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Galois Theory of $B_{\mathrm{d}R}^{+}$

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Abstract. We formulate and prove necessary and sufficient conditions for the Galois correspondence to hold for the ring of p-adic periods $B_{\mathrm{d}R}^{+}$.

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0. Introduction

Let us fix a prime number p. The notation $B_{\mathrm{d}R}^{+}$ refers to the ring of p-adic periods of algebraic varieties over local $(p$-adic$)$ fields as defined by J.-M. Fontaine in [Fo]. It is a topological local ring with residue field $\mathbb{C}_p$ (see the section Notations) and it is endowed with a canonical, continuous action of $G := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, where $\overline{\mathbb{Q}}_p$ is the algebraic closure of $\mathbb{Q}_p$ in $\mathbb{C}_p$. Let us denote by I its maximal ideal and $B_n := B_{\mathrm{d}R}^{+}/I^n$. Then $B_{\mathrm{d}R}^{+}$ (and B_n for each $n \geqslant 1$) is canonically a $\overline{\mathbb{Q}}_p$-algebra and, moreover, $\overline{\mathbb{Q}}_p$ is dense in $B_{\mathrm{d}R}^{+}$ (and in each B_n, respectively) if we consider the 'canonical topology' on $B_{\mathrm{d}R}^{+}$ which is finer than the I-adic topology.

Let now L be any algebraic extension of $\mathbf{Q}_p$ contained in $\overline{\mathbf{Q}}_p$ and $G_L := \mathrm{Gal}(\overline{\mathbf{Q}}_p/L)$. Then it is a classical result of J. Tate [T], J. Ax [Ax] and S. Sen [S] that L is dense in $(\mathbb{C}_p)^{G_L}$. Moreover, the map $H \to (\mathbb{C}_p)^H$ gives a bijection between the set of closed subgroups of G and the set of complete subfields of $\mathbb{C}_p$ (see [I–Z]). We then say that we have a Galois theory for $\mathbb{C}_p$. Since $\overline{\mathbf{Q}}_p$ is dense in $B_{\mathrm{d}R}^{+}$ (and in B_n for each n respectively) it makes sense to ask whether we have a Galois theory for $B_{\mathrm{d}R}^{+}$ (and for B_n for each n, respectively). For instance, it makes sense to ask the question: is L dense in $(B_{\mathrm{d}R}^{+})^{G_L}$? (or is L dense in $(B_n)^{G_L}$ for different n's?). Although this was known for finite extensions L (actually for extensions L such that the ramification degree of $L/\mathbb{Q}_p$ is finite) it is not true in general. A counterexample, which was pointed out to us by P. Colmez, is presented in Section 8. If L is dense in $(B_{\mathrm{d}R}^{+})^{G_L}$ (or in $(B_n)^{G_L}$ for some n) we shall say that L satisfies the Galois correspondence for $B_{\mathrm{d}R}^{+}$ (or it satisfies the

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Galois correspondence at level n, respectively). The main purpose of this paper is to characterize the algebraic extensions L of $K := \mathbb{Q}_p^{ur}$ (the maximal unramified extension of $\mathbb{Q}_p$) which satisfy the Galois correspondence for $B_{\mathrm{d}R}^+$ and at different levels. A concept that plays an important role in what follows is that of *deeply ramified* extension, which was introduced by J. Coates and R. Greenberg in [C–G] (see Section 1 below). We will prove the following:

THEOREM 0.1. *If L is an algebraic extension of K which is not deeply ramified then L satisfies the Galois correspondence for $B_{\mathrm{d}R}^+$ and at any level n.*

The situation changes dramatically if L is deeply ramified. To study this case we bring into play the modules of nth differential forms $\Omega^{(n)}(L/K)$, $n \in \mathbb{N}$, defined by P. Colmez (see Section 1 below). In this context we make the following definition.

Let L be a deeply ramified extension of K. Then we say that L is deeply ramified at a given level $n \geqslant 2$ if $\Omega^{(n-1)}(L/K)$ is not annihilated by a finite power of p.

We also consider the groups $H_{\mathrm{d}R}^{(n)}(L/K) := \Omega^{(n)}(L/K)/d_n(\mathcal{O}_L^{(n-1)})$ which we call the nth de Rham cohomology group of L/K (see Remark 6.1). We say that L is de Rham at level n if $H_{\mathrm{d}R}^{(n-1)}(L/K) = 0$. Then we have the following.

THEOREM 0.2. *Let L be a deeply ramified algebraic extension of K and $n \in \mathbb{N}$, $n \geqslant 2$. Then*

(i) *The following are equivalent*:
  - (a) *L satisfies the Galois correspondence at any level $m \leqslant n$.*
  - (c) *L is deeply ramified and de Rham at any level $m \leqslant n$.*
  - (e) *The topological closure $\overset{\wedge n}{L}$ of L in B_n contains a uniformizer of B_n.*

(ii) *The following are equivalent*:
  - (a) *L satisfies the Galois correspondence in $B_{\mathrm{d}R}^+$.*
  - (c) *L is deeply ramified and de Rham at all levels.*
  - (d) *The topological closure $\overset{\wedge \infty}{L}$ of L in $B_{\mathrm{d}R}^+$ contains a uniformizer of $B_{\mathrm{d}R}^+$.*

The plan of the paper is the following: In Section 1 we recall the main constructions and results which will be used in the sequel. In Section 2, we examine the situation at level 2, namely we give a characterization of deeply ramified extensions using 1-differential forms which will be used later. We present here all the concepts and ideas which appear naturally at level 2 and will be generalized later. Section 3 is devoted to the computation of the Galois cohomology of $\mathbb{C}_p(n)$ and the cohomology groups in positive degrees of $B_{\mathrm{d}R}^+$ and B_n for all n. The rest of the paper will be spent on analyzing H^0. Theorem 0.1 is proved in Section 4. In Section 5 we study deeply ramified extensions at different levels n using the nth differential forms. One of the main results of this section is that if L is deeply ramified at level n we have 'almost' Galois correspondence for the $n-1$ differential forms (here 'almost' has the sense defined by G. Faltings in [Fa]). In Section 6 we study de Rham extensions and then prove Theorem 0.2. Section 7 contains the statements of the main results proven before, including a more complete version of Theorem 0.2. In the last section we propose some problems and give some examples of deeply ramified extensions which are not de Rham at level 2 and deeply ramified extensions which are de Rham at level 2. We do not have nontrivial examples of deeply ramified and/or de Rham extensions at higher levels. Within this context, let us remark that any example of a field L which satisfies the Galois correspondence in $B_{\rm dR}^{+}$ automatically produces a whole class of examples. In fact as a consequence of the above Theorems 0.1 and 0.2, one has the following corollary:

COROLLARY 0.1. *Let L be an algebraic extension of K which satisfies the Galois correspondence in $B_{\rm dR}^{+}$. Then the Galois correspondence in $B_{\rm dR}^{+}$ is satisfied either by all the subextensions of L or by all the extensions of L.*

Indeed, if L is not deeply ramified, then any subextension of L is not deeply ramified and the result follows from Theorem 0.1, while if L is deeply ramified, then from Theorem 0.2 we know that $\overset{\wedge\infty}{L}$ contains a uniformiser of $B_{\rm dR}^{+}$ and any extension of L will have this property.

Notations

Let p be a positive prime integer, $K = \mathbb{Q}_p^{ur}$ the maximal unramified extension of $\mathbb{Q}_p$, $\overline{K}$ a fixed algebraic closure of K and $\mathbb{C}_p$ the completion of $\overline{K}$ with respect to the unique extension v of the p-adic valuation on $\mathbb{Q}_p$ (normalized such that $v(p) = 1$). All the algebraic extensions of K considered in this paper will be contained in $\overline{K}$. Let L be such an algebraic extension. We denote by $G_L := \mathrm{Gal}(\overline{K}/L)$, $\hat{L}$ the (topological) closure of L in $\mathbb{C}_p$, $\mathcal{O}_L$ the ring of integers in L and m_L its maximal ideal. If $K \subset L \subset F \subset \overline{K}$, and F is a finite extension of L, $\Delta_{F/L}$ denotes the different of F over L.

If A and B are commutative rings and $\phi: A \to B$ is a ring homomorphism we denote by $\Omega_{B/A}$ the B-module of Kähler differentials of B over A, and $d: B \to \Omega_{B/A}$ the structural derivation.

If M is an Abelian group we denote for $k \in \mathbb{N}$,

$$M[p^k] = \{x \in M \mid p^k x = 0\} \quad \text{and} \quad T_p M = \varprojlim M[p^k].$$

Let $\mathcal{A}$ be a Banach space whose norm is given by the valuation w and suppose that the sequence $\{a_m\}$ converges in $\mathcal{A}$ to some α. We will write this as $a_m \xrightarrow{w} \alpha$.

1. Review of Some Constructions, Definitions and Results

We will first recall the construction of $B_{\rm dR}^{+}$, which is due to J.-M. Fontaine in [Fo]. Let R denote the set of sequences $x = (x^{(n)})_{n\geqslant 0}$ of elements of $\mathcal{O}_{C_p}$ which verify the relation $(x^{(n+1)})^p = x^{(n)}$. Let us define $v_R(x) := v(x^{(0)})$ and $x + y = s$, where $s^{(n)} = \lim_{n\to\infty}(x^{(n+m)} + y^{(n+m)})^{p^m}$ and $xy = t$, where $t^{(n)} =$

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$x^{(n)}y^{(n)}$. With these operations R becomes a perfect ring of characteristic p on which v_R is a valuation. R is complete with respect to v_R. Let $W(R)$ be the ring of Witt vectors with coefficients in R and if $x\in R$ we denote by $[x]$ its Teichmüller representative in $W(R)$. Denote by θ the homomorphism $\theta\colon W(R)\to \mathcal{O}_{C_p}$ which sends $(x_0,x_1,\ldots,x_n,\ldots)$ to $\sum_{n=0}^{\infty} p^n x_n^{(n)}$. Then θ is surjective and its kernel is principal. Let also θ denote the map $W(R)[p^{-1}]\to \mathbb{C}_p$. We denote $B_{\mathrm{dR}}^{+} := \varprojlim W(R)[p^{-1}]/(\mathrm{Ker}(\theta))^n$. Then θ extends to a continuous, surjective ring homomorphism $\theta=\theta_{\mathrm{dR}}\colon B_{\mathrm{dR}}^{+}\to \mathbb{C}_p$ and we denote $I:=\mathrm{Ker}(\theta_{\mathrm{dR}})$ and $I_+ := I\cap W(R)$. Let $\varepsilon=(\varepsilon^{(n)})_{n\geqslant 0}$ be an element of R, where $\varepsilon^{(n)}$ is a primitive p^nth root of unity such that $\varepsilon^{(0)}=1$ and $\varepsilon^{(1)}\neq 1$. Then the power series $\sum_{n=1}^{\infty}(-1)^{n-1}([\varepsilon]-1)^n/n$ converges in B_{dR}^{+}, and its sum is denoted by $t:=\log[\varepsilon]$. It is proved in [Fo] that t is a generator of the ideal I, and as $G_K:=\mathrm{Gal}(\overline{K}/K)$ acts on t by multiplication with the cyclotomic character, we have $I^n/I^{n+1}\cong \mathbb{C}_p(n)$, where the isomorphism is $\mathbb{C}_p$-linear and G_K-equivariant. Therefore, for each integer $n\geqslant 2$, if we denote $B_n := B_{\mathrm{dR}}^{+}/I^n$ we have an exact sequence of G_K-equivariant homomorphisms $0\to \mathbb{C}_p(n)\to B_{n+1}\xrightarrow{\phi_n} B_n\to 0$ which will be called 'the fundamental exact sequence'.

Let us now review P. Colmez's differential calculus with algebraic numbers as in the Appendix of [F-C]. We should point out that as our K is unramified over $\mathbb{Q}_p$ and so $W(R)$ is canonically an $\mathcal{O}_K$ as well as an $\mathcal{O}_{\hat{K}}$-algebra, we will work with $W(R)$ instead of A_{inf}. For each nonnegative integer k, we set $A_{\mathrm{inf}}^{k} := W(R)/I_{+}^{k+1}$. We define recurrently the sequence of subrings $\mathcal{O}_{\bar{K}}^{(k)}$ of $\mathcal{O}_{\bar{K}}$ and of $\mathcal{O}_{\bar{K}}$-modules $\Omega^{(k)}$ setting: $\mathcal{O}_{\bar{K}}^{(0)}=\mathcal{O}_{\bar{K}}$ and if $k\geqslant 1$ $\Omega^{(k)} := \mathcal{O}_{\bar{K}}\otimes_{\mathcal{O}_{\bar{K}}^{k-1}} \Omega^{1}_{\mathcal{O}_{\bar{K}}^{(k-1)}/\mathcal{O}_K}$ and $\mathcal{O}_{\bar{K}}^{k}$ is the kernel of the canonical derivation $d^{(k)}\colon \mathcal{O}_{\bar{K}}^{(k-1)}\to \Omega^{(k)}$. Then we have

THEOREM 1.1 (Colmez, Appendix of [F-C], Theorem 1). (i) *If* $k\in\mathbb{N}$, *then* $\mathcal{O}_{\bar{K}}^{(k)} = \overline{K}\cap(W(R)+I^{k+1})$ *and for all* $n\in\mathbb{N}$ *the inclusion of* $\mathcal{O}_{\bar{K}}^{(k)}$ *in* $W(R)+I^{k+1}$ *induces an isomorphism*

$$\frac{A_{\mathrm{inf}}^{k}}{p^n A_{\mathrm{inf}}^{k}} \cong \frac{\mathcal{O}_{\bar{K}}^{(k)}}{p^n\mathcal{O}_{\bar{K}}^{(k)}}.$$

(ii) *If* $k\geqslant 1$, *then* $d^{(k)}$ *is surjective and* $\Omega^{(k)}\cong(\overline{K}/\mathbf{a}^k)(k)$, *where* $\mathbf{a}$ *is the fractional ideal of* $\overline{K}$ *whose inverse is the ideal generated by* $\varepsilon^{(1)}-1$ (*recall* $\varepsilon^{(1)}$ *is a fixed primitive* pth *root of unity.*)

Some consequences of this theorem are gathered in the following corollary:

COROLLARY 1.1. (i) $A_{\mathrm{inf}}^{(n)}\cong \varprojlim(\mathcal{O}_{\bar{K}}^{(n)}/p^i O_{\bar{K}}^{(n)})$ *and* $A_{\mathrm{inf}}^{(n)}\otimes_{\mathbf{Z}_p}\mathbb{Q}_p\cong B_{n+1}$ *for all* $m\geqslant 0$.

(ii) $\Omega^{(n)}$ *is a* p*-divisible and a* p*-torsion* $\mathcal{O}_{\bar{K}}$*-module.*

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We would now like to recall the Coates–Greenberg concept of deeply ramified extensions. Let L be an algebraic extension of $\mathbb{Q}_p$, contained in $\overline{K}$. Then we have

THEOREM 1.2 (Coates and Greenberg, [C-G]). *The following conditions are equivalent*

(i) L *does not have a finite conductor* (*which means that* L *is not fixed by any of the ramification subgroups of* $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$).
(ii) *The set* $\{v(\Delta_{F/K})\|$ *where* F *is any finite extension of* $\mathbb{Q}_p$ *contained in* $L\}$ *is unbounded*.
(iii) *For every* L' *finite extension of* L, *we have* $m_L \subset \mathrm{Tr}_{L'/L}(m_{L'})$.

Remark 1.1. There are more equivalent conditions in [C-G], but we won't use them here.

DEFINITION 1.1 (Coates and Greenberg, [C-G]). We say that L is deeply ramified if it satisfies the equivalent conditions of the above theorem.

