ANZIAM J. 57(2016), 222–243
doi:10.1017/S1446181115000450

APPROXIMATE SOLUTIONS FOR THE BRITISH PUT OPTION AND ITS OPTIMAL EXERCISE BOUNDARY

JOANNA GOARD[1]

(Received 7 November, 2014; accepted 5 March, 2015; first published online 22 January 2016)

Abstract

British put options are financial derivatives with an early exercise feature whereby on payoff, the holder receives the best prediction of the European put payoff under the hypothesis that the true drift of the stock price is equal to a contract drift. In this paper, we derive simple analytic approximations for the optimal exercise boundary and the option valuation, valid for short expiry times – which is a common feature of most options traded in the market. Empirical results show that the approximations provide accurate results for expiries of at least up to two months.

2010 *Mathematics subject classification*: 91G20.

Keywords and phrases: British options, free boundary problems, analytic approximations.

1. Introduction

Put options are financial derivative contracts between a holder and a writer that allows the put option holder to sell an underlying security for a predetermined exercise (or strike) price K at an expiry time $t = T$. If the value of the underlying security at expiry is S_T, then the value of the contract at expiry, called its payoff, is $\max(K - S_T, 0)$. A put option that can only be exercised at expiry is called a *European* put option and these options can be priced using the famous Black–Scholes option pricing formula [2]. A put option that can be exercised at any time up to the expiry time is known as an *American* put option. These are considerably harder to price as the possibility of early exercise leads to a free boundary problem, where the free boundary separates the region between where it is optimal to hold the option and where to exercise the option. This boundary is known as the optimal exercise boundary. Zhu [10] found an analytic solution to the American put problem when the underlying asset pays no dividends. Although the solution involves an infinite sum of double integrals and can be very computationally time intensive for accurate results, the result constituted

[1]School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia; e-mail: joanna@uow.edu.au.

a significant breakthrough in the search for a closed-form solution to the American put pricing problem. There is an enormous amount of literature on the pricing of American options under different approaches (for example, binomial, Monte Carlo, finite-difference, analytic approximations, numerical methods for integral equations etc). For a detailed exposition, we refer the reader to the comprehensive review article by Zhu [11] and the exhaustive list of references in the book by Detemple [3].

Recently, Peskir and Samee [7] introduced a new put option contract called a *British put option* defined as "a financial contract between a seller/hedger and a buyer/holder entitling the latter to exercise at any stopping time t' prior to expiry $t = T$, whereupon his payoff (deliverable immediately) is the 'best prediction' of the European put payoff $\max(K - S_T, 0)$, given all the information up to time t' under the hypothesis that the true drift of the stock price is set equal to a predetermined contract drift μ". The authors in [7] explain that from the viewpoint of a "true buyer" who does not wish to sell or hedge the option, if the true drift of the stock price μ_{real} is greater than the risk-free rate r, then the return from the investment in an American or European put option is unfavourable. With a British put option, this issue is addressed so that under the circumstance that the put holder believes $\mu_{\text{real}} > r$, then he can effectively substitute the unfavourable drift with a contract drift and minimize his losses.

Assuming that the underlying stock price S evolves as $dS = \mu_{\text{real}} S\,dt + \sigma S\,dW$, with W a standard Wiener process, the price of a British put option $V(S,t)$ with exercise price K and expiry T, in the continuation region $\{(S,t) \in (0,\infty) \times [0,T], S > b(t)\}$, where $b(t)$ is the optimal stopping boundary (to be determined), satisfies a free boundary problem for a partial differential equation (PDE). Assuming that the contract drift $\mu > r$, the mathematical formulation is (see the article by Peskir and Samee [7])

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rS V_S - rV = 0, \tag{1.1a}$$

$$V(S,t) \to 0 \quad \text{as } S \to \infty, \tag{1.1b}$$

$$V(b(t),t) = G^{\mu}(b(t),t), \tag{1.1c}$$

$$V_S(b(t),t) = G^{\mu}_S(b(t),t), \tag{1.1d}$$

$$V(S,T) = \max(K - S, 0), \tag{1.1e}$$

where in (1.1c) and (1.1d)

$$G^{\mu}(S,t) = KN(f_1) - S e^{\mu(T-t)} N(f_2), \tag{1.2a}$$

$$f_1 = -\frac{[\ln(S/K) + (\mu - \sigma^2/2)(T-t)]}{\sigma\sqrt{T-t}}, \tag{1.2b}$$

$$f_2 = -\frac{[\ln(S/K) + (\mu + \sigma^2/2)(T-t)]}{\sigma\sqrt{T-t}} \tag{1.2c}$$

and $N(x) = (1/\sqrt{2\pi}) \int_{-\infty}^{x} e^{-y^2/2}\,dy$ is the standard normal distribution function.

https://doi.org/10.1017/S1446181115000450 Published online by Cambridge University Press

Peskir and Samee [7] showed that the value of the optimal exercise boundary at expiry is given by $b(T) = rK/\mu$. We note that $e^{-r(T-t)}G^r(S,t)$ is the Black–Scholes formula for the arbitrage-free price of a European put with strike price K and expiry T. Hence, if the contract drift μ could be less than the risk-free rate r, then the British put option could be considered as an American option on an undiscounted European put option on a stock that pays a dividend $\delta = r - \mu \geq 0$. However, for a British put option, the contract drift μ satisfies $\mu > r$.

In the stopping region, where $S \leq b(t)$, the price is set as $V(S,t) = G^\mu(S,t)$. Peskir and Samee [7] showed that the arbitrage-free price admits the early exercise premium representation

$$V(S,t) = e^{-r(T-t)}G^r(S,t) + \int_t^T J(S,t,v,b(v))\,dv, \tag{1.3a}$$

where

$$J(x,t,v,z) = -e^{-r(v-t)}\int_0^z H^\mu(v,y) f(v-t,x,y)\,dy, \tag{1.3b}$$

with

$$\begin{aligned} f(v-t,x,y) &= \frac{1}{\sigma y\sqrt{v-t}}\phi\Big(\frac{1}{\sigma\sqrt{v-t}}\Big[\log\Big(\frac{y}{x}\Big) - \Big(r - \frac{\sigma^2}{2}\Big)(v-t)\Big]\Big), \\ H^\mu(t,x) &= \mu x e^{\mu(T-t)} N\Big(\frac{1}{\sigma\sqrt{T-t}}\Big[\log\Big(\frac{K}{x}\Big) - \Big(\mu + \frac{\sigma^2}{2}\Big)(T-t)\Big]\Big) \\ &\quad - rKN\Big(\frac{1}{\sigma\sqrt{T-t}}\Big[\log\Big(\frac{K}{x}\Big) - \Big(\mu - \frac{\sigma^2}{2}\Big)(T-t)\Big]\Big), \end{aligned}$$

and where $\phi(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ is the standard normal density function. Furthermore, the rational exercise boundary of the British put option is given in [7] as the unique continuous solution to the nonlinear integral equation

$$F(b(t),t) = \int_t^T J(b(t),t,v,b(v))\,dv, \tag{1.4}$$

where $F(x,t) = G^\mu(x,t) - e^{-r(T-t)}G^r(x,t)$ and J is given in (1.3b). The *implicit* form of the optimal exercise boundary in equation (1.4), involving a double integration, makes finding this boundary computationally and numerically intensive (see, for example, the book by Kwok [5] for numerical procedures for approximating the free boundary for American options). The optimal exercise boundary also needs to be found in order to compute the option values. In this paper, we provide quick, simple and accurate approximations for the optimal exercise boundary for short expiry times, such as one or two months. In fact, options with short tenor dominate the options markets. We also derive an analytic approximation for the option value that is independent of the exercise boundary values, and which is valid for short times to expiry and for stock spot prices in the vicinity of Kr/μ.

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2. The approximate solution

MAIN RESULT. An approximate solution for the optimal exercise boundary $b(t)$ valid for short times to expiry in the vicinity of Kr/μ is given by

$$b(t) = \frac{Kr}{\mu} \exp\Big(\sum_{i=1}^{\infty} b_i \tau^{i/2}\Big), \tag{2.1}$$

where $\tau = \sigma^2(T-t)/2$ and b_i, for $i = 1, 2, \ldots, 5$, are listed in Appendix A.

