $\exp\left(\log([\epsilon]) \cdot \frac{\partial}{\partial \log(X_u)}\right)$ defined by the formula (5.15.1) and viewed as a ring endomorphism of $D_\Psi(A)$ with the action of the generator $[\epsilon]$ of the uth copy of $\mathbb{Z}_p(1)$ on $\mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty})$. In particular, letting γ_u denote this generator, we may use the same formula as in (5.16.1) to define the morphism of complexes

$$\Omega^\bullet_{D_\Psi(A)/B^+_{\mathrm{dR}}} = K_{D_\Psi(A)}\left(\left(\frac{\partial}{\partial \log(X_u)}\right)_{u\in\Psi}\right) \to K_{\mathbb{B}^+_{\mathrm{dR}}(A^+_{\Psi,\infty})}((\gamma_u - 1)_{u\in\Psi}), \tag{6.8.3}$$

whose formation is functorial in Ψ and, after passing to the direct limit over all Ψ, also in $\mathrm{Spa}(A, A^\circ)$. The almost purity theorem identifies the cohomology of the sheaf of complexes determined by the target of (6.8.3) with $R\Gamma_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{A}_{\mathrm{inf},\mathfrak{X}_C^{\mathrm{ad}}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+$ (see [BMS18, proof of 13.1]). The cohomology of the sheaf of complexes determined by the source of (6.8.3) is, by definition, $R\Gamma_{\mathrm{cris}}(\mathfrak{X}_C^{\mathrm{ad}}/B_{\mathrm{dR}}^+)$ (see § 6.2). Therefore, by passing to the direct limit over all Ψ, sheafifying, and forming cohomology, the maps (6.8.3) produce the aforementioned composition f defined in [BMS18, proof of 13.1].

The same construction gives the morphisms (6.8.3) for the objects $\mathrm{Spa}(A, A^\circ)$ of the basis of the étale topology of $\mathfrak{X}_C^{\mathrm{ad}}$ considered in § 6.2. Due to (6.2.7), this leads to the same map f. In addition, we may generalize the construction of the morphisms (6.8.3) further by using the basis for the étale topology of $\mathfrak{X}_C^{\mathrm{ad}}$ considered in § 6.3: the cover (6.8.1) gets replaced by the cover

$$\mathrm{Spa}(A_{\Psi,\Xi,\infty}, A^+_{\Psi,\Xi,\infty}) \to \mathrm{Spa}(A, A^\circ)$$

that is the base change of the perfectoid

$$\left(\textstyle\prod_\Psi \mathbb{Z}_p(1) \times \prod_\Xi \mathbb{Z}_p(1)\right)\text{-cover} \quad C\langle (X_u^{\pm 1/p^\infty})_{u\in\Psi}, (X_a^{1/p^\infty})_{a\in\Xi}\rangle \quad \text{of} \quad C\langle (X_u^{\pm 1})_{u\in\Psi}, (X_a)_{a\in\Xi}\rangle$$

for large enough $\Psi \subset (A^\circ)^\times$ and $\Xi \subset A^\circ \cap A^\times$, and the rest is (mildly) modified accordingly. Due to (6.3.12), this variant of the construction gives the same map f.

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In conclusion, since the construction of f may be carried out in the setting of § 6.3 and follows the same pattern as the construction of the map (5.40.1), namely, is based on the map as in (5.16.1), all we need to check is that, in the notation of § 6.5, the following diagram commutes.

$$\begin{array}{ccc} D_{\Sigma,\Lambda} & \xrightarrow{(6.5.5)} & D_{\Psi,\Xi}(A) \\ {\scriptstyle (5.38.1)}\downarrow & & \downarrow{\scriptstyle 6.8.2} \\ \mathbb{A}_{\mathrm{cris}}(R_{\Sigma,\Lambda,\infty}) & \xrightarrow{(5.36.1)} & \mathbb{B}^{+}_{\mathrm{dR}}(A^{+}_{\Psi,\Xi,\infty}) \end{array} \tag{6.8.4}$$

For this desired commutativity, we may first replace $\mathbb{B}^{+}_{\mathrm{dR}}(A^{+}_{\Psi,\Xi,\infty})$ by $\mathbb{B}^{+}_{\mathrm{dR}}(A^{+}_{\Psi,\Xi,\infty})/\xi^{n}$ for a variable $n>0$, then replace $D_{\Sigma,\Lambda}$ by $D_{j_{\lambda_0}}$ for some $\lambda_0\in\Lambda$, and, finally, since $\mathbb{B}^{+}_{\mathrm{dR}}(A^{+}_{\Psi,\Xi,\infty})/\xi^{n}$ is a $\mathbb{Q}$-algebra and $D_{j_{\lambda_0}}$ is generated by divided powers, replace $D_{j_{\lambda_0}}$ by

$$(A^{\square}_{\Sigma,\Lambda}\otimes_{A_{\mathrm{inf}}}A^{0}_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}].$$

However, each X_τ of (5.19.2) with either $\tau=\sigma$ for $\sigma\in\Sigma$ or $\tau=(\lambda,i)$ for $\lambda\in\Lambda$ and $1\leqslant i\leqslant d$ maps to the unit $[X_\tau^{1/p^\infty}]\in(\mathbb{B}^{+}_{\mathrm{dR}}(A^{+}_{\Psi,\Xi,\infty}))^{\times}$ under either of the two maps from

$$(A^{\square}_{\Sigma,\Lambda}\otimes_{A_{\mathrm{inf}}}A^{0}_{\mathrm{cris}})\otimes_{\mathbb{Z}[Q]}\mathbb{Z}[P_{\lambda_0}]\quad\text{to}\quad\mathbb{B}^{+}_{\mathrm{dR}}(A^{+}_{\Psi,\Xi,\infty})/\xi^{n}$$

supplied by the diagram (6.8.4), so these two maps indeed agree, as desired. □

7. The A_{inf}-cohomology modules $H^{i}_{A_{\mathrm{inf}}}(\mathfrak{X})$ and their specializations

In this section, we define and analyze the A_{inf}-cohomology groups $H^{i}_{A_{\mathrm{inf}}}(\mathfrak{X})$ of an $\mathcal{O}_C$-proper $\mathfrak{X}$. We show that each $H^{i}_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a Breuil–Kisin–Fargues module (see Theorem 7.4) and deduce that, loosely speaking, the p-adic étale cohomology of $\mathfrak{X}^{\mathrm{ad}}_{C}$ has at most the amount of torsion contained in the logarithmic crystalline cohomology of $\mathfrak{X}_k$ or the logarithmic de Rham cohomology of $\mathfrak{X}$ (see Theorems 7.9 and 7.12). Most of these results are variants of their analogues established in the smooth case in [BMS18]. Their proofs, granted inputs from § 2 and §§ 4–5, are generally similar to those of [BMS18] and in large part rely on commutative algebra over A_{inf}.