Finally, we will recall an approximation result due to Ax [A]. Let L be an algebraic extension of $\mathbb{Q}_p$ contained in $\overline{K}$ and $a \in \overline{K}$. Then we have

THEOREM 1.3. *There exists a constant* c_0 (*it does not depend on* L *or* a) *such that there exists* $\alpha \in L$ *with the property*

$$v(a-\alpha) + c_0 \geqslant \inf_{\sigma \in G_L} \{v(\sigma(a) - a)\}.$$

Remark 1.2. c_0 in the above theorem may be taken as $p/(p-1)$.

2. The Level Two Case

In the notations of the previous section let $\mathcal{O} := \mathcal{O}_{\bar{K}}^{(1)}$, $\Omega := \Omega^{(1)}$, $d := d^{(1)}$ and $A := A_{\mathrm{inf}}^{(1)}$. Also if L is an algebraic extension of K we denote $\mathcal{O}(L/K) := \mathrm{Ker}(d\colon \mathcal{O}_L \to \Omega_{O_L/O_K})$.

Let $a \in \mathcal{O}_{\bar{K}}$. Let F be a finite extension of K which contains a, π a uniformizer of F and $f \in \mathcal{O}_K[X]$ such that $a = f(\pi)$. Then we set

$$\delta(a) := \min\left(v\left(\frac{f'(\pi)}{\Delta_{F/K}}\right), 0\right).$$

It is not hard to see that δ does not depend on π, f or F, so it defines a function $\delta\colon \mathcal{O}_{\bar{K}} \to (\infty, 0]$.

Properties of δ

(a) If $a, b \in \mathcal{O}_{\bar{K}}$ then $\delta(a+b) \geqslant \min(\delta(a), \delta(b))$ and if $\delta(a) \neq \delta(b)$ then we have equality.
(b) $\delta(ab) \geqslant \min(\delta(a) + v(b), \delta(b) + v(a))$.

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(c) If $f \in \mathcal{O}_K[X]$ and $\theta \in \mathcal{O}_{\bar{K}}$ then $\delta(f(\theta)) = \min(v(f'(\theta)) + \delta(\theta), 0)$.
(d) If $x, y \in \mathcal{O}_{\bar{K}}$ then $x\mathrm{d}y = 0$ if and only if $v(x) + \delta(y) \geqslant 0$.
(e) for $a \in \mathcal{O}_{\bar{K}}$, $\delta(a) = 0$ is equivalent to $a \in \mathcal{O}$.
(f) The formula $\delta(a\mathrm{d}b) := \min(v(a) + \delta(b), 0)$ is well-defined and gives a map $\delta\colon \Omega \to (-\infty, 0]$, which makes the obvious diagram commutative.

Now we define another map, w, which is a valuation on $\overline{K}$, namely if $a \in \overline{K}$ we set $w(a) = \sup\{m \in \mathbb{Z} \parallel a \in p^m\mathcal{O}\}$.

Properties of w

(a) $w(a + b) \geqslant \min(w(a), w(b))$ and if $w(a) \neq w(b)$ then we have equality, for all $a, b \in \overline{K}$.
(b) $w(ab) \geqslant w(a) + w(b)$ for all a, b.
(c) $w(a) = \infty$ if and only if $a = 0$.
(d) $v(a) \geqslant w(a)$ for all $a \in \overline{K}$.

The relationship between w and δ is as follows: for any $a \in \mathcal{O}_{\bar{K}} - \mathcal{O}$ we have $w(a) = [\delta(a)]$ (where $[\,\cdot\,]$ denotes the integral part function).

From Theorem 1.1 it follows that the completion of $\overline{K}$ with respect to w is B_2, w extends to a valuation on B_2 which will be also called w and A is its ring of integers (i.e. $A = \{x \in B_2 || w(x) \geqslant 0\}$). Let us denote by $\widetilde{\mathcal{O}}_{\bar{K}}$ the completion of $\mathcal{O}_{\bar{K}}$ with respect to the valuation w restricted to $\mathcal{O}_{\bar{K}}$. As d is continuous with respect to the topology defined by w on $\mathcal{O}_{\bar{K}}$ and the discrete topology on Ω, it extends uniquely to an $\mathcal{O}_K$-linear map, also called $d\colon \widetilde{\mathcal{O}}_{\bar{K}} \to \Omega$. If we denote $J := \mathrm{Ker}(\theta\colon B_2 \to \mathbb{C}_p)$ then we have

LEMMA 2.1. $J \subset \widetilde{\mathcal{O}}_{\bar{K}}$.

Proof. Let $x \in B_2$ be such that $\theta(x) = 0$. Let $(a_n)_n$, $a_n \in \overline{K}$ be a sequence such that $a_n \xrightarrow{w} x$. Then $a_n \xrightarrow{v} \theta(x) = 0$, so for n big enough $a_n \in \mathcal{O}_{\bar{K}}$. □

We want now to characterize the class of deeply ramified extensions of K (see Definition 1.1) using differentials. For this we need

LEMMA 2.2. *Let* $a, b \in \mathcal{O}_{\bar{K}}$ *be such that* $\delta(a) \leqslant \delta(b)$. *Then there exists* $c \in \mathcal{O}_{K[a,b]}$ *such that* $c\,\mathrm{d}a = \mathrm{d}b$.

Proof. Let π be a uniformizer of $K[a, b]$ and $h_1, h_2 \in \mathcal{O}_K[X]$ such that $a = h_1(\pi)$ and $b = h_2(\pi)$. Then $\mathrm{d}a = h_1'(\pi)\,\mathrm{d}\pi$ and $\mathrm{d}b = h_2'(\pi)\,\mathrm{d}\pi$. If $\delta(b) = 0$ then we can choose $c = 0$ and if $\delta(b) < 0$ we have $\delta(a) = v(h_1'(\pi)) + \delta(\pi) \leqslant v(h_2'(\pi) + \delta(\pi) = \delta(b)$. It follows that we can choose $c = h_2'(\pi)/h_1'(\pi) \in \mathcal{O}_{K[a,b]}$. □

PROPOSITION 2.1. *Let L be an algebraic extension of K. Then the following conditions are equivalent*

(a) *the set of real numbers $\delta(\mathcal{O}_L)$ is unbounded (from below).*
(b) *For every algebraic extension F of L we have $\Omega_{\mathcal{O}_F/\mathcal{O}_K} = \mathcal{O}_F \cdot \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ as subgroups of Ω.*

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Proof. Let us prove that (a) implies (b). Let $x \in \mathcal{O}_F$. Then there is $y \in \mathcal{O}_L$ such that $\delta(y) \leqslant \delta(x)$. From Lemma 2.2 we deduce the existence of $z \in \mathcal{O}_F$ such that $\mathrm{d}x = z\,\mathrm{d}y \in \mathcal{O}_F \cdot \Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Conversely, let us suppose that $\delta(\mathcal{O}_L)$ is bounded and let $N \in \mathbb{N}$ be such that $-N \leqslant \inf(\delta(a) \| a \in \mathcal{O}_L)$. Then $p^N \Omega_{\mathcal{O}_L/\mathcal{O}_K} = 0$. If we choose $\alpha \in \mathcal{O}_{\bar{K}}$ such that $\delta(\alpha) < -N$, then $\mathrm{d}\alpha$ cannot be in $\mathcal{O}_{\bar{K}} \cdot \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ as $p^N\,\mathrm{d}\alpha \neq 0$. □

Let us recall that we denoted by $G_K := \mathrm{Gal}(\overline{K}/K)$. Then if $\sigma \in G_K$ and $a, b \in \mathcal{O}_{\bar{K}}$ the formula $\sigma(a\,\mathrm{d}b) := \sigma(a)\,\mathrm{d}(\sigma(b))$ is well-defined and it gives a continuous semilinear action of G_K on Ω. Then we have

THEOREM 2.1. *There exists an absolute constant* c_0 (*which can be taken the same as in Theorem* 1.3) *such that if* L *is a deeply ramified extension of* K *then* $p^{c_0}(\Omega^{G_L}/\Omega_{\mathcal{O}_L/\mathcal{O}_K}) = 0$.

Proof. We clearly have $\Omega_{\mathcal{O}_L/\mathcal{O}_K} \subset \Omega^{G_L}$. Let $b \in \mathcal{O}_{\bar{K}}$ such that $\mathrm{d}b \in \Omega^{G_L}$, so for each $\sigma \in G_L$ $\mathrm{d}(\sigma(b)) = \mathrm{d}b$. Obviously, if L is deeply ramified $\delta(\mathcal{O}_L)$ is unbounded, we get from Proposition 2.1 that $\Omega = \mathcal{O}_{\bar{K}} \cdot \Omega_{\mathcal{O}_L/\mathcal{O}_K}$. So let $c \in \mathcal{O}_{\bar{K}}$ and $a \in \mathcal{O}_L$ be such that $\mathrm{d}b = c\,\mathrm{d}a$. Therefore, for each $\sigma \in G_L$ we have $\sigma(c)\,\mathrm{d}a = c\,\mathrm{d}a$ or $v(\sigma(c) - c) \geqslant -\delta(a)$. From Theorem 1.3 we deduce that there exists $\alpha \in \mathcal{O}_L$ such that

$$v(c - \alpha) \geqslant -c_0 + \min_{\sigma \in G_L} (v(\sigma(c) - c) \geqslant -c_0 - \delta(a).$$

Thus, $v(p^{c_0}(c - \alpha)) \geqslant -\delta(a)$ or $p^{c_0}\,\mathrm{d}b = p^{c_0} c\,\mathrm{d}a = p^{c_0} \alpha\,\mathrm{d}a \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$. □

Remark 2.1. In Section 5 we improve Theorem 2.1 for deeply ramified extensions (Proposition 5.3). Using that result Theorem 2.1 becomes: If L is deeply ramified then $m_L \cdot (\Omega^{G_L}/\Omega_{\mathcal{O}_L/\mathcal{O}_K}) = 0$, i.e. the inclusion $\Omega_{\mathcal{O}_L/\mathcal{O}_K} \subset (\Omega)^{G_L}$ is an almost isomorphism. We would not need this strong form of Theorem 2.1 in this section.

Remark 2.2. Theorem 2.1 and Remark 2.1 say that for deeply ramified extensions we have ‘almost’ Galois correspondence for differentials.

Now we are finally able to formulate the main result of this section, namely the characterization of deeply ramified extensions of K using differentials.

THEOREM 2.2. *Let* L *be an algebraic extension of* K. *The following conditions are equivalent*

(1) L *is deeply ramified.*
(2) $\delta(\mathcal{O}_L)$ *is unbounded.*
(3) $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ *is non-zero and* p*-divisible.*
(4) *For every algebraic extension* F *of* L *we have* $\Omega_{\mathcal{O}_F/\mathcal{O}_L} = 0$.
(5) *For every algebraic extension* F *of* L *we have* $\Omega_{\mathcal{O}_F/\mathcal{O}_K} = \mathcal{O}_F \cdot \Omega_{\mathcal{O}_L/\mathcal{O}_K}$.

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(6) $d(J^{G_L}) = \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ *(here we use the notations in the discussion before Lemma 2.1).*

Proof. (1) and (2) are clearly equivalent and it was proved in Proposition 2.1 that (2) and (5) are equivalent. On the other hand, if F is any algebraic extension of L, the inclusions $\mathcal{O}_K \subset \mathcal{O}_L \subset \mathcal{O}_F$ induce the exact sequence

$$\mathcal{O}_F \otimes_{O_L} \Omega_{\mathcal{O}_L/\mathcal{O}_K} \xrightarrow{f} \Omega_{\mathcal{O}_F/\mathcal{O}_K} \to \Omega_{\mathcal{O}_F/\mathcal{O}_L} \to 0$$

and as the image of f is $\mathcal{O}_F \cdot \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ we get that (4) is equivalent to (5).

Let us now prove that (2) implies (3). Let $u\,\mathrm{d}v \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, let us choose $\beta \in \mathcal{O}_L$ such that $\delta(\beta) \leqslant \delta(v) - 1$ and let us apply Lemma 2.2. So there exists $c \in \mathcal{O}_L$ such that $\mathrm{d}v = c\,\mathrm{d}\beta$. Then $v(c) \geqslant 1$ so if denote $c_1 = c/p \in \mathcal{O}_L$ we get that $u\,\mathrm{d}v = puc_1$ $\mathrm{d}\beta \in p\Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Conversely, let us assume (3) and suppose that $\delta(\mathcal{O}_L)$ is bounded. Then if $\varepsilon := \inf(\delta(a) \| a \in \mathcal{O}_L) < 0$ let $x \in \mathcal{O}_L$ be such that $0 \leqslant \delta(x) - \varepsilon \leqslant 1/2$ and $\delta(x) < 0$. As $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ was supposed p-divisible there are $a, b \in \mathcal{O}_L$ such that $\mathrm{d}x = pa\,\mathrm{d}b$. Then $\delta(x) = \delta(pa\,\mathrm{d}b) = 1 + v(a) + \delta(b)$. So $\delta(b) \leqslant \delta(x) - 1 \leqslant \varepsilon - 1/2$ which contradicts the definition of ε. So we have proved that (2) is equivalent to (3).

Let us now prove that (1) implies (6). Let $\beta \in J^{G_L}$. Let $(b_n)_n$, with $b_n \in \mathcal{O}_{\bar{K}}$ be a sequence such that $b_n \xrightarrow{w} \beta$. Then, for any $\sigma \in G_L$, the sequence $w(\sigma(b_n) - b_n) \to \infty$ uniformly in σ. It follows that $v(\sigma(b_n) - b_n) \to \infty$ uniformly in σ, so from Theorem 1.3, for large n's we can write $b_n = x_n + p^{c_0}\gamma_n$, where $x_n \in \mathcal{O}_L$, $\gamma_n \in \mathcal{O}_{\bar{K}}$ and $\gamma_n \xrightarrow{v} 0$. Then $w(\sigma(b_n) - b_n) = w(\sigma(\gamma_n) - \gamma_n) + c_0$ so $w(\sigma(\gamma_n) - \gamma_n) \to \infty$ uniformly in σ. In particular, it follows that $d\gamma_n \in \Omega^{G_L}$ for large n. From Theorem 2.1 it follows that $p^{c_0}\,\mathrm{d}\gamma_n \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, hence for large n we have $\mathrm{d}\beta = \mathrm{d}b_n = \mathrm{d}x_n + p^{c_0}\,\mathrm{d}\gamma_n \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$. This proves that $d(J^{G_L}) \subset \Omega_{\mathcal{O}_L/\mathcal{O}_K}$. Let us prove the other inclusion. For this let $0 \neq u_0\,\mathrm{d}v_0 \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ and let us choose $u_n, v_n \in \mathcal{O}_L$ such that $u_n\,\mathrm{d}v_n = pu_{n+1}\,\mathrm{d}v_{n+1}$ for all $n \geqslant 0$. (This is possible as we have shown that (1) and (3) are equivalent.) Then let $\alpha_n \in \mathcal{O}_{\bar{K}}$ be such that $\mathrm{d}\alpha_n = u_n\,\mathrm{d}v_n$ for all n, so $d(p\alpha_{n+1} - \alpha_n) = 0$ and therefore the sequence $\beta_n := p^n\alpha_n$ is Cauchy with respect to the valuation w. Then $\beta_n \xrightarrow{w} \beta \in B_2$, $\theta(\beta) = 0$ as $\beta_n \xrightarrow{v} 0 = \theta(\beta)$, so $\beta \in J$ and $d(\beta) = u_0\,\mathrm{d}v_0$. On the other hand, for any $\sigma \in G_L$, $\sigma(u_n) = u_n$ and $\sigma(v_n) = v_n$ so $w(\sigma(\alpha_n) - \alpha_n) \geqslant 0$ so $\sigma(\beta) = \beta$, i.e. $\beta \in J^{G_L}$. This finishes the proof of (1) implies (6). We will end the proof of the theorem showing that (6) implies (3). But this is obvious as J^{G_L} is a $\hat{L}$-vector space so obviously p-divisible. □

COROLLARY 2.1. *If L is a deeply ramified extension of K then*

(a) $T_p(\Omega_{\mathcal{O}_L/\mathcal{O}_K}) \neq 0$.
(b) $(T_p(\Omega))^{G_L} \neq 0$.