An approximate solution for the British put option $V(S,t)$ with exercise price K and expiry T, valid for short times to expiry and in the vicinity of Kr/μ in the continuation region $\{(S,t) \in (0,\infty) \times [0,T], S > b(t)\}$, is given by

$$V(S,t) = KW\Big[\ln\Big(\frac{S}{K}\Big), \frac{\sigma^2}{2}(T-t)\Big] + G^{\mu}(S,t), \tag{2.2a}$$

where

$$W(x,\tau) = \sum_{j=0}^{\infty} \tau^{(3+j)/2} Q_j\Big(\frac{x - \ln(r/\mu)}{\sqrt{\tau}}\Big), \tag{2.2b}$$

$$Q_j(\eta) = \sum_{i=0}^{3+j} \gamma_j[i]\eta^i + \operatorname{erf}\Big(\frac{\eta}{2}\Big) \sum_{i=0}^{3+j} \beta_j[i]\eta^i + e^{-\eta^2/4} \sum_{i=0}^{2+j} \alpha_j[i]\eta^i \tag{2.2c}$$

and $G^{\mu}(S,t)$ is given in (1.2a)–(1.2c). For $j = 0, 1, \ldots, 4$, the coefficients of $Q_j(\eta)$ in (2.2c), namely, $\gamma_j[i], i = 0, 1, \ldots, 3+j, \beta_j[i], i = 0, 1, \ldots, 3+j, \alpha_j[i], i = 0, 1, \ldots, 2+j$, are listed in Appendix B.

We note that expansions of the type (2.1) for the free boundary $\ln b(t)$ have been used in heat conduction by Tao [8], in which $b(t)$ is the solidification boundary, and have also been used in option pricing by Alobaidi and Mallier [1] and by Goard [4] for the optimal 'shout' boundary for shout options.

PROOF. First, to simplify the boundary conditions, we let

$$V(S,t) = KW(S,t) + G^{\mu}(S,t),$$

which transforms the problem (1.1a)–(1.1e) to

$$W_t + \frac{\sigma^2 S^2}{2} W_{SS} + rSW_S - rW + f(S,t) = 0,$$

where

$$f(S,t) = \frac{(r-\mu)}{\sigma\sqrt{T-t}}\Big[-N'(f_1) + \frac{S}{K}e^{\mu(T-t)}N'(f_2)\Big] - rN(f_1) + \frac{\mu S}{K}e^{\mu(T-t)}N(f_2),$$

subject to

$$W(S,t) \to 0 \quad \text{as } S \to \infty,$$
$$W(b(t),t) = 0, \quad W_S(b(t),t) = 0, \quad W(S,T) = 0.$$

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Now, setting $S = Ke^x, t = T - 2\tau/\sigma^2$, the problem reduces to

$$W_\tau = W_{xx} + (\bar{k} - 1)W_x - \bar{k}W + \mathfrak{F}(x, \tau), \tag{2.3a}$$

where

$$\mathfrak{F}(x, \tau) = \frac{2}{\sigma^2}\frac{(r-\mu)}{\sqrt{2\tau}}[-N'(f_1) + e^x e^{2\mu/\sigma^2\tau} N'(f_2)] - \bar{k}N(f_1) + \frac{2\mu}{\sigma^2} e^x e^{2\mu/\sigma^2\tau} N(f_2) \tag{2.3b}$$

and

$$f_1 = -\frac{[x + (\mu - \sigma^2/2)2\tau/\sigma^2]}{\sqrt{2\tau}}, \quad f_2 = -\frac{[x + (\mu + \sigma^2/2)2\tau/\sigma^2]}{\sqrt{2\tau}}, \quad \bar{k} = \frac{2r}{\sigma^2}. \tag{2.3c}$$

Equation (2.3a) needs to be solved in the continuation region where $x > B(\tau) = \ln(b(T - 2\tau/\sigma^2)/K)$, subject to

$$\begin{aligned} W(x, \tau) \to 0 \quad &\text{as } x \to \infty, \\ W(B(\tau), \tau) &= 0, \\ W_x(B(\tau), \tau) &= 0, \\ W(x, 0) &= 0. \end{aligned}$$

Note that $B(0) = b_0 = \ln(r/\mu) < 0$, as $\mu > r$.

Since, at expiry, the value of the exercise boundary is not the strike price, we perform a local analysis in the vicinity of $x = b_0 = \ln(r/\mu)$ and $\tau = 0$, and introduce the new variables

$$x - b_0 = \nu X, \tag{2.4a}$$

$$\tau = ay, \tag{2.4b}$$

$$W = \varepsilon\psi(X, y). \tag{2.4c}$$

Substituting (2.4a)–(2.4c) into (2.3a) yields

$$\frac{\varepsilon}{a}\psi_y = \frac{\varepsilon}{\nu^2}\psi_{XX} + (\bar{k} - 1)\frac{\varepsilon}{\nu}\psi_X - \bar{k}\varepsilon\psi + \mathfrak{F}\Big(\ln\Big(\frac{r}{\mu}\Big) + \nu X, ay\Big). \tag{2.5}$$

In order to balance the leading terms, we need $\varepsilon = \nu^3, a = \nu^2$ and so, from equations (2.5) and (2.3b), (2.3c),

$$\psi_y = \psi_{XX} + (\bar{k} - 1)\nu\psi_X - \bar{k}\nu^2\psi + \Omega(X, y; \nu), \tag{2.6a}$$

where

$$\begin{aligned} \Omega(X, y; \nu) = \Big[\bar{k}X &+ \frac{\nu}{2}\Big(\bar{k}X^2 + \frac{4\bar{k}\mu}{\sigma^2}y\Big) + \frac{\nu^2}{3!}\Big(\bar{k}X^3 + \frac{12\bar{k}X\mu y}{\sigma^2}\Big) \\ &+ \frac{\nu^3}{4!}\Big(\bar{k}X^4 + \frac{24\bar{k}\mu}{\sigma^2}X^2y + \frac{48\bar{k}\mu^2}{\sigma^4}y^2\Big) \\ &+ \frac{\nu^4}{5!}\Big(\bar{k}X^5 + \frac{40\bar{k}\mu}{\sigma^2}X^3y + \frac{240\bar{k}\mu^2}{\sigma^4}Xy^2\Big) + \cdots\Big] \end{aligned} \tag{2.6b}$$

to be solved on $\{(X, y) : X > X_f(y) = B(\tau) - b_0/\nu, y \in [0, \sigma^2T/2\nu^2]\}$.

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The boundary and initial conditions are now

$$\psi(X,0)=0,\quad \psi(X_f(y),y))=0,\quad \psi_X(X_f(y),y))=0.$$

Letting

$$\psi(X,y)=\psi_0(X,y)+\nu\psi_1(X,y)+\nu^2\psi_2(X,y)+\cdots, \tag{2.7}$$

from (2.6a),

$$\begin{aligned}(\psi_0(X,y)&+\nu\psi_1(X,y)+\nu^2\psi_2(X,y)+\cdots)_y\\ &=(\psi_0(X,y)+\nu\psi_1(X,y)+\nu^2\psi_2(X,y)+\cdots)_{XX}\\ &\quad+(\bar{k}-1)\nu(\psi_0(X,y)+\nu\psi_1(X,y)+\nu^2\psi_2(X,y)+\cdots)_X\\ &\quad-\bar{k}\nu^2(\psi_0(X,y)+\nu\psi_1(X,y)+\nu^2\psi_2(X,y)+\cdots)+\Omega(x,y;\nu)\end{aligned} \tag{2.8a}$$

to be solved, subject to

$$\psi_0(X,0)+\nu\psi_1(X,0)+\nu^2\psi_2(X,0)+\cdots=0, \tag{2.8b}$$

$$\psi_0(X_f(y),y)+\nu\psi_1(X_f(y),y)+\nu^2\psi_2(X_f(y),y)+\cdots=0, \tag{2.8c}$$

$$\frac{\partial}{\partial X}(\psi_0)(X_f(y),y)+\nu\frac{\partial}{\partial X}(\psi_1)(X_f(y),y)+\nu^2\frac{\partial}{\partial X}(\psi_2)(X_f(y),y)+\cdots=0. \tag{2.8d}$$