7.1 Properness of $\mathfrak{X}$

Throughout § 7, we assume that $\mathfrak{X}$ is proper and $\mathfrak{X}_k$ is purely d-dimensional.

7.2 The A_{inf}-cohomology $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$

We use the object $A\Omega_{\mathfrak{X}}\in D^{\geqslant 0}(\mathfrak{X}_{\text{ét}},A_{\mathrm{inf}})$ of § 2.2 to set

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}):=R\Gamma(\mathfrak{X}_{\text{ét}},A\Omega_{\mathfrak{X}})\in D^{\geqslant 0}(A_{\mathrm{inf}})\quad\text{and}\quad H^{i}_{A_{\mathrm{inf}}}(\mathfrak{X}):=H^{i}(R\Gamma(\mathfrak{X}_{\text{ét}},A\Omega_{\mathfrak{X}}))\quad\text{for } i\in\mathbb{Z}.$$

Since $L\eta$ commutes with pullback along a flat morphism of ringed topoi (see [BMS18, 6.14]), the object $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is contravariantly functorial in $\mathfrak{X}$: an $\mathcal{O}_C$-morphism $\mathfrak{X}'\to\mathfrak{X}$ induces a morphism

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})\to R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}')\quad\text{in}\quad D^{\geqslant 0}(A_{\mathrm{inf}}),\quad\text{so also}\quad H^{i}_{A_{\mathrm{inf}}}(\mathfrak{X})\to H^{i}_{A_{\mathrm{inf}}}(\mathfrak{X}')\quad\text{for}\quad i\in\mathbb{Z}.$$

Corollary 4.20 ensures that $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is perfect, that is, isomorphic to a bounded complex of finite free A_{inf}-modules. Moreover, by (2.3), (4.18.1), and (5.43.2), we have the following

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identifications:

$$\begin{aligned} R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right] &\cong R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right]; \\ R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}},\theta} \mathcal{O}_C &\cong R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C); \\ R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} W(k) &\cong R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)). \end{aligned} \tag{7.2.1}$$

If $\mathfrak{X}_k$ is k-smooth, then we may drop 'log' from the subscripts (compare with (5.44.2)).

The Frobenius morphism (2.2.5) gives rise to the Frobenius morphism

$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \to R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \quad \text{in} \quad D^{\geqslant 0}(A_{\mathrm{inf}})$$

that becomes an isomorphism after inverting $\varphi(\xi)$ (see (2.2.6)). Consequently the cohomology modules $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ come equipped with the A_{inf}-module morphism

$$\varphi\colon H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}} \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$$

that becomes an isomorphism after inverting $\varphi(\xi)$. We will prove in Theorem 7.4 that these morphisms make each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ a Breuil–Kisin–Fargues module in the following sense of [BMS18, 4.22].

7.3 Breuil–Kisin–Fargues modules

A *Breuil–Kisin–Fargues module* is a finitely presented A_{inf}-module M equipped with an $A_{\mathrm{inf}}[\frac{1}{\varphi(\xi)}]$-module isomorphism

$$\varphi_M\colon (M \otimes_{A_{\mathrm{inf}},\varphi} A_{\mathrm{inf}})\left[\tfrac{1}{\varphi(\xi)}\right] \xrightarrow{\sim} M\left[\tfrac{1}{\varphi(\xi)}\right]$$

such that $M[\frac{1}{p}]$ is $A_{\mathrm{inf}}[\frac{1}{p}]$-free. By [BMS18, 4.9(i)], any such M is perfect as an A_{inf}-module, that is, M has a finite resolution by finite free A_{inf}-modules. A morphism of Breuil–Kisin–Fargues modules is an A_{inf}-module morphism that commutes with the isomorphisms φ_M.

Theorem 7.4. *Each $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}), \varphi)$ is a Breuil–Kisin–Fargues module which vanishes unless $i \in [0, 2d]$. In particular, each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is perfect as an A_{inf}-module and each $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\frac{1}{p}]$ is $A_{\mathrm{inf}}[\frac{1}{p}]$-free.*

Proof. Due to the relation with $R\Gamma_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$, each $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\frac{1}{p\mu}]$ is a free $A_{\mathrm{inf}}[\frac{1}{p\mu}]$-module. Moreover, by Corollary 5.43, the cohomology modules of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} A_{\mathrm{cris}}[\frac{1}{p}]$ are free over $A_{\mathrm{cris}}[\frac{1}{p}]$. Therefore, [BMS18, 4.20] applies and proves that each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a finitely presented A_{inf}-module that becomes free after inverting p, so $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}), \varphi)$ is a Breuil–Kisin–Fargues module.

Since $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ is perfect, its top degree cohomology is finitely presented and of formation compatible with base change. Thus, by the de Rham specialization of (7.2.1) and the Nakayama lemma, $H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) = 0$ for $i > 2d$. The same holds for $i < 0$ because $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \in D^{\geqslant 0}(A_{\mathrm{inf}})$. □

For completeness sake, we mention the following corollary, which may also be proved more directly.

Corollary 7.5. *For each $i \in \mathbb{Z}$, the rank of the finitely presented $\mathbb{Z}_p$-module $H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ is equal to the rank of the finitely presented $W(k)$-module $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$, and is also equal to the rank of the finitely presented $\mathcal{O}_C$-module $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) := R^i\Gamma(\mathfrak{X}_{\text{ét}}, \Omega^\bullet_{\mathfrak{X}/\mathcal{O}_C, \log})$ (see also (7.10.1) below).*

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Proof. The finite presentation assertions follow, for instance, from the perfectness of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$, the comparisons (7.2.1), and the coherence of the ring $\mathcal{O}_C$. Due to Theorem 7.4 and the comparisons (7.2.1), all the ranks in question are equal to the rank of the free $A_{\mathrm{inf}}[\frac{1}{p}]$-module $(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))[\frac{1}{p}]$. □

7.6 Base change for individual $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$

Since $A_{\mathrm{inf}}[\frac{1}{\mu}]$ is A_{inf}-flat, equation (7.2.1) implies that

$$(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))\left[\tfrac{1}{\mu}\right] \cong H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right] \quad \text{for each} \quad i \in \mathbb{Z}. \tag{7.6.1}$$

In particular, since μ is a unit in $W(C^\flat)$ and $W(C^\flat)$ is A_{inf}-flat (the localization of A_{inf} at the prime ideal (p) is a discrete valuation ring whose completion is $W(C^\flat)$)

$$H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(C^\flat) \cong H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(C^\flat). \tag{7.6.2}$$