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Proof. From Theorem 2.2 (3), $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is p-divisible, so (a) follows. We have $T_p(\Omega_{\mathcal{O}_L/\mathcal{O}_K}) \subset (T_p(\Omega))^{G_L}$ so (b) follows. □

3. Galois Cohomology

Let $n \in \mathbb{Z}$ and let $\mathbb{C}_p(n)$ be the one-dimensional $\mathbb{C}_p$-vector space on which G_K acts continuously and semilinearly via the nth power of the cyclotonic character χ. Let L be any algebraic extension of K. Then G_L acts on $\mathbb{C}_p(n)$ by restricting χ^n, and we want to compute the continuous cohomology groups $H^i(G_L, \mathbb{C}_p(n)),\ i \geqslant 0$. (The continuous cohomology is computed using continuous cocycles and coboundaries with respect to the Krull topology on G_L and valuation topology on $\mathbb{C}_p(n)$.) We distinguish two cases: L deeply ramified and L not deeply ramified.

PROPOSITION 3.1. *If L is deeply ramified, then $(\mathbb{C}_p(n))^{G_L}$ is a one-dimensional $\hat{L}$-vector space and $H^i(G_L, \mathbb{C}_p) = 0$ for all $i > 0$.*

Proof. For $n = 0$ we have $\mathbb{C}_p^{G_L} = \hat{L}$ from the main results of [T] and [Ax]. For $n = 1$ we recall from the previous section the result in [F-C] that $\Omega \cong (\overline{K}/\mathbf{a})(1)$ (as G_K-modules) and so $T_p(\Omega)[1/p] \cong \mathbb{C}_p(1)$ as G_K-modules. On the other hand if L is deeply ramified it follows from Corollary 2.1 that $0 \neq T_p(\Omega_{\mathcal{O}_L/\mathcal{O}_K} \subset (T_p(\Omega))^{G_L}$. As $T_p(\Omega)$ is torsion free it follows that

$$(\mathbb{C}_p(1))^{G_L} \cong (T_p(\Omega)[1/p])^{G_L} \neq 0.$$

Let us now suppose that there are two $\hat{L}$-linearly independent elements $a, b \in (\mathbb{C}_p(1))^{G_L}$. Then $a = \beta b$, for some $\beta \in \mathbb{C}_p$ but not in $\hat{L}$. So there is $\sigma \in G_L$ such that $\sigma(\beta) \neq \beta$. Moreover, as a, b are G_L-invariant, we have $a = \beta b = \sigma(\beta) b$ so $b = 0$ which contradicts the assumptions. Therefore $(\mathbb{C}_p)^{G_L}$ is a one-dimensional $\hat{L}$-vector space. It easily follows that $(\mathbb{C}_p(n))^{G_L}$ is a one-dimensional $\hat{L}$-vector space for all n. The statement about the H^i's for $i > 0$ can be proved following exactly the same arguments as in Corollary 1, Corollary 2 and Proposition 10 of Section 3 of [T]. □

PROPOSITION 3.2. *If L is not deeply ramified, then*

(i) $(\mathbb{C}_p)^{G_L} = \hat{L}$ *and* $(\mathbb{C}_p(n))^{G_L} = 0$ *for* $n \neq 0$.
(ii) $H^1(G_L, \mathbb{C}_p)$ *is a one-dimensional* $\hat{L}$ *vector space and* $H^1(G_L, \mathbb{C}_p(n)) = 0$ *for* $n \neq 0$.
(iii) $H^i(G_L, \mathbb{C}_p(n)) = 0$ *for* $i \geqslant 2$ *and all* n.

Proof. Let $\{L_n\}_{n \geqslant 0}$ be a sequence of finite extensions of K such that $L_n \subseteq L_{n+1}$ and $L = \cup_{n \geqslant 0} L_n$. We will apply Tate's theory as in Section 3 of [T] to each of the L_n's. We point out that although the L_n's are not complete everything works fine, finite extensions, degrees, differents and the Galois groups are all preserved

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by taking completions. Let K_∞ be a $\mathbb{Z}_p$ extension of K and denote K_i as the fixed field of the subgroup $p^i\mathbb{Z}_p$ of $\mathbb{Z}_p$, and $K_{n,i} := L_nK_i$ for all $n, i \in \mathbb{N} \cup \{\infty\}$ (where $L_\infty := L$). As L is not deeply ramified, $L \cap K_\infty$ is a finite extension of K, so without loss of generality we may assume that $L \cap K_\infty = L_1$. Then L_n and K_i are linearly disjoint over L_1 for $n > 1$ and $i > 1$.

LEMMA 3.1. *$v(\Delta_{K_{n,i/L_n}}) = c_n + i + (\alpha_{i,n}/p^i)$, where c_n does not depend on i and is bounded with respect to n and $\alpha_{i,n}$ is bounded with respect to both i and n.*

Proof. We have the following diagram

$$\begin{array}{ccc} L_n & \text{---} & K_{n,i} \\ | & & | \\ L_1 & \text{---} & K_{1,i} \\ | & & \\ K & & \end{array}$$

From the multiplicativity of the different we have

$$v(\Delta_{K_{n,i/L_n}}) = v(\Delta_{K_{n,i}/K_{1,i}}) + v(\Delta_{K_{1,i/L_1}}) - v(\Delta_{L_n/L_1}).$$

But $v(\Delta_{L_n/L_1})$ does not depend on i, and $v(\Delta_{K_{1,i/L_1}}) = c + i + (a_i/p^i)$ where c is a constant with respect to both i and n and a_i is bounded with respect to i and does not depend on n ([T] Section 3 Proposition 5). In order to evaluate $v(\Delta_{K_{n,i}/K_{1,i}})$ we will use the Coates–Greenberg [C-G] integral formula

$$\begin{aligned} &v(\Delta_{K_{n,i}/K_{1,i}}) \\ &\quad = v(\Delta_{K_{n,i}/L_1}) - v(\Delta_{K_{1,i}/L_1}) \\ &\quad = \frac{1}{[L_1 \colon K]} \int_{-1}^{\infty} \left(\frac{1}{[K_{1,i} \colon K_{1,i} \cap L_1^\omega]} - \frac{1}{[K_{n,i} : K_{n,i} \cap L_1^\omega]} \right) \mathrm{d}\omega, \end{aligned}$$

where $L_1^\omega := \overline{K}^{G_{L_1}^{(\omega)}}$, and $G_{L_1}^{(\omega)}$ is the ω-ramification subgroup of G_{L_1} in upper numbering. Let ω_0 be such that $L \subseteq L_1^{\omega_0}$ (this is possible since L is *not deeply ramified* over K, so it is not deeply ramified over L_1). Then if $\omega \geqslant \omega_0$, we have

$$[K_{n,i} \colon K_{n,i} \cap L_1^\omega] = [K_{1,i} \colon K_{1,i} \cap L_1^\omega]$$

as $K_{1,i}$ and L_1^ω are linearly disjoint over $K_{1,i} \cap L_1^\omega$, hence

$$[K_{1,i} \colon K_{1,i} \cap L_1^\omega] = [K_{1,i} \cdot (K_{n,i} \cap L_1^\omega) \colon K_{n,i} \cap L_1^\omega]$$

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and $K_{1,i}\cdot(K_{n,i}\cap L_1^{\omega}) = K_{n,i}$. Then

$$\begin{aligned} 0 &\leqslant v(\Delta_{K_{n,i}/K_{1,i}}) \\ &= \frac{1}{[L_1\colon K]}\int_{-1}^{\omega_0}\left(\frac{1}{[K_{1,i}\colon K_{1,i}\cap L_1^{\omega}]}-\frac{1}{[K_{n,i}\colon K_{n,i}\cap L_1^{\omega}]}\right)\mathrm{d}\omega \\ &\leqslant \frac{1}{[L_1\colon K]}\int_{-1}^{\omega_0}\frac{1}{[K_{1,i}\colon K_{1,i}\cap L_1^{\omega}]}\,\mathrm{d}\omega \end{aligned}$$

but $K_{1,i}\cap L_1^{\omega}\subset K_{1,i}\cap L_1^{\omega_0}$ for all $\omega\in[-1,\omega_0]$. Therefore

$$\int_{-1}^{\omega_0}\frac{1}{[K_{1,i}\colon K_{1,i}\cap L_1^{\omega}]}\,\mathrm{d}\omega \leqslant (\omega_0+1)\frac{1}{[K_{1,i}\colon L_1^{\omega_0}]}\leqslant\frac{\omega_0+1}{p^i}.$$

It follows that $v(\Delta_{K_{n,i}/K1,i}) = \beta_{n,i}/p^i$, where $\beta_{n,i}$ is bounded with respect to both i and n (for example by $(\omega_0+1)/[L_1\colon K]$). Then $v(\Delta_{K_{n,i}/L_n}) = c_n+i+(\alpha_{i,n}/p^i)$, where $\alpha_{i,n} = a_i+\beta_{i,n}$ and $c_n = c - v(\Delta_{L_n/L_1})$. Since L is not deeply ramified the sequence $(c_n)_n$ is bounded. This proves the Lemma. □

Let us continue the proof of Proposition 3.2.

We define (following Section 3 of [T]) $t_n\colon K_{n,\infty}\to L$ $(n\in\mathbb{N}\cup\{\infty\})$ to be $t_n(x) = p^{-i}Tr_{K_{n,i}/L_n}(x)$ if $x\in K_{n,i}$. This is a well-defined L_n-linear operator and

$$t_\infty|_{K_{n,\infty}} = t_{n+1}|_{K_{n,\infty}} = t_n \quad \text{for all } n\in\mathbb{N} \text{ and } t_\infty = \varinjlim t_n.$$

LEMMA 3.2. *If σ is a topological generator of* $\mathrm{Gal}(K_{\infty,\infty}/L) = \mathrm{Gal}(K_{n,\infty}/L_n)$ *for all $n>1$, there exists a real number $d>0$, such that*

$$|x-t_\infty(x)|\leqslant d|\sigma(x)-x| \quad \text{for all} \quad x\in K_{\infty,\infty}.$$

Proof. From Proposition 6, Section 3 of [T], for each $n\in\mathbb{N}$, there exists a real number $d_n>0$ such that

$$|x-t_n(x)|\leqslant d_n|\sigma(x)-x|, \quad \text{for all} \quad x\in K_{\infty,n}.$$

From the proof of Proposition 6 of Section 3 of [T] and Lemma 3.1 above it follows that the d_n's are bounded with respect to n, hence $d=\sup_n d_n$ will do. □

Remark. 3.1. We have from Lemma 3.1 $v(\Delta_{K_{n,i}/L_n}) = c_n+i+\alpha_{n,i}/p^i$ and although c_n depends on n this does not matter as c_n cancels in the calculations of d_n (see also the Remark after Proposition 6 in [T]).

From Lemma 3.2 it follows that t_∞ is a continuous linear operator. Moreover, if we define by $X=\hat{K}_{\infty,\infty}$ (this is a Banach space over $\hat{L}$), we can extend $t := t_\infty$

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by continuity to X, so $t\colon X \to \hat{L}$ is a continuous linear operator, and we denote $X_0 = \ker t$.

LEMMA 3.3. (a) *X is a direct sum of $\hat{L}$ and X_0.*

(b) *The operator $\sigma - 1$ annihilates $\hat{L}$ and is bijective with a continuous inverse on X_0.*

(c) *Let λ be a unit in $\hat{L}$ such that $\lambda \equiv 1 \bmod m_{\hat{L}}$) which is not a root of unity. Then $\sigma - \lambda$ is bijective with a continuous inverse on X.*

Proof. The proof follows identically the proof of Proposition 7 of Section 3 of [T]. □

Finally, one can conclude the proof of (i) and (ii) of Proposition 3.2 following the proof of Proposition 8 and Theorem 1 of Section 3 of [T]. (iii) of Proposition 3.2 follows from the following facts:

(a) the inflation map $H^i(\mathrm{Gal}(K_{\infty,\infty}/L), \hat{K}_{\infty,\infty}(n)) \to H^i(G_L, \mathbb{C}_p(n))$ is an isomorphism for all $i \geqslant 0$ and all n as shown in [H] Lemma (3–5).

(b) Gal $(K_{\infty,\infty}/L) \cong \mathbb{Z}_p$. □

Now we would like to use the results of Proposition 3.1 and Proposition 3.2 to compute the Galois cohomology of the B_n's and $B_{\mathrm{d}R}^+$. We have seen in Section 2 that we can define valuation w on $\overline{K}$ such that B_2 is the completion of $\overline{K}$ with respect to this valuation. Theorem 1.1 allows us to define such a valuation for each $n \geqslant 1$. Namely, for each $n \geqslant 1$ let $\mathcal{O}_{\bar{K}}^{(n)}$ be the subring of $\mathcal{O}_{\bar{K}}$ defined in Section 1. For $a \in K^*$ we define $w_n(a) := \max\{m \in \mathbb{Z} \| a \in p^m \mathcal{O}_{\bar{K}}^{(n-1)}\}$. Our old $w = w_2$ and for each n, w_n has the same formal properties as w, namely:

Properties of w_n

(a) $w_n(a+b) \geqslant \min(w_n(a), w_n(b))$ and if $w_n(a) \neq w_n(b)$ then we have equality, for all $a, b \in \overline{K}$.

(b) $w_n(ab) \geqslant w_n(a) + w_n(b)$ for all a, b.

(c) $w_n(a) = \infty$ if and only if $a = 0$.

(d) $v(a) \geqslant w_{n-1}(a) \geqslant w_n(a)$ for all $a \in \overline{K}$ and $n \geqslant 3$.

(e) For each $n \geqslant 2$ the completion of $\overline{K}$ with respect to w_n is canonically isomorphic to B_n.

Remark 3.2. If we define the norm $\|a\|_n := p^{-w_n(a)}$ for all $a \in \overline{K}$, then w_n and $\|\cdot\|_n$ extend naturally to B_n which becomes a Banach algebra over $\hat{K}$. Furthermore, the canonical maps $\phi_n\colon B_n \to B_{n-1}$ are continuous Banach algebra homomorphisms of norm 1. As p-adic Banach spaces are orthonormalizable (i.e. have orthonormal basis [Se]) the map ϕ_n has a continuous additive section, for all $n \geqslant 1$.

