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Real Hypersurfaces in Complex Projective Space Whose Structure Jacobi Operator Is of Codazzi Type

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Abstract. We prove the non existence of real hypersurfaces in complex projective space whose structure Jacobi operator is of Codazzi type.

1 Introduction

Let $\mathbb{C}P^m$, $m \geq 2$, be a complex projective space endowed with the metric g of constant holomorphic sectional curvature 4. Let M be a connected real hypersurface of $\mathbb{C}P^m$ without boundary. Let J denote the complex structure of $\mathbb{C}P^m$ and N a locally defined unit normal vector field on M. Then $-JN = \xi$ is a tangent vector field to M called the structure vector field on M. We also call $\mathbb{D}$ the maximal holomorphic distribution on M, that is, the distribution on M given by all vectors orthogonal to ξ at any point of M.

The study of real hypersurfaces in nonflat complex space forms is a classical topic in differential geometry. The classification of homogeneous real hypersurfaces in $\mathbb{C}P^m$ was obtained by Takagi, see [15–17], and is given by the following list:

A$_1$ Geodesic hyperspheres.
A$_2$ Tubes over totally geodesic complex projective spaces.
B Tubes over complex quadrics and $\mathbb{R}P^m$.
C Tubes over the Segre embedding of $\mathbb{C}P^1 \times \mathbb{C}P^n$, where $2n + 1 = m$ and $m \geq 5$.
D Tubes over the Plücker embedding of the complex Grassmann manifold $G(2, 5)$. In this case $m = 9$.
E Tubes over the cannonical embedding of the Hermitian symmetric space $SO(10)/U(5)$. In this case $m = 15$.

Other examples of real hypersurfaces are ruled real ones, introduced by Kimura [6]. Take a regular curve γ in $\mathbb{C}P^m$ with tangent vector field X. At each point of γ there is a unique complex projective hyperplane cutting γ so as to be orthogonal not only to X but also to JX. The union of these hyperplanes is called a ruled real hypersurface. It will be an embedded hypersurface locally, although globally it will, in general, have self-intersections and singularities. Equivalently, a ruled real hypersurface is such that $\mathbb{D}$ is integrable or $g(A\mathbb{D}, \mathbb{D}) = 0$, where A denotes the shape operator of the immersion. For further examples of ruled real hypersurfaces, see [8].

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Except for these real hypersurfaces, there are very few examples of real hypersurfaces in $\mathbb{C}P^n$.

On the other hand, Jacobi fields along geodesics of a given Riemannian manifold $(\tilde{M}, \tilde{g})$ satisfy a very well-known differential equation. This classical differential equation naturally inspires the so-called Jacobi operator. That is, if $\tilde{R}$ is the curvature operator of $\tilde{M}$ and X is any tangent vector field to $\tilde{M}$, the Jacobi operator (with respect to X) at $p \in M$, $\tilde{R}_X \in \operatorname{End}(T_p\tilde{M})$, is defined as $(\tilde{R}_X Y)(p) = (\tilde{R}(Y, X)X)(p)$ for all $Y \in T_p\tilde{M}$, being a selfadjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$. Clearly, each tangent vector field X to $\tilde{M}$ provides a Jacobi operator with respect to X.

The study of Riemannian manifolds by means of their Jacobi operators has been developed following several ideas. For instance, Chi [1] pointed out that (locally) symmetric spaces of rank 1 (among them complex space forms) satisfy that all the eigenvalues of $\tilde{R}_X$ have constant multiplicities and are independent of the point and the tangent vector X. The converse is a well-known problem which has been studied by many authors, although it is still open.

Let M be a real hypersurface in a complex projective space, and let ξ be the structure vector field on M. We will call the Jacobi operator on M with respect to ξ, the structure Jacobi operator on M. Then the structure Jacobi operator $R_\xi \in \operatorname{End}(T_pM)$ is given by $(R_\xi(Y))(p) = (R(Y, \xi)\xi)(p)$ for any $Y \in T_pM$, $p \in M$, where R denotes the curvature operator of M in $\mathbb{C}P^m$. Some papers devoted to studying several conditions on the structure Jacobi operator of a real hypersurface in $\mathbb{C}P^m$ are [2–4].

Recently, we proved the non-existence of real hypersurfaces in $\mathbb{C}P^m$ with parallel structure Jacobi operator [10]. We have studied distinct conditions on the structure Jacobi operator (Lie parallelism, Lie ξ-parallelism, $\mathbb{D}$-parallelism, and so on) [11–14].