A similar de Rham comparison consists of the following exact sequences that result from (7.2.1) and [SP, 0662]:

$$0 \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \to H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \to (H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X}))[\xi] \to 0 \quad \text{for each} \quad i \in \mathbb{Z}. \tag{7.6.3}$$

Similarly, by Theorem 7.4 and [BMS18, 4.9], we have a Frobenius-equivariant exact sequence

$$0 \to H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k) \to H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \to \mathrm{Tor}^1_{A_{\mathrm{inf}}}(H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X}), W(k)) \to 0 \tag{7.6.4}$$

for each $i \in \mathbb{Z}$. In particular, we have the top degree base changes

$$H^{2d}_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}},\theta} \mathcal{O}_C \cong H^{2d}_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \quad \text{and} \quad H^{2d}_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k) \cong H^{2d}_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)).$$

Due to Theorem 7.4, the injections in the sequences (7.6.3)–(7.6.4) become isomorphisms after inverting p. The same holds without inverting p in the case when $H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X})$ is A_{inf}-free. For such freeness, we have the following consequence of Theorem 7.4 and [BMS18, § 4.2].

PROPOSITION 7.7. *For each $i \in \mathbb{Z}$, the $\mathcal{O}_C$-module $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is p-torsion free (equivalently, free) if and only if the $W(k)$-module $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ is p-torsion free (equivalently, free), in which case $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is free as an A_{inf}-module and $H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ is free as a $\mathbb{Z}_p$-module.*

Proof. Due to Theorem 7.4, we may apply [BMS18, 4.18] and combine it with (7.2.1) to conclude that $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ is p-torsion free if and only if so is $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$. When these conditions hold, the freeness of $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and $H^i_{\text{ét}}(\mathfrak{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ follows from [BMS18, 4.17] and (7.6.1). □

Remark 7.8. As was observed by Jesse Silliman and Ravi Fernando during the Arizona Winter School 2017, the first assertion of Proposition 7.7 may be strengthened as follows: for each $i \in \mathbb{Z}$,

$$\dim_k(H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)_{\mathrm{tors}} \otimes_{\mathcal{O}_C} k) = \dim_k(H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))_{\mathrm{tors}} \otimes_{W(k)} k), \tag{7.8.1}$$

that is, $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ have the same number of cyclic summands (in the sense of (7.10.1) below). Indeed, by Corollary 7.5, the ranks of $H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$ agree and, by [Bei13a, (1.8.1)], so do the k-fibers of $R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)$ and $R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))$, so the claim follows by descending induction on i from the following exact sequences supplied by [SP, 0662]:

$$0 \to H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes_{\mathcal{O}_C} k \to H^i(R\Gamma_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \otimes^{\mathbb{L}}_{\mathcal{O}_C} k) \to \mathrm{Tor}^{\mathcal{O}_C}_1(H^{i+1}_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C), k) \to 0,$$
$$0 \to H^i_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \otimes_{W(k)} k \to H^i(R\Gamma_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} k) \to H^{i+1}_{\log \mathrm{cris}}(\mathfrak{X}_k/W(k))[p] \to 0.$$

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The following variant of [BMS18, 14.5(ii)] strengthens the relationship between the freeness of $H^i_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p)$ and that of $H^i_{\log\text{cris}}(\mathfrak{X}_k/W(k))$ supplied by Proposition 7.7.

Theorem 7.9. *For every $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geqslant 0}$, we have*

$$\begin{aligned} \operatorname{length}_{\mathbb{Z}_p}((H^i_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p)_{\text{tors}})/p^n) &\leqslant \operatorname{length}_{W(k)}((H^i_{\log\text{cris}}(\mathfrak{X}_k/W(k))_{\text{tors}})/p^n), \\ \operatorname{length}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}/p^n\mathbb{Z})) &\leqslant \operatorname{length}_{W(k)}(H^i_{\log\text{cris}}(\mathfrak{X}_k/W_n(k))). \end{aligned} \tag{7.9.1}$$

Proof. The proof of the first inequality is analogous to the proof of [BMS18, 14.5(ii)]. Namely, by Corollary 7.5, we may drop the subscripts 'tors' and, by Theorem 7.4, (7.6.2), and [BMS18, 4.15(ii)], we have

$$\operatorname{length}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p)/p^n) \leqslant \operatorname{length}_{W(k)}((H^i_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}} W(k))/p^n). \tag{7.9.2}$$

Since $\operatorname{length}_{W(k)}(Q/p^n) = \operatorname{length}_{W(k)}(\operatorname{Tor}_1^{W(k)}(Q, W(k)/p^n))$ for every $W(k)$-module Q that is finite and torsion, the short exact sequence (7.6.4) yields the inequality

$$\operatorname{length}_{W(k)}((H^i_{A_{\text{inf}}}(\mathfrak{X}) \otimes_{A_{\text{inf}}} W(k))/p^n) \leqslant \operatorname{length}_{W(k)}(H^i_{\log\text{cris}}(\mathfrak{X}_k/W(k))/p^n),$$

and the first inequality in (7.9.1) follows. Due to the short exact sequences

$$\begin{aligned} 0 \to H^i_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p)/p^n \to H^i_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}/p^n\mathbb{Z}) \to (H^{i+1}_{\text{ét}}(\mathfrak{X}_C^{\text{ad}}, \mathbb{Z}_p))[p^n] \to 0, \\ 0 \to H^i_{\log\text{cris}}(\mathfrak{X}_k/W(k))/p^n \to H^i_{\log\text{cris}}(\mathfrak{X}_k/W_n(k)) \to (H^{i+1}_{\log\text{cris}}(\mathfrak{X}_k/W(k)))[p^n] \to 0 \end{aligned}$$

that result from [SP, 0662], the second inequality in (7.9.1) follows from the first. □

The de Rham analogue of Theorem 7.9 is Theorem 7.12 below and uses the following formalism.