As a consequence of Remark 3.1, if L is any algebraic extension of K we get a long G_L-continuous cohomology sequence from the 'fundamental exact sequence' $0 \to \mathbb{C}_p(n) \to B_{n+1} \to B_n \to 0$. Applying the results of Proposition 3.1 and Proposition 3.2 we get

THEOREM 3.1. (a) *If L is a deeply ramified extension of K then $H^i(G_L, B_n) = H^i(G_L, B_{\mathrm{dR}}^{+}) = 0$, for all $n \geqslant 2$ and all $i \geqslant 1$.*

(b) *If L is an algebraic extension of K which is not deeply ramified then the canonical maps $B_n \to \mathbb{C}_p$ and $B_{\mathrm{dR}}^{+} \to \mathbb{C}_p$ induce isomorphisms $H^i(G_L, B_n) \cong H^i(G_L, \mathbb{C}_p) \cong H^i(G_L, B_{\mathrm{dR}}^{+})$ for all $n \geqslant 2$ and $i \geqslant 1$, i.e., $H^1(G_L, B_n) \cong H^1(G_L, B_{\mathrm{dR}}^{+})$ is a one-dimensional $\hat{L}$-vector space and $H^i(G_L, B_n) = H^i(G_L, B_{\mathrm{dR}}^{+}) = 0$ for $i \geqslant 2$ and all $n \geqslant 2$.*

The rest of this paper will be devoted to the computations of $H^0(G_L, B_n)$ and $H^0(G_L, B_{\mathrm{dR}}^{+})$, for L an algebraic extension of K.

4. The Nondeeply Ramified Case

Using the results of the previous section we can easily deal with the nondeeply ramified extensions. We have

THEOREM 4.1. *If L is not deeply ramified then $B_n^{G_L} = \hat{L}$ for all $n \geqslant 1, n \in \mathbb{N}$ and $(B_{\mathrm{dR}}^{+})^{G_L} = \hat{L}$.*

Proof. The statement is true for $n = 1$. Suppose it is true for n, and let us prove it for $n+1$. We have the exact sequence $0 \to \mathbb{C}_p(n) \to B_{n+1} \to B_n \to 0$. Hence, we get an exact sequence

$$0 \to (\mathbb{C}_p(n))^{G_L} \to (B_{n+1})^{G_L} \to (B_n)^{G_L} \to H^1(G_L, \mathbb{C}_p(n)).$$

But $(\mathbb{C}_p(n))^{G_L} = 0$ from Proposition 3.2 (i), $H^1(G_L, \mathbb{C}_p(n)) = 0$ by Proposition 3.2 (ii) and $(B_n)^{G_L} \cong \hat{L}$ by the inductive hypothesis. Therefore ϕ_{n+1} induces a continuous ring isomorphism between $(B_{n+1})^{G_L} \cong \hat{L}$. But ϕ_{n+1} is the restriction of a morphism of Banach spaces of norm 1, so its inverse is also continuous.

COROLLARY 4.1. *If L is not deeply ramified then for all $n \in \mathbb{N}, n \geqslant 2$ $\overset{\wedge_n}{L} = \hat{L}$.*

From the above corollary it follows that the topologies induced by the restrictions of the valuations w_n to L (if L is not deeply ramified over K) are the same as the p-adic topology. In the rest of this section we'll show that actually the restrictions of w_n's to L are equivalent valuations, and we'll give estimates for how far apart they are. These estimates are not going to be used in the rest of this paper, so the reader might want to skip the rest of this section at the first reading.

We start by introducing another sequence of maps, the higher level analogues of δ defined in Section 2. Let $n \geqslant 2$ be an integer and $\omega \in \Omega^{(n)}$. We define

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$\mathrm{Ann}(\omega) = \{a \in O_{\bar{K}} | a\omega = 0\}$. Then $\mathrm{Ann}(\omega)$ is an ideal in $\mathcal{O}_{\bar{K}}$. Let $\delta_n(\omega) = -v(\mathrm{Ann}(\omega)) = -\inf\{v(a) | a \in \mathrm{Ann}(\omega)\}$. This defines a map $\delta_n\colon \Omega^{(n)} \to (-\infty, 0]$. It is easy to see that δ_1 is the same as δ defined in Section 2.

LEMMA 4.1. *The maps δ_n have the following properties:*

(i) $\delta_n(\omega) = 0$ *if and only if* $m_{\bar{K}} \cdot \omega = 0$.
(ii) *if* $\alpha \in O_{\bar{K}}$ *and* $\omega \in \Omega^{(n)}$ *then* $\delta_n(\alpha\omega) = \min(0, v(\alpha) + \delta_n(\omega))$.

Proof. (i) is clear from the definition.

(ii) If $\min(0, v(\alpha) + \delta_n(\omega)) = 0$ then $v(\alpha) \geqslant -\delta_n(\omega) = v(\mathrm{Ann}(\omega))$ so $m_{\bar{K}}\alpha \subseteq \mathrm{Ann}(\omega)$ and $\delta_n(\alpha\omega) = 0$.

Now if min $(0, v(\alpha) + \delta_n(\omega)) < 0$ then $v(\alpha) < -\delta_n(\omega) = v(\mathrm{Ann}(\omega))$.

Let $\beta \in \mathrm{Ann}(\alpha\omega)$. Then $\alpha\beta \in \mathrm{Ann}\,\omega$, hence $v(\beta) \geqslant -\delta_n(\omega) - v(\alpha)$ and so $\delta_n(\alpha\omega) \leqslant \delta_n(\omega) + v(\alpha)$. Let $\gamma \in \mathrm{Ann}(\omega)$, then $v(\gamma) > v(\alpha)$ so $\gamma/\alpha \in \mathrm{Ann}(\alpha\omega)$ and so $\delta_n(\alpha\omega) \geqslant \delta_n(\omega) + v(\alpha)$. □

Now we will introduce a new sequence of subrings of $\overline{K}$ denoted $\{\mathcal{A}^{(n)}\}_n$ and a new sequence of derivations. The $\{\mathcal{A}^{(n)}\}_n$'s are defined as follows $\mathcal{A}^{(0)} = \mathcal{O}_{\bar{K}}$ and if $n \geqslant 1$, $\mathcal{A}^{(n)} = \{h(\theta) | h \in \mathcal{O}_K[x], \theta \in \mathcal{O}_{\bar{K}}$ and $h'(\theta) = h''(\theta) = \cdots = h^{(n)}(\theta) = 0\}$ where $h^{(i)}(x) := d^i h(x)/dx^i$. Hence: $\mathcal{A}^{(0)} \supseteq \mathcal{A}^{(1)} \supseteq \mathcal{A}^{(2)} \supseteq \cdots \supseteq \mathcal{A}^{(n)} \supseteq \cdots \supseteq O_K$.

LEMMA 4.2. *For each $n \geqslant 0$, $\mathcal{A}^{(n)}$ is a ring.*

Proof. For each $\theta \in \mathcal{O}_{\bar{K}}$, we denote $\mathcal{A}_\theta^{(n)} = \{h(\theta) | h \in O_K[x]\ \mathrm{and}\ h'(\theta) = h''(\theta) = \cdots = h^{(n)}(\theta) = 0\}$. It is clear that for each θ, $\mathcal{A}_\theta^{(n)}$ is a subring of $\mathcal{O}_{\bar{K}}$. Moreover, if $\theta \in \mathcal{O}_K[\eta]$, $\eta \in \mathcal{O}_{\bar{K}}$, then $\theta = g(\eta)$, $g \in \mathcal{O}_K[x]$. Let $h \in O_K[x]$, and $h(\theta) \in \mathcal{A}_\theta^{(n)}$, then $h_1(\eta) = h(g(\eta))$ has the property that for all $1 \leqslant i \leqslant n$, $h_1^{(i)}(\eta) = 0$, hence $h_1(\eta) \in \mathcal{A}_\eta^{(n)}$ and so $\mathcal{A}_\theta^{(\eta)} \subseteq \mathcal{A}_\eta^{(n)}$. Now we clearly have $\mathcal{A}^{(n)} = \bigcup_{\theta \in O_E} \mathcal{A}_\theta^{(n)}$ and let $x, y \in \mathcal{A}^{(n)}$. Then $x \in \mathcal{A}_{\theta_1}^{(n)}$, $y \in \mathcal{A}_{\theta_2}^{(n)}$. Let $\eta \in \mathcal{O}_{\bar{K}}$ be such that $\theta_1, \theta_2 \in O_K[\eta]$. Then from the previous discussion $x, y \in \mathcal{A}_\eta^{(n)}$ and so $x + y, x \cdot y \in \mathcal{A}_\eta^{(n)} \subseteq \mathcal{A}^{(n)}$. □

PROPOSITION 4.1. *For $n \geqslant 0$, if denote $n^* = 3^n - 1/2$ we have $\mathcal{A}^{(n^*)} \subseteq \mathcal{O}^{(n)}$.*

Proof. Let $\alpha \in \mathcal{A}^{(n^*)}, \pi$ a uniformizer of $K(\alpha)$ and $r_0 \geqslant v(\Delta_{K(\alpha)/K})$. Let $h \in \mathcal{O}_K[x]$ be such that $h(\pi) = \alpha$ and $h^{(i)}(\pi) = 0, 1 \leqslant i \leqslant n^*$. Consider the Eisenstein equation over $K(\alpha)$: $y^{p^{s+t}} + p^s y - \pi = 0$ with 'large' s and t which will be specified later. Let β be any root of this equation and denote $\theta = \beta^{p^{s+t}}$. We have $v(\theta - \pi) > s$ and from Krasner's lemma it follows that for s big enough we have: $K(\alpha) = K(\pi) \subseteq K(\beta)$. As clearly $v(\Delta_{K(\beta)/K(\alpha)}) = s$ we have that $v(\Delta_{K(\beta)/K}) = v(P'_\beta(\beta)) \leqslant s + r_0$ (where P_β is the minimal polynomial of β over K). Now we apply Lemma 2 of the Appendix of [F-C]. We denote

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$r_n := (3^n - 1/2)(s + r_0)$ and choose $a = p^{s+t}$, then $r_n(a) := \inf(r_n, s + t) = r_n$ for t big enough with respect to s. Lemma 2 of the Appendix of [F-C] tells us that $\theta = \beta^{p^{s+t}} = p^{r_n - r_n(a)}\beta^a \in \mathcal{O}_{\bar{K}}^{(n)}$. Obviously $h(\theta) \in \mathcal{O}_{\bar{K}}^{(n)}$ as well. But $h(\theta) = \alpha + \sum_{i \geqslant n^*+1}(\theta - \alpha)^i (h^{(i)}(\pi)/i!)$, where $h^{(i)}(\pi)/i! \in O_{K(\pi)}$ so we have $v(h(\theta) - \alpha) \geqslant (n^* + 1)v(\theta - \pi) \geqslant (n^* + 1)s \geqslant r_n + 1$, for s big enough. Applying again Lemma 2 of the Appendix of [F-C], we get $h(\theta) - \alpha \in \mathcal{O}^{(n)}$, hence $\alpha \in \mathcal{O}_{\bar{K}}^{(n)}$. □

For any L algebraic extension of K, we denote $\mathcal{A}_L^{(n)} = \bigcup_{\theta \in \mathcal{O}_L} \mathcal{A}_\theta^{(n)}$. Then we have the following proposition:

PROPOSITION 4.2. *Let F be a finite extension of K, let π be a uniformizer of F. Let $\alpha \in \mathcal{A}_F^{(n)}$, $\alpha = h(\pi)$, with $h^{(i)}(\pi) = 0$, $1 \leqslant i \leqslant n$. Then we have*

(i) $v(h^{(n+1)}(\pi)) \geqslant v(n!(\Delta_{F/K})^n)$

(ii) *If $h_1(\pi) = \alpha = h_2(\pi)$ with $h_1^{(i)}(\pi) = h_2^{(i)}(\pi) = 0$ for $i \leqslant i \leqslant n$ then $h_1^{(n+1)}(\pi) \equiv h_2^{(n+1)}(\pi)$ (mod $(n+1)!(\Delta_{F/K})^{n+1}$).*

Proof. Let us denote $P(x) \in \mathcal{O}_K[x]$ the minimal polynomial of π over K. Then, as $h'(\pi) = \cdots = h^{(n)}(\pi) = 0$ we have that $P^{(n)}(x)$ divides $h'(x)$ so $h'(x) = P^{(n)}(x) \cdot g(x)$, where $g \in \mathcal{O}_K[x]$. Hence, (i) follows. For (ii), we notice that $H(x) = h_1(x) - h_2(x)$ has the properties: $H(\pi) = H'(\pi) = \cdots = H^{(n)}(\pi) = 0$ hence $P^{(n+1)}(x)$ divides $H(x)$ or $H(x) = P^{(n+1)}(x) \cdot g_1(x)$, where $g_1 \in \mathcal{O}_K[x]$.□

We continue to work in the hypothesis of Proposition 4.2. We define the application

$$D_{n,F,\pi}\colon \mathcal{A}_F^{(n)} \to (n!(\Delta_{F/K})^n)\mathcal{O}_F/((n+1)!(\Delta_{F/K})^{n+1})\mathcal{O}_F$$

by: for any $\alpha \in \mathcal{A}_F^{(n)}, \alpha = h(\pi)$, with $h^{(i)}(\pi) = 0$ for $1 \leqslant i \leqslant n$, $D_{n,F,\pi}(\alpha) = h^{(n+1)}(\pi)$ (mod $(n+1)(\Delta_{F/K})^{n+1}\mathcal{O}_F$). Proposition 4.2 guarantees that $D_{n,F,\pi}$ is well defined.

PROPOSITION 4.3. *$D_{n,L,\pi}$ is a derivation and* Ker $(D_{n,F,\pi}) = \mathcal{A}_F^{(n+1)}$.

Proof. It is clear that $D_{n,F,\pi}$is a derivation and that $\mathcal{A}_F^{(n+1)} \subseteq \mathrm{Ker}(D_{n,F,\pi})$. In order to prove the converse, let $\alpha \in \mathrm{Ker}(D_{n,F,\pi})$. Then $\alpha = h(\pi)$, for $h \in \mathcal{O}_K[x]$ such that $h^{(i)}(\pi) = 0$ for all $1 \leqslant i \leqslant n$. As $D_{n,F,\pi}(\alpha) = 0$, we have $h^{(n+1)}(\pi) = (n+1)!(\Delta_{F/K})^{n+1}\beta$ where $\beta \in \mathcal{O}_F$. Hence, there is a polynomial $f \in \mathcal{O}_K[x]$ such that $f(\pi) = \beta$. Therefore, if we denote $h_1(x) = h(x) - P^{n+1}(x)f(x)$, where $P(x)$ is the minimal polynomial of π over K, we have: $\alpha = h_1(\pi)$ and $h_1^{(i)}(\pi) = 0$ for all $1 \leqslant i \leqslant n + 1$, and so $\alpha \in \mathcal{A}_F^{(n+1)}$ □

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Now we define $\overline{\delta}_{n,F,\pi}\colon A_F^{(n)} \to (-\infty, 0] \cap \mathbb{Q}$ by

$$\overline{\delta}_{n,F,\pi}(\alpha) = \min(v(D_{n,F,\pi}(\alpha)) - v((n+1)!(\Delta_{F/K})^{n+1}); 0).$$

We are in the same hypothesis of Proposition 4.2.

PROPOSITION 4.4. *We have*

(i) $\overline{\delta}_{n,F,\pi}$ *does not depend on* π, *hence we denote it* $\overline{\delta}_{n,F}$.

(ii) *If* $F_1 \subseteq F_2$ *are finite extensions of* K *we have:* $\overline{\delta}_{n,F_2}|_{\mathcal{A}_{F_1}^{(n)}} = \overline{\delta}_{n,F_1}$. *Therefore we define* $\overline{\delta}_n$ *on the whole of* $\mathcal{A}^{(n)}$ *as the inductive limit of the* $\overline{\delta}_{n,F}$ *over all the finite extensions* F *of* K.

(iii) *We have* $\overline{\delta}_0 = \delta$ *on* $\mathcal{O}_{\bar{K}}$ *where* δ *has been defined in Section 2.*

(iv) *Let* $\alpha \in \mathcal{A}_F^{(n)}$. *Then* $\overline{\delta}_{n,L}(\alpha) = 0$ *if and only if* $D_{n,K(\alpha),\pi}(\alpha) = 0$ *for some uniformizer* π *of* $K(\alpha)$ *if and only if* $\alpha \in \mathcal{A}_L^{(n+1)}$, *where* L *is any algebraic extension of* K *and* $\overline{\delta}_{n,L}$ *is the restriction of* $\overline{\delta}_n$ *to* $\mathcal{A}_L^{(n)}$.