A type $(1,1)$ tensor T on a real hypersurface M of $\mathbb{C}P^m$ is of *Codazzi type* if it satisfies the Codazzi equation, that is, $(\nabla_X T)Y = (\nabla_Y T)X$ for any X, Y tangent to M. Naturally, this is a weaker condition than T being parallel. In [7] the authors studied the so-called real hypersurfaces M with harmonic curvature in $\mathbb{C}P^m$. These real hypersurfaces satisfy that their Ricci tensor S is of Codazzi type. They obtain that there exist no such real hypersurfaces when the structure vector field ξ is principal. See also [5].

The purpose of the present paper is to study real hypersurfaces of $\mathbb{C}P^m$ whose structure Jacobi operator is of Codazzi type. That is,

$$(\nabla_X R_\xi)Y = (\nabla_Y R_\xi)X \tag{1.1}$$

for any X, Y tangent to M. Concretely we prove the following.

Theorem *There exist no real hypersurfaces in $\mathbb{C}P^m$, $m \geq 3$, with Codazzi type structure Jacobi operator.*

2 Preliminaries.

Throughout this paper, all manifolds, vector fields, *etc.,* will be considered of class C^∞ unless otherwise stated. Let M be a connected real hypersurface in $\mathbb{C}P^m$, $m \geq 2$, without boundary. Let N be a locally defined unit normal vector field on M. Let ∇ be the Levi–Civita connection on M and (J, g) the Kaehlerian structure of $\mathbb{C}P^m$.

For any vector field X tangent to M, we write $JX = \phi X + \eta(X)N$, and $-JN = \xi$. Then (ϕ, ξ, η, g) is an almost contact metric structure on M. That is, we have

$$(2.1) \qquad \phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any tangent vectors X, Y to M. From (2.1) we obtain $\phi\xi = 0$, $\eta(X) = g(X, \xi)$. From the parallelism of J we get $(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi$ and $\nabla_X \xi = \phi AX$ for any X, Y tangent to M, where A denotes the shape operator of the immersion. As the ambient space has holomorphic sectional curvature 4, the equations of Gauss and Codazzi are given, respectively, by

$$(2.2) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z + g(AY, Z)AX - g(AX, Z)AY,$$

and

$$(2.3) \qquad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi$$

for any tangent vectors X, Y, Z to M, where R is the curvature tensor of M.

In the sequel we need the following results.

Lemma 2.1 ([9]) *If ξ is a principal curvature vector with corresponding principal curvature α and $X \in \mathbb{D}$ is principal with principal curvature λ, then ϕX is principal with principal curvature $(\alpha\lambda + 2)/(2\lambda - \alpha)$.*

Lemma 2.2 ([10]) *There exist no real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 3$, such that the shape operator is given by $A\xi = \xi + \beta U$, $AU = \beta\xi + (\beta^2 - 1)U$, $A\phi U = -\phi U$, $AX = -X$, for any tangent vector X orthogonal to* span$\{\xi, U, \phi U\}$, *where U is a unit vector field in $\mathbb{D}$ and β is a nonvanishing smooth function defined on M.*

3 Some Lemmas

We first prove some lemmas which we will need in the proof of the theorem.

Lemma 3.1 *Let M be a real hypersurface of CP^m, $m \geq 2$, satisfying $(\nabla_\xi R_\xi)X = (\nabla_X R_\xi)\xi$ for any X tangent to M. Then $R_\xi \phi A = -A\phi R_\xi$.*

Proof As R_ξ is self-adjoint with respect to g, then $\nabla_\xi R_\xi$ is also self-adjoint. Thus $g((\nabla_\xi R_\xi)X, Y) = g(X, (\nabla_\xi R_\xi)Y)$ for any X, Y tangent to M. Therefore, in the conditions of the lemma $g((\nabla_X R_\xi)\xi, Y) = g(X, (\nabla_Y R_\xi)\xi)$. This yields $g(R_\xi(\phi AX), Y) = g(X, R_\xi(\phi AY))$ for any X, Y tangent to M and the lemma follows. ∎

Lemma 3.2 *There exist no Hopf real hypersurfaces M in $\mathbb{C}P^m$, $m \geq 2$, satisfying $R_\xi \phi A = -A\phi R_\xi$.*

