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Measure-theoretic sequence entropy pairs and mean sensitivity

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Abstract. We characterize measure-theoretic sequence entropy pairs of continuous actions of abelian groups using mean sensitivity. This addresses an open question of Li and Yu [On mean sensitive tuples. *J. Differential Equations* **297** (2021), 175–200]. As a consequence of our results, we provide a simpler characterization of Kerr and Li's independence sequence entropy pairs (μ-IN-pairs) when the measure is ergodic and the group is abelian.

Key words: sequence entropy pairs, mean sensitivity, entropy pairs, IN-pairs
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1. *Introduction*

Given a dynamical system (or a group action) with positive entropy, one might wonder which points of the phase space contribute to the entropy (and which do not). One approach to answer this question was given in Blanchard's seminal paper [**2**], where entropy pairs were introduced. In this approach, the answer to this question is not a subset of the phase space X but a subset of the product space X^2. It turns out that a system has positive topological entropy if and only if there is an entropy pair. This sets the basis of what is now known as local entropy theory (see the surveys [**10**, **12**]). This theory has provided ground for building new bridges from dynamics to other areas of mathematics like combinatorics [**16**], point-set topology [**6**], group theory, operator algebras [**1**, **4**], and descriptive set theory [**5**]. Of particular interest is Kerr and Li's characterization of entropy pairs of continuous actions of amenable groups using a combinatorial notion of independence [**16**].

Positive (measure) entropy can also be localized using measure entropy pairs for a measure (as defined in [**3**]). Furthermore, these pairs can also be characterized using a different notion of combinatorial independence [**17**]. However, the definition of independence used for this characterization is quite more involved and technical than the topological one (see §2.4). The aim of this paper is to bring more clarity to the measure-theoretic theory of independence pairs in the particular case of sequence entropy.

Sequence entropy was introduced by Kushnirenko, and it provided the first link between the functional analytic ergodic theory of von Neumann and the entropy-related ergodic theory of Kolmogorov's school by proving that a system has pure point spectrum if and only if it has zero sequence entropy [18]. As in the classic case, sequence entropy pairs appear exactly when a system has positive measure sequence entropy [15]. By using arbitrarily large sets instead of positive density, Kerr and Li also characterized sequence entropy pairs using independence [16, 17].

We characterize measure sequence entropy pairs of abelian group actions (Theorem 3.6) using the so-called mean sensitivity pairs (introduced in [8]). This addresses an open question mentioned in the introduction of [21]. Mean sensitivity with respect to a measure is a statistical version of sensitivity to initial conditions used in chaotic dynamics. As an application of our result, we provide a simpler version of independence pairs for sequence entropy (Theorem 3.8).

There are previous results that indicate that sequence entropy pairs satisfy some type of sensitivity with respect to a measure [14, 20]. Nonetheless, those previous studied notions were too weak to induce a characterization.

Theorem 3.6 is a local version of [8, Theorem 38]. Nonetheless, an important difference is that while the global result holds for general Borel invariant measures [13, Theorem 4.7], the local characterization may fail if the measure is not ergodic [19, Theorem 1.6]. Theorem 3.6 was obtained independently in the case where $G = \mathbb{N}$ in [19] using different tools.

The paper is organized as follows. In §2, we give definitions of the main concepts used in the paper (such as entropy, independence, and sensitivity). In §3, we prove the main technical lemma (Lemma 3.5) and the main results of the paper (Theorems 3.6 and 3.8). Finally, in §4, we introduce a stronger definition, diam-mean sensitivity pairs; these pairs are always sequence entropy pairs but we do not know if the converse holds.

2. *Preliminaries*

Throughout this paper, X represents a compact metrizable space with a compatible metric d. We denote by $\mathcal{B}(X)$ the set of all Borel sets of X. Given μ, a Borel probability measure on X, we denote the set of all Borel sets with positive measure by $\mathcal{B}^+_\mu(X)$.

2.1. *Amenable groups.*

In this paper, G denotes a countable group with identity e. The inverse of a point $g \in G$ will be denoted by g^{-1}. A sequence $\{F_n\}_{n\in\mathbb{N}}$ of non-empty finite subsets of G is called a *Følner sequence* if

$$\lim_{n\to\infty} |sF_n \Delta F_n|/|F_n| \to 0$$

for every $s \in G$. In general, we will simply denote this sequence with $\{F_n\}$. A group is called *amenable* if it admits a Følner sequence. Every abelian group is amenable.