Proof. (i) Let π_1 and π_2 be uniformizers of F, then $\pi_1 = g(\pi_2)$ where $g(x) \in \mathcal{O}_K[x]$ and $g'(\pi_2) = u$ is a unit in $\mathcal{O}_F$. If $\alpha \in \mathcal{A}_F^{(n)}$, then we have $D_{n,F,\pi_1}(\alpha) = D_{n,F,\pi_2}(\alpha)u^{n+1}$, hence $\overline{\delta}_{n,F,\pi_1}(\alpha) = \overline{\delta}_{n,F,\pi_2}(\alpha)$

(ii) Let $F_1 \subseteq F_2$ and π_1 and π_2 be uniformizers of F_1 and F_2 respectively. Then $\pi_1 = g(\pi_2)$ for some $g(x) \in \mathcal{O}_K[x]$. Let $\alpha \in \mathcal{A}_{n,F_1}$ and $h(x) \in \mathcal{O}_K[x]$ be such that $\alpha = h(\pi_1)$ and $h^{(i)}(\pi_1) = 0$ for all $1 \leqslant i \leqslant n$.

Let $f(x) = h(g(x))$, then $\alpha = f(\pi_2)$, $f^{(i)}(\pi_2) = 0$ for all $1 \leqslant i \leqslant n$ and $f^{(n+1)}(\pi_2) = h^{(n+1)}(\pi_1)(g'(\pi_2))^{n+1}$. But $v(g'(\pi_2)) = v(\Delta_{F_2/F_1})$, hence $\overline{\delta}_{n,F_1}(\alpha) = \overline{\delta}_{n,F_2}(\alpha)$.

(iii) This follows from known facts about d_1 ([F-C]).

(iv) This proof is similar to the proof of Proposition 4.3. □

Finally, we have the following proposition:

PROPOSITION 4.5. *If* L *is any algebraic extension of* K, *then* $\mathcal{A}^{(n)} \cap \mathcal{O}_L = \mathcal{A}_L^{(n)}$.

Proof. We will prove the statement by induction over the n's. For $n = 1$, from (iii) and (iv) above, we have that $\mathcal{A}_L^{(1)} = \mathcal{O}_L^{(1)} = \mathcal{O}^{(1)} \cap \mathcal{O}_L = \mathcal{A}^{(1)} \cap \mathcal{O}_L$. Suppose now that the statement is true for n and let us prove it for $n+1$. The inclusion $\mathcal{A}_L^{(n+1)} \subseteq \mathcal{A}^{(n+1)} \cap \mathcal{O}_L$ is trivial so let $\alpha \in \mathcal{A}^{(n+1)} \cap \mathcal{O}_L$. Then in particular $\alpha \in \mathcal{A}^{(n)} \cap \mathcal{O}_L = \mathcal{A}_L^{(n)}$, hence can apply $\overline{\delta}_{n,L}$ to α. But $\overline{\delta}_{n,L}(\alpha) = \overline{\delta}_n(\alpha) = 0$ as $\alpha \in \mathcal{A}^{(n+1)}$. Therefore, $\alpha \in \mathcal{A}_L^{(n+1)}$. □

Remark. 4.1. This last property of the $\mathcal{A}^{(n)}$'s makes them easier to handle than the $\mathcal{O}^{(n)}$'s.

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We can use the $\mathcal{A}^{(n)}$'s in order to prove the following theorem:

THEOREM 4.2. *Let L be an algebraic extension of K which is* not deeply *ramified. Then, for each $n \in \mathbb{N}, n \geqslant 2$, the valuations $w_n|_L$ and $v|_L$ are equivalent.*

Proof. We claim that in order to prove the theorem it would be enough to show that for each $n \in \mathbb{N}$, $n \geqslant 2$, there exists $k_n \in \mathbb{N}$ such that $p^{k_n}\mathcal{O}_L \subseteq \mathcal{O}_L^{(n)}$ (k_n depends only on L and n). Let us show that granted the claim the equivalence of $w_n|_L$ and $v|_L$ follows. Let $x \in L$, $x \neq 0$, and denote $m = w_n(x)$. Hence $x \in p^m\mathcal{O}_L^{(n)} - p^{m+1}\mathcal{O}_L^{(n)}$. But $p^m\mathcal{O}_L^{(n)} \subseteq p^m\mathcal{O}_L$ and $p^{m+k_n+1}\mathcal{O}_L \subseteq p^{m+1}\mathcal{O}_L^{(n)}$. Hence, for all $x \in L$, $w_n(x) \leqslant v(x) \leqslant w_n(x) + k_n + 1$ and we are done. Let us now prove *the claim*. Let $n \geqslant 2, n \in \mathbb{N}$ be given. Then if we denote $n^* = (3^n - 1)/2$ we have that $\mathcal{A}^{(n^*)} \subseteq \mathcal{O}^{(n)}$ and therefore $\mathcal{A}_L^{(n^*)} = \mathcal{O}_L \cap \mathcal{A}^{(n^*)} \subseteq \mathcal{O}_L^{(n)}$. So it would be enough to show that there exists $k_n \in \mathbb{N}$ such that $p^{k_n} \cdot \mathcal{O}_L \subseteq \mathcal{A}_L^{(n^*)}$. For this, we notice that for each $F \subseteq L$, finite over K and $0 \leqslant k \leqslant n^*$ $(k!(\Delta_{F/K})^k)\mathcal{O}_F \subseteq \mathcal{A}_F^{(k-1)}$, hence $((n^*+1)!(\Delta_{F/K})^{n^*+1})\mathcal{O}_F \subseteq \mathcal{A}_F^{(n^*)}$. Let $k_n \in \mathbb{N}$ be greater than or equal to $\sup_F v((n^*+1)!(\Delta_{F/K})^{n^*+1})$, where the supremum is taken over all finite extensions F of K, contained in L (the supremum is finite as L is not deeply ramified). Then:

$$p^{k_n} \cdot \mathcal{O}_F \subseteq \mathcal{A}_F^{(n^*)} \quad \text{for all } F, \text{ hence} \quad p^{k_n}\mathcal{O}_L \subseteq \mathcal{A}_L^{(n^*)} \subseteq \mathcal{O}_L^{(n)}. \qquad \square$$

5. Deeply Ramified Extensions at Higher Levels

We want to calculate $(B_m)^{G_L}$, and $(B_{\mathrm{dR}}^{+})^{G_L}$ for all $m \geqslant 1$ and all deeply ramified extensions L of K. We have $\overset{\wedge n}{L} \subseteq (B_n)^{G_L}$ and $\overset{\wedge \infty}{L} \subseteq (B_{\mathrm{dR}}^{+})^{G_L}$. As was pointed out in the introduction these inclusions may be strict. We want to describe all deeply ramified extensions L of K for which the above inclusions are equalities, i.e., for which the Galois correspondence holds for L at level n or in B_{dR}^{+} respectively.

At this point, the first thing we want to clarify is the relationship between the Galois correspondence at finite levels and the Galois correspondence in B_{dR}^{+}. We have:

THEOREM 5.1. *For a deeply ramified extension L over K the following are equivalent:*

(a) *L satisfies the Galois correspondence in B_{dR}^{+}.*
(b) *L satisfies the Galois correspondence at all levels $n \in \mathbb{N}$.*

We will be able to prove one implication in this section. The other will be proved in Section 6.

We denote $\theta_n : B_{\mathrm{dR}}^{+} \to B_n$, $\eta_n : B_n \to \mathbb{C}_p$ and $\phi_n : B_n \to B_{n-1}$ the canonical projections. We denote by J_n the kernel of ϕ_n.

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Proof of Theorem 5.1 implication (b) $\Rightarrow$ (a).

For any extension L of K we have $\overset{\wedge\infty}{L} \subseteq (B^+_{\mathrm{dR}})^{G_L}$.

Let $\beta \in (B^+_{\mathrm{dR}})^{G_L}$. Then for each $n \in \mathbb{N}$ $\theta_n(\beta) \in B_n^{G_L} = \overset{\wedge n}{L}$. Let $\alpha_n \in L$ be such that $w_n(\alpha_n - \theta_n(\beta)) \geqslant n$.

We claim that the sequence $\{\alpha_n\}$ is Cauchy in B^+_{dR}. In order to show this, let us compute $w_{n-1}(\alpha_n - \alpha_{n-1}) = w_{n-1}(\alpha_n - \theta_{n-1}(\beta) + \theta_{n-1}(\beta) - \alpha_n) \geqslant \min (w_{n-1}(\alpha_n - \theta_{n-1}(\beta)), w_{n-1}(\theta_{n-1}(\beta) - \alpha_{n-1}))$.

But $w_{n-1}(\alpha_n - \theta_{n-1}(\beta)) = w_{n-1}(\phi_{n-1}(\alpha_n - \theta_n(\beta))) \geqslant w_n(\alpha_n - \theta_n(\beta)) \geqslant n$.

Therefore $w_{n-1}(\alpha_n - \alpha_{n-1}) \geqslant n-1$ and the claim is proved.

Let $\alpha \in B^+_{\mathrm{dR}}$ be $\alpha = \lim_{n\to\infty} \alpha_n$.

We want to show that $\alpha = \beta$.

For this we fix $n_0 \in \mathbb{N}$ and let $n \geqslant n_0$. Then $w_{n_0}(\theta_{n_0}(\alpha - \beta)) - w_{n_0}(\phi_{n_0} \phi_{n_0+1} \ldots \phi_n(\theta_n(\alpha - \beta))) \geqslant w_n(\theta_n(\alpha-\beta)) = w_n(\theta_n(\alpha) - \alpha_n + \alpha_n - \theta_n(\beta)) \geqslant n$.

Hence $\alpha = \beta$. □

Let $n \geqslant 2$ be an integer and denote A_n the topological closure of $O_{\bar{K}}^{(n-2)}$ in B_n.

LEMMA 5.1. *d_{n-1} is continuous with respect to w_n (we consider the discrete topology on $\Omega^{(n-1)}$) so it extends to an $\mathcal{O}_K$-linear map from A_n to $\Omega^{(n-1)}$.*

Proof. The proof is obvious. □

LEMMA 5.2. $J_n \subseteq A_n$

Proof. Let us consider $x \in J_n$ and $a_m \in \overline{K}$ such that $a_m \overset{w_n}{\to} x$. Then $a_m = \phi_n(a_m) \overset{w_{n-1}}{\to} \phi_n(x) = 0$ so $w_{n-1}(a_m) \geqslant 0$ for m large enough. Therefore $a_m \in \mathcal{O}_{\bar{K}}^{(n-2)}$ for large m and so $x \in A_n$. □

PROPOSITION 5.1. *Let $n > 1$ and $\omega_1, \omega_2 \in \Omega^{(n)}$. If $\delta_n(\omega_1) \leqslant \delta_n(\omega_2) < 0$ then there exists $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ such that $\omega_2 = \alpha\omega_1$.*

Proof. For the proof we need the following lemma:

LEMMA 5.3. $d_{n-1}\colon J_n \to \Omega^{(n-1)}$ *is surjective for all* $n \geqslant 2$.

Proof. Let $0 \neq \omega_0 \in \Omega^{(n-1)}$. Since by Corollary 1.1 $\Omega^{(n-1)}$ is p-divisible we can choose $\{\omega_r\}_{r\geqslant 1}$ such that $\omega_r \in \Omega^{(n-1)}$ and $\omega_r = p\omega_{r+1}$ for all $r \geqslant 0$. From Theorem 1.1, it follows that there are $\alpha_r \in \mathcal{O}_{\bar{K}}^{(n-2)}$ such that $\omega_r = d_{n-1}\alpha_r$ for all r. The sequence $\{p^r\alpha_r\}_r$ is Cauchy (with respect to w_n), hence $p^r\alpha_r \overset{w_n}{\to} x$ for some $x \in B_n$. Clearly $x \in J_n$ and $d_{n-1}x = \omega_0$. □

We now return to Proposition 5.1. Let $x_1, x_2, \in J_{n+1}$ be such that $d_n(x_i) = \omega_i$, $i = 1, 2$, and let $a \in \mathbb{C}_p$ such that $x_2 = ax_1$. Suppose $a \notin \mathcal{O}_{\mathbb{C}_p}$. Then $x_1 = a^{-1}x_2$ and $v(a^{-1}) > 0$. Hence $\delta_n(\omega_1) = \delta_n(d_n(x_1)) = \delta_n(a^{-1}\omega_2) = \min(0, v(a^{-1}) + \delta_n(\omega_2))$ which contradicts the assumption on ω_1, ω_2. Therefore

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$a \in \mathcal{O}_{\mathbb{C}_p}$ and we are done as $\omega_2 = d_n x_2 = a d_n x_1 = a\omega_1$. □

PROPOSITION 5.2. (i) *If* $n \geqslant 2,\ n \in \mathbb{N}$, *for all* $y \in B_{n-1}$ *there exists* $x \in B_n$ *with* $\phi_n(x) = y$ *such that if* $n \geqslant 3$, *we have* $w_n(x) = w_{n-1}(y)$ *and if* $n = 2$, *we have* $w_2(x) = [v(y)]$.

(ii) *If* $n \geqslant 2$ $n \in \mathbb{N}$, *for all* $y \in \mathbb{C}_p$ *there exists* $x \in B_n$ *such that* $\eta_n(x) = y$ *and* $w_n(x) > v(y) - 1$.

(iii) *For all* $a \in \mathbb{C}_p^*$ *and* $u \in J_n$ $u \neq 0$ *we have* $|w_n(au) - w_n(u) - v(a)| \leqslant 1$.

Proof. (i) Let $n \geqslant 3$, let $y \in B_{n-1}$ and choose $\alpha \in B_n$ such that $y = \phi_n(\alpha)$. Suppose $w_{n-1}(y) > w_n(\alpha)$. Multiplying if necessary by a suitable power of p, we may assume that $w_{n-1}(y) = 0$. Let $\{\alpha_m\}_{m\geqslant 0}$ be a sequence of elements in $\overline{K}$ such that $\alpha_m \underset{m}{\overset{w_n}{\to}} \alpha$. Then $\alpha_m \overset{w_{n-1}}{\to} y$, hence $w_{n-1}(\alpha_m) = 0$ for m sufficiently large. Also $d_{n-1}\alpha = d_{n-1}\alpha_m$ for m large enough. By Lemma 5.3 there exists $u \in J_n$ such that $d_{n-1}\alpha = d_{n-1}u$. Then $x = \alpha - u$ satisfies the required properties. Clearly the same proof works for $n = 2$, except $v(y)$ is not necessarily an integer, so we work with its integral part.

For (ii) we apply (i) several times.

(iii) Let $b \in B_n$ be such that $\eta_n(b) = a$ and $w_n(b) \geqslant v(a) - 1$. Then we have

$$w_n(au) = w_n(bu) \geqslant w_n(b) + w_n(u) \geqslant v(a) + w_n(u) - 1.$$

And similarly

$$w_n(u) = w_n\left(\frac{1}{a} \cdot au\right) \geqslant v\left(\frac{1}{a}\right) + w_n(au) - 1. \qquad \square$$

COROLLARY 5.1. *If* $\alpha, a_m \in \mathcal{O}_{\mathbb{C}_p}$ *are such that* $a_m \overset{v}{\to} \alpha$ *and* $x \in J_n$ *then* $(a_m \cdot x) \overset{w_n}{\to} \alpha x$.