7.10 The normalized length

Let $\mathfrak{o}$ be a valuation ring of rank 1 and mixed characteristic $(0, p)$. We normalize its valuation $\operatorname{val}_{\mathfrak{o}}$ by requiring that $\operatorname{val}_{\mathfrak{o}}(p) = 1$. By the structure theorem [SP, 0ASP] (see also [GR03, 6.1.14]), every finitely presented $\mathfrak{o}$-module M is of the form

$$M \cong \textstyle\bigoplus_{i=1}^n \mathfrak{o}/(a_i) \quad \text{with} \quad a_i \in \mathfrak{o}. \tag{7.10.1}$$

If M is, in addition, torsion, to the effect that the a_i are nonzero, then we set

$$\operatorname{val}_{\mathfrak{o}}(M) := \textstyle\sum_{i=1}^n \operatorname{val}(a_i).$$

More intrinsically, $\operatorname{val}_{\mathfrak{o}}(M)$ is the valuation of any generator of the 0th Fitting ideal $\operatorname{Fitt}_0(M) \subset \mathfrak{o}$ of M, so it depends only on M. If $\mathfrak{o}$ is a discrete valuation ring for which p is a uniformizer, then $\operatorname{val}_{\mathfrak{o}}(M) = \operatorname{length}_{\mathfrak{o}}(M)$. In general, $\operatorname{val}_{\mathfrak{o}}$ has the advantage of being invariant under the extension of scalars to a larger $\mathfrak{o}$. Any short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of finitely presented, torsion $\mathfrak{o}$-modules gives rise to the equality $\operatorname{Fitt}_0(M_2) = \operatorname{Fitt}_0(M_1)\operatorname{Fitt}_0(M_3)$ (see [GR03, 6.3.1 and 6.3.5(i)]), so the assignment $\operatorname{val}_{\mathfrak{o}}(-)$ satisfies

$$\operatorname{val}_{\mathfrak{o}}(M_2) = \operatorname{val}_{\mathfrak{o}}(M_1) + \operatorname{val}_{\mathfrak{o}}(M_3). \tag{7.10.2}$$

The following lemma is the de Rham version of [BMS18, 4.14], which gave the inequality (7.9.2).

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LEMMA 7.11. *For a finitely presented $W_n(\mathcal{O}_C^\flat)$-module M for some $n \geqslant 1$, we have*

$$\mathrm{val}_{W(C^\flat)}(M \otimes_{A_{\mathrm{inf}}} W(C^\flat)) = \mathrm{val}_{\mathcal{O}_C}(M/\xi M) - \mathrm{val}_{\mathcal{O}_C}(M[\xi]). \tag{7.11.1}$$

Proof. Since the ring $W_n(\mathcal{O}_C^\flat)$ is coherent (see [BMS18, 3.24]), the $W_n(\mathcal{O}_C^\flat)$-module $M[\xi]$ is finitely presented. Moreover, due to (7.10.2), the flatness of $A_{\mathrm{inf}} \to W(C^\flat)$ (see § 7.6), and the snake lemma, both sides of (7.11.1) are additive in short exact sequences. Therefore, we may assume that $n = 1$ and, due to the structure theorem [SP, 0ASP], that $M = \mathcal{O}_C^\flat/(x)$ for some $x \in \mathcal{O}_C^\flat$.

If $x = 0$, then both sides of (7.11.1) are equal to 1. If $x \neq 0$, then the left-hand side vanishes, and so does the right-hand side because $M[\xi] \cong \mathrm{Tor}_{\mathcal{O}_C^\flat}^1(M, \mathcal{O}_C/p)$ and the following sequence is exact:

$$0 \to \mathrm{Tor}_{\mathcal{O}_C^\flat}^1(\mathcal{O}_C^\flat/(x), \mathcal{O}_C/p) \to \mathcal{O}_C/p \xrightarrow{\theta([x])} \mathcal{O}_C/p \to M/\xi M \to 0. \qquad \square$$

THEOREM 7.12. *For every $i \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geqslant 0}$, we have (recall from § 7.10 that $\mathrm{val}_{\mathbb{Z}_p} = \mathrm{length}_{\mathbb{Z}_p}$)*

$$\begin{aligned} \mathrm{val}_{\mathbb{Z}_p}((H^i_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p)_{\mathrm{tors}})/p^n) &\leqslant \mathrm{val}_{\mathcal{O}_C}((H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)_{\mathrm{tors}})/p^n), \\ \mathrm{val}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}/p^n\mathbb{Z})) &\leqslant \mathrm{val}_{\mathcal{O}_C}(R^i\Gamma(\mathfrak{X}_{\mathcal{O}_C/p^n, \text{ét}}, \Omega^\bullet_{\mathfrak{X}_{\mathcal{O}_C/p^n}/(\mathcal{O}_C/p^n), \log})). \end{aligned} \tag{7.12.1}$$

Proof. The proof is analogous to that of Theorem 7.9. Namely, by Corollary 7.5, we may drop the subscripts 'tors' and, by Theorem 7.4, (7.6.2), and Lemma 7.11, we have

$$\mathrm{val}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p)/p^n) \leqslant \mathrm{val}_{\mathcal{O}_C}(H^i_{A_{\mathrm{inf}}}(\mathfrak{X})/(p^n, \xi)).$$

The presentation (7.10.1) implies that $\mathrm{val}_{\mathcal{O}_C}(Q/p^n) = \mathrm{val}_{\mathcal{O}_C}(\mathrm{Tor}_1^{\mathcal{O}_C}(Q, \mathcal{O}_C/p^n))$ for every finitely presented, torsion $\mathcal{O}_C$-module Q, so the short exact sequence (7.6.3) yields the inequality

$$\mathrm{val}_{\mathcal{O}_C}(H^i_{A_{\mathrm{inf}}}(\mathfrak{X})/(p^n, \xi)) \leqslant \mathrm{val}_{\mathcal{O}_C}(H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)/p^n).$$

This proves the first inequality in (7.12.1) and, analogously to the proof of Theorem 7.9, the second inequality follows from the first. $\square$

The results above, specifically, (7.8.1) and Theorems 7.9 and 7.12 prompt the following question.

Question 7.13. Are there examples of $\mathcal{O}_C$-proper $\mathfrak{X}$ satisfying the assumptions of § 1.5 for which

$$\mathrm{val}_{W(k)}(H^i_{\log \mathrm{cris}}(\mathfrak{X}/W(k))_{\mathrm{tors}}) \neq \mathrm{val}_{\mathcal{O}_C}(H^i_{\log \mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C)_{\mathrm{tors}})?$$

8. A functorial lattice inside the de Rham cohomology

To a proper, smooth scheme X over a complete, discretely valued extension K of $\mathbb{Q}_p$ with a perfect residue field, in Example 8.6 we functorially associate an $\mathcal{O}_K$-lattice

$$L^i_{\mathrm{dR}}(X) \subset H^i_{\mathrm{dR}}(X/K) \quad \text{for every} \quad i \in \mathbb{Z}.$$

In fact, $L^i_{\mathrm{dR}}(X)$ functorially depends only on $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ and its construction, which relies on the theory of Breuil–Kisin–Fargues modules, proceeds along familiar lines of integral p-adic Hodge theory; compare, for instance, with [Liu18, § 4]. The work of the preceding sections allows

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us to interpret $L^i_{\mathrm{dR}}(X)$ geometrically: we show in Theorem 8.7 that if X has a proper, flat, semistable $\mathcal{O}_K$-model $\mathcal{X}$ for which $H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ are $\mathcal{O}_K$-free, then

$$L^i_{\mathrm{dR}}(X) = H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \quad \text{inside} \quad H^i_{\mathrm{dR}}(X/K).$$

We do not know whether the same holds 'modulo torsion' if one drops the $\mathcal{O}_K$-freeness assumption.