Let $\{F_n\}$ be a Følner sequence for G and $S \subseteq G$. We define the lower density of S as

$$\underline{D}(S) = \liminf_{n\to\infty} \frac{|S \cap F_n|}{|F_n|},$$

and the upper density of S as

$$\overline{D}(S) = \limsup_{n\to\infty} \frac{|S \cap F_n|}{|F_n|}.$$

Definition 2.1. Let $\{F_n\}$ be a Følner sequence for G. We say that $\{F_n\}$ is *tempered* if there exists $C > 0$ such that

$$\left| \bigcup_{k=1}^{n-1} F_k^{-1} F_n \right| \le C|F_n| \quad \text{for all } n > 1.$$

Every Følner sequence has a subsequence that is tempered [22, Proposition 1.5].

2.2. *Group actions and sequence entropy.* By an *action* of the group G on X, we mean a map $\alpha\colon G \times X \to X$ such that $\alpha(s, \alpha(t, x)) = \alpha(st, x)$ and $\alpha(e, x) = x$ for all $x \in X$ and $s, t \in G$. For simplicity, we refer to the action as $G \curvearrowright X$ and we denote the image of a pair (s, x) as sx.

Given a continuous group action $G \curvearrowright X$, we say a Borel probability measure is *ergodic* if it is G-invariant and every G-invariant measurable set has measure 0 or 1. Throughout this paper, every measure is a Borel probability measure and we will omit writing this.

THEOREM 2.2. [22, Theorem 1.2] *Let $G \curvearrowright X$ be a continuous action, μ an ergodic measure, f an integrable function, and $\{F_n\}$ a tempered Følner sequence. Then,*

$$\lim_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} f(sx) = \int f(x)d\mu(x) \quad \mu\text{-almost everywhere.}$$

Let A be a measurable set. We say x is a *generic point for A* if it satisfies the pointwise ergodic theorem for $f = 1_A$.

Let $G \curvearrowright X$ be a continuous action, μ a G-invariant measure, $S = \{s_i\}_{i=0}^{\infty}$ a sequence in G, and $\mathcal{P}$ a finite measurable partition of X. The *Shannon entropy* of $\mathcal{P}$ is given by

$$H_\mu(\mathcal{P}) = -\sum_{A\in\mathcal{P}} \mu(A) \log \mu(A).$$

The *sequence entropy of $G \curvearrowright X$ along S with respect to μ and $\mathcal{P}$ is defined by*

$$h_\mu^S(X, G, \mathcal{P}) = \limsup_{n\to\infty} \frac{1}{n} H_\mu\bigg(\bigvee_{i=0}^{n-1} s_i^{-1}\mathcal{P}\bigg).$$

The *sequence entropy of $G \curvearrowright X$ along S with respect to μ is*

$$h_\mu^S(X, G) = \sup_{\mathcal{P}} h_\mu^S(X, G, \mathcal{P}),$$

where the supremum is taken over all finite measurable partitions $\mathcal{P}$.

Sequence entropy can be studied locally using sequence entropy pairs, introduced in [15].

Definition 2.3. Let $G \curvearrowright X$ be a continuous group action and μ an invariant measure. We say that $(x, y) \in X^2$ is a *μ-sequence entropy pair* if $x \neq y$ and for any finite measurable

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partition $\mathcal{P}$, such that there is no $P \in \mathcal{P}$ with $x, y \in \overline{P}$, there exists a sequence, S, in G with $h_\mu^S(X, \mathcal{P}) > 0$.

2.3. *Mean sensitivity.* Sensitivity with respect to a measure was introduced in [14]. Mean sensitivity with respect to a measure is a statistical form of sensitivity introduced in [8]. In contrast with μ-sensitivity, μ-mean sensitivity is invariant under measure isomorphism.