LEMMA 5.4. *For each* $n \in \mathbb{N},\ n \geqslant 1, \Omega^{(n)}$ *has a natural structure of* $\mathcal{O}_{\mathbb{C}_p}$ *module and* $d_n\colon J_{n+1} \to \Omega^{(n)}$ *defined in Lemmas* 5.1 *and* 5.2 *is an* $\mathcal{O}_{\mathbb{C}_p}$*-module homomorphism.*

Proof. $\Omega^{(n)}$ is a p-torsion $\mathcal{O}_{\bar{K}}$-module (see Section 1). Let $\omega \in \Omega^{(n)}$. Then the map $\mathcal{O}_{\bar{K}} \to \Omega^{(n)}$ defined by $a \mapsto a \cdot \omega$ is continuous (consider the v-topology on $\mathcal{O}_{\bar{K}}$ and discrete topology on $\Omega^{(n)}$) hence can be extended canonically to $\mathcal{O}_{\mathbb{C}_p}$. We now show that d_n is an $\mathcal{O}_{\mathbb{C}_p}$-module homomorphism. From the above corollary it follows that we only need to show that d_n is $\mathcal{O}_{\bar{K}}$-linear. So let $x \in J_{n+1}$ and $a \in \mathcal{O}_{\bar{K}}$. We can find $k \in \mathbb{N}$ such that $p^k a \in \mathcal{O}_{\bar{K}}^{(n)}$ and a sequence $b_m \in \mathcal{O}_{\bar{K}}^{(n-1)}$ such that $b_m \overset{w_{n+1}}{\to} x$. Thus one can choose m large enough such that $d_n(ax) = d_n(ab_m)$, $d_n(x) = d_n(b_m)$ and $w_n(b_m) \geqslant k$. Therefore we have: $d_n(ax) = d_n(ab_m) = d_n(p^k a \cdot b_m/p^k) = p^k a d_n(b_m/p^k) + b_m/p^k \cdot d_n(p^k a) = a d_n(b_m) = a d_n(x)$. □

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COROLLARY 5.2. *Let* $\omega \in \Omega^{(n)}$ *and* $\alpha \in \mathcal{O}_{\mathbb{C}_p}$. *Then*

$$\delta_n(\alpha\omega) = \min(0, v(\alpha) + \delta_n(\omega)).$$

The following result completes our characterization of deeply ramified extensions using differential forms (see Theorem 2.2).

THEOREM 5.2. *Let* L *be an algebraic extension of* K. *Then the following are equivalent:*

(i) L *is deeply ramified*.
(ii) $J_n^{G_L} \neq 0$ *for all* $n \geqslant 2$.
(iii) $B_n^{G_L}$ *contains a generator of the maximal ideal of* B_n *for all* $n \in \mathbb{N},\ n \geqslant 2$ *(a generator of the maximal ideal of* B_n *will be called a uniformizer of* B_n*).*
(iv) $(B_{\mathrm{dR}}^+)^{G_L}$ *contains a generator of the maximal ideal of* B_{dR}^+.

Proof. We first show that (ii) $\Leftrightarrow$ (i). As $J_n = T_p\Omega^{(n-1)} \otimes \mathbb{Q}_p$ we have $J_2^{G_L} = (T_p(\Omega_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}))^{G_L} \otimes \mathbb{Q}_p$. But (i) is equivalent to (iii) of Theorem 2.2 which is equivalent to the statement $T_p\Omega_{\mathcal{O}_L/\mathcal{O}_K} \neq 0$. So (i) $\Rightarrow$ $J_2^{G_L} \neq 0$. If L is not deeply ramified since $J_2 \cong \mathbb{C}_p(1)$ as G_L-modules, we get from Proposition 3.2 (i) that $J_2^{G_L} = (\mathbb{C}_p(1))^{G_L} = 0$. Hence (i) $\Leftrightarrow$ $J_2^{G_L} \neq 0$. Now, let $t \in B_{\mathrm{dR}}^+$ be a generator of the maximal ideal such that $\sigma t \equiv t \pmod{t^2}$ (such a t can be found as $J_2^{G_L} \neq 0$) for all $\sigma \in G_L$. Then $\sigma(t^n) \equiv t^n \pmod{t^{n+1}}$ for all $\sigma \in G_L$. Hence $0 \neq t^n (\mathrm{mod}\, t^{n+1}) \in J_{n+1}^{G_L}$ for all n and (ii) is proved. (ii) clearly implies (iii), (iv) implies (iii) and (iii) implies (i) from Theorem 4.1. Finally we prove that (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv). Let L be deeply ramified and $n \in \mathbb{N},\ n \geqslant 2$. We want to show that $B_n^{G_L}$ contains a uniformizer of B_n. We shall proceed by induction. The assumption is true for $n = 2$, as it was proved that (i) $\Rightarrow$ (ii). So, let us suppose that $n > 2$ and $B_n^{G_L}$ contains a uniformizer say y. Let z be a uniformizer of B_{n+1} such that $\phi_{n+1}(z) = y$. Since $\sigma(y) = y$ for all $\sigma \in G_L$ we have $\phi_{n+1}(\sigma(z) - z) = 0$. For all $\sigma \in G_L$ there exists a unique $\zeta(\sigma) \in \mathbb{C}_p$ such that $\sigma(z) - z = \zeta(\sigma) \cdot z^n$ so $\zeta: G_L \to \mathbb{C}_p$ is a well defined map. We claim that ζ is a continuous 1-cocycle. Let $\sigma, \tau \in G_L$. Then $(\sigma \cdot \tau)(z) - z = \sigma((\tau)(z) - z) + \sigma(z) - z = \sigma(\zeta(\tau) \cdot z^n) + \zeta(\sigma) \cdot z^n = \sigma(\zeta(\tau)) \cdot \sigma(z^n) + \zeta(\sigma) \cdot z^n$. But $\sigma z = z + \zeta(\sigma) \cdot z^n$. If we raise to the nth power we get $\sigma(z^n) = z^n$ (as $z^{n+1} = 0$). Hence $\zeta(\sigma \cdot \tau) = \sigma\zeta(\tau) + \zeta(\sigma)$. Now let's see the continuity. Let $f: G_L \to J_n$ be defined by $f(\sigma) = \sigma z - z$ and $g: \mathbb{C}_p \to J_{n+1}$ be defined by $g(a) = a \cdot z^n$. Then $f = g \circ \zeta$ and g is a homeomorphism from Proposition 5.2. Hence, in order to prove that ζ is continuous, it would be enough to show that f is also continuous. But this is obvious as G_L acts continuously on B_{n+1}.

So finally if we denote $[\zeta]$ its cohomology class, $[\zeta] \in H^1(G_L, \mathbb{C}_p) = 0$. Hence we can find $\varepsilon \in \mathbb{C}_p$ such that $\zeta(\sigma) = \sigma(\varepsilon) - \varepsilon$. Now we put $z' = z - \varepsilon z^n \in B_n^{G_L}$, and z' is a uniformizer of B_{n+1} and $\phi_{n+1}(z') = \phi_{n+1}(z)$. So the statement is proved for all B_n's.

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The proof above shows that we can find a sequence $\{z_n\}_{n\geqslant 2}$, $z_n \in B_n^{G_L}$, z_n uniformizer of B_n and $\phi_n(z_n) = z_{n-1}$. Denote $\tilde{z} := (z_n)_n \in \lim_{\leftarrow} B_n = B_{\mathrm{dR}}^{+}$. Then $\tilde{z}$ is a uniformizer of B_{dR}^{+} and $\sigma(\tilde{z}) = \tilde{z}$ for all $\sigma \in G_L$. □

We would like to consider the Coates–Greenberg notion of deep ramification as the level two deep ramification and define deep ramification at level n for all n as follows:

DEFINITION 5.1. Let $n \geqslant 2$ and L a deeply ramified extension. We will say that L is deeply ramified at level n if $\Omega^{(n-1)}(L/K)$ is not annihilated by a finite power of p.

Although, by the above definition we only ask that $\delta_{n-1}(\Omega^{(n-1)}(L/K))$ is unbounded, we can show that in this case $\Omega^{(n-1)}(L/K)$ is almost p-divisible.

PROPOSITION 5.3. *Let $n \in \mathbb{N}^*$ and L be deeply ramified at level $n+1$. Then*

(i) $\Omega^{(n)} = \mathcal{O}_{\bar{K}} \cdot \Omega^{(n)}(L/K)$.
(ii) $\Omega^{(n)}\ (L/K)$ *has a nonzero p-divisible submodule* $\Omega_0^{(n)}\ (L/K)$ *such that* $m_L(\Omega^{(n)}\ (L/K)/\Omega_0^{(n)}\ (L/K)) = 0$.
(iii) $d_n(\mathcal{O}_L^{(n-1)}) + \Omega_0^{(n)}(L/K) = \Omega^{(n)}(L/K)$.

We will see later that, for certain n's and deeply ramified extensions L, the Galois correspondence at level n might fail. But if L is deeply ramified at level n, then one has an approximate Galois correspondence for the $(n-1)$-differential forms namely:

THEOREM 5.3. *Let L be a deeply ramified extension. Then*

(i) $m_L((\Omega^{(n-1)})^{G_L}/d_{n-1}(J_n^{G_L})) = 0$ *for all* $n \in \mathbb{N},\ m \geqslant 2$.
(ii) *If L is deeply ramified at level n, then* $m_L((\Omega^{(n-1)})^{G_L}/\Omega^{(n-1)}(L/K)) = 0$.

In order to prove the results just stated, we need to derive first a technical result. For any algebraic extension L of K and any $\alpha \in \overline{K}$ we denote

$$c(L, \alpha) = \min_{\sigma \in G_L} v(\sigma(\alpha) - \alpha) - \sup_{x \in L} v(\alpha - x).$$

One clearly has $c(L, \alpha) \geqslant 0$. On the other hand $c(L, \alpha)$ is bounded from above (e.g. by $p/(p-1)$, cf. [Ax] and [Sen]) and this result was crucial in the proof of the Galois correspondence of $\mathbb{C}_p$. Let us denote $c(L) = \sup_{\alpha \in \overline{K}} c(L, \alpha)$ and call it the Ax-Sen number of L. Any $\sigma \in G_L$ extends to a continuous automorphism of $\mathbb{C}_p$ therefore in the above definitions we could take $\alpha \in \mathbb{C}_p$. This will not change $c(L)$ since $c(L, \alpha)$ is clearly continuous as a function of α.

PROPOSITION 5.4. *The Ax–Sen number of any deeply ramified extension of $\mathbb{Q}_p$ is zero.*

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Proof. Let L be deeply ramified and $\alpha \in \overline{\mathbb{Q}}_p$. Let $(L_n)_{n\geqslant 0}$. be an increasing sequence of finite extensions of $\mathbb{Q}_p$ such that $\bigcup_n L_n = L$. We have $\lim_{n\to\infty} c(L_n, \alpha) = c(L, \alpha)$. On the other hand, from Theorem 1.2 it follows easily that $\lim_{n\to\infty} v(\Delta_{L_n(\alpha)/L_n}) = 0$. Then the same holds for the relative discriminants:

$$\lim_{n\to\infty} v(\mathcal{D}_{L_n(\alpha)/L_n}) = 0.$$

Let F be one of the L_n's. We want to find an integral basis of $F(\alpha)/F$ expressed in terms of α so that we can relate $v(\mathcal{D}_{F(\alpha)/F})$ to $c(F, \alpha)$. Popescu–Zaharescu in [P-Z] constructed such a basis $\{\theta_1, \theta_2, \ldots, \theta_m\}, m = [F(\alpha): F]$, with $\theta_1 = 1, \theta_2 = (\alpha - a)/\pi^s$ where π is a uniformizer of F, $a \in F$ is such that $v(\alpha - a) = \sup_{x\in F} v(\alpha - x)$ and $s \in \mathbb{Z}$ is such that $0 \leqslant v(\theta_2) < v(\pi)$. θ_r for $r > 2$ are certain polynomials in α, irrelevant for our discussion. Let M denote the matrix $(\sigma(\theta_i))_{\sigma,i}$, where σ runs over the embeddings of $F(\alpha)$ in $\overline{\mathbb{Q}}_p$ over F. Then $v(\mathcal{D}_{F(\alpha)/F}) = 2v(\det M)$. If we subtract the first row from the others, we get a $(m-1)(m-1)$ determinant which has as first column

$$\begin{pmatrix} \dfrac{\sigma(\alpha)-\alpha}{\pi^s} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}.$$

Therefore

$$v\left(\det M\right) \geqslant \min_\sigma v\left(\frac{\sigma(\alpha)-\alpha}{\pi^s}\right) \geqslant \min_\sigma v(\sigma(\alpha)-\alpha) - v(\alpha - a) = c(F, \alpha).$$

We now take $F = L_n$ and derive $c(L, \alpha) = 0$. □

Proof of Proposition 5.3. (i) Let $\omega \in \Omega^{(n)}$. As $\delta_n(\Omega^{(n)}(L/K))$ is unbounded there exists $\omega_1 \in \Omega^{(n)}(L/K)$ such that $\delta_n(\omega_1) \leqslant \delta_n(\omega)$. Now from Proposition 5.1 we find $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ such that $\omega = \alpha\omega_1 \in \mathcal{O}_{\mathbb{C}_p} \cdot \Omega^{(n)}(L/K) = \mathcal{O}_{\bar{K}} \cdot \Omega^{(n)}(L/K)$.

(ii) We define $\Omega_0^{(n)}(L/K) := \{\omega \in \Omega^{(n)}(L/K)|$ there exists a sequence $(\omega_n)_{n\geqslant 0}$ such that $\omega_n \in \Omega^{(n)}(L/K), \omega_0 = \omega$, and $\omega_m = p\omega_{m+1}$ for all $m\}$.

Then $\Omega_0^{(n)}(L/K)$ is the maximal p-divisible submodule of $\Omega^{(n)}(L/K)$. First of all let us notice that $\Omega^{(n)}(L/K) \neq 0$. Otherwise from (i) it would follow that $\Omega^{(n)} = 0$ which is not the case (see Theorem 1.1). Now fix $\beta \in m_L$ and let $\omega \in \Omega^{(n)}(L/K)$. There exists $\omega_0 \in \Omega^{(n)}$ such that $\omega = p\omega_0$. From (i) we get that $\omega_0 = \sum_{i=1}^n a_i\omega_i$, with $a_i \in \mathcal{O}_{\bar{K}}$ and $\omega_i \in \Omega^{(n)}(L/K)$. We denote u_1 one of the ω_i's above such that $\delta_n(u_1)$ is minimum and apply Proposition 5.1. We get $\omega = pa_1 \cdot u_1$, with $a_1 \in \mathcal{O}_{\bar{K}}$. Similarly, for any $m > 1$, we find $a_m \in \mathcal{O}_{\bar{K}}$ and $u_m \in \Omega^{(n)}(L/K)$ such that $u_{m-1} = pa_m u_m$. We apply $\sigma \in G_L$ to this last equality

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and we get $(\sigma(a_m) - a_m) \cdot p \cdot u_m = 0$. Hence $v(\sigma(a_m) - a_m) \geqslant -\delta_n(pu_m)$. Now we use Proposition 5.4: for any $m \in \mathbb{N}^*$ and $\beta \in m_L$ there exists $r_m \in \mathcal{O}_L$ such that $v(a_m - r_m) > -\delta_n(pu_m) - v(\beta)$. Then $\beta(a_m - r_m) \cdot pu_m = 0$ and $\beta u_{m-1} = \beta pa_m u_m = p\beta r_m u_m$. We multiply this equality by $r_1 \cdot r_2 \ldots r_{m-1}$ and get

$$\beta\omega = pv_1, \ldots, v_m = pv_{m+1} \quad \text{where} \quad v_m = \beta r_1 \ldots r_m u_m \in \Omega^{(n)}(L/K).$$

So $\beta\omega \in \Omega_0^{(n)}(L/K)$ for any $\omega \in \Omega^{(n)}(L/K)$ and any $\beta \in m_L$. It now also follows that $\Omega_0^{(n)}(L/K) \neq 0$.