We assume that the free boundary $X_f(y)$ is of the form

$$X_f(y)=\sum_{i=1}^{\infty}b_i\nu^{i-1}y^{i/2}=b_1\sqrt{y}+b_2\nu y+b_3\nu^2y^{3/2}+\cdots$$

(so that $B(\tau)=b_0+b_1\sqrt{\tau}+b_2\tau+b_3\tau^{3/2}+\cdots$). To determine the functions $\psi_i(X,y)$ in (2.7), we equate coefficients of ν in (2.8a)–(2.8d). In general, we find that the governing PDE for $\psi_n(X,y)$ takes the form

$$(\psi_n)_y=(\psi_n)_{XX}+(\bar{k}-1)(\psi_{n-1})_X-\bar{k}\psi_{n-2}+\sum_{j=0}^{M}c_{nj}X^{1+n-2j}y^j,\quad n\geq 2, \tag{2.9}$$

where

$$M=\begin{cases}\dfrac{n}{2} & n\text{ even},\\[2mm] \dfrac{n-1}{2} & n\text{ odd}\end{cases}$$

and the c_{nj} are constants. When $n=0$, the governing PDE is similar to (2.9), but does not contain the second and third terms on the right-hand side (RHS); and when $n=1$, the governing PDE is again similar to (2.9), but does not contain the third term on the RHS. By symmetry reduction, ψ_n can be written as $\psi_n=y^{(n+3)/2}Q_n(\eta)$, where $\eta=X/\sqrt{y}$, reducing the PDE to an ordinary differential equation (ODE) for $Q_n(\eta)$. We note that this corresponds to PDE (2.9), possessing a classical symmetry with generator

$$\Gamma=X\frac{\partial}{\partial X}+2y\frac{\partial}{\partial y}+(n+3)\psi_n\frac{\partial}{\partial\psi_n},$$

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which generates a scaling transformation $\bar{x} = e^{\epsilon}x, \bar{y} = e^{2\epsilon}y, \bar{\psi}_n = e^{(n+3)\epsilon}\psi_n$ that can be seen to leave (2.9) invariant.

The free boundary under this transformation is of the form

$$\eta_f(y) = \frac{X_f(y)}{\sqrt{y}} = b_1 + b_2\nu\sqrt{y} + b_3\nu^2 y + \cdots$$

and the boundary conditions (2.8c) and (2.8d) would be as follows, to be evaluated at $\eta = b_1$:

$$\begin{aligned} &y^{3/2}Q_0 + \nu y^2[b_2Q_0' + Q_1] + \nu^2 y^{5/2}\Big[\frac{b_2^2}{2}Q_0'' + b_3Q_0' + b_2Q_1' + Q_2\Big] \\ &\quad + \nu^3y^3\Big[b_2Q_2' + b_4Q_0' + \frac{b_2^3}{6}Q_0''' + b_3b_2Q_0'' + Q_3 + \frac{b_2^2}{2}Q_1'' + b_3Q_1'\Big] \\ &\quad + \nu^4y^{7/2}\Big[b_2b_4Q_0'' + \frac{b_2^2b_3}{2}Q_0''' + \frac{b_3^2}{2}Q_0'' + \frac{b_2^4}{24}Q_0'''' + b_2Q_3' + \frac{b_2^2}{2}Q_2'' \\ &\quad + b_3Q_2' + b_4Q_1' + \frac{b_2^3}{6}Q_1''' + b_2b_3Q_1'' + Q_4 + b_5Q_0'\Big] + \cdots = 0 \end{aligned} \tag{2.10a}$$

and

$$\begin{aligned} &yQ_0' + \nu y^{3/2}[b_2Q_0'' + Q_1'] + \nu^2y^2\Big[\frac{b_2^2}{2}Q_0''' + b_3Q_0'' + b_2Q_1'' + Q_2'\Big] \\ &\quad + \nu^3y^{5/2}\Big[b_2Q_2'' + b_4Q_0'' + \frac{b_2^3}{6}Q_0'''' + b_2b_3Q_0''' + Q_3' + \frac{b_2^2}{2}Q_1''' + b_3Q_1''\Big] \\ &\quad + \nu^4y^3\Big[b_2b_4Q_0''' + \frac{b_2^2b_3}{2}Q_0'''' + \frac{b_3^2}{2}Q_0''' + \frac{b_2^4}{24}Q_0''''' + b_2Q_3'' + \frac{b_2^2}{2}Q_2''' \\ &\quad + b_3Q_2'' + b_4Q_1'' + \frac{b_2^3}{6}Q_1'''' + b_2b_3Q_1''' + Q_4' + b_5Q_0''\Big] + \cdots = 0. \end{aligned} \tag{2.10b}$$

We now equate coefficients of $\nu^i, i = 0, 1, \ldots, 4$, to determine the functions $\psi_i(X, y), i = 0, 1, \ldots, 4$. In the following, algebraic equations are solved using the mathematical package MAPLE [6].

- *Equating coefficients of* ν^0: From (2.8a)–(2.8d), we find the problem for $\psi_0(X, y)$ as

  $$(\psi_0)_y = (\psi_0)_{XX} + \bar{k}Xr,$$

  subject to

  $$\psi_0(X, 0) = 0, \quad \psi_0(b_1, y) = 0, \quad (\psi_0)_X(b_1, y) = 0.$$

  Using the transformation $\psi_0 = y^{3/2}Q_0(\eta)$, where $\eta = X/\sqrt{y}$, the problem reduces to

  $$Q_0''(\eta) + \frac{\eta}{2}Q_0'(\eta) - \frac{3}{2}Q_0(\eta) + \bar{k}\eta = 0, \tag{2.11a}$$

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subject to

$$Q_0(b_1) = Q_0'(b_1) = 0 \quad \text{and} \tag{2.11b}$$

$$\lim_{y\to 0} y^{3/2} Q_0\Big(\frac{X}{\sqrt{y}}\Big) = 0. \tag{2.11c}$$

The solution to (2.11a)–(2.11c) can be written as

$$Q_0(\eta) = \bar{k}\eta + A_1(\eta^3 + 6\eta) + A_2\Big[e^{-\eta^2/4}(\eta^2+4) + \frac{\sqrt{\pi}}{2}(\eta^3+6\eta)\mathrm{erf}\Big(\frac{\eta}{2}\Big)\Big]. \tag{2.12}$$

In order to satisfy equation (2.11c), we need only consider the case $X > 0$, because when $y = 0$ in the continuation region, $X > 0$. So, from (2.12), we require $A_1 = -A_2\sqrt{\pi}/2$, so that

$$Q_0(\eta) = \bar{k}\eta + A_2\Big[-\frac{\sqrt{\pi}}{2}(\eta^3+6\eta) + e^{-\eta^2/4}(\eta^2+4) + \frac{\sqrt{\pi}}{2}(\eta^3+6\eta)\mathrm{erf}\Big(\frac{\eta}{2}\Big)\Big].$$

From the two conditions in (2.11b) at the free boundary, $b_1 = -0.9034465979$ and $A_2 = 0.07536083712\bar{k}$.

The solution for $Q_0(\eta)$ can be written as

$$Q_0(\eta) = e^{-\eta^2/4}[\alpha_0[0] + \alpha_0[2]\eta^2] + \mathrm{erf}\Big(\frac{\eta}{2}\Big)[\beta_0[1]\eta + \beta_0[3]\eta^3] + [\gamma_0[1]\eta + \gamma_0[3]\eta^3],$$

where the coefficients are listed in Appendix B.