Proof If M is Hopf, then $A\xi = \alpha\xi$, where α is a locally constant function on M. Let $X \in \mathbb{D}$ such that $AX = \lambda X$. As $R_\xi(\phi AX) = -A\phi R_\xi(X)$, we get $\lambda\phi X + \alpha\lambda A\phi X = -A\phi X - \alpha\lambda A\phi X$. As by Lemma 2.1 $A\phi X = ((\alpha\lambda + 2)/(2\lambda - \alpha))\phi X$, this yields

$$\lambda(1 + \alpha((\alpha\lambda + 2)/(2\lambda - \alpha))) = -(1 + \alpha\lambda)((\alpha\lambda + 2)/(2\lambda - \alpha)).$$

Thus $\alpha \neq 0$ and $(1 + \alpha^2)\lambda^2 + 2\alpha\lambda + 1 = 0$. As such a λ cannot exist, we have a contradiction. ∎

Proposition 3.3 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$, satisfying* (1.1). *Then its shape operator is given by $A\xi = \alpha\xi + \beta U$, $AU = \beta\xi + ((\beta^2 - 1)/\alpha)U$, $A\phi U = -(1/\alpha)\phi U$, $AX = \lambda X$, where $\lambda^2 + 2\alpha\lambda + 1 = 0$, α and β are nonnull functions on M, $\alpha^2 \neq 1$, U is a unit vector field in $\mathbb{D}$ and X is any unit vector field in $\mathbb{D}_U = \operatorname{span}\{\xi, U, \phi U\}^\perp$.*

Proof By Lemma 3.2, M cannot be Hopf. Thus, at least locally, there exist a unit $U \in \mathbb{D}$ and functions α, β on M, β being nonnull, such that $A\xi = \alpha\xi + \beta U$. From now on, all the computations are made in a neighbourhood of any point.

From Lemma 3.1, $R_\xi(\phi A\xi) = 0$. Then by the Gauss equation (2.2),

$$R_\xi(\phi U) = 0, \quad \alpha \neq 0, \quad A\phi U = -(1/\alpha)\phi U. \tag{3.1}$$

Then (3.1) and Lemma 3.1 give $R_\xi(\phi A\phi U) = 0$, and developing this equality we have

$$R_\xi(U) = 0, \quad AU = \beta\xi + ((\beta^2 - 1)/\alpha)U. \tag{3.2}$$

From (3.1) and (3.2) we conclude that $\mathbb{D}_U$ is A-invariant. Let $X \in \mathbb{D}_U$ such that $AX = \lambda X$. From Lemma 3.1 we get

$$(2\alpha\lambda + 1)A\phi X = -\lambda\phi X. \tag{3.3}$$

If there exists $X \in \mathbb{D}_U$ for which λ vanishes at some point of M, then $AX = 0$ on a neighbourhood of such a point. From (3.3) also $A\phi X = 0$. The Codazzi equation gives $(\nabla_X A)\phi X - (\nabla_{\phi X} A)X = -2\xi$. If we develop it and take its scalar product with ξ, we get

$$g([X, \phi X], U) = 2/\beta. \tag{3.4}$$

From (1.1) we have $\nabla_X R_\xi(\phi X) - R_\xi(\nabla_X \phi X) = \nabla_{\phi X} R_\xi(X) - R_\xi(\nabla_{\phi X} X)$. Taking its scalar product with U and bearing in mind (3.2), we obtain $g([X, \phi X], U) = 0$, which contradicts (3.4). Thus from (3.3), if $X \in \mathbb{D}_U$ is such that $AX = \lambda X$, then $\lambda \neq 0$, and $A\phi X = -(\lambda/(2\alpha\lambda + 1))\phi X$.

We could have $-\lambda = \lambda/(2\alpha\lambda + 1)$. This yields $\alpha\lambda = -1$. Starting with the same Codazzi equation as above and taking its scalar product with ξ, respectively U, implies $g([X, \phi X], U) = -2/(\alpha^2\beta)$, respectively $g([X, \phi X], U) = -2/\beta$, and we can conclude $\alpha^2 = 1$. If $\alpha = 1$, then $AX = -X$ and $A\phi X = -\phi X$. If there exists another $Y \in \mathbb{D}_U$ such that $AY = \mu Y$, then $A\phi Y = -(\mu/(2\mu + 1))\phi Y$. The Codazzi equation gives $(\nabla_Y A)\phi Y - (\nabla_{\phi Y} A)Y = -2\xi$. If we develop this equality and take its scalar product with ξ, we get

$$g([Y,\phi Y],U) = 2(2\mu^2 + 2\mu + 1)/\beta(2\mu + 1), \tag{3.5}$$

and its scalar product with U yields

$$\begin{aligned} -\mu((1/(2\mu+1))g(\nabla_Y\phi Y,U) + g(\nabla_{\phi Y}Y,U)) \\ = (\beta^2 - 1)g([Y,\phi Y],U) - \beta\mu + (\beta\mu)/(2\mu+1). \end{aligned} \tag{3.6}$$