Definition 2.4. Let $\{F_n\}$ be a Følner sequence for G, $G \curvearrowright X$ a continuous group action, and μ a measure on X. We say that $G \curvearrowright X$ is *μ-mean sensitive* if there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}_\mu^+(X)$, there exist $x, y \in A$ such that

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} d(sx, sy) > \varepsilon.$$

Remark 2.5. Let μ be an ergodic measure and $\{F_n\}$ a Følner sequence for G. An action is μ-mean sensitive if and only if it is not μ-mean equicontinuous [8, Theorem 26] (while the result is stated for actions of $\mathbb{Z}^d$, the same proof holds for any action of an amenable group). Furthermore, an action is μ-mean equicontinuous with respect to a tempered Følner sequence if and only if it has discrete spectrum [8, Corollary 39], [23, Theorem 1.3]. Since discrete spectrum does not depend on the choice of the Følner sequence, we conclude that if μ-mean sensitivity holds for one tempered Følner sequence (and μ ergodic), it must hold for all. The role of non-tempered Følner sequences was studied for mean equicontinuity in [7]. Such study has not been done for μ-mean equicontinuity.

Now we will define the local notion of the previous definition.

Definition 2.6. Let $\{F_n\}$ be a Følner sequence for G, $G \curvearrowright X$ a continuous group action, and μ an invariant measure. We say that $(x, y) \in X^2$ is a *μ-mean sensitivity pair* if $x \neq y$ and for all open neighborhoods U_x of x and U_y of y, there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}_\mu^+(X)$, there exist $p, q \in A$ such that

$$\overline{D}(\{s \in G : sp \in U_x \text{ and } sq \in U_y\}) > \varepsilon.$$

We denote the set of μ-mean sensitivity pairs by $S_\mu^m(X, G)$.

We will now define sensitivity with respect to an L^2 function.

Definition 2.7. Let $\{F_n\}$ be a Følner sequence for G, $G \curvearrowright X$ a continuous group action, μ a measure on X, and $f \in L^2(X, \mu)$. We define

$$d_f(x, y) = \limsup_{n\to\infty} \left(\frac{1}{|F_n|} \sum_{s\in F_n} |f(sx) - f(sy)|^2 \right)^{1/2}.$$

Definition 2.8. Let $\{F_n\}$ be a Følner sequence for G, $G \curvearrowright X$ a continuous group action, μ an invariant measure, and $f \in L^2(X, \mu)$. We say that $G \curvearrowright X$ is *μ-f-mean sensitive* if there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}_\mu^+(X)$, there exist $x, y \in A$ such that $d_f(x, y) > \varepsilon$.

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In this case, we say that f is *μ-mean sensitive*. We denote the set of μ-mean sensitive functions with H_{ms}.

Remark 2.9. We have that $G \curvearrowright X$ is μ-1_B-mean sensitive if and only if there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}^+_\mu(X)$, there exist $p, q \in A$ such that $\overline{D}(\{s \in G : sp \in B$ and $sq \in B^c\}) > \varepsilon$.

2.4. *Independence.* The following definitions were introduced in [17].

Definition 2.10. Let (A_1, A_2) be a pair of subsets of X and $E : G \to 2^X$ a function. We say that a set $I \subseteq G$ is an *independence set for* (A_1, A_2) *relative to* E if for every non-empty finite subset $F \subseteq I$ and any map $\sigma : F \to \{1, 2\}$, we have $\bigcap_{s\in F}(E(s) \cap s^{-1}A_{\sigma(s)}) \neq \emptyset$.

We say that I is a *independence set for* (A_1, A_2) *relative to* $D \subseteq X$ if it is an independence set for (A_1, A_2) relative to the constant function given by $E(s) = D$ for all $s \in G$.

We denote by $\mathcal{B}'_\mu(X, \varepsilon)$ the set of all maps $E : G \to \mathcal{B}(X)$ such that $\mu(E(s)) \geq 1 - \varepsilon$ for all $s \in G$.

Definition 2.11. Let $G \curvearrowright X$ be a continuous group action and μ an invariant measure. For $A_1, A_2 \subseteq X$ and $\varepsilon > 0$, we say that (A_1, A_2) has (ε, μ)-*independence over arbitrarily large finite sets* if there exists $c > 0$ such that for every $N > 0$, there is a finite set $F \subseteq G$ with $|F| > N$ such that for every $E \in \mathcal{B}'_\mu(X, \varepsilon)$, there is an independence set $I \subseteq F$ for (A_1, A_2) relative to E with $|I| \geq c|F|$.