- *Equating coefficients of* ν^1: From (2.8a), we find the governing PDE for $\psi(X, y)$ as

$$(\psi_1)_y = (\psi_1)_{XX} + (\bar{k}-1)(\psi_0)_X + \Big(\frac{\bar{k}X^2}{2} + \frac{2\bar{k}\mu}{\sigma^2}y\Big) \tag{2.13}$$

and, using the transformation $\psi_1 = y^2 Q_1(\eta)$, where $\eta = X/\sqrt{y}$, this PDE reduces to the ODE

$$Q_1''(\eta) + \frac{\eta}{2}Q_1'(\eta) - 2Q_1(\eta) + (\bar{k}-1)Q_0'(\eta) + \Big(\frac{\bar{k}\eta^2}{2} + \frac{2\bar{k}\mu}{\sigma^2}\Big) = 0 \tag{2.14a}$$

and, from (2.8b) and (2.10a)–(2.10b), the conditions are

$$Q_1(b_1) = 0, \tag{2.14b}$$

$$b_2 Q_0''(b_1) + Q_1'(b_1) = 0 \quad \text{and} \tag{2.14c}$$

$$\lim_{y\to 0} y^2 Q_1\Big(\frac{X}{\sqrt{y}}\Big) = 0. \tag{2.14d}$$

We write the solution to the equations (2.14a)–(2.14d) in the form

$$Q_1(\eta) = e^{-\eta^2/4}\sum_{i=0}^{3}\alpha_1[i]\eta^i + \mathrm{erf}\Big(\frac{\eta}{2}\Big)\sum_{i=0}^{4}\beta_1[i]\eta^i + \sum_{i=0}^{4}\gamma_1[i]\eta^i. \tag{2.15}$$

Substitution of this form into (2.14a) yields $\alpha_1[0] = \alpha_1[2] = 0, \beta_1[1] = \beta_1[3] = 0$, $\gamma_1[1] = \gamma_1[3] = 0$ as well as a set of six equations in eight unknowns, so that

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we can write the solutions for the coefficients in (2.15) in terms of two free parameters, $\alpha_1[3], \gamma_1[4]$. From the limit condition at $y = 0$, that is, from (2.14d), $\gamma_1[4] = -\sqrt{\pi}/2\alpha_1[3]$. Then, from (2.14b) and (2.14c),

$$b_2 = -0.2898271390299\bar{k} - \frac{1.42034572194\mu}{\sigma^2}$$

and the value of $\alpha_1[3]$. The nonzero coefficients of equation (2.15) are listed in Appendix B.

- *Equating coefficients of ν^2*: From (2.8a), we get the following governing PDE for $\psi_2(X, y)$:

$$(\psi_2)_y = (\psi_2)_{XX} + (\bar{k} - 1)(\psi_1)_X - \bar{k}\psi_0 + \frac{1}{3!}\left(\bar{k}X^3 + \frac{12\bar{k}\mu X}{\sigma^2}y\right).$$

Using the transformation $\psi_2 = y^{5/2}Q_2(\eta)$, where $\eta = X/\sqrt{y}$, the PDE reduces to the ODE

$$Q_2''(\eta) + \frac{\eta}{2}Q_2'(\eta) - \frac{5}{2}Q_2(\eta) + (\bar{k} - 1)Q_1'(\eta) - \bar{k}Q_0(\eta) + \left(\frac{\bar{k}\eta^3}{6} + \frac{2\bar{k}\mu}{\sigma^2}\eta\right) = 0. \tag{2.16a}$$

From (2.8b), (2.10a), (2.10b), (2.11b), (2.14b) and (2.14c), we get the conditions

$$Q_2(b_1) - \frac{b_2^2}{2}Q_0''(b_1) = 0, \tag{2.16b}$$

$$\frac{b_2^2}{2}Q_0'''(b_1) + b_3Q_0''(b_1) + b_2Q_1''(b_1) + Q_2'(b_1) = 0 \quad \text{and} \tag{2.16c}$$

$$\lim_{y\to 0} y^{5/2}Q_0\left(\frac{X}{\sqrt{y}}\right) = 0. \tag{2.16d}$$

We can write the solution to (2.16a)–(2.16d) in the form

$$Q_2(\eta) = e^{-\eta^2/4}\sum_{i=0}^{4}\alpha_2[i]\eta^i + \operatorname{erf}\left(\frac{\eta}{2}\right)\sum_{i=0}^{5}\beta_2[i]\eta^i + \sum_{i=0}^{5}\gamma_2[i]\eta^i. \tag{2.17}$$

Substitution of (2.17) into (2.16a) yields $\alpha_2[1] = \alpha_2[3] = 0$, $\beta_2[0] = \beta_2[2] = \beta_2[4] = 0$, $\gamma_2[1] = \gamma_2[3] = \gamma_2[4] = 0$, as well as a set of seven equations in nine unknowns, so that we have solutions for the coefficients in terms of two free parameters, $\alpha_2[4], \gamma_2[5]$. From (2.16d), we have $\gamma_2[5] = -\sqrt{\pi}/2\alpha_2[4]$. Then, from (2.16b) and (2.16c),

$$\begin{aligned} b_3 = {} & \frac{0.118637288618\bar{k}\mu}{\sigma^2} + 0.019602516263 + 0.083527050341\bar{k} \\ & - 0.02965932215\bar{k}^2 - \frac{0.1186372886179\mu^2}{\sigma^4} + \frac{0.167054100682\mu}{\sigma^2} \end{aligned}$$

and the value of $\alpha_2[4]$. The nonzero coefficients in (2.17) are listed in Appendix B.

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- *Equating coefficients of* v^3: From (2.8a), we get the following governing PDE for $\psi_3(X, y)$:

$$(\psi_3)_y = (\psi_3)_{XX} + (\bar{k} - 1)(\psi_2)_X - \bar{k}\psi_1 + \frac{1}{4!}\Big(\bar{k}X^4 + \frac{24\bar{k}\mu X^2}{\sigma^2}y + \frac{48\bar{k}\mu^2}{\sigma^4}y^2\Big).$$

Using the transformation $\psi_3 = y^3 Q_3(\eta)$, with $\eta = X/\sqrt{y}$, reduces the PDE to the ODE

$$Q_3''(\eta) + \frac{\eta}{2}Q_3'(\eta) - 3Q_3(\eta) + (\bar{k} - 1)Q_2'(\eta) - \bar{k}Q_1(\eta) + \Big(\frac{\bar{k}\eta^4}{4!} + \frac{\bar{k}\mu}{\sigma^2}\eta^2 + \frac{2\bar{k}\mu^2}{\sigma^4}\Big) = 0. \tag{2.18a}$$

From (2.8b), (2.10a), (2.10b) and (2.14c), we get the conditions

$$\text{at } \eta = b_1, \quad Q_2' b_2 + \frac{Q_0'''}{6}b_2^3 + Q_3 + \frac{Q_1''}{2}b_2^2 = 0, \tag{2.18b}$$

$$b_2 Q_2'' + b_4 Q_0'' + \frac{b_2^3}{6}Q_0'''' + b_2 b_3 Q_0''' + Q_3' + \frac{Q_1'''}{2}b_2^2 + Q_1'' b_3 = 0 \tag{2.18c}$$

$$\text{and} \quad \lim_{y\to 0} y^3 Q_3\Big(\frac{X}{\sqrt{y}}\Big) = 0. \tag{2.18d}$$

We can write the solution to (2.18a)–(2.18d) in the form

$$Q_3(\eta) = e^{-\eta^2/4}\sum_{i=0}^{5}\alpha_3[i]\eta^i + \operatorname{erf}\Big(\frac{\eta}{2}\Big)\sum_{i=0}^{6}\beta_3[i]\eta^i + \sum_{i=0}^{6}\gamma_3[i]\eta^i. \tag{2.19}$$

Substitution of (2.19) into (2.18a) yields $\alpha_3[0] = \alpha_3[2] = \alpha_3[4] = 0$, $\beta_3[1] = \beta_3[3] = \beta_3[5] = 0$, $\gamma_3[1] = \gamma_3[3] = \gamma_3[5] = 0$, as well as a set of nine equations in 11 unknowns, so that we have solutions for the coefficients in terms of two free parameters, $\alpha_3[5], \gamma_3[6]$. The limit condition at $y = 0$, that is, equation (2.18d), yields $\gamma_3[6] = -\sqrt{\pi}/2\alpha_3[5]$. From (2.18b) and (2.18c), we get b_4 (which we list in Appendix A) and the value of $\alpha_3[5]$. The nonzero coefficients in (2.19) are listed in Appendix B.