On the other hand, $\nabla_{\phi Y}R_\xi(Y) - R_\xi(\nabla_{\phi Y}Y) = \nabla_Y R_\xi(\phi Y) - R_\xi(\nabla_Y\phi Y)$. Taking its scalar product with U and bearing in mind (3.1), we obtain

$$(1+\mu)(g(\nabla_{\phi Y}Y,U) - (1/(2\mu+1))g(\nabla_Y\phi Y,U)) = 0. \tag{3.7}$$

From (3.7), if $\mu \neq -1$,

$$g(\nabla_{\phi Y}Y,U) = (1/(2\mu+1))g(\nabla_Y\phi Y,U). \tag{3.8}$$

From (3.6) and (3.8) we get

$$\beta(2\mu+1)g(\nabla_{\phi Y}Y,U) = \mu. \tag{3.9}$$

From (3.5) and (3.9) we have

$$g(\nabla_Y\phi Y,U) = (4\mu^2 + 5\mu + 2)/\beta(2\mu+1). \tag{3.10}$$

Now (3.8), (3.9) and (3.10) imply $\mu = (4\mu^2+5\mu+2)/(2\mu+1)$. Thus $\mu^2+2\mu+1 = 0$. Its unique solution is $\mu = -1$, but from Lemma 2.2 this kind of real hypersurface does not exist.

A similar reasoning gives the same result if $\alpha = -1$. Therefore, for $X \in \mathbb{D}_U$, we have $AX = \lambda X$, $\lambda \neq 0$, $\lambda\alpha \neq 1$ and $A\phi X = -(\lambda/(2\alpha\lambda+1))\phi X$.

The Codazzi equation applied to X and ϕX after taking its scalar product with ξ, respectively U, yields

$$g([X,\phi X],U) = 2((1+\alpha^2)\lambda^2 + 2\alpha\lambda + 1)/\beta(2\alpha\lambda+1), \tag{3.11}$$

respectively,

$$\begin{aligned} -\lambda((1/2\alpha\lambda+1)g(\nabla_X\phi X,U) + g(\nabla_{\phi X}X,U)) \\ = ((\beta^2-1)/\alpha)g([X,\phi X],U) - \beta\lambda + (\beta\lambda/(2\alpha\lambda+1)). \end{aligned} \tag{3.12}$$

From (1.1), $\nabla_{\phi X}R_\xi(X) - R_\xi(\nabla_{\phi X}X) = \nabla_X R_\xi(\phi X) - R_\xi(\nabla_X\phi X)$. Taking its scalar product with U and bearing in mind (3.1), we get

$$(1+\alpha\lambda)(g(\nabla_{\phi X}X,U) - (1/(2\alpha\lambda+1)g(\nabla_X\phi X,U)) = 0. \tag{3.13}$$

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But as $1+\alpha\lambda \neq 0$, (3.13) gives

$$g(\nabla_{\phi X}X, U) = (1/(2\alpha\lambda+1))g(\nabla_X \phi X, U). \tag{3.14}$$

From (3.12) and (3.14) we obtain

$$g(\nabla_{\phi X}X, U) = \alpha\lambda/\beta(2\alpha\lambda+1). \tag{3.15}$$

From (3.11) and (3.15) we have $g(\nabla_X\phi X, U) = (2(1+\alpha^2)\lambda^2+5\alpha\lambda+2)/\beta(2\alpha\lambda+1)$. From 3.14 and 3.15, this yields $2(1+\alpha^2)\lambda^2 + 5\alpha\lambda + 2 = (2\alpha\lambda+1)\alpha\lambda$. From this, $\lambda^2 + 2\alpha\lambda + 1 = 0$, and this finishes the proof. ■