Definition 2.12. Let $G \curvearrowright X$ be a continuous group action and μ an invariant measure. We say that $(x, y) \in X^2$ is a *μ-IN pair* if for every product neighborhood $U_x \times U_y$ of (x, y), there exists $\varepsilon > 0$ such that (U_x, U_y) has (ε, μ)-independence over arbitrarily large finite sets.

THEOREM 2.13. [17, Theorem 4.9] *Let $G \curvearrowright X$ be a continuous group action and μ an invariant measure. A pair $(x, y) \in X^2$ with $x \neq y$ is an μ-IN pair if and only if it is a μ-sequence entropy pair.*

2.5. *Almost periodicity.* Let $G \curvearrowright X$ be a continuous group action and μ an invariant measure. We define the *Koopman representation* $\kappa : G \to \mathcal{B}(L^2(X, \mu))$ by $\kappa(s)f(x) = f(s^{-1}x)$ for all $s \in G$, $f \in L^2$, and $x \in X$, where $\mathcal{B}(L^2(X, \mu))$ is the space of all bounded linear operators on $L^2(X, \mu)$.

Definition 2.14. Let $G \curvearrowright X$ be a continuous group action, μ an invariant measure, and $f \in L^2(X, \mu)$. We say that f is an *almost periodic* function if $\overline{\kappa(G)(f)}$ is compact as a subset of $L^2(X, \mu)$. We denote by H_{ap} the set of almost periodic functions.

THEOREM 2.15. [11, Theorem 1.15] *Let G be an abelian group, $\{F_n\}$ a tempered Følner sequence, $G \curvearrowright X$ a continuous group action, and μ an ergodic measure. We have that $H_{ms} = H^c_{ap}$.*

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3. *Characterization of sequence entropy pairs*

3.1. *Mean sensitivity pairs*

LEMMA 3.1. [18] *Let (X, μ) be a probability space and $\{\xi_n\}$ be a sequence of two-set measurable partitions of X, with $\xi_n = \{P_n, P_n^c\}$. The closure of $\{1_{P_1}, 1_{P_2}, \ldots\} \subseteq L^2(X, \mu)$ is compact if and only if for all increasing sequences $\{m_i\}_{i\in\mathbb{N}}$ of integers,*

$$\lim_{n\to\infty} \frac{1}{n} H\bigg(\bigvee_{i=1}^{n} \xi_{m_i}\bigg) = 0.$$

The next theorem is obtained directly from Lemma 3.1.

THEOREM 3.2. *Let $G \curvearrowright X$ be a continuous group action, μ an invariant measure, and $B \in \mathcal{B}_\mu^+(X)$. Then $1_B \in H_{ap}$ if and only if $h_\mu^S(X, \{A, A^c\}) = 0$ for any sequence $S \subseteq G$.*

PROPOSITION 3.3. *Let G be an abelian group, $\{F_n\}$ a tempered Følner sequence for G, $G \curvearrowright X$ a continuous group action, and μ an ergodic measure. If $(x, y) \in X^2$ is a μ-mean sensitivity pair, then it is a μ-sequence entropy pair.*

Proof. Let (x, y) be a μ-mean sensitivity pair and $\mathcal{P} = \{P, P^c\}$ a finite partition, such that $x \in P \setminus \overline{P^c}$ and $y \in P^c \setminus \overline{P}$. This implies that there exist neighborhoods U_x of x and U_y of y, such that $U_x \subseteq P$ and $U_y \subseteq P^c$. Note that $G \curvearrowright X$ is μ-1_P-mean sensitive. Hence, by Theorems 2.15 and 3.2, we obtain that there exists $S \subseteq G$ such that $h_\mu^S(X, \mathcal{P}) > 0$. Thus, (x, y) is a μ-sequence entropy pair. □

The following lemma is standard, e.g. see [14, Proposition 5.8].

LEMMA 3.4. *Let $(X, \mathcal{B}(X), \mu)$ be a Borel probability space, $a > 0$, and $\{E_s\}_{s\in G}$ a sequence of measurable sets with $\mu(E_s) \geq a$ for all $s \in G$. There exists $N \in \mathbb{N}$ such that for any set $F \subseteq G$ with $|F| \geq N$, there exist $s_F, t_F \in F$ such that $s_F \neq t_F$ and $\mu(E_{s_F} \cap E_{t_F}) > 0$.*

The following lemma is the main tool used to provide a connection between independence over finite sets and sensitivity.