- *Equating coefficients of* v^4: From (2.8a), we get the following governing PDE for $\psi_4(X, y)$:

$$(\psi_4)_y = (\psi_4)_{XX} + (\bar{k} - 1)(\psi_3)_X - \bar{k}\psi_2 + \frac{1}{5!}\Big(\bar{k}X^5 + \frac{40\bar{k}\mu X^3}{\sigma^2}y + \frac{240\bar{k}\mu^2 X}{\sigma^4}y^2\Big).$$

Using the transformation $\psi_4 = y^{7/2}Q_4(\eta)$, with $\eta = X/\sqrt{y}$, reduces the PDE to the ODE

$$Q_4''(\eta) + \frac{\eta}{2}Q_4'(\eta) - \frac{7}{2}Q_4(\eta) + (\bar{k} - 1)Q_3'(\eta) - \bar{k}Q_2(\eta) + \Big(\frac{\bar{k}\eta^5}{5!} + \frac{\bar{k}\mu}{3\sigma^2}\eta^3 + \frac{2\bar{k}\mu^2}{\sigma^4}\eta\Big) = 0. \tag{2.20a}$$

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From (2.8b) and on equating coefficients of ν^4 in (2.10a) and (2.10b), we get the conditions

$$\text{at } \eta = b_1, \quad b_2 b_4 Q_0'' + \frac{b_2^2 b_3}{2} Q_0''' + \frac{Q_0''}{2} b_3^2 + \frac{Q_0''''}{24} b_2^4 + b_2 Q_3' + Q_2'' \frac{b_2^2}{2} + b_3 Q_2' + b_4 Q_1' + \frac{b_2^3}{6} Q_1''' + b_2 b_3 Q_1'' + Q_4 = 0, \tag{2.20b}$$

$$b_2 b_4 Q_0''' + \frac{b_2^2 b_3}{2} Q_0'''' + \frac{b_3^2}{2} Q_0''' + \frac{b_2^4}{24} Q_0''''' + b_2 Q_3'' + \frac{b_2^2}{2} Q_2''' + Q_2'' b_3 + Q_1'' b_4 + \frac{b_2^3}{6} Q_1'''' + b_2 b_3 Q_1''' + Q_4' + b_5 Q_0'' = 0 \tag{2.20c}$$

and

$$\lim_{y \to 0} y^{7/2} Q_4\Big(\frac{X}{\sqrt{y}}\Big) = 0. \tag{2.20d}$$

We write the solution to (2.20a)–(2.20d) in the form

$$Q_4(\eta) = e^{-\eta^2/4} \sum_{i=0}^{6} \alpha_4[i]\eta^i + \operatorname{erf}\Big(\frac{\eta}{2}\Big) \sum_{i=0}^{7} \beta_4[i]\eta^i + \sum_{i=0}^{7} \gamma_4[i]\eta^i. \tag{2.21}$$

Substitution of (2.21) into (2.20a) yields

$$\alpha_4[1] = \alpha_4[3] = \alpha_4[5] = 0, \quad \beta_4[0] = \beta_4[2] = \beta_4[4] = \beta_4[6] = 0,$$
$$\gamma_4[0] = \gamma_4[2] = \gamma_4[4] = \gamma_4[6] = 0,$$

as well as a set of 10 equations in 12 unknowns, so that we have solutions for the coefficients in terms of two free parameters, $\alpha_4[6], \gamma_4[7]$. The limit condition at $y = 0$, (2.20d), yields $\gamma_4[7] = -\sqrt{\pi}/2\alpha_4[6]$. Then, from (2.20b) and (2.20c), we get b_5 (which we list in Appendix A) and the value of $\alpha_4[6]$. The nonzero coefficients in (2.21) are listed in Appendix B.

Undoing the change of variables, we get the optimal exercise boundary as given in (2.1) and the option valuation (2.2a)–(2.2c), giving the required result. □

Sample plots of the optimal exercise boundary for various parameter values are given in Figure 1. As shown, at expiry, the boundary value is Kr/μ, from which it decreases slowly with time to expiry.

Since American options are well known, it is interesting to compare valuations for American and British options. A comparison of American and British optimal exercise boundaries with $r = 0.04, \mu = 0.05, \sigma = 0.3, K = 1$ is plotted in Figure 2. The American boundary was computed using the analytic approximation with dividend yield $q = 0$ given by Zhang and Li [9], which is valid for short times to expiry. When $r > q$, at expiry the American optimal exercise boundary is at the exercise price K. For small times to expiry, its value remains above that of the British optimal exercise boundary, although the difference between the two narrows with increasing time to expiry.

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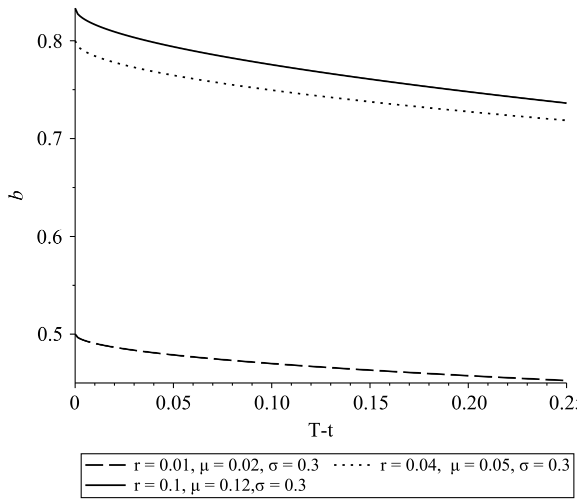

FIGURE 1. Optimal exercise boundary $b(t)$ using equation (2.1) with $K = 1$.

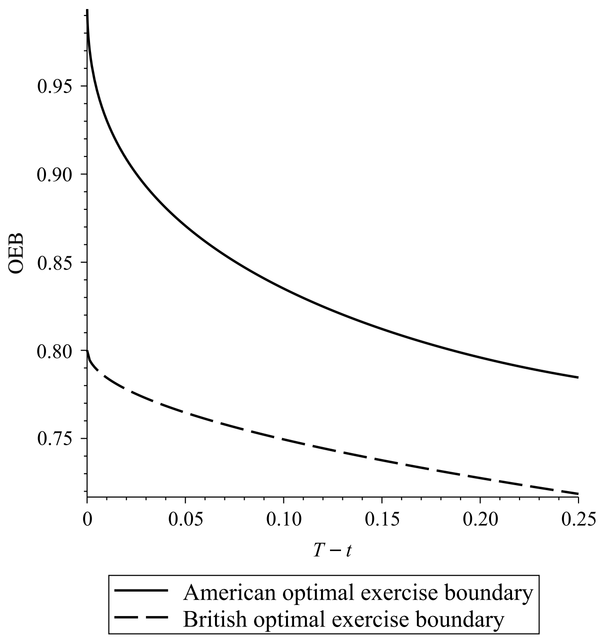

FIGURE 2. Comparison of American and British optimal exercise boundaries (OEBs). Parameters used: $r = 0.04, \mu = 0.05, K = 1, \sigma = 0.3$.

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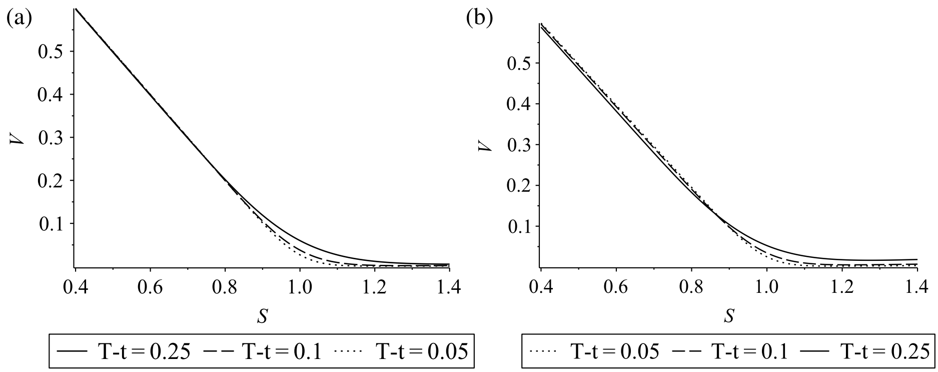

FIGURE 3. British put option curves using (a) $r = 0.01, \mu = 0.02, \sigma = 0.3, K = 1$ and (b) $r = 0.1, \mu = 0.12, \sigma = 0.3, K = 1$.

