Solution of a Problem proposed by Dr Muir.

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The problem is to show that the expression

$$\{(a^{\frac{1}{4}}+b^{\frac{1}{4}})^4-c\}\{(a^{\frac{1}{4}}+\omega b^{\frac{1}{4}})^4-c\}\{(a^{\frac{1}{4}}+\omega^2 b^{\frac{1}{4}})^4-c\}\{(a^{\frac{1}{4}}+\omega^3 b^{\frac{1}{4}})^4-c\},$$

where ω is an imaginary fourth root of unity, is symmetrical with respect to a, b, and c.

First Method.—Denote the above expression by P. Then, by means of the identity $x^4-y^4=(x+y)(x+\omega y)(x+\omega^2 y)(x+\omega^3 y)$, P may be expressed as the product of sixteen factors, namely, the sixteen values of $a^{\frac{1}{4}}+\omega^r b^{\frac{1}{4}}+\omega^s c^{\frac{1}{4}}$, where r and s have successively the values 1, 2, 3, 4. This product is obviously symmetrical, and the symmetry is not destroyed to the eye by a re-combination of the factors in sets of four, the form thus obtained being

$$(a+b+c-2a^{\frac{1}{2}}b^{\frac{1}{2}}-2b^{\frac{1}{2}}c^{\frac{1}{2}}-2c^{\frac{1}{2}}a^{\frac{1}{2}})(a+b+c-2a^{\frac{1}{2}}b^{\frac{1}{2}}+2b^{\frac{1}{2}}c^{\frac{1}{2}}+2c^{\frac{1}{2}}a^{\frac{1}{2}})$$
$$\times(a+b+c+2a^{\frac{1}{2}}b^{\frac{1}{2}}-2b^{\frac{1}{2}}c^{\frac{1}{2}}+2c^{\frac{1}{2}}a^{\frac{1}{2}})(a+b+c+2a^{\frac{1}{2}}b^{\frac{1}{2}}+2b^{\frac{1}{2}}c^{\frac{1}{2}}-2c^{\frac{1}{2}}a^{\frac{1}{2}}).$$

Second Method.—Since

$$(a^{\frac{1}{4}}+\omega b^{\frac{1}{4}})^4-c=\omega^{12}(a^{\frac{1}{4}}+\omega b^{\frac{1}{4}})^4-c=(\omega^3 a^{\frac{1}{4}}+b^{\frac{1}{4}})^4-c,$$

P is evidently symmetrical with respect to a and b. Further, since the change of $b^{\frac{1}{4}}$ into $\omega b^{\frac{1}{4}}$ does not alter P, P is rational in b and therefore also in a. When $b=0$, P becomes $(a-c)^4$ which is symmetrical in a and c. Hence P has been proved symmetrical in a and b, and symmetrical in a and c except in those terms which contain all the three letters. Thus

$$P=\Sigma a^4+A\Sigma a^3 b+B\Sigma a^2 b^2+C(a^2bc+ab^2c)+Dabc^2.$$

It remains to prove $C=D$.

The co-efficient of c^2 in P is $\Sigma(a^{\frac{1}{4}}+b^{\frac{1}{4}})^4(a+\omega b^{\frac{1}{4}})^4$ and D is equal to the value of this when $a=b=1$, less twice the value when $a=1$, $b=0$, that is $-112-2\times 6=-124$.

The co-efficient of c in P is $-\Sigma(a^{\frac{1}{4}}+b^{\frac{1}{4}})^4(a^{\frac{1}{4}}+\omega b^{\frac{1}{4}})^4(a^{\frac{1}{4}}+\omega^2 b^{\frac{1}{4}})^4$ and 2C is equal to the value of this when $a=b=1$, less twice the value when $a=1$, $b=0$, that is $-(256-2\times 4)=-248=2D$.

Therefore $C=D$.

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