LEMMA 3.5. *Let $G \curvearrowright X$ be a continuous group action, μ an ergodic measure, $U_x, U_y \subseteq X$ open sets, and $\varepsilon > 0$. If (U_x, U_y) has (ε, μ)-independence over arbitrarily large finite sets, then for every $A \in \mathcal{B}_\mu^+(X)$, there exists $(s_A, t_A) \in R_A$ such that $\mu(s_A^{-1}U_x \cap t_A^{-1}U_y) > \varepsilon$, where*

$$R_A = \{(s, t) \in G^2 : \mu(s^{-1}A \cap t^{-1}A) > 0\}.$$

Proof. We will prove this by contradiction. Assume that there exists $A \in \mathcal{B}_\mu^+(X)$ such that

$$\mu(s^{-1}U_x \cap t^{-1}U_y) \leq \varepsilon$$

for all $(s, t) \in R_A$. This implies that $\mu((s^{-1}U_x \cap t^{-1}U_y)^c) \geq 1 - \varepsilon$ for all $(s, t) \in R_A$.

By the independence hypothesis, there exists $c > 0$ which satisfies the definition of (ε, μ)-independence over arbitrarily large finite sets (Definition 2.11). Using Lemma 3.4

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on $\{E_s\}_{s\in G}$ with $E_s = s^{-1}A$, we conclude that there is $N_0 > 0$ such that for any finite set $F \subseteq G$ with $|F| \geq N_0$, there exist $s_F, t_F \in F$ such that $s_F \neq t_F$ and

$$\mu(s_F^{-1}A \cap t_F^{-1}A) > 0.$$

Using Definition 2.11, there exists a finite set F_0 with $|F_0| \geq N_0/c$ such that for every $E \in \mathcal{B}'_\mu(X, \varepsilon)$, there is an independence set $I \subseteq F_0$ such that $|I| \geq c|F_0| \geq N_0$. This implies that for every $\sigma : I \to \{x, y\}$, we have that

$$\bigcap_{g\in I} E(g) \cap g^{-1}U_{\sigma(g)} \neq \emptyset. \tag{1}$$

Furthermore, since $|I| \geq N_0$, there exists $s_I, t_I \in I$ such that $s_I \neq t_I$ and

$$\mu(s_I^{-1}A \cap t_I^{-1}A) > 0.$$

Let $E : G \to 2^X$ be the constant function with

$$E(g) = (s_I^{-1}U_x \cap t_I^{-1}U_y)^c.$$

(The fact that it is constant will be important for Theorem 3.8.) Note that $E \in \mathcal{B}'_\mu(X, \varepsilon)$. Let $\sigma : I \to \{x, y\}$ with $\sigma(s_I) = x$ and $\sigma(t_I) = y$. Then,

$$\begin{aligned}\bigcap_{g\in I} E(g) \cap g^{-1}U_{\sigma(g)} &\subseteq (E(s_I) \cap s_I^{-1}U_x) \cap (E(t_I) \cap t_I^{-1}U_y)\\ &= (s_I^{-1}U_x \cap t_I^{-1}U_y)^c \cap s_I^{-1}U_x \cap t_I^{-1}U_y\\ &= \emptyset.\end{aligned}$$

This is a contradiction to equation (1). Therefore, there exists $(s_A, t_A) \in R_A$ such that $\mu(s_A^{-1}U_x \cap t_A^{-1}U_y) > \varepsilon$. □

In [14, Theorem 5.9], it was shown that every sequence entropy pair is a μ-sensitivity pair. In [20, Theorem 1.8], it was shown that sequence entropy pairs are density-sensitivity pairs.

Now we provide a characterization.