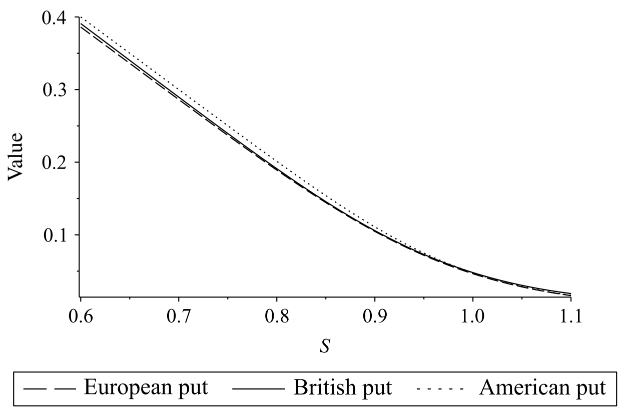

FIGURE 4. Comparison of European, British and American option curves with $r = 0.07, \mu = 0.08, \sigma = 0.3, K = 1, T - t = 0.2$.

In Figure 3, we use (2.2a)–(2.2c) to plot British put option curves for various parameter values. As with European put options, for small stock prices the British put option values decrease slightly with expiry time but, at a certain stock price which is less than the exercise price, there is a cross-over and the longer-dated options become more expensive.

In Figure 4, we use the integral representation of the American put value given by Zhang and Li [9], the Black–Scholes formula [2] for a European put and our approximation (2.2a)–(2.2c) to the British put to compare their prices using parameter values $r = 0.07, \mu = 0.075, \sigma = 0.3, K = 1$ and time to expiry 0.2. As expected, the British put option has a value between the European and American values – hence the name 'British' as given by Peskir and Samee [7].

TABLE 1. Errors in equation (3.1) using formula (2.1).

$T-t$	$r = 0.01, \mu = 0.02$	$r = 0.1, \mu = 0.12$	$r = 0.04, \mu = 0.05$
0.001	2.33×10^{-8}	2.569×10^{-7}	9.65×10^{-8}
0.025	1.56×10^{-7}	5.53×10^{-6}	1.23×10^{-6}
0.05	3.16×10^{-7}	1.21×10^{-5}	2.69×10^{-6}
0.1	7.08×10^{-7}	3.31×10^{-5}	6.97×10^{-6}
0.15	1.17×10^{-6}	7.39×10^{-5}	1.61×10^{-5}
0.2	1.67×10^{-6}	1.36×10^{-4}	3.20×10^{-5}
0.25	2.20×10^{-6}	2.16×10^{-4}	5.52×10^{-5}

3. Empirical results

3.1. The optimal exercise boundary In order to check the accuracy of our optimal exercise boundary (2.1), using (2.1) we compute values of $b(t)$ for $T = 1$ and $t = 0.999$ to 0.75 in steps of -0.001 and find the errors in the calculation of (1.4), that is,

$$\text{error} = F(b(t), t) - \int_t^T J(b(t), t, v, b(v))\, dv. \tag{3.1}$$

The errors are naturally cumulative, as computation of the error at time t relies on estimates of b at times t to T. Other parameters used are $\sigma = 0.3, K = 1$. As can be seen from Table 1, the errors are very small, even when the time to expiry is 0.25. This shows that equation (2.1) provides good approximations to the optimal exercise boundary for small times to expiry.

3.2. The British put option valuations Here, we test the accuracy of our solution (2.2a)–(2.2c) using various parameter values. We calculate signed percentage errors, that is,

$$\frac{(V_{\text{exact}} - V_{\text{est}})}{V_{\text{exact}}} \times 100\%,$$

where V_{est} is our analytic approximation (2.2a)–(2.2c) and V_{exact} is the exact solution given by Peskir and Samee [7], namely (1.3a), where we used our optimal boundary values. Other parameter values used were $K = 1, \sigma = 0.3$. The results are listed in Table 2.

From Table 2, we see that the percentage errors are very small for S values up to the strike price. The best approximations are, as expected, for the smaller time to expiry and for S values near Kr/μ. While the percentage errors may look large for the out-of-the-money options when $S = 1.1$, the absolute errors are quite small as the actual option values themselves are very small. For example, the relative error -13.09% for $r = 0.01, \mu = 0.02, T - t = 0.1$ corresponds to an absolute error of 1.05×10^{-3} and the relative error -46.98% for $r = 0.1, \mu = 0.12, T - t = 0.1$ corresponds to an absolute error of 3.13×10^{-3}. We also found that the percentage errors with the new approximation formula did not change very much with different values of σ when the other parameter values were held constant. This again suggests that the

TABLE 2. Signed percentage errors in equations (2.2a)–(2.2c).

S	$r = 0.01, \mu = 0.02$, $Kr/\mu = 0.5$: $T - t = 0.1$ (%)	$r = 0.01, \mu = 0.02$, $Kr/\mu = 0.5$: $T - t = 0.2$ (%)	$r = 0.1, \mu = 0.12$, $Kr/\mu = 0.8\dot{3}$: $T - t = 0.1$ (%)	$r = 0.1, \mu = 0.12$, $Kr/\mu = 0.8\dot{3}$: $T - t = 0.2$ (%)	$r = 0.04, \mu = 0.05$, $Kr/\mu = 0.8$: $T - t = 0.1$ (%)	$r = 0.04, \mu = 0.05$, $Kr/\mu = 0.8$: $T - t = 0.2$ (%)
0.5	(-6.01×10^{-8})	(-8.53×10^{-5})	-5.65	-0.13	-1.85	-2.38
0.6	(-2.75×10^{-7})	(-5.97×10^{-6})	-1.96	-0.12	-0.54	-0.48
0.7	(-8.96×10^{-6})	(-2.49×10^{-3})	-0.34	-0.08	-0.06	-0.03
0.8	(-4.57×10^{-3})	-0.05	-0.02	-0.15	-0.01	2.35
0.9	-0.13	-0.41	-0.4	-2.67	-0.16	-0.56
1.0	-1.51	-2.26	-4.78	-8.84	-1.95	-3.19
1.1	-13.09	-10.31	-46.98	-17.06	-18.55	-15.72

analytic approximation in (2.2a)–(2.2c) provides relatively accurate results for British put options with short tenor.

4. Conclusion

Peskir and Samee [7] introduced the British put option, which offers the put option holder protection when the holder believes that the actual drift of the underlying asset is greater than the risk-free interest rate. They derived a nonlinear integral equation for the optimal exercise boundary and a double integral, an early exercise premium representation for the value of the option. In this paper, we have derived a simple analytical formula for the optimal exercise boundary which has been shown to generate fast and accurate answers for options with expiry times up to 0.2 years. We note that options with short tenor are extremely popular in the market. Further, we have derived a simple formula for the option valuation, independent of the exercise boundary, which again has been shown to be accurate for options with short tenor. Both formulas found only involve sums of standard functions and are easily implemented in mathematical computer packages.

Appendix A

In this appendix we list the first five coefficients b_i in the optimal exercise boundary (2.1) in terms of $\bar{k} = 2r/\sigma^2, \mu$ and σ.

$$
\begin{aligned}
b_1 &= -0.9034465979,\\
b_2 &= -0.2898271390\bar{k} - \frac{1.4203457219\mu}{\sigma^2},\\
b_3 &= \frac{0.1186372886\bar{k}\mu}{\sigma^2} + 0.0196025162 + 0.0835270503\bar{k} - 0.0296593222\bar{k}^2\\
&\quad - \frac{0.1186372886\mu^2}{\sigma^4} + \frac{0.1670541007\mu}{\sigma^2},
\end{aligned}
$$