8.1 The base field K

Throughout § 8, we assume that $C \cong \widehat{\overline{K}}$ for a fixed complete, discretely valued field K that is of mixed characteristic $(0,p)$ and has a perfect residue field k_0. We set

$$G := \mathrm{Gal}(\overline{K}/K),$$

so that G acts continuously on C, and hence also on A_{inf}. The continuous maps φ and θ are G-equivariant, and the ideals (ξ), $(\varphi(\xi))$, and (μ) of A_{inf} are G-stable (see § 2.1).

If $\mathcal{X}$ is a p-adic formal $\mathcal{O}_K$-scheme for which $\mathfrak{X} := \mathcal{X}\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_C$ satisfies the assumptions of § 1.5, then, by the functoriality of $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$ (see § 7.2), G acts A_{inf}-semilinearly on each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$.

8.2 The Fargues equivalence

By [BMS18, 4.26], for any Breuil–Kisin–Fargues module (M,φ_M) (see § 7.3), its *étale realization*, namely,

$$M_{\text{ét}} := (M \otimes_{A_{\mathrm{inf}}} W(C^\flat))^{\varphi_M\otimes\varphi=1},$$

is a finitely generated $\mathbb{Z}_p$-module that comes equipped with an identification

$$M \otimes_{A_{\mathrm{inf}}} W(C^\flat) \cong M_{\text{ét}} \otimes_{\mathbb{Z}_p} W(C^\flat) \quad \text{under which} \quad M \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right] \cong M_{\text{ét}} \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}\left[\tfrac{1}{\mu}\right].$$

Thus, $M_{\text{ét}}$ is $\mathbb{Z}_p$-free if M is A_{inf}-free and, for any (M,φ_M), we have $M\otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}} \cong M_{\text{ét}}\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$, so that $M_{\text{ét}}$ comes equipped with a B^+_{dR}-sublattice (recall that $M[\frac{1}{p}]$ is $A_{\mathrm{inf}}[\frac{1}{p}]$-free, see § 7.3)

$$M \otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}} \subset M_{\text{ét}} \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

By a theorem of Fargues [BMS18, 4.28], the category of A_{inf}-free Breuil–Kisin–Fargues modules (M,φ_M) is equivalent to that of pairs (T,Ξ) consisting of a finite free $\mathbb{Z}_p$-module T and a B^+_{dR}-lattice $\Xi \subset T\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ via the functor

$$(M,\varphi_M) \mapsto (M_{\text{ét}}, M\otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}}).$$

8.3 Breuil–Kisin–Fargues G-modules

Due to the origin of our C (see § 8.1), we may consider *Breuil–Kisin–Fargues G-modules*, that is, Breuil–Kisin–Fargues modules (M,φ_M) equipped with an A_{inf}-semilinear G-action on M for which φ_M is G-equivariant. A morphism of Breuil–Kisin–Fargues G-modules is a G-equivariant A_{inf}-module morphism that commutes with the isomorphisms φ_M.

For instance, if an $\mathcal{X}$ as in § 8.1 is proper, then each $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a Breuil–Kisin–Fargues G-module (see Theorem 7.4). The étale realization $M_{\text{ét}}$ of a Breuil–Kisin–Fargues G-module (M,φ_M) carries the induced $\mathbb{Z}_p$-linear G-action.

Proposition 8.4. *The category of A_{inf}-free Breuil–Kisin–Fargues G-modules (M,φ_M) is equivalent to that of pairs (T,Ξ) consisting of a finite free $\mathbb{Z}_p$-module T equipped with a G-action and a G-stable B^+_{dR}-lattice $\Xi \subset T\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ via the functor*

$$(M,\varphi_M) \mapsto (M_{\text{ét}}, M\otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}}).$$

Proof. The claim follows from the Fargues equivalence reviewed in § 8.2. □

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8.5 An étale lattice determines a de Rham lattice

Let T be a finite free $\mathbb{Z}_p$-module endowed with a continuous G-action for which the G-representation $T[\frac{1}{p}]$ is de Rham, so that there is a G-equivariant identification

$$T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}} \cong D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}, \quad \text{where} \quad D_{\mathrm{dR}}(T) := (T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^G.$$

For such T, the B_{dR}^+-lattice $D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}^+$ is evidently G-stable in $T \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$. Thus, by Proposition 8.4, the pair $(T, D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}^+)$, so T, determines an A_{inf}-free Breuil–Kisin–Fargues G-module

$$(M(T), \varphi_{M(T)})$$

that depends functorially on T. The de Rham realization

$$M(T)_{\mathrm{dR}} := M(T) \otimes_{A_{\mathrm{inf}}, \theta} \mathcal{O}_C \quad \text{of} \quad (M(T), \varphi_{M(T)})$$

is an $\mathcal{O}_C$-lattice in

$$(M(T) \otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}}^+)/\xi \cong (D_{\mathrm{dR}}(T) \otimes_K B_{\mathrm{dR}}^+)/\xi \cong D_{\mathrm{dR}}(T) \otimes_K C.$$

Therefore, functorially in T, we obtain the $\mathcal{O}_K$-lattice

$$(M(T)_{\mathrm{dR}})^G \quad \text{inside the } K\text{-vector space} \quad D_{\mathrm{dR}}(T).$$

Example 8.6. We fix a K-scheme X (or even a K-rigid space, which we view as an adic space, see [Hub96, 1.1.11(d)]) that is proper and smooth, and set

$$L^i_{\text{ét}}(X) := H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)/H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)_{\mathrm{tors}} \cong H^i_{\text{ét}}(X_C, \mathbb{Z}_p)/H^i_{\text{ét}}(X_C, \mathbb{Z}_p)_{\mathrm{tors}} \quad \text{for} \quad i \geqslant 0.$$