Lemma 3.4 *Let M be a real hypersurface in $\mathbb{C}P^m$, $m \geq 3$ satisfying* (1.1)*. Then*

$$\operatorname{grad}(\alpha) = ((3\beta/\alpha) + \alpha\beta - \beta g(\nabla_\xi \phi U, U))\phi U,$$
$$\operatorname{grad}(\beta) = ((\beta^2-1)/\alpha^2) + \beta^2 - (\beta^2/\alpha)g(\nabla_\xi\phi U, U))\phi U,$$

where α, β and U are as in Proposition 3.3*.*

Proof Let $X \in \mathbb{D}_U$ such that $AX = \lambda X$ and $A\phi X = -(\lambda/(2\alpha\lambda+1))\phi X$. Then from Proposition 3.3, $A\phi X = -(1/\lambda)\phi X$. The Codazzi equation yields $(\nabla_X A)\xi - (\nabla_\xi A)X = -\phi X$. Developing this equality and taking its scalar product with ξ, respectively U, we get

$$X(\alpha) + \beta g(\nabla_\xi X, U) = 0, \tag{3.16}$$

$$X(\beta) + (((\beta^2-1)/\alpha) - \lambda)g(\nabla_\xi X, U) = 0. \tag{3.17}$$

As $(\nabla_U R_\xi)\xi = (\nabla_\xi R_\xi)U$, from (3.1) and (3.2) we have $R_\xi(\nabla_\xi U) = 0$. Thus $0 = g(\nabla_\xi U, R_\xi(X)) = (1+\alpha\lambda)g(\nabla_\xi U, X)$. As $\lambda\alpha \neq -1$, we obtain

$$g(\nabla_\xi U, X) = 0. \tag{3.18}$$

From (3.16), (3.17) and, (3.18) we have

$$X(\alpha) = X(\beta) = 0 \tag{3.19}$$

for any $X \in \mathbb{D}_U$. As $(\nabla_\xi R_\xi)X = (\nabla_X R_\xi)\xi$, we get $\nabla_\xi((1+\alpha\lambda)X) - R_\xi(\nabla_\xi X) = -\lambda R_\xi(\phi X)$, and taking its scalar product with X, we obtain

$$\lambda\xi(\alpha) + \alpha\xi(\lambda) = 0. \tag{3.20}$$

But from Proposition 3.3, $\lambda^2 + 2\alpha\lambda + 1 = 0$. Thus

$$\lambda\xi(\lambda) + \lambda\xi(\alpha) + \alpha\xi(\lambda) = 0. \tag{3.21}$$

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From (3.20) and (3.21) we get

$$\xi(\alpha) = \xi(\lambda) = 0. \tag{3.22}$$

Once more, the Codazzi equation gives $(\nabla_\xi A)U - (\nabla_U A)\xi = \phi U$. Developing it, from Proposition 3.3 and taking its scalar product with ξ, we have

$$\xi(\beta) = U(\alpha). \tag{3.23}$$

As $(\nabla_X R_\xi)U = (\nabla_U R_\xi)X$, if we take its scalar product with $X \in \mathbb{D}_U$ we obtain

$$(1 + \alpha\lambda)g(\nabla_X U, X) = -\lambda U(\alpha) - \alpha U(\lambda), \tag{3.24}$$

and from Proposition 3.3,

$$(1 + \alpha\lambda)g(\nabla_X U, X) = \lambda U(\lambda). \tag{3.25}$$

From the Codazzi equation, $(\nabla_U A)X - (\nabla_X A)U = 0$. Its scalar product with X yields

$$U(\lambda) + ((\alpha\lambda - \beta^2 + 1)/\alpha)g(\nabla_X U, X) = 0. \tag{3.26}$$

Now (3.25) and (3.26) imply $(\alpha + \alpha^2\lambda + \alpha\lambda^2 - \lambda\beta^2 + \lambda)U(\lambda) = 0$. If $U(\lambda) = 0$, then from (3.25), $g(\nabla_X U, X) = 0$, and from (3.24), $U(\alpha) = 0$. Therefore, (3.23) also gives $\xi(\beta) = 0$. If $U(\lambda) \neq 0$, then $\alpha + \alpha^2\lambda + \alpha\lambda^2 - \lambda\beta^2 + \lambda = 0$. As, from Proposition 3.3, $\lambda^2 = -2\alpha\lambda - 1$, we have $\lambda(\alpha^2 + \beta^2 - 1) = 0$. This means $\alpha^2 + \beta^2 = 1$. Thus $\alpha\xi(\alpha) + \beta\xi(\beta) = 0$, and from (3.22), $\xi(\beta) = 0$. So we have proved that always