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$$
\begin{aligned}
b_4 &= -\frac{0.0168388832\mu^3}{\sigma^6} + 0.0021048604\bar{k}^3 + 0.0313406909\bar{k}^2 - \frac{0.0083468988\mu}{\sigma^2} \\
&\quad - \frac{0.1253627636\mu^2}{\sigma^4} + \frac{0.0252583248\bar{k}\mu^2}{\sigma^4} - \frac{0.0126291624\bar{k}^2\mu}{\sigma^2} + 0.0041734494\bar{k}, \\
b_5 &= -\frac{0.0044941011\bar{k}\mu^3}{\sigma^6} + \frac{0.0033705758\bar{k}^2\mu^2}{\sigma^4} - \frac{0.0011235253\bar{k}^3\mu}{\sigma^2} \\
&\quad - \frac{0.00602187120\bar{k}\mu^2}{\sigma^4} - \frac{0.0030109356\bar{k}^2\mu}{\sigma^2} - \frac{0.0159250152\bar{k}\mu}{\sigma^2} \\
&\quad + \frac{0.0022470506\mu^4}{\sigma^8} + 0.0001404407\bar{k}^4 - 0.0057207135\bar{k}^2 + 0.00150546782\bar{k}^3 \\
&\quad - 0.003558808\bar{k} + \frac{0.0120437424\mu^3}{\sigma^6} - \frac{0.0228828542\mu^2}{\sigma^4} \\
&\quad - \frac{0.0071176157\mu}{\sigma^2} - 0.0007732477.
\end{aligned}
$$

Appendix B

Here, we list the nonzero coefficients of the function $Q_j(\eta)$, $j = 0, \ldots, 4$, in (2.2c) in terms of $\bar{k} = 2r/\sigma^2, \mu$ and σ.

$$
\begin{aligned}
\alpha_0[0] &= 0.3014433484\bar{k}, \\
\alpha_0[2] &= 0.0753608371\bar{k}, \\
\beta_0[1] &= 0.4007208178\bar{k}, \\
\beta_0[3] &= 0.0667868030\bar{k}, \\
\gamma_0[1] &= 0.5992791822\bar{k}, \\
\gamma_0[3] &= -0.0667868030\bar{k}, \\
\alpha_1[1] &= 0.0922844581\bar{k}^2 + 0.1507216742\bar{k} + \frac{0.2675961065\bar{k}\mu}{\sigma^2}, \\
\alpha_1[3] &= -0.0133798053\bar{k}^2 + 0.0376804186\bar{k} + \frac{0.0267596107\bar{k}\mu}{\sigma^2}, \\
\beta_1[0] &= 0.2584302930\bar{k}^2 + \frac{0.2845810497\bar{k}\mu}{\sigma^2}, \\
\beta_1[2] &= 0.0580698841\bar{k}^2 + 0.2003604089\bar{k} + \frac{0.2845810497\bar{k}\mu}{\sigma^2}, \\
\beta_1[4] &= -0.0118575437\bar{k}^2 + 0.0333934015\bar{k} + \frac{0.0237150875\bar{k}\mu}{\sigma^2}, \\
\gamma_1[0] &= 0.2415697070\bar{k}^2 + \frac{0.7154189503\bar{k}\mu}{\sigma^2}, \\
\gamma_1[2] &= -0.0580698841\bar{k}^2 + 0.2996395911\bar{k} - \frac{0.2845810497\bar{k}\mu}{\sigma^2},
\end{aligned}
$$

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$$\gamma_1[4] = 0.0118575437\bar{k}^2 - 0.0333934015\bar{k} - \frac{0.0237150875\bar{k}\mu}{\sigma^2},$$

$$\alpha_2[0] = 0.0682292065\bar{k}^3 + \frac{0.2032594488\bar{k}^2\mu}{\sigma^2} + \frac{0.2248943216\bar{k}^2\mu}{\sigma^4} - 0.0120391558\bar{k} - 0.09498252197\bar{k}^2 + \frac{0.1114783045\bar{k}\mu}{\sigma^2},$$

$$\alpha_2[2] = -0.0218934573\bar{k}^3 - \frac{0.0194646133\bar{k}^2\mu}{\sigma^2} + \frac{0.1265030559\bar{k}\mu^2}{\sigma^4} + 0.0544586550\bar{k} + 0.0398150836\bar{k}^2 + \frac{0.1965045995\bar{k}\mu}{\sigma^2},$$

$$\alpha_2[4] = 0.0017569869\bar{k}^3 - \frac{0.0070279475\bar{k}^2\mu}{\sigma^2} + \frac{0.0070279476\bar{k}\mu^2}{\sigma^4} + 0.0113989072\bar{k} - 0.0049480542\bar{k}^2 + \frac{0.0168635023\bar{k}\mu}{\sigma^2},$$

$$\beta_2[1] = 0.0214751205\bar{k}^2 + 0.0092047045\bar{k}^3 + \frac{0.1954607184\bar{k}^2\mu}{\sigma^2} + 0.0050398811\bar{k} + \frac{0.3275312906\bar{k}\mu}{\sigma^2} + \frac{0.373701381\bar{k}\mu^2}{\sigma^4},$$

$$\beta_2[3] = 0.0265150015\bar{k}^2 - 0.0162883932\bar{k}^3 - \frac{0.0297067771\bar{k}^2\mu}{\sigma^2} + 0.0684667634\bar{k} + \frac{0.2040374468\bar{k}\mu}{\sigma^2} + \frac{0.124567127\bar{k}\mu^2}{\sigma^4},$$

$$\beta_2[5] = 0.0015570891\bar{k}^3 - \frac{0.0062283564\bar{k}^2\mu}{\sigma^2} + \frac{0.0062283564\bar{k}\mu^2}{\sigma^4} + 0.0101020185\bar{k} - 0.0043850988\bar{k}^2 + \frac{0.0149448898\bar{k}\mu}{\sigma^2},$$

$$\gamma_2[1] = -0.0050398811\bar{k} - 0.0214751205\bar{k}^2 - 0.0092047045\bar{k}^3 + \frac{0.6724687094\bar{k}\mu}{\sigma^2} - \frac{0.1954607184\bar{k}^2\mu}{\sigma^2} - 0.373701381\bar{k}^2\mu^2,$$

$$\gamma_2[3] = 0.0981999033\bar{k} - 0.0265150015\bar{k}^2 + 0.0162883932\bar{k}^3 - \frac{0.2040374468\bar{k}\mu}{\sigma^2} + \frac{0.0297067771\bar{k}^2\mu}{\sigma^2} - \frac{0.124567127\bar{k}\mu^2}{\sigma^4},$$

$$\gamma_2[5] = -0.0015570899\bar{k}^3 + \frac{0.0062283564\bar{k}^2\mu}{\sigma^2} - \frac{0.0062283564\bar{k}\mu^2}{\sigma^4} - 0.0101020185\bar{k} + 0.0043850988\bar{k}^2 - \frac{0.0149448898\bar{k}\mu}{\sigma^2},$$

$$\alpha_3[1] = -0.0060195779\bar{k} + \frac{0.1923132585\bar{k}\mu^3}{\sigma^6} + \frac{0.0902730697\bar{k}\mu}{\sigma^2} + \frac{0.142016522\bar{k}^2\mu}{\sigma^2}$$