As is well known and follows from (6.7.2), the G-representation $(L^i_{\text{ét}}(X))[\frac{1}{p}]$ is de Rham and

$$D_{\mathrm{dR}}(L^i_{\text{ét}}(X)) \cong (L^i_{\text{ét}}(X) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}})^G \overset{(6.7.2)}{\cong} (H^i_{\mathrm{dR}}(X/K) \otimes_K B_{\mathrm{dR}})^G \cong H^i_{\mathrm{dR}}(X/K) \tag{8.6.1}$$

functorially in X. Thus, using the discussion of § 8.5, we obtain the $\mathcal{O}_K$-lattice

$$L^i_{\mathrm{dR}}(X) := (M(L^i_{\text{ét}}(X))_{\mathrm{dR}})^G \subset H^i_{\mathrm{dR}}(X/K)$$

that is functorial in X (even in $L^i_{\text{ét}}(X)$). Its definition implies that for a finite Galois extension K'/K,

$$L^i_{\mathrm{dR}}(X) = (L^i_{\mathrm{dR}}(X_{K'}))^{\mathrm{Gal}(K'/K)} \quad \text{inside} \quad H^i_{\mathrm{dR}}(X/K) = (H^i_{\mathrm{dR}}(X_{K'}/K'))^{\mathrm{Gal}(K'/K)}.$$

If X extends to a proper, flat, semistable $\mathcal{O}_K$-scheme $\mathcal{X}$ such that $H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ are $\mathcal{O}_K$-free (where $\mathcal{X}$ is endowed with the log structure $\mathcal{O}_{\mathcal{X},\text{ét}} \cap (\mathcal{O}_{\mathcal{X},\text{ét}}[\frac{1}{p}])^\times$), then, by the following Theorem 8.7 (and GAGA techniques, similarly to Remark 4.19),

$$L^i_{\mathrm{dR}}(X) = H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \quad \text{inside} \quad H^i_{\mathrm{dR}}(X/K);$$

in particular, if $\mathcal{X}'$ is another such $\mathcal{O}_K$-model of X, then

$$H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) = H^i_{\log\mathrm{dR}}(\mathcal{X}'/\mathcal{O}_K) \quad \text{inside} \quad H^i_{\mathrm{dR}}(X/K). \tag{8.6.2}$$

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Theorem 8.7. *Let $\mathcal{X}$ be a proper, flat p-adic formal $\mathcal{O}_K$-scheme endowed with the log structure $\mathcal{O}_{\mathcal{X},\text{ét}} \cap (\mathcal{O}_{\mathcal{X},\text{ét}}[\frac{1}{p}])^\times$ such that $\mathcal{X}$ has an étale cover by affines $\mathcal{U}$ each of which has an étale morphism*

$$\mathcal{U} \to \operatorname{Spf}(\mathcal{O}_K\{t_0,\dots,t_r,t_{r+1},\dots,t_d\}/(t_0\cdots t_r-\pi)) \quad \text{for some nonunit} \quad \pi \in \mathcal{O}_K \setminus \{0\} \tag{8.7.1}$$

(where d, r, and π depend on $\mathcal{U}$). If $H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ and $H^{i+1}_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ are $\mathcal{O}_K$-free, then

$$L^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K) = H^i_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \quad \text{inside} \quad H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K/K); \tag{8.7.2}$$

in fact, then, setting $\mathfrak{X} := \mathcal{X}\widehat{\otimes}_{\mathcal{O}_K}\mathcal{O}_C$, we have the identification

$$M(L^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_K)) \cong H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \tag{8.7.3}$$

of Breuil–Kisin–Fargues G-modules.

Proof. By working locally on $\mathcal{U}$, in the target of (8.7.1) we may replace each t_i by $t_i^{\pm 1}$ for $r+1 \leqslant i \leqslant d$, so $\mathfrak{X}$ satisfies the assumptions of § 1.5. Moreover, by the Grothendieck comparison theorem and flat base change (compare with Remark 4.19), for $j = i$ and $j = i+1$, we have

$$H^j_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C) \cong H^j_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)\otimes_{\mathcal{O}_K}\mathcal{O}_C, \quad \text{so} \quad H^j_{\log\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K) \cong (H^j_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C))^G. \tag{8.7.4}$$

Thus, by Proposition 7.7, the Breuil–Kisin–Fargues G-modules $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and $H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X})$ (see § 8.3) are A_{inf}-free. By (7.6.2), we have the G-equivariant identification of the étale realization:

$$(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}))_{\text{ét}} \cong H^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p),$$

which, consequently, is $\mathbb{Z}_p$-free. By Proposition 6.8, the B_{dR}-base change of this identification agrees with the identification $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})\otimes_{A_{\mathrm{inf}}} B_{\mathrm{dR}} \cong H^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_C,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}$ that results by combining

$$H^i_{A_{\mathrm{inf}}}(\mathfrak{X})\otimes_{A_{\mathrm{inf}}} B^+_{\mathrm{dR}} \overset{(6.6.2)}{\cong} H^i_{\mathrm{cris}}(\mathcal{X}^{\mathrm{ad}}_C/B^+_{\mathrm{dR}}) \overset{(6.2.8)}{\cong} H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K/K)\otimes_K B^+_{\mathrm{dR}}$$

and

$$H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_K/K)\otimes_K B_{\mathrm{dR}} \overset{(6.7.2)}{\cong} H^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_C,\mathbb{Z}_p)\otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

This compatibility and § 8.5 (see also (8.6.1)) supply the desired G-equivariant identification (8.7.3):

$$M(L^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_K)) \cong H^i_{A_{\mathrm{inf}}}(\mathfrak{X}).$$

Under this identification, by Theorem 6.6 and the sentence after (6.2.8), the identifications

$$M(L^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_K))\otimes_{A_{\mathrm{inf}},\theta} C \overset{(8.6.1)}{\cong} H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_C/C) \quad \text{and} \quad H^i_{A_{\mathrm{inf}}}(\mathfrak{X})\otimes_{A_{\mathrm{inf}},\theta} C \overset{(4.18.1)}{\cong} H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_C/C)$$

agree. Thus, (7.6.3) implies the following equality inside $H^i_{\mathrm{dR}}(\mathcal{X}^{\mathrm{ad}}_C/C)$:

$$M(L^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_K))_{\mathrm{dR}} = M(L^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_K))\otimes_{A_{\mathrm{inf}},\theta}\mathcal{O}_C = H^i_{A_{\mathrm{inf}}}(\mathfrak{X})\otimes_{A_{\mathrm{inf}},\theta}\mathcal{O}_C = H^i_{\log\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C),$$

which, together with the second identification in (8.7.4), gives the desired (8.7.2). □

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Remark 8.8. In the proof above, we have seen that both $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ and $H^{i+1}_{A_{\mathrm{inf}}}(\mathfrak{X})$ are A_{inf}-free, so, by (7.6.4), we have the G-equivariant and Frobenius-equivariant identifications

$$H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k) \cong H^i_{\log\,\mathrm{cris}}(\mathfrak{X}_k/W(k)) \overset{(5.44.1)}{\cong} H^i_{\log\,\mathrm{cris}}(\mathcal{X}_k/W(k)),$$

and hence also the Frobenius-equivariant identification

$$(H^i_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(k))^G \cong H^i_{\log\,\mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0)). \tag{8.8.1}$$

In particular, (8.7.3) and (8.8.1) show that, under the assumptions of Theorem 8.7, the integral p-adic étale cohomology $H^i_{\text{ét}}(\mathcal{X}^{\mathrm{ad}}_C, \mathbb{Z}_p)$ endowed with its Galois action functorially determines the integral logarithmic crystalline cohomology $H^i_{\log\,\mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0))$ endowed with its Frobenius.

