Canad. Math. Bull. Vol. **33** (3), 1990

ON THE NON-ISOMORPHISM OF $C^*_\lambda(G) \otimes_{\max} C^*_\rho(G)$ AND $C^*(C^*_\lambda(G), C^*_\rho(G))$

BY

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ABSTRACT. We show that for a discrete ICC group G, which is not inner amenable and is such that $C^*_\lambda(G)$ has a weakly outer automorphism, $C^*_\lambda(G) \otimes_{\max} C^*_\rho(G)$ is not isomorphic to $C^*(C^*_\lambda(G), C^*_\rho(G))$.

Let G be a discrete group. On $\ell^2(G)$ we have two representations: for $\xi \in \ell^2(G)$ and $s, t \in G$, $\lambda_s \xi(t) = \xi(s^{-1}t)$ and $\rho_s \xi(t) = \xi(ts)$. Let $C^*_\lambda(G)$ and $C^*_\rho(G)$ be the C^*-algebras generated by these representations. These C^*-algebras are spatially isomorphic via the unitary $w : w\xi(t) = \xi(t^{-1})$, $w^* \lambda_s w = \rho_s$. We denote by ξ_0 the function in $\ell^2(G)$ which is 0 everywhere but at the identity of G where it is 1. An automorphism of $C^*_\lambda(G)$ is called *weakly outer* if it extends to an automorphism of $C^*_\lambda(G)''$ and is an outer automorphism of this von Neumann algebra. $C^*(C^*_\lambda(G), C^*_\rho(G))$ denotes the C^*-algebra on $\ell^2(G)$ generated by $C^*_\lambda(G)$ and $C^*_\rho(G)$. For convenience we shall denote it by $C^*_{\lambda,\rho}(G)$. For the definition of the maximal and minimal tensor products see Takesaki [8, IV.4].

$C^*_\lambda(G) \otimes_{\max} C^*_\rho(G)$ has two canonical representations which we shall call π_λ and π_t. On $\ell^2(G) \otimes \ell^2(G)$ we define $\pi_\lambda(\sum x_i \otimes y_i) = \sum \lambda(x_i) \otimes \rho(y_i)$ for $x_i \in C^*_\lambda(G)$ and $y_i \in C^*_\rho(G)$. We define π_t to be a representation on $\ell^2(G)$ by setting $\pi_t(\sum x_i \otimes y_i) = \sum x_i y_i$.

Takesaki [7] gave the first example of a non-nuclear C^*-algebra by showing that when $G = \mathbb{F}_2$ there is no map $\tilde{\pi}_t$ making the following diagram commute.

$$\begin{tikzcd}
C^*_\lambda(G) \otimes_{\max} C^*_\rho(G) \arrow[rr, "\pi_\lambda"] \arrow[dr, "\pi_t"'] & & C^*_\lambda(G) \otimes_{\min} C^*_\rho(G) \arrow[dl, "\tilde{\pi}_t"] \\
& C^*_{\lambda,\rho}(G) &
\end{tikzcd}$$

If π_λ were an isomorphism then a $\tilde{\pi}_t$ would certainly exist, so if no $\tilde{\pi}_t$ exists then $C^*_\lambda(G)$ is not nuclear.

Received by the editors October 24, 1988 and, in revised form, April 1, 1989.

Research supported by the Natural Sciences and Engineering Research Council of Canada.

1980 Mathematics Subject Classification 47A60.

An easy way to see that no such $\tilde{\pi}_t$ exists is to observe that there is a unital inclusion $\Delta : C^*_\lambda(G) \longrightarrow C^*_\lambda(G) \otimes_{\min} C^*_\rho(G)$ given by $\lambda_s \longmapsto \lambda_s \otimes \rho_s$. This extends to a bounded map as $\lambda \otimes \rho \sim \lambda \otimes i$, as representations of G (Dixmier [3, 13.11.3]). Now $(\tilde{\pi}_t \circ \Delta(\lambda_s)\xi_0 \mid \xi_0) = (\tilde{\pi}_t(\lambda_s \otimes \rho_s)\xi_0 \mid \xi_0) = (\lambda_s \rho_s \xi_0 \mid \xi_0) = 1$. So $\tilde{\pi}_t \circ \Delta$ is the trivial representation. This implies that the trivial representation of G is weakly contained in the left regular, which implies that G is amenable, (see Greenleaf [5, Theorem 3.52]).

