ON THE DIMENSION OF VEBLEN-WEDDERBURN SYSTEMS

by CARLTON J. MAXSON

(Received 11 March, 1969)

1. Introduction. In [**1**, p. 97], Bruck and Bose ask the question " Has every (right) Veblen-Wedderburn system finite dimension over its left operator skew-field? " It is the purpose of this note to show that, in general, this question has a negative answer.

We recall that, in [**1**], the *left operator skew-field* of a Veblen-Wedderburn system $\langle R, +, .\rangle$ is defined to be the subsystem $\langle F, +, .\rangle$ consisting of those elements $x \in R$ satisfying, for all $a, b \in R$,

(i) $x.(a+b) = x.a + x.b$,

(ii) $x.(a.b) = (x.a).b$

The Veblen-Wedderburn systems considered in this paper will be (right) near-fields $\langle F, +, .\rangle$ with the additional property

(P) for all $a, b, c \in F$, $a \neq b$, there exists one and only one element $x \in F$ such that $xa = xb + c$.

We recall that a near-field is an algebraic system $\langle F, +, .\rangle$ such that $+$ and . are associative binary operations on F, $\langle F, +\rangle$ is a group with identity 0 (say), $\langle F - \{0\}, .\rangle$ is a group and $(a+b).c = a.c + b.c$ for all $a, b, c \in F$. As usual $a.c$ will be written ac and the multiplicative identity denoted by 1.

The near-fields satisfying property (P) are called *planar* by Zemmer [**4**] and *projective* by Kerby [**2**].

The *kern of a near-field* $\langle F, +, .\rangle$ is defined to be the set $K(F) = \{a \in F \mid a(b+c) = ab + ac$ for all $b, c \in F\}$. $\langle K(F), +, .\rangle$ is a subskew-field of F and $\langle F, +\rangle$ is a (left) vector space over $K(F)$. In particular, the kern of F is the left operator skew-field of F (considering F as a Veblen-Wedderburn system). Moreover, if the dimension of $\langle F, +\rangle$ over $K(F)$ (i.e. $[F : K(F)]$) is finite, then $\langle F, +, .\rangle$ is a planar near-field (see [**2**] and [**4**]).

Both Kerby and Zemmer give examples of near-fields not satisfying property (P). Kerby also gives an example of an infinite near-field satisfying (P). We use the methods of Kerby to construct infinite planar near-fields (hence Veblen-Wedderburn systems) which are infinite dimensional over $K(F)$ (i.e., over their left operator skew-fields).

In our construction of infinite planar near-fields, we use the concept of " coupling map " defined in [3]. For the sake of completeness, we give this definition.

DEFINITION. Let $\langle R, +, .\rangle$ be a ring and $\mathrm{End}_0\, R$ be the semigroup of ring endomorphisms of R with 0_R adjoined. A function $\phi : R \to \mathrm{End}_0\, R$ $(a \to \phi_a)$ is said to be a *coupling map* of R if $\phi_0 = 0_R$ and $\phi_a \circ \phi_b = \phi_{a\phi_b \cdot b}$, for all $a, b \in R$.

2. Results. Let H be a field and T an arbitrary but fixed automorphism of H, $T \neq I_H$. T induces an automorphism T^* on $H((x))$, the field of formal power series over H; that is, for $\alpha = \sum_h^\infty \alpha_i x^i \in H((x))$, $\alpha T^* = \sum_h^\infty (\alpha_i)Tx^i$. The mapping $\phi : H((x)) \to \mathrm{End}_0\langle H((x)), +, .\rangle$

https://doi.org/10.1017/S0017089500000951 Published online by Cambridge University Press

given by

$$\alpha\phi = \begin{cases} \alpha(T^*)^{\delta(\alpha)}, & \alpha \neq 0, \\ 0 \quad , & \alpha = 0, \end{cases}$$

where $\delta(\alpha) = \text{Ord}\,\alpha$ (= smallest index for which $\alpha_i \neq 0$), is a coupling map for $H((x))$ and therefore (see [3], p. 6) $\langle H((x)), +, \circ\rangle$ is a near-field. We recall that the multiplication $\circ$ is given by

$$\alpha \circ \beta = \begin{cases} 0 \quad , & \beta = 0, \\ \alpha(T^*)^{\delta(\beta)}.\beta, & \beta \neq 0. \end{cases}$$

Thus, if $\alpha = \sum_r^\infty \alpha_i x^i$ and $0 \neq \beta = \sum_t^\infty \beta_j x^j$, then $\alpha \circ \beta = \sum_r^\infty (\alpha_i) T^{\delta(\beta)} x^i . \sum_t^\infty \beta_j x^j$.

Kerby [2] has shown that $\langle H((x)), +, \circ\rangle$ is a near-field with property (P).