9. The semistable comparison isomorphism

Our final goal is to deduce the semistable comparison isomorphism for suitable proper, 'semistable' formal schemes (see Theorem 9.5). This extends [BMS18, 1.1(i)], which treated the good reduction case (see also [TT15, 1.2] for a result 'with coefficients' over an absolutely unramified base), and is similar to the semistable comparison established by Colmez–Niziol [CN17, 5.26]. More precisely, [CN17, 5.26] also includes cases in which the log structures are not 'vertical'.

9.1 The ring B_{st}

We consider the log PD thickenings A_{cris}/p^n of $\mathcal{O}_C/p$ (see § 5.2) and set

$$J_n := \mathrm{Ker}(A_{\mathrm{cris}}/p^n \twoheadrightarrow \mathcal{O}_C/p) \quad \text{and} \quad J := \varprojlim_{n\geqslant 1} J_n \cong \mathrm{Ker}(A_{\mathrm{cris}} \twoheadrightarrow \mathcal{O}_C/p).$$

The element $p \in \mathcal{O}_C \setminus \{0\}$ belongs to the log structure of $\mathcal{O}_C/p$ (see § 1.6 (1)), so its preimage in the log structure of A_{cris}/p^n is a $(1+J_n, \times)$-torsor, which is necessarily trivial[20] (compare with [Bei13a, § 1.1, Exercises, (iii)]). Consequently, as n varies, these torsors comprise a trivial $(1+J, \times)$-torsor τ_0, whose base change along the logarithm map $(1+J, \times) \to (J, +) \subset (A_{\mathrm{cris}}, +)$ furnished by the divided power structure on J is a trivial $(A_{\mathrm{cris}}, +)$-torsor τ, the *Fontaine–Hyodo–Kato torsor* (compare with [Bei13a, § 1.15, p. 23]). The functor which to an A_{cris}-algebra A assigns the underlying set of the $(A, +)$-torsor $\tau \times_{(A_{\mathrm{cris}}, +)} (A, +)$ is represented by the A_{cris}-algebra A_{st}, so A_{st} is the initial A_{cris}-algebra over which the Fontaine–Hyodo–Kato torsor is canonically trivialized.

We may noncanonically trivialize τ_0 (for instance, $[p^{1/p^\infty}]$ is a trivialization, see (5.2.1)) to obtain an isomorphism $A_{\mathrm{st}} \simeq A_{\mathrm{cris}}[T]$, which, upon adjusting the trivialization by an $a \in 1+J$, gets postcomposed with the A_{cris}-automorphism $A_{\mathrm{cris}}[T] \xrightarrow{T \mapsto T + \log(a)} A_{\mathrm{cris}}[T]$. The A_{cris}-derivation $-d/dT$ respects these automorphisms, so it induces a canonical A_{cris}-derivation, the *monodromy operator*,

$$N\colon A_{\mathrm{st}} \to A_{\mathrm{st}} \quad \text{for which} \quad (A_{\mathrm{st}})^{N=0} = A_{\mathrm{cris}}$$

[20] Quasi-coherent cohomology of affine schemes vanishes, so, for a finitely generated, and hence nilpotent, ideal $J' \subset J_n$ of A_{cris}/p^n, the étale sheaf on $\mathrm{Spec}(A_{\mathrm{cris}}/p^n)$ associated to $(1+J', \times)$ has no nontrivial torsors. The filtered direct limit of these sheaves is the analogous sheaf associated to $(1+J_n, \times)$, so it, too, has no nontrivial torsors.

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(our N agrees with that of [Bei13a], see [Bei13a, § 1.15, Remarks (i)]; compare also with [Tsu99, 4.1.1]).

By [Bei13a, (1.15.2)], the Frobenius pullback of τ_0 is isomorphic to the p-fold self-product of τ_0, and hence likewise for the base change τ of τ_0 to $(A_{\mathrm{cris}}, +)$. Thus, we have an A_{cris}-semilinear Frobenius

$$\varphi \colon A_{\mathrm{st}} \to A_{\mathrm{st}} \tag{9.1.1}$$

that in terms of an isomorphism $A_{\mathrm{st}} \simeq A_{\mathrm{cris}}[T]$ obtained by trivializing τ_0 is described by $T \mapsto pT$. The interaction of φ and N is described by the formula $N\varphi = p\varphi N$.

Since μ and $\log([\epsilon])$ are unit multiples of each other in A_{cris} (see § 5.14) and

$$\varphi(\log([\epsilon])) = p \log([\epsilon]),$$

the Frobenius (9.1.1) and, evidently, also the derivation N induce their counterparts on

$$B_{\mathrm{st}}^{+} := A_{\mathrm{st}}\left[\tfrac{1}{p}\right] \quad \text{and} \quad B_{\mathrm{st}} := A_{\mathrm{st}}\left[\tfrac{1}{p\mu}\right].$$

The relation $N\varphi = p\varphi N$ continues to hold for B_{st}^{+} and B_{st}. As is explained in [Bei13a, § 1.17], the A_{cris}-algebras B_{st}^{+} and B_{st} reviewed above agree with those constructed in [Fon94, § 3].

For us, the significance of the period ring B_{st}^{+} lies in the following comparison between the logarithmic crystalline cohomology of $\mathfrak{X}_k$ over $W(k)$ and of $\mathfrak{X}_{\mathcal{O}_C/p}$ over A_{cris} (compare with [BMS18, 13.21]).