On the other hand when G is ICC (Takesaki [8, V.7.10]) we always have a map $\tilde{\pi}_\lambda : C^*_{\lambda,\rho}(G) \longrightarrow C^*_\lambda(G) \otimes_{\min} C^*_\rho(G)$ given by $\sum x_i y_i \longmapsto \sum x_i \otimes y_i$ (Takesaki [8, IV.4.20 and IV.4.9]). This yields the following diagram.

$$\begin{tikzcd}
C^*_\lambda(G) \otimes_{\max} C^*_\rho(G) \arrow[rr, "\pi_\lambda"] \arrow[dr, "\pi_t"'] & & C^*_\lambda(G) \otimes_{\min} C^*_\rho(G) \\
& C^*_{\lambda,\rho}(G) \arrow[ur, "\pi_\lambda"'] &
\end{tikzcd}$$

Thus when G ICC, G is amenable (i.e., π_λ is an isomorphism) if and only if $\tilde{\pi}_\lambda$ is an isomorphism. This raises the problem of deciding when π_t is an isomorphism.

In [4] Effros introduced a notion which he called inner amenability. All amenable groups are inner amenable but not conversely (see Bedos and de la Harpe [1] for a discussion of inner amenability). The fact about inner amenability that we shall need was found by Paschke [6].

Let $\alpha_s = \lambda_s \rho_s$ for $s \in G$. Then α is a representation of G on $\ell^2(G)$. Let $C^*_\alpha(G)$ be the C^*-algebra generated by $\alpha(G)$. Let p_0 be the projection onto the one dimensional subspace spanned by ξ_0.

PROPOSITION 1[6]. *Let G be an infinite discrete group. Then G is inner amenable if and only if $p_0 \notin C^*_\alpha(G)$.*

COROLLARY 2. *If G is an infinite discrete ICC group and is not inner amenable, then K, the algebra of compact operators on $\ell^2(G)$, is contained in $C^*_\alpha(G)$.*

PROOF. When G is ICC, $\mathcal{R}(G) = C^*_\lambda(G)''$ is a factor and so $C^*_{\lambda,\rho}(G)$ is irreducible and contains a compact operator. So by Kadison's transitivity theorem (for example), $K \subseteq C^*_{\lambda,\rho}(G)$. □

COROLLARY 3. *If G is a discrete non inner amenable group and ϑ is a weakly outer automorphism of $C^*_\lambda(G)$ then $\Theta(\sum x_i y_i) = \sum \vartheta(x_i)y_i$ for $x_i \in C^*_\lambda(G)$, $y_i \in C^*_\rho(G)$ does not extend to an automorphism of $C^*_{\lambda,\rho}(G)$.*

PROOF. Suppose that Θ does extend to an automorphism. Regard $\Theta : C^*_{\lambda,\rho}(G) \longrightarrow B(\ell^2(G))$ as an irreducible representation. As $\Theta|_K \neq 0$, $\Theta|_K$ is irreducible (Dixmier [3.4.1.5]). Thus there is $u \in B(\ell^2(G))$ with $\Theta|_K = \mathrm{Ad}_u|_K$. Then Θ and Ad_u agree on $C^*_{\lambda,\rho}(G)$ (Dixmier [3,4.1.5]). But $u \in C^*_\rho(G)' = \mathcal{R}(G)$, and so ϑ is weakly inner.

The idea of extending ϑ to Θ was used by Connes [2, Theorem 2.1]. The argument above also shows that if $K \subseteq C^*(M, M')$ for a II_1 factor M, then $\mathrm{Int}(M) = \overline{\mathrm{Int}(M)}$. □

PROPOSITION 4. *Let G be a non inner amenable discrete ICC group such that $C^*_\lambda(G)$ has a weakly outer automorphism. Then $\pi_t : C^*_\lambda(G) \otimes_{\max} C^*_\rho(G) \longrightarrow C^*_{\lambda,\rho}(G)$ is not an isomorphism.*

PROOF. Suppose π_t is an isomorphism. Let ϑ be a weakly outer automorphism of $C^*_\lambda(G)$. Define $\pi_\vartheta : C^*_\lambda(G) \otimes_{\max} C^*_\rho(G) \longrightarrow C^*_{\lambda,\rho}(G)$ by $\pi_\vartheta(\sum x_i \otimes y_i) = \sum \vartheta(x_i) y_i$. Then $\pi_\vartheta \circ \pi_t^{-1}(\sum x_i y_i) = \sum \vartheta(x_i) y_i$. So Θ extends to an automorphism of $C^*_{\lambda,\rho}(G)$. This contradicts Corollary 3. □

We conclude by showing that there are plenty of weakly outer automorphisms.