(iii) Let $\omega \in \Omega^{(n)}(L/K), \omega = \sum_{j=1}^{r} \alpha_j d_n \beta_j$, with $\alpha_j \in \mathcal{O}_L,\ \beta_j \in \mathcal{O}_L^{(n-1)}$. There are $a_j \in \mathcal{O}_K,\ b_j \in m_L$ such that $\alpha_j = a_j + b_j,\ j = 1, \ldots, r$. Then $\sum_{j=1}^{r} a_j d_n \beta_j = d_n(\sum_{j=1}^{r} a_j \beta_j) \in d_n(\mathcal{O}_L^{(n-1)})$ and $\sum_{j=1}^{r} b_j d_n \beta_j \in \Omega_0^{(n)}(L/K)$. □

Proof of Theorem 5.3. (ii) Suppose L is deeply ramified at level n. We have $\Omega^{(n-1)}(L/K) \subseteq (\Omega^{(n-1)})^{G_L}$. So let $\omega \in (\Omega^{(n-1)})^{G_L}$, we use Proposition 5.3(i) to write $\omega = \sum_{i=1}^{n} a_i \omega_i, a_i \in \mathcal{O}_{\bar{K}}$ and $\omega_i \in \Omega^{(n-1)}(L/K)$. Now we follow the same argument as in the proof of Proposition 5.3(ii).

(i) We have $d_{m-1}(J_m^{G_L}) \subseteq (\Omega^{(m-1)})^{G_L}$.

Let now $\beta \in m_L$ and $\omega \in (\Omega^{(m-1)})^{G_L}$. From Lemma 5.3 we get $x \in J_m$ such that $\omega = d_{m-1}x$. We use Theorem 5.2(ii) to get $0 \neq y \in J_m^{G_L}$ so we can find $\alpha \in \mathbb{C}_p$ such that $x = \alpha y$. Without loss of generality, we may suppose that $\alpha \in \mathcal{O}_{\mathbb{C}_p}$ (if not write $\alpha = \alpha'/p^k, \alpha' \in \mathcal{O}_{\mathbb{C}_p}$ and $x = \alpha y = \alpha'(y/p^K)$ and $y/p^K \in J_m^{G_L}$ as well). Thus $\alpha d_{m-1} y = d_{m-1}(\alpha y) = d_{m-1}x = \omega \in (\Omega^{(m-1)})^{G_L}$. Hence $(\sigma\alpha - \alpha) d_{m-1} y = 0$ for all $\sigma \in G_L$ and the trick used in the proof of Proposition 5.3 gives us a $\gamma \in \mathcal{O}_L$ such that $\beta\omega = \beta\gamma d_{m-1} y = d_{m-1}(\beta\gamma y)$ and $\beta\gamma y \in J_m^{G_L}$. □

6. De Rham Extensions

All over this section L will denote a deeply ramified extension of K.

DEFINITION 6.1. For all $n \in \mathbb{N}$ define $H_{\mathrm{dR}}^{(n)}(L/K) := \Omega^{(n)}(L/K)/d_n(\mathcal{O}_L^{(n-1)})$.

Remark. 6.1. Let us consider $d\colon \Omega^{(n)}(L/K) \to \Omega^{(1)}(L/K) \wedge \Omega^{(n)}(L/K)$, where d is the obvious derivation. If L is deeply ramified we saw that $\Omega^{(1)}(L/K)$ is p-divisible while $\Omega^{(n)}(L/K)$ is a p-torsion module. Hence their wedge product is zero and we could think of $\Omega^{(n)}(L/K)$ as consisting of 'closed' n-forms. Therefore we may think of $H_{\mathrm{dR}}^{(n)}(L/K)$ as being the quotient 'closed n-forms/exact n-forms'.

Remark. 6.2. We have

$$H_{\mathrm{dR}}^{(n)}(L/K) = \Omega_0^{(n)}(L/K)/(d_n(\mathcal{O}_L^{(n-1)}) \cap \Omega_0^{(n)}(L/K)),$$

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where $\Omega_0^{(n)}(L/K)$ is the maximal p-divisible submodule of $\Omega^{(n)}(L/K)$ as defined in the proof of Proposition 5.3. This motivates the next definition.

DEFINITION 6.2. We say that L has property $(*)$ at level n if $T_p(d_{n-1}(\mathcal{O}_L^{(n-2)})) \neq 0$.

Remark. 6.3. Definition 6.2 is equivalent to the following: there exists a sequence $\{\alpha_r\}_r, \alpha_r \in \mathcal{O}_L^{(n-2)}$ such that $0 \neq d_{n-1}\alpha_r = pd_{n-1}\alpha_{r+1}$ for all r.

DEFINITION 6.3. We say that L is a de Rham extension of K at level n if $H_{\mathrm{dR}}^{(n-1)}(L/K) = 0$.

LEMMA 6.1. *If L is de Rham and deeply ramified at level n, then $\Omega_0^{(n-1)}(L/K) = d_{n-1}(J_n \cap \overset{\wedge n}{L})$, where $\Omega_0^{(n-1)}(L/K)$ was defined in the proof of Proposition 5.3.*

Proof. Since $J_n \cap \overset{\wedge n}{L}$ is p-divisible it is clear that $d_{n-1}(J_n \cap \overset{\wedge n}{L}) \subseteq \Omega_0^{(n-1)}(L|K)$. Let now $0 \neq \omega_0 \in \Omega_0^{(n-1)}(L/K)$ and choose $\{\omega_r\}_{r \geqslant 1}$ such that $\omega_r \in \Omega^{(n-1)}(L/K)$ and $\omega_r = p\omega_{r+1}$ for all $r \geqslant 0$. We can find $\alpha_r \in \mathcal{O}_L^{(n-2)}$ such that $\omega_r = d_{n-1}\alpha_r$ for all r. Now the sequence $\{p^r\alpha_r\}_r$ is Cauchy (with respect to w_n), hence $p^r\alpha_r \xrightarrow{w_n} x$ for some $x \in B_n$. Then $x \in J_n \cap \overset{\wedge n}{L}$ and $d_{n-1}x = \omega_0$. □

THEOREM 6.1. *Let L be a deeply ramified extension and $n \geqslant 2$. Then the following are equivalent*:

(i) *L is deeply ramified and de Rham at level n.*
(ii) *L has property $(*)$ at level n.*
(iii) $J_n \cap \overset{\wedge n}{L} \neq 0$.
(iv) $J_n \cap \overset{\wedge n}{L} = J_n^{G_L}$.

Proof. (iii) ⇔ (iv) follows from the fact that $J_n \cap \overset{\wedge n}{L} \subseteq J_n^{G_L}$ both are $\hat{L}$-vector spaces and $\dim_{\hat{L}} J_n^{G_L} = 1$.

(ii) ⇒ (iii) Let $\{\alpha_r\}_r$ be a sequence as in Remark 6.1.

Then the sequence $\{p^r\alpha_r\}_r$ converges in w_n to some nonzero element of $J_n \cap \overset{\wedge n}{L}$.

(iii) ⇒ (ii) $J_n \cap \overset{\wedge n}{L}$ is clearly p-divisible so if $0 \neq x_0 \in J_n \cap \overset{\wedge n}{L}$ for any $r \in \mathbb{N}^*$ we can choose $\alpha_r \in \mathcal{O}_L^{(n-2)}$ such that $d_{n-1}\alpha_r = d_{n-1}(x_0/p^r)$. Then the α_r's give the property $(*)$. ($d_{n-1}\alpha_r \neq 0$ for $r > w_n(x_0)$).

(i) ⇒ (iii) follows from Lemma 6.1.

(ii) clearly implies that L is deeply ramified at level n.

We show that (iv) implies that L is de Rham at level n.

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For this we use Theorem 5.3(i) to derive:

$$\Omega_0^{(n-1)}(L/K) = p\Omega_0^{(n-1)}(L/K) \subseteq p\Omega^{(n-1)}(L/K) \subseteq p(\Omega^{(n-1)})^{G_L} \subseteq$$
$$\subseteq d_{n-1}(J_n^{G_L}) = d_{n-1}(J_n \cap \overset{\wedge n}{L}) \subseteq d_{n-1}(\mathcal{O}_L^{(n-2)}).$$

From Proposition 5.3(iii) it follows that $d_{n-1}(\mathcal{O}_L^{(n-2)}) = \Omega^{(n-1)}(L/K)$. □

PROPOSITION 6.1. *Let L be a deeply ramified extension of K and $n \geqslant 2$, $n \in \mathbb{N}$ such that $J_n \cap \overset{\wedge n}{L} \neq 0$.*

Then: (i) *If $n > 2$ and $\overset{\wedge n}{L} \overset{\phi_n}{\to} \overset{\wedge \infty}{L}$ is the canonical map, let $y \in$ Im ϕ_n. Then there exists $x \in \overset{\wedge n}{L}$ such that $\phi_n(x) = y$ and $w_n(x) \geqslant w_{n-1}(y) - 1$.*

(ii) *If $n = 2$ and $\overset{\wedge 2}{L} \overset{\phi_2}{\to} \hat{L}$ is the canonical map, let $y \in$ Im ϕ_2. Then there exists $x \in \overset{\wedge 2}{L}$ such that $\phi_2(x) = y$ and $w_2(x) \geqslant [v(y)] - 1$, where [ ] denotes the integral part function.*

(iii) $\phi_n(\overset{\wedge n}{L}) = \overset{\wedge n-1}{L}$.

Proof. (i) Let $\alpha \in \overset{\wedge n}{L}$ and $y = \phi_n(\alpha) \in \overset{\wedge \infty}{L}$ and suppose $w_{n-1}(y) > w_n(\alpha) + 1$. Without loss of generality, we may suppose that $w_{n-1}(y) = 0$ (if not multiply with a suitable power of p). Let $\{\alpha_m\}_{m \geqslant 0}$ be a sequence such that $\alpha_m \in L$ for all m and $\alpha_m \underset{m}{\overset{w_n}{\to}} \alpha$. Then $\alpha_m \overset{w_{n-1}}{\to} y$ hence $w_{n-1}(\alpha_m) = 0$ for m sufficiently large.

Therefore $d_{n-1}\alpha = d_{n-1}\alpha_m$ for m big enough. Let $u \in J_n \cap \overset{\wedge n}{L}$ be such that $p \cdot d_{n-1}\alpha = pd_{n-1}u$ (we apply Theorem 5.3(i) and Theorem 6.1), hence $w_n(\alpha - u) + 1 \geqslant 0$ and $\phi_n(\alpha - u) = \phi_n(\alpha) = y$.

Finally $0 = w_{n-1}(y) \leqslant w_n(\alpha - u) + 1$ so set $x = \alpha - u$.

(ii) The proof is identical except we cannot make $v(y)$ zero by multiplying a power of p, we can make it $0 \leqslant v(y) < 1$.

(iii) Suppose $n \geqslant 2$. Let $y \in \overset{\wedge n-1}{L}$ and $\{\alpha_m\}_{m \geqslant 0}$ be a sequence in L such that $\alpha_m \overset{w_{n-1}}{\longrightarrow} y$. Let $\beta_0 \in \overset{\wedge n}{L}$ be such that $\phi_n(\beta_0) = \alpha_0$ and $w_n(\beta_0) + 2 > w_{n-1}(\alpha_0)$. Let now $\beta_1 \in \overset{\wedge n}{L}$ be such that $\phi_n(\beta_1) = \alpha_1$ and $w_n(\beta_1 - \beta_0) > w_{n-1}(\alpha_1 - \alpha_0) - 2$. Construct inductively $\{\beta_m\}$ such that $\phi_m(\beta_m) = \alpha_m$ and $w_n(\beta_m - \beta_{m-1}) > w_{n-1}(\alpha_m - \alpha_{m-1}) - 2$. Then β_m is Cauchy in w_n and let $\beta = \lim_{w_n} \beta_m$. Then $\phi_n(\beta) = \alpha$. □

PROPOSITION 6.2. *Let L be deeply ramified. Then the map*

$$\phi_n\colon B_n^{G_L} \to B_{n-1}^{G_L}$$

is surjective.

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Proof. We consider the long exact cohomology sequence coming from the fundamental exact sequence (Section 1):

$$0 \to (\mathbb{C}_p(n-1))^{G_L} \to B_n^{G_L} \stackrel{\phi_n}{\to} B_{n-1}^{G_L} \to H^1(G_L, \mathbb{C}_p(n-1)).$$

As L is deeply ramified $H^1(G_L, \mathbb{C}_p(n-1)) = 0$ as proved in Proposition 3.1. □

PROPOSITION 6.3. *Let L be a deeply ramified extension of K and $n \in \mathbb{N}$, $n \geqslant 2$. Then the following are equivalent*

(i) $B_n^{G_L} = \overset{\wedge n}{L}$.

(ii) $\overset{\wedge n}{L}$ *contains a uniformizer of* B_n.

Proof. (i) ⇒ (ii) follows from Theorem 5.2.

(ii) ⇒ (i). Let $n = 2$. From Propositions 6.1 and 6.2 we get a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_2^{G_L} & \longrightarrow & B_2^{G_L} & \longrightarrow & \hat{L} & \longrightarrow & 0 \\
 & & \cup & & \cup & & \| & & \\
0 & \longrightarrow & J_2 \cap \overset{\wedge 2}{L} & \longrightarrow & \overset{\wedge 2}{L} & \longrightarrow & \hat{L} & \longrightarrow & 0.
\end{array}$$

We know $J_2 \cap \overset{\wedge 2}{L} = J_2^{G_L}$, therefore $B_2^{G_L} = \overset{\wedge 2}{L}$.

Let us suppose that the statement is true for $m \leqslant n$ and assume that $\overset{\wedge n+1}{L}$ contains a uniformizer of B_{n+1}, say z. Then $\phi_{n+1}(z) \in \overset{\wedge n}{L}$ and it is a uniformizer of B_n. Hence $0 \neq z_n \in J_{n+1} \cap \overset{\wedge n+1}{L}$ and therefore we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & J_{n+1}^{G_L} & \longrightarrow & B_{n+1}^{G_L} & \longrightarrow & B_n^{G_L} & \longrightarrow & 0 \\
 & & \| & & \cup & & \| & & \\
0 & \longrightarrow & J_{n+1} \cap \overset{\wedge n+1}{L} & \longrightarrow & \overset{\wedge n+1}{L} & \longrightarrow & \overset{\wedge n}{L} & \longrightarrow & 0.
\end{array}$$

Hence the statement is true for all n. □

PROPOSITION 6.4. *Let L be a deeply ramified extension of K. Then the following are equivalent:*

(i) $(B_{\mathrm{dR}}^+)^{G_L} = \overset{\wedge \infty}{L}$.

(ii) $\overset{\wedge \infty}{L}$ *contains a uniformizer of* B_{dR}^+.

Proof. (i) ⇒ (ii) follows from Theorem 5.2.