$$\xi(\beta) = U(\alpha) = 0. \tag{3.27}$$

But $g((\nabla_\xi A)U - (\nabla_U A)\xi, U) = 0$. From (3.22) and (3.27) we get

$$U(\beta) = 0. \tag{3.28}$$

Now the Codazzi equation implies $(\nabla_\xi A)\phi U - (\nabla_{\phi U} A)\xi = -U$. If we take its scalar product with ξ, respectively U, we obtain

$$(\phi U)(\alpha) = (3\beta/\alpha) + \alpha\beta - \beta g(\nabla_\xi \phi U, U), \tag{3.29}$$

$$(\phi U)(\beta) = ((\beta^2 - 1)/\alpha^2) + \beta^2 - (\beta^2/\alpha)g(\nabla_\xi \phi U, U). \tag{3.30}$$

The proof finishes if we look at (3.19), (3.22), (3.27), (3.28), (3.29) and (3.30). ■

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4 Proof of the Theorem

From Lemma 3.4 we have $\nabla_X \operatorname{grad}(\alpha) = X(\delta)\phi U + \delta\nabla_X\phi U$, for any X tangent to M, where $\delta = (3\beta/\alpha) + \alpha\beta - \beta g(\nabla_\xi\phi U, U)$. Thus

$$\begin{aligned} 0 &= g(\nabla_X \operatorname{grad}(\alpha), Y) - g(\nabla_Y \operatorname{grad}(\alpha), X) \\ &= X(\delta)g(\phi U, Y) - Y(\delta)g(\phi U, X) + \delta(g(\nabla_X\phi U, Y) - g(\nabla_Y\phi U, X)) \end{aligned}$$

for any X, Y tangent to M. If we take $Y = \xi$, we get

$$0 = -\xi(\delta)g(\phi U, X) + \delta(g(\nabla_X\phi U, \xi) - g(\nabla_\xi\phi U, X))$$

for any X tangent to M. Now take $X = U$. We have

$$\delta(((1-\beta^2)/\alpha) - g(\nabla_\xi\phi U, U)) = 0.$$

Thus either $\delta = 0$ or $g(\nabla_\xi\phi U, U) = (1-\beta^2)/\alpha$. Thus

(4.1) $$g(\nabla_\xi\phi U, U) = (\alpha^2+3)/\alpha \quad \text{or} \quad g(\nabla_\xi\phi U, U) = (1-\beta^2)/\alpha.$$

The same reasoning applied to $\operatorname{grad}(\beta)$ gives

(4.2) $$g(\nabla_\xi\phi U, U) = (\alpha^2\beta^2+\beta^2-1)/\alpha\beta^2 \quad \text{or} \quad g(\nabla_\xi\phi U, U) = (1-\beta^2)/\alpha.$$

If we suppose $g(\nabla_\xi\phi U, U) \neq (1-\beta^2)/\alpha$, then from (4.1) and (4.2) we have $2\beta^2+1 = 0$, which is impossible. Thus $g(\nabla_\xi\phi U, U) = (1-\beta^2)/\alpha$. Then (3.29) and (3.30) become

(4.3) $$(\phi U)(\alpha) = (\alpha^2+\beta^2+2)\beta/\alpha,$$

(4.4) $$(\phi U)(\beta) = (\beta^4+\alpha^2\beta^2-1)/\alpha^2.$$

From the Codazzi equation, $(\nabla_\xi A)U - (\nabla_U A)\xi = \phi U$. Its scalar product with ϕU, bearing in mind (4.3), yields

(4.5) $$g(\nabla_U\phi U, U) = (2\beta^2-\beta^4-1)/\alpha^2\beta.$$

As from the Codazzi equation $(\nabla_U A)\phi U - (\nabla_{\phi U} A)U = -2\xi$, if we take its scalar product with U, from (4.3) and (4.4) we get

(4.6) $$g(\nabla_U\phi U, U) = (\beta^2-\beta^4-4\alpha^2)/\alpha^2\beta.$$

From (4.5) and (4.6) we have

(4.7) $$4\alpha^2+\beta^2 = 1.$$

From (4.7) we obtain $4\alpha(\phi U)(\alpha) + \beta(\phi U)(\beta) = 0$. Then (4.3) and (4.4) imply

(4.8) $$\beta^2-\alpha^2 = 1.$$

From (4.7) and (4.8) we obtain $\alpha = 0$, which contradicts Proposition 3.3 and finishes the proof. ■

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