THEOREM 3.6. *Let G be an abelian group, $G \curvearrowright X$ a continuous group action, μ an ergodic measure, and $(x, y) \in X^2$ with $x \neq y$. The following are equivalent:*

(1) *$(x, y) \in X^2$ is a μ-sequence entropy pair;*
(2) *$(x, y) \in X^2$ is a μ-mean sensitivity pair with respect to a tempered Følner sequence;*
(3) *$(x, y) \in X^2$ is a μ-mean sensitivity pair with respect to every tempered Følner sequence.*

Proof. Given item (2), we obtain item (1) with Proposition 3.3. It only remains to prove that item (1) implies item (3). Let $(x, y) \in X^2$ be a μ-sequence entropy pair, U_x and U_y neighborhoods of x and y, and $A \in \mathcal{B}^+_\mu(X)$. From Theorem 2.13, there exists $\varepsilon > 0$ such that (U_x, U_y) has (ε, μ)-independence over arbitrarily large finite sets. Using Lemma 3.5, there exists $(s, t) \in R_A$ such that

$$\mu(s^{-1}U_x \cap t^{-1}U_y) > \varepsilon.$$

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Let $z \in s^{-1}A \cap t^{-1}A$ be a generic point of $s^{-1}U_x \cap t^{-1}U_y$, $p = sz$, and $q = tz$. Note that $p, q \in A$. If $gz \in s^{-1}U_x \cap t^{-1}U_y$, then $gp = gsz \in U_x$ and $gq = gtz \in U_y$. Therefore, for any tempered Følner sequence, we have that

$$\overline{D}(\{g \in G : gp \in U_x \text{ and } gq \in U_y\}) = \mu(s^{-1}U_x \cap t^{-1}U_y) \geq \varepsilon. \qquad \square$$

3.2. *Independence.* As a consequence of our techniques, we provide a simpler characterization of μ-IN pairs when μ is an ergodic measure.

We define $\mathcal{B}_\mu(X, \varepsilon) = \{D \in \mathcal{B}(X) : \mu(D) \geq 1 - \varepsilon\}$.

The following definition is very similar to Kerr and Li's definition of measure IN-pairs, but we use $\mathcal{B}$ instead of $\mathcal{B}'$.

Definition 3.7. Let $G \curvearrowright X$ be a continuous group action and μ an invariant measure. For $A_1, A_2 \subseteq X$ and $\varepsilon > 0$, we say that (A_1, A_2) has $\mathcal{B}$-(ε, μ)*-independence over arbitrarily large finite sets* if there exists $c > 0$ such that for every $N > 0$, there is a finite set $F \subseteq G$ with $|F| > N$ such that for every $D \in \mathcal{B}_\mu(X, \varepsilon)$, there is an independence set $I \subseteq F$ for (U_x, U_y) relative to D with $|I| \geq c|F|$.

We say that (x, y) is a $\mathcal{B}$-μ*-IN pair* if for every neighborhood U_x of x and U_y of y, there exists $\varepsilon > 0$ such that (U_x, U_y) has $\mathcal{B}$-(ε, μ)-independence over arbitrarily large finite sets.

THEOREM 3.8. *Let G be an abelian group, $G \curvearrowright X$ a continuous group action, $x \neq y \in X$, and μ an ergodic measure. Then (x, y) is an μ-IN pair if and only if it is a $\mathcal{B}$-μ-IN pair.*

Proof. One direction is trivial. The other can be obtained by noting that in the proof of Lemma 3.5, we actually only use $\mathcal{B}$-μ-IN pairs. $\square$

4. *Diam-mean sensitivity pairs*

Diam-mean sensitivity was introduced in [8] and can be used to characterize when a maximal equicontinuous factor is 1-1 for μ-almost every point [9]. In this section, we introduce the measure theoretic version of this concept. We provide some basic results which are adaptations from results in [8, 14]. Nonetheless, this new notion is still somewhat mysterious to the authors since we do not know if it is invariant under isomorphism or not.

Definition 4.1. Let $\{F_n\}$ be a Følner sequence for G, $G \curvearrowright X$ a continuous group action, and μ an invariant measure. We say that $G \curvearrowright X$ is μ*-diam-mean sensitive* if there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}^+_\mu(X)$, we have that

$$\limsup_{n\to\infty} \frac{1}{|F_n|} \sum_{s\in F_n} \operatorname{diam}(sA) > \varepsilon.$$

Note that if $G \curvearrowright X$ is μ-mean sensitive, then it is μ-diam-mean sensitive.

We do not know if there exists a μ-diam mean sensitive system with discrete spectrum.