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(ii) $\Rightarrow$ (i) If $\overset{\wedge\infty}{L}$ contains a uniformizer of B_{dR}^{+}, then $\overset{\wedge n}{L}$ contains a uniformizer of B_n for all n, hence $B_n^{G_L} = \overset{\wedge n}{L}$ for all n by Proposition 6.3. Therefore $(B_{\mathrm{dR}}^{+})^{G_L} = \overset{\wedge\infty}{L}$ by the implication already proved in Theorem 5.1. (b) $\Rightarrow$ (a). □

But we have now proved the remaining implication of Theorem 5.1 and also

COROLLARY 6.1. *Let* $n \in \mathbb{N}$, $n \geqslant 2$. *Then if* $B_n^{G_L} = \overset{\wedge n}{L}$ *we have that* $B_k^{G_L} = \overset{\wedge k}{L}$ *for all* $2 \leqslant k \leqslant n$.

COROLLARY 6.2. *The following are equivalent for* $n \in \mathbb{N}, n \geqslant 2$.

(i) $J_m \cap \overset{\wedge m}{L} \neq 0$ *for all* $m \leqslant n$.
(ii) $B_m^{G_L} = \overset{\wedge m}{L}$ *for all* $m \leqslant n$.

Proof. (i) $\Rightarrow$ (ii). If $J_m \cap \overset{\wedge m}{L} \neq 0$ for all $m \leqslant n$ by induction we can show that $B_m^{G_L} = \overset{\wedge m}{L}$ for all $m \leqslant n$ (as in the proof of Proposition 6.3).

(ii) $\Rightarrow$ (i). If $B_m^{G_L} = \overset{\wedge m}{L}$, then $\overset{\wedge m}{L}$ contains a uniformizer of B_m, say z. But then $0 \neq z^{m-1} \in J_m \cap \overset{\wedge m}{L}$. □

COROLLARY 6.3. *Let* L *be deeply ramified. Then the following are equivalent:*

(i) $J_m \cap \overset{\wedge m}{L} \neq 0$ *for all* m.
(ii) $B_m^{G_L} = \overset{\wedge m}{L}$ *for all* m.
(iii) $(B_{\mathrm{dR}}^{+})^{G_L} = \overset{\wedge\infty}{L}$.

Proof. The proof follows from Corollary 6.2 and Theorem 5.1 □

COROLLARY 6.4. *Let* L *be deeply ramified and* $n \in \mathbb{N}, n \geqslant 2$ *and* M *be an algebraic extension of* L. *So if* $B_n^{G_L} = \overset{\wedge n}{L}$, *then* $B_n^{G_M} = \overset{\wedge n}{M}$ *and if* $(B_{\mathrm{dR}}^{+})^{G_L} = \overset{\wedge\infty}{L}$ *then* $(B_{\mathrm{dR}}^{+})^{G_M} = \overset{\wedge\infty}{M}$.

Proof. We have that $\overset{\wedge n}{L} \subseteq \overset{\wedge n}{M}$ so if $\overset{\wedge n}{L}$ contains a uniformizer of B_n so does $\overset{\wedge n}{M}$. Also $\overset{\wedge\infty}{L} \subseteq \overset{\wedge\infty}{M}$ so if $\overset{\wedge\infty}{L}$ contains a uniformizer of B_{dR}^{+} so does $\overset{\wedge\infty}{M}$. □

7. Main Results

We have proved:

THEOREM 7.1. *Let* L *be an algebraic extension of* K *and* $n \in \mathbb{N}, n \geqslant 2$. *Then*

(i) *If* L *is not deeply ramified then* $B_n^{G_L} = \overset{\wedge n}{L} = \hat{L}$.
(ii) *If* L *is deeply ramified the following are equivalent*

 (a) $B_m^{G_L} = \overset{\wedge m}{L}$ *for all* $m \leqslant n$.

(b) *L has property $(*)$ at all levels $m \leqslant n$.*
(c) *L is deeply ramified and de Rham for all levels $m \leqslant n$.*
(d) *$J_m \cap \overset{\wedge m}{L} \neq 0$ for all $m \leqslant n$.*
(e) *$\overset{\wedge n}{L}$ contains a uniformizer of B_n.*

THEOREM 7.2. *Let L be an algebraic extension of K. Then*

(i) *If L is not deeply ramified then $(B_{\mathrm{dR}}^{+})^{G_L} = \overset{\wedge \infty}{L} = \hat{L}$.*
(ii) *If L is deeply ramified the following are equivalent*

(a) *$(B_{\mathrm{dR}}^{+})^{G_L} = \overset{\wedge \infty}{L}$.*
(b) *L has property $(*)$ at all levels.*
(c) *L is deeply ramified and de Rham at all levels.*
(d) *$\overset{\wedge \infty}{L}$ contains a uniformizer of B_{dR}^{+}*

8. Examples

I. EXAMPLES OF DEEPLY RAMIFIED EXTENSIONS WHICH ARE NOT DE RHAM AT LEVEL TWO

First example
Because we will work at level 2 in this section, we will denote d, w, δ, Ω by respectively, $d_1, w_1, \delta_1, \Omega^{(1)}$.

Also property $(*)$ will denote property $(*)$ at level 2. We have:

PROPOSITION 8.1. *If $L = \bigcup_n L_n$, where L_n are finite extensions of K such that $v(\Delta_{L_n/K}) \equiv -1/([L_n\colon K]) \pmod 1$ then $\delta(\Omega(L|K)) \cap \mathbb{Z} = \{0\}$.*

Proof. Let $\alpha \in \mathcal{O}_L$ with $\delta(\mathrm{d}\alpha) < 0$ and choose n such that $\alpha \in L_n$. Then $\mathrm{d}\alpha = h'(\pi_n) \cdot \mathrm{d}\pi_n$ where π_n is a uniformizer of L_n and $h \in \mathcal{O}_K[x]$ is such that $\alpha = h(\pi_n)$ and $\deg h < [L_n\colon K]$. Then

$$\begin{aligned} \delta(\mathrm{d}\alpha) &= v(h'(\pi_n)) - v(\Delta_{L_n/K}) \\ &\equiv v(h'(\pi_n)) + \frac{1}{[L_n\colon K]} = v(\pi_n \cdot h'(\pi_n)) \pmod 1. \end{aligned}$$

But $x \cdot h'(x)$ has no terms of degree multiple of $[L_n\colon K]$. □

COROLLARY 8.1. *$L = \bigcup_n K(\sqrt[p^n]{p})$ is not de Rham at level two (it is deeply ramified though), where $\sqrt[p^n]{p}$ is a root of $X^{p^n} - p = 0$ such that $(\sqrt[p^n]{p})^p = \sqrt[p^{n-1}]{p}$ chosen at step $n-1$.*

Proof. We have $[K(\sqrt[p^n]{p})\colon K] = p^n$ and $v(\Delta_{K(\sqrt[p^n]{p})/K}) = n + 1 - (1/p^n)$. Hence $\delta(d\mathcal{O}_L) \cap \mathbb{Z} = \{0\}$. On the other hand $\delta(\Omega(L|K)) \supseteq \mathbb{Z}$ (e.g. $\delta(\pi_1^{p-1} d\pi_1) =$

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-1 where $\pi_1^p = p$) hence $H_{\mathrm{dR}}^{(1)}(L|K) \neq 0$. □

Second example (We owe this example to P. Colmez)

PROPOSITION 8.2. *K_∞ is not a de Rham extension at level 2.*

Proof. We will prove that if $n > 0$ is an integer and ζ is a primitive p^nth root of unity, then $\mathrm{d}\zeta/\zeta$ is not in $d(\mathcal{O}_{K_\infty})$. Suppose not, and let $a \in \mathcal{O}_{K_m}$ for some $m \geqslant n$ be such that $\mathrm{d}a = \mathrm{d}\zeta/\zeta$. Let η be a primitive p^mth root of unity such that $\eta^{p^{m-n}} = \zeta$. Then $\mathrm{d}a = \mathrm{d}\zeta/\zeta = p^{m-n}(\mathrm{d}\eta/\eta)$ If denote $\pi = \eta - 1$ then

$$\frac{1}{\eta} = \sum_{i=0}^{\infty}(-1)^i\pi^i = \sum_{i=0}^{d-1} a_i\pi^i,$$

where $d = p^m - p^{m-1} = [K_m\colon K]$, and $a_i \in \mathcal{O}_K$ for all i and $a_{d-1} \equiv 1 \pmod{p}$. We also have $a = \sum_{i=0}^{d-1} b_i\pi^i$, with $b_i \in \mathcal{O}_K$ for all i. Then

$$\begin{aligned} 0 \;&=\; da - p^{m-n}\frac{\mathrm{d}\eta}{\eta} = \left(\sum_{i=1}^{d-1} ib_i\pi^{i-1} - p^{m-n}\sum_{i=0}^{d-1} a_i\pi^i\right)\mathrm{d}\pi \\ &= \left(\left(\sum_{i=0}^{d-2}((i+1)b_{i+1} - p^{m-n}a_i)\pi^i\right) - p^{m-n}a_{d-1}\pi^{d-1}\right)\mathrm{d}\pi. \end{aligned}$$

Let us denote by $M := \sum_{i=0}^{d-2}((i+1)b_{i+1} - p^{m-n}a_i)\pi^i - p^{m-n}a_{d-1}\pi^{d-1}$. Then we have $v(M) \geqslant v(\Delta_{K_m/K})$ hence $v(M) \geqslant m$. On the other hand, if we compute directly $v(M) \leqslant v(p^{m-n}a_{d-1}\pi^{d-1}) < m-n+1$ so we get a contradiction. □

II. EXAMPLES OF DEEPLY-RAMIFIED EXTENSIONS WHICH ARE DE RHAM AT LEVEL TWO

Let $\varepsilon_0 > 0$ be a real number and L an algebraic extension of K.

DEFINITION 8.1. We say (L, ε_0) has property $(**)$ if there exists $\{\beta_n\}_{n\geqslant 1}$ with $\beta_n \in \mathcal{O}_L$ such that $d\beta_n \neq 0$ and $\delta(\beta_n - p\beta_{n+1}) \geqslant \min\,(0, \delta(\beta_n) + \varepsilon_0)$ for all $n \geqslant 1$.

PROPOSITION 8.3. *The following are equivalent*:

(1) *L has property $(*)$.*
(2) *for all $\ell > 0 (L, \ell)$ has property $(**)$.*
(3) *there exists $\varepsilon_0 > 0$ such that (L, ε_0) has property $(**)$.*

COROLLARY 8.2. *Let $q = p^n, n \geqslant 2$, then if we denote F_q the unique unramified extension of $\mathbb{Q}_p$ with residue field with q elements, $(F_q)^{ab}$ has property $(*)$, where $(F_q)^{ab}$ denotes the maximal Abelian extension of F_q.*

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Proof. Let us consider the Lubin–Tate extensions of F_q given by roots of $\phi_m(x) = (f^m(x)/f^{m-1}(x)$, where

$$f(x) = x^q + px \quad \text{and} \quad f^m(x) = \underbrace{f \circ f \circ \cdots \circ f}_{\text{m times}}(x).$$

Choose for all m a root β_m of ϕ_m such that $\beta_m = f(\beta_{m+1})$ for $m \geqslant 1$. From the equality $\beta_m^q + p\beta_m = \beta_{m-1}$ we get $q \cdot \beta_m^{q-1} d\beta_m + pd\beta_m = d\beta_{m-1}$. But p^2/q, hence $\delta(q\beta_m^{q-1}d\beta_m) > \delta(pd\beta_m) = \delta(\beta_{m-1})$. Hence $\delta(\beta_{m-1} - p\beta_m) = \delta(q\beta_m^{q-1}d\beta_m) \geqslant$ min $\{0, 1 + \delta(\beta_{m-1})\}$. Therefore $((F_q)^{ab}, 1)$ has property $(**)$. □

Proof of Proposition 8.3. Clearly (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). We will show that (3) $\Rightarrow$ (1). Let $\varepsilon_0 > 0$ be such that (L, ε_0) has property $(**)$. It is clear that $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is p-divisible (as if $udv \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, then there exists n_0 such that $\delta(\beta_{n_0}) < \delta(v) - 1$ hence $dv = \gamma_0 d\beta_{n_0}$ with $v(\gamma_0) \geqslant 1$. So we have $udv = p(u\gamma_1 d\beta_{n_0})$ when $\gamma_1 = (\gamma_0/p) \in \mathcal{O}_L$). It would be enough to show that L is de Rham or in other words that $d\colon \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective. We will show using induction on $r \in \mathbb{N}^*$ that every $udv \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ with $\delta(udv) \geqslant -r\varepsilon_0$ is in Imd. $r = 1$: Let $udv \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ be such that $\delta(udv) \geqslant -\varepsilon_0$. Choose n_0 such that $\delta(v) > \delta(\beta_{n_0})$ and $\gamma_0 \in \mathcal{O}_L$ such that $dv = \gamma_0 d\beta_{n_0}$. So $udv = u\gamma_0 d\beta_{n_0}$. But $\delta(udv) \geqslant -\varepsilon_0$, hence $\delta(\beta_{n_0}) \geqslant -\varepsilon_0 - v(u\gamma_0)$. Now (L, ε_0) has property $(**)$ so $\delta(\beta_{n_0} - p\beta_{n_0+1}) \geqslant$ min $(0, -v(u\gamma_0))$ and so $u\gamma_0(d\beta_{n_0} - pd\beta_{n_0+1}) = 0$. We get $udv = pu\gamma_0 d\beta_{n_0+1}$. Use the same reasoning several times and get

$$udv = p^2 u\gamma_0 d\beta_{n_0+2} = \cdots = p^m u\gamma_0 d\beta_{n_0+m}$$

until $p^m u\gamma_0 \in \mathcal{O}_L^{(1)}$ (or $d(p^m u\gamma_0) = 0$). But now $udv = d(p^m ur_0 \beta_{n_0+m})$ and the case $r = 1$ is proved. Suppose we have proved the statement for r and let us prove it for $r+1$. Let $udv \in \Omega_{\mathcal{O}_l/\mathcal{O}_K}$ be such that $\delta(udv) \geqslant -(r+1)\varepsilon_0$. Choose γ_0, n_0 such that $udv = u\gamma_0 d\beta_{n_0}$. $\delta(\beta_{n_0}) \geqslant -(r+1)\varepsilon_0 - v(u\gamma_0)$. Hence $\delta(\beta_{n_0} - p\beta_{n_0+1}) \geqslant$ min $(0, -r\varepsilon_0 - v(u\gamma_0))$. So $\delta(u\gamma_0(d\beta_{n_0} - pd\beta_{n_0+1})) \geqslant -r\varepsilon_0$. Hence, by the induction hypothesis, there exists $z_1 \in \mathcal{O}_L$ such that $u\gamma_0(d\beta_{n_0} - pd\beta_{n_0+1}) = dz_1$. So as before $udv = dz_1 + u\gamma_0 pd\beta_{n_0+1} = \cdots = dz_1 + \cdots + dz_m + u\gamma_0 p^m d\beta_{n_0+m}$ with $u\gamma_0 p^m \in \mathcal{O}_L^{(1)}$. Hence $udv \in \mathrm{Im}\, d$. □

III. SOME OPEN PROBLEMS

Among the numerous problems which might become subjects for further work, we state the following two:

(1) Is there any connection between the deep ramification property at different levels?

In particular, are there proper subsets $\mathcal{N}$ of $\mathbb{N}$ with the following property: If we assume in Theorem 0.2(ii) (b) only that L is de Rham at all levels and that it is deeply ramified at any level n in $\mathcal{N}$, then L is deeply ramified at all levels?

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(2) Corollary 8.2 shows that if F_q is an unramified extension of Q_p, $F_q \neq Q_p$, then the maximal Abelian extension of F_q satisfies the Galois correspondence at level 2. Does $(F_q)^{ab}$ satisfy the Galois correspondence at higher levels provided, say, F_q is large enough?

This would imply that K^{ab} satisfies the Galois correspondence in B_{dR}^{+}.

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