PROPOSITION 5. *Let G be a discrete ICC group.*

(1) *If ϑ is an outer automorphism of G, then $\sum \alpha_s \lambda_s \longmapsto \sum \alpha_s \lambda_{\vartheta(s)}$ extends to a weakly outer automorphism of $C^*_\lambda(G)$.*

(2) *If χ is non trivial character of G then $\sum \alpha_s \lambda_s \longmapsto \sum \alpha_s \chi(s) \lambda_s$ extends to a weakly outer automorphism of $C^*_\lambda(G)$.*

For the reader's convenience we provide a proof of (1). The proof of (2) is similar, easier, and better known.

PROOF OF (1). Let $u(\sum \alpha_s \delta_s) = \sum \alpha_s \delta_{\vartheta(s)}$, (where $\delta_s = \lambda_s \xi_0$). Then u extends to a unitary on $\ell^2(G)$ and $u \sum \alpha_s \lambda_s u^* = \sum \alpha_s \lambda_{\vartheta(s)}$. So $u C^*_\lambda(G) u^* = C^*_\lambda(G)$, and defines an automorphism of $C^*_\lambda(G)$ which we denote ϑ. Moreover ϑ extends to an automorphism of $\mathcal{R}(G)$.

Suppose that ϑ is weakly inner, i.e. there is a unitary $v \in \mathcal{R}(G)$ such that $\vartheta(x) = v x v^*$, $\forall x \in \mathcal{R}(G)$. Let $\eta = v \xi_0$. We shall reach a contradiction by showing that $\eta = \delta_y$, for some $y \in G$. This implies that that $v = \lambda_y$ and thus that ϑ is an inner automorphism of G. Now $\lambda_{\vartheta(x)} \eta = \vartheta(\lambda_x) \eta = v \lambda_x \xi_0 = v \rho_x^{-1} \xi_0 = \rho_x^{-1} \eta$. Hence $\forall x, y \in G$, $\eta(\vartheta(x^{-1}) y x) = \eta(y)$. Hence for each $y \in G$ such that $\eta(y) \neq 0$, $\{\vartheta(x^{-1}) y x \mid x \in G\}$ must be finite as η is square summable.

Suppose that $\eta(y_1), \eta(y_2) \neq 0$. We shall show that $y_1 = y_2$. Let $N_1 = \{x \in G \mid \vartheta(x)^{-1} y_1 x = y_1\}$ and $N_2 = \{x \in G \mid \vartheta(x)^{-1} y_2 x = y_2\}$. It is straightforward to verify that N_1 and N_2 are subgroups of G. Now $x_1 x_2^{-1} \in N_1$ if and only if $\vartheta(x_1)^{-1} y_1 x_1 = \vartheta(x_2)^{-1} y_1 x_2$. So the index of N_1 in G equals the cardinality of $\{\vartheta(x)^{-1} y_1 x \mid x \in G\}$, which is finite. Thus N_1 and N_2 have finite index and hence so does $N_1 \cap N_2$, (this is sometimes known as Poincaré's Theorem). Now $N_1 \cap N_2 \subseteq C(y_1 y_2^{-1}) = \{x \in G \mid x y_1 y_2^{-1} = y_1 y_2^{-1} x\}$. So the index of $C(y_1 y_2^{-1})$ is finite. But the index of $C(y_1 y_2^{-1})$ equals the cardinality of the conjugacy class of $y_1 y_2^{-1}$, which is infinite unless $y_1 = y_2$. Hence ϑ is not weakly inner. □

The automorphism which permutes the generators of a free group is outer. Free products of finite groups have non trivial characters. Neither of these groups nor groups with property (T) are inner amenable, (see Bedos and de la Harpe [1.§3]).

https://doi.org/10.4153/CMB-1990-043-8 Published online by Cambridge University Press

ACKNOWLEDGEMENT. The author wishes to thank Professor M.-D. Choi for suggesting this problem.

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