Proposition 9.2. *If $\mathfrak{X}$ is $\mathcal{O}_C$-proper, then*

$$R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_k/W(k)) \otimes^{\mathbb{L}}_{W(k)} B_{\mathrm{st}}^{+} \cong R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B_{\mathrm{st}}^{+}, \tag{9.2.1}$$

where the log structures are those of §§ 1.6 (2), 5.2, and 5.42. In particular, if $\mathfrak{X}$ is $\mathcal{O}_C$-proper and $\mathcal{Y}$ is a descent of $\mathfrak{X}_{\mathcal{O}_C/p}$ to a proper, log smooth, fine log $\mathcal{O}/p$-scheme of Cartier type for some discrete valuation subring $\mathcal{O} \subset \mathcal{O}_C$ with a perfect residue field k_0 and $C \cong (\overline{\mathcal{O}[\frac{1}{p}]})\widehat{\ }$ (where $\mathcal{O}/p$ is equipped with the log structure associated to the chart $\mathcal{O} \setminus \{0\} \to \mathcal{O}/p$), then we have the following identification that is compatible with the actions of φ and N (described in the proof):

$$R\Gamma_{\log\mathrm{cris}}(\mathcal{Y}_{k_0}/W(k_0)) \otimes^{\mathbb{L}}_{W(k_0)} B_{\mathrm{st}}^{+} \cong R\Gamma_{\log\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B_{\mathrm{st}}^{+}, \tag{9.2.2}$$

where $W(k_0)$ is endowed with the log structure associated to $\mathbb{N}_{\geqslant 0} \xrightarrow{1 \mapsto 0,\, 0 \mapsto 1} W(k_0)$.

Proof. A descent $\mathcal{Y}$ exists (see the proof of Corollary 5.43), so (9.2.2) follows from [Bei13a, (1.16.2) and (1.18.5)] and, due to (5.44.1), it implies (9.2.1). On the left-hand side of (9.2.2), the operator N combines the monodromy of $R\Gamma_{\log\mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0))$ and B_{st}^{+}, so is '$N \otimes 1 + 1 \otimes N$'; on the right-hand side, N is the monodromy of B_{st}^{+}. On both sides of (9.2.2), the Frobenius φ acts on both factors. □

Remark 9.3. One may eliminate the dependence of (9.2.2) on the choice of $\mathcal{Y}$ by forming a direct limit over all the possible $\mathcal{Y}$, see [Bei13a, § 1.18, Remarks (i)].

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9.4 The base field K

For the rest of § 9, we assume that $C = \widehat{\overline{K}}$ for a fixed complete, discretely valued subfield $K \subset C$ with a perfect residue field k_0, set $G := \mathrm{Gal}(\overline{K}/K)$, and endow $\mathcal{O}_K$ (respectively, $\mathcal{O}_K/p$) with the log structure associated to the chart $\mathcal{O}_K \setminus \{0\} \hookrightarrow \mathcal{O}_K$ (respectively, its base change). By functoriality, G acts on A_{cris}, A_{st}, B_{st}^+, and, since the ideal (μ) does not depend on the choice of ϵ (see § 2.1), also on B_{st}. These G-actions commute with φ and N. When $\mathcal{O}$ of Proposition 9.2 is our $\mathcal{O}_K$, the identification (9.2.2) is G-equivariant granted that G acts on both sides by functoriality.

THEOREM 9.5. *Let $\mathcal{X}$ be a proper p-adic formal $\mathcal{O}_K$-scheme that has an étale cover by affines $\mathcal{U}$ each of which has an étale $\mathcal{O}_K$-morphism*

$$\mathcal{U} \to \mathrm{Spf}(\mathcal{O}_K\{t_0, \dots, t_r, t_{r+1}, \dots, t_d\}/(t_0 \cdots t_r - \pi)) \quad \text{for some nonunit} \quad \pi \in \mathcal{O}_K \setminus \{0\},$$

(where d, r, and π depend on $\mathcal{U}$) and endow $\mathcal{X}$ with the log structure $\mathcal{O}_{\mathcal{X},\text{ét}} \cap (\mathcal{O}_{\mathcal{X},\text{ét}}[\frac{1}{p}])^\times$. There is the following natural, G-equivariant isomorphism compatible with the actions of φ and N:

$$R\Gamma_{\text{ét}}(\mathcal{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}} \cong R\Gamma_{\log\,\mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0)) \otimes^{\mathbb{L}}_{W(k_0)} B_{\mathrm{st}}, \tag{9.5.1}$$

where $W(k_0)$ is endowed with the log structure associated to $\mathbb{N}_{\geqslant 0} \xrightarrow{1 \mapsto 0,\, 0 \mapsto 1} W(k_0)$. In particular,

$$\text{the } G\text{-representation} \quad H^i_{\text{ét}}(\mathcal{X}_C^{\mathrm{ad}}, \mathbb{Q}_p) \quad \text{is semistable for every} \quad i \in \mathbb{Z}.$$

Proof. We set $\mathfrak{X} := \mathcal{X} \widehat{\otimes}_{\mathcal{O}_K} \mathcal{O}_C$, so that $\mathfrak{X}$ meets the requirements of § 1.5. By Claims 1.6.1 and 1.6.3 and [Kat89, 4.8], the base change $\mathcal{X}_{\mathcal{O}_K/p}$ is fine, log smooth, and of Cartier type over $\mathcal{O}_K/p$, so Proposition 9.2 applies to it and gives the G-equivariant (see § 9.4) identification

$$R\Gamma_{\log\,\mathrm{cris}}(\mathcal{X}_{k_0}/W(k_0)) \otimes^{\mathbb{L}}_{W(k_0)} B_{\mathrm{st}}^+ \cong R\Gamma_{\log\,\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/A_{\mathrm{cris}}) \otimes^{\mathbb{L}}_{A_{\mathrm{cris}}} B_{\mathrm{st}}^+ \overset{(5.43.2)}{\cong} R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{st}}^+$$

that is compatible with φ and N. In addition, by (2.3.1), we have the G-equivariant identification

$$R\Gamma(\mathfrak{X}_{\text{ét}}, A\Omega_{\mathfrak{X}}) \otimes^{\mathbb{L}}_{A_{\mathrm{inf}}} B_{\mathrm{st}} \cong R\Gamma(\mathfrak{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}} \cong R\Gamma(\mathcal{X}_C^{\mathrm{ad}}, \mathbb{Z}_p) \otimes^{\mathbb{L}}_{\mathbb{Z}_p} B_{\mathrm{st}}$$

that is trivially compatible with N and is compatible with φ by the discussion after Theorem 2.3. The desired (9.5.1) follows by combining the displayed identifications. □

Remark 9.6. The isomorphism (9.5.1) is compatible with filtrations in the following sense: by [Fon94, § 4.2], there is a (noncanonical) A_{cris}-algebra homomorphism $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ and, by the proof above and Proposition 6.8, the B_{dR}-base change of the isomorphism (9.5.1) is identified with the de Rham comparison isomorphism (6.7.2) (with $X_0 = \mathcal{X}_K^{\mathrm{ad}}$) that is compatible with filtrations.

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