In particular, let H be the field $k(x)$ of rational functions in one indeterminate over a field k of characteristic zero. Let $T: k(x) \to k(x)$ be the automorphism of $k(x)$ given by $x \to x+1$. We denote the coupled near-field $\langle k(x)((t)), +, \circ\rangle$ by F. We proceed to show that $[F : K(F)]$ is not finite.

LEMMA *Let* $\alpha = \sum_h^\infty \alpha_i t^i \in F$; *then* $\alpha \in K(F)$ *if and only if* $\alpha_i T = \alpha_i$ *for all* i.

Proof. If $\alpha \in K(F)$, then $\alpha \circ (1+t) = \alpha \circ 1 + \alpha \circ t = \alpha + \alpha \circ t$. Hence $\alpha \circ (1+t) = \alpha . (1+t) = \alpha + \sum_h^\infty \alpha_i T t^{i+1}$, which in turn implies that $\sum_h^\infty \alpha^i t^{i+1} = \sum_h^\infty \alpha_i T t^{i+1}$. Thus $\alpha_i = \alpha_i T$ for all i. Conversely, if this is the case, then, for all $n \in \mathbf{Z}$, $\alpha_i T^n = \alpha_i$. Hence $\alpha \circ \beta = \alpha . \beta$ for all $\beta \in F$, and so $\alpha \in K(F)$.

COROLLARY. *Let* $\alpha = \sum_h^\infty \alpha_i t^i \in F$; *then* $\alpha \in K(F)$ *if and only if* $\alpha_i \in k$, *for all* i.

Proof. Let $q \in k(x)$, $q = f(x)/g(x)$, where $f(x)$, $g(x) \in k[x]$, $g(x) \neq 0$; we may assume without loss of generality that g.c.d. $\{f(x), g(x)\} = 1$. We must verify that $qT = q$ is equivalent to $q \in k$. Clearly $q \in k$ implies that $qT = q$. Conversely, $f(x)/g(x) = f(x+1)/g(x+1)$ implies that $f(x)g(x+1) = f(x+1)g(x)$. Hence $f(x) \mid f(x+1)$ and so $f(x+1) = rf(x)$, where $r \in k$. Let $f(x) = \sum_{i=0}^\infty a_i x^i$, where $a_n \neq 0$. Assume that $n \geqq 1$. From $f(x+1) = rf(x)$, by equating the coefficients of x^n and x^{n+1}, one obtains

(i) $a_n = ra_n$,

(ii) $na_n + a_{n-1} = ra_{n-1}$.

From (i), $r = 1$, since $a_n \neq 0$. But then, from (ii), $na_n = 0$, which is a contradiction since k is of characteristic zero. Hence $f(x) = a_0 \in k$. If $a_0 = 0$, then $q \in k$. If $a_0 \neq 0$, we obtain $g(x) = g(x+1)$ and then find that $g(x) = b_0 \in k$. Hence $q \in k$, as desired.

Since $k(x)$ is a simple transcendental extension of k, $[k(x) : k] = \infty$. Let $B = \{b_\alpha \mid \alpha \in \Lambda\}$ be a basis for $k(x)$ over k. Since $B \subseteq k(x)$, we have $B \subseteq k(x)((t))$. For any finite subset

https://doi.org/10.1017/S0017089500000951 Published online by Cambridge University Press

$\{b_{\alpha_i} \mid i = 1, 2, \ldots, r\}$ of B, let $0 = b_{\alpha_1} f_1 + b_{\alpha_2} f_2 + \ldots + b_{\alpha_r} f_r$, where $f_i \in K(F)$, $i = 1, 2, \ldots, r$. Hence $f_i = \sum_{h_i}^{\infty} a^i_j t^j$, $a^i_j \in k$. For each $j \geqq h = \min\{h_i \mid i = 1, 2, \ldots, r\}$,

$$0 = b_{\alpha_1} a^1_j + b_{\alpha_2} a^2_j + \ldots + b_{\alpha_r} a^r_j$$

and since $[k(x) : k] = \infty$, we have $a^i_j = 0$ for $i = 1, 2, \ldots, r$ and all j. Hence B is an independent set over $K(F)$ and consequently $[F : K(F)] \geqq [k(x) : K(F)] = \infty$.

We have established the following

THEOREM. *There exist Veblen-Wedderburn systems having infinite dimension over their left operator skew-fields.*

REFERENCES

1. R. H. Bruck and R. C. Bose, The construction of translation planes from projective spaces, *J. Algebra* **1** (1964), 85–102.

2. W. Kerby, Projective und nicht-projective Fastkörper, *Abh. Math. Sem. Univ. Hamburg*, **32** (1968), 20–24.

3. C. J. Maxson, Dickson near-rings, *J. Algebra* **14** (1970), 152–169.

4. J. L. Zemmer, Near-fields, planar and non-planar, *The Math. Student*, **31** (1964), 145–150.

TEXAS A and M UNIVERSITY
COLLEGE STATION, TEXAS 77843