Definition 4.2. Let $\{F_n\}$ be a Følner sequence for G, $G \curvearrowright X$ a continuous group action, and μ an invariant measure. We say that $(x, y) \in X^2$ is a μ*-diam-mean sensitivity pair* if $x \neq y$ and for all open neighborhoods U_x of x and U_y of y, there exists $\varepsilon > 0$ such that for every $A \in \mathcal{B}^+_\mu(X)$, there exists $S \subseteq G$ with $\overline{D}(S) > \varepsilon$ such that for every $s \in S$, there exist

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$p, q \in A$ such that $sp \in U_x$ and $sq \in U_y$. We denote the set of μ-diam-mean sensitivity pairs by $S_\mu^{dm}(X, G)$.

PROPOSITION 4.3. *Let $G \curvearrowright X$ a continuous group action and μ an ergodic measure. We have that $G \curvearrowright X$ is μ-diam-mean sensitive if and only if $S_\mu^{dm}(X, G) \neq \emptyset$.*

Proof. Suppose that $G \curvearrowright X$ is μ-diam-mean sensitive with sensitive constant $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$. We consider the following compact set:

$$X^\varepsilon = \{(x, y) \in X^2 : d(x, y) \geq \varepsilon\}.$$

Suppose that $S_\mu^{dm}(X, G) = \emptyset$. This implies that for every $(x, y) \in X^\varepsilon$, there exist open neighborhoods of x and y, $U_{x,y}$ and $V_{x,y}$, such that for every $\delta > 0$, there exists $A_\delta(x, y) \in \mathcal{B}_\mu^+(X)$ such that

$$\overline{D}(\{s \in G : \text{ there exists } p, q \in A_\delta(x, y) \text{ such that } sp \in U_{x,y}, sq \in V_{x,y}\}) \leq \delta.$$

There exists a finite set $F \subset X^\varepsilon$ such that

$$X^\varepsilon \subseteq \bigcup_{(x,y)\in F} U_{x,y} \times V_{x,y}.$$

Let $\delta = \varepsilon/|F|$ and $S(x, y) = \{s \in G : \text{ there exists } p, q \in A_\delta(x, y) \text{ such that } (sp, sq) \in U_{x,y} \times V_{x,y}\}$forevery $(x, y) \in F$. Since μ is ergodic, for every $(x, y) \in F$, there exists $t(x, y) \in G$ such that $A_0 := \bigcap_{(x,y)\in F} t(x, y)A_\delta(x, y) \in \mathcal{B}_\mu^+(X)$. Let $A = \bigcup_{(x,y)\in F} t(x, y)^{-1}A_0$. Note that $A \in \mathcal{B}_\mu^+(X)$, and $A \subseteq A_\delta(x, y)$ for every $(x, y) \in F$. Then

$$\begin{aligned} S :&= \{s \in G : \text{there exists } p, q \in A \text{ such that } (sp, sq) \in X^\varepsilon\} \\ &\subseteq \bigcup_{(x.y)\in F} \{s \in G : \text{there exists } p, q \in A \text{ such that } (sp, sq) \in U_{x,y} \times V_{x,y}\} \\ &\subseteq \bigcup_{(x,y)\in F} S(x, y). \end{aligned}$$

Hence, $\overline{D}(S) \leq |F|\delta = \varepsilon$.

However, since $G \curvearrowright X$ is μ-diam-mean sensitive and ε is smaller than the sensitive constant, we have that $\overline{D}(S) > \varepsilon$. This is a contradiction.

Now we prove the other direction.

Suppose that $S_\mu^{dm}(X, G) \neq \emptyset$. Let $(x, y) \in S_\mu^{dm}(X, G)$. There exist neighborhoods U_x of x and U_x of y, with $d(U_x, U_y) > 0$, and $\varepsilon > 0$, such that for every $A \in \mathcal{B}_\mu^+(X)$, there exists $S \subseteq G$ with $\overline{D}(S) > \varepsilon$ such that for every $s \in S$, there exist $p, q \in A$ such that $sp \in U_x$ and $sq \in U_y$. This implies that $\overline{D}(\{s \in G : \text{diam}(sA) > d(U_x, U_y)\}) \geq \overline{D}(S) > \varepsilon$. Therefore, $G \curvearrowright X$ is μ-diam-mean sensitive. □

A consequence of Theorem 3.6 is the following corollary.

COROLLARY 4.4. *Let G be an abelian group, $G \curvearrowright X$ a continuous group action, and μ an ergodic measure. Every μ-sequence entropy pair is a μ-diam-mean sensitivity pair.*

We do not know if the converse of the previous corollary holds.

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