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$$
\begin{aligned}
&- \frac{0.0466047697\bar{k}^3\mu}{\sigma^2} + \frac{0.0629274898\bar{k}^2\mu^2}{\sigma^4} - 0.0164686449\bar{k}^4 \\
&- 0.0223387205\bar{k}^2 + \frac{0.3493774826\bar{k}\mu^2}{\sigma^4} + 0.0158322102\bar{k}^3, \\
\alpha_3[3] = {}& 0.014669188\bar{k} + \frac{0.0407937215\bar{k}\mu^3}{\sigma^6} + \frac{0.0796289629\bar{k}\mu}{\sigma^2} - \frac{0.0056936269\bar{k}^2\mu}{\sigma^2} \\
&- \frac{0.0045444466\bar{k}^3\mu}{\sigma^2} - \frac{0.0260508445\bar{k}^2\mu^2}{\sigma^4} + 0.0036857193\bar{k}^4 \\
&+ 0.0108529091\bar{k}^2 + \frac{0.113509475\bar{k}\mu^2}{\sigma^4} - 0.0081120702\bar{k}^3, \\
\alpha_3[5] = {}& 0.0025594187\bar{k} + \frac{0.0014569186\bar{k}\mu^3}{\sigma^6} + \frac{0.0058552312\bar{k}\mu}{\sigma^2} - \frac{0.0035139738\bar{k}^2\mu}{\sigma^2} \\
&+ \frac{0.001092689\bar{k}^3\mu}{\sigma^2} - \frac{0.0021853779\bar{k}^2\mu^2}{\sigma^4} - 0.0001821148\bar{k}^4 \\
&- 0.0011857671\bar{k}^2 + \frac{0.0053089005\bar{k}\mu^2}{\sigma^4} + 0.0004297618\bar{k}^3, \\
\beta_3[0] = {}& \frac{0.0270841149\bar{k}^3\mu}{\sigma^2} + \frac{0.0429502409\bar{k}^2\mu}{\sigma^2} + \frac{0.0105752886\bar{k}\mu}{\sigma^2} \\
&- 0.0843161342\bar{k}^3 - 0.0014457843\bar{k}^4 - 0.0002477633\bar{k}^2 \\
&+ \frac{0.1908854824\bar{k}\mu^2}{\sigma^4} + \frac{0.1412924885\bar{k}^2\mu^2}{\sigma^4} + \frac{0.1549392616\bar{k}\mu^3}{\sigma^6}, \\
\beta_3[2] = {}& -\frac{0.0571041868\bar{k}^3\mu}{\sigma^2} + \frac{0.1406806001\bar{k}^2\mu}{\sigma^2} + \frac{0.1796285782\bar{k}\mu}{\sigma^2} \\
&+ 0.0025199405\bar{k} - 0.0033942628\bar{k}^3 - 0.0067710287\bar{k}^4 + 0.0078459748\bar{k}^3 \\
&+ \frac{0.4731789140\bar{k}\mu^2}{\sigma^4} + \frac{0.0250880423\bar{k}^2\mu^2}{\sigma^4} + \frac{0.2324088924\bar{k}\mu^3}{\sigma^6}, \\
\beta_3[4] = {}& -\frac{0.0020906702\bar{k}^3\mu}{\sigma^2} - \frac{0.01127420180\bar{k}^2\mu}{\sigma^2} + \frac{0.080947451\bar{k}\mu}{\sigma^2} \\
&+ 0.0175366809\bar{k} - 0.0064274021\bar{k}^3 + 0.0029435935\bar{k}^4 + 0.0075164228\bar{k}^2 \\
&+ \frac{0.1100049341\bar{k}\mu^2}{\sigma^4} - \frac{0.0269604414\bar{k}^2\mu^2}{\sigma^4} + \frac{0.0387348154\bar{k}\mu^3}{\sigma^6}, \\
\beta_3[6] = {}& 0.0022682258\bar{k} + \frac{0.0012911605\bar{k}\mu^3}{\sigma^6} + \frac{0.0051890636\bar{k}\mu}{\sigma^2} - \frac{0.0031141782\bar{k}^2\mu}{\sigma^2} \\
&+ \frac{0.0009683704\bar{k}^3\mu}{\sigma^2} - \frac{0.0019367408\bar{k}^2\mu^2}{\sigma^4} - 0.0001613951\bar{k}^4 \\
&- 0.0010508587\bar{k}^2 + \frac{0.0047048903\bar{k}\mu^2}{\sigma^4} + 0.0003808665\bar{k}^3, \\
\gamma_3[0] = {}& -\frac{0.1549392616\bar{k}\mu^3}{\sigma^6} - \frac{0.0105752886\bar{k}\mu}{\sigma^2} - \frac{0.0429502409\bar{k}^2\mu}{\sigma^2}
\end{aligned}
$$

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$$- \frac{0.0270841149\bar{k}^3\mu}{\sigma^2} - \frac{0.1412924885\bar{k}^2\mu^2}{\sigma^4} + 0.0014457843\bar{k}^4$$
$$+ 0.0002477633\bar{k}^2 + \frac{0.4757811843\bar{k}\mu^2}{\sigma^4} - 0.0823505325\bar{k}^3,$$
$$\gamma_3[2] = -0.0025199405\bar{k} - \frac{0.2324088925\bar{k}\mu^3}{\sigma^6} + \frac{0.3203714218\bar{k}\mu}{\sigma^2}$$
$$- \frac{0.1406806001\bar{k}^2\mu}{\sigma^2} + \frac{0.0571041868\bar{k}^3\mu}{\sigma^2} - \frac{0.0250880423\bar{k}^2\mu^2}{\sigma^4}$$
$$+ 0.0067710287\bar{k}^4 - 0.0078459748\bar{k}^2 - \frac{0.4731789140\bar{k}\mu^2}{\sigma^4} + 0.0033942628\bar{k}^3,$$
$$\gamma_3[4] = 0.0241299858\bar{k} - \frac{0.0387348154\bar{k}\mu^3}{\sigma^6} - \frac{0.0809474581\bar{k}\mu}{\sigma^2} + \frac{0.0020906702\bar{k}^3\mu}{\sigma^2}$$
$$+ \frac{0.0269604414\bar{k}^2\mu^2}{\sigma^4} - 0.0029435935\bar{k}^4 - 0.0075164228\bar{k}^2$$
$$- \frac{0.1100049341\bar{k}\mu^2}{\sigma^4} + 0.0064274021\bar{k}^3 + \frac{0.0112742018\bar{k}^2\mu}{\sigma^2},$$
$$\gamma_3[6] = -0.0022682258\bar{k} - \frac{0.0012911605\bar{k}\mu^3}{\sigma^6} - \frac{0.0051890636\bar{k}\mu}{\sigma^2}$$
$$+ \frac{0.0031141782\bar{k}^2\mu}{\sigma^2} - \frac{0.0009683704\bar{k}^3\mu}{\sigma^2} + \frac{0.0019367408\bar{k}^2\mu^2}{\sigma^4}$$
$$+ 0.0001613951\bar{k}^4 + 0.0010508587\bar{k}^2 - \frac{0.0047048905\bar{k}\mu^2}{\sigma^4} - 0.0003808665\bar{k}^3,$$
$$\alpha_4[0] = 0.0005805601\bar{k} + \frac{0.2156789654\bar{k}\mu^3}{\sigma^6} - \frac{0.0026624681\bar{k}\mu}{\sigma^2} - \frac{0.0162015851\bar{k}^3\mu}{\sigma^2}$$
$$+ \frac{0.1243392791\bar{k}^2\mu^2}{\sigma^4} - 0.0236372729\bar{k}^4 + 0.0046883439\bar{k}^2$$
$$+ \frac{0.0956713641\bar{k}\mu^4}{\sigma^8} + 0.0264278432\bar{k}^3 - \frac{0.0095035228\bar{k}^2\mu}{\sigma^2}$$
$$+ \frac{0.0569428780\bar{k}\mu^2}{\sigma^4} + \frac{0.0883856478\bar{k}^2\mu^3}{\sigma^6} - \frac{0.0121188595\bar{k}^4\mu}{\sigma 2}$$
$$+ \frac{0.0050323842\bar{k}^3\mu}{\sigma^4} - 0.0028062279\bar{k}^5,$$
$$\alpha_4[2] = -0.0017189986\bar{k} + \frac{0.2916156917\bar{k}\mu^3}{\sigma^6} + \frac{0.0383690519\bar{k}\mu}{\sigma^2}$$
$$- \frac{0.0275978736\bar{k}^3\mu}{\sigma^2} + \frac{0.0643347672\bar{k}^2\mu^2}{\sigma^4} - 0.0038997463\bar{k}^4$$
$$- 0.0041020107\bar{k}^2 + \frac{0.0867021737\bar{k}\mu^4}{\sigma^8} + 0.0025362970\bar{k}^3$$
$$+ \frac{0.0563492461\bar{k}^2\mu}{\sigma^2} + \frac{0.2175082900\bar{k}\mu^2}{\sigma^4} - \frac{0.0160571359\bar{k}^2\mu^3}{\sigma^6}$$

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$$+\frac{0.0043798463\bar{k}^4\mu}{\sigma^2}-\frac{0.0356880812\bar{k}^3\mu^2}{\sigma^4}+0.0033203533\bar{k}^5,$$

$$\alpha_4[4]=0.0031158039\bar{k}+\frac{0.0428634196\bar{k}\mu^3}{\sigma^6}+\frac{0.0225811415\bar{k}\mu}{\sigma^2}-\frac{0.0043748893\bar{k}^3\mu}{\sigma^2}$$
$$-\frac{0.0170031154\bar{k}^2\mu^2}{\sigma^4}+0.0010802960\bar{k}^4+0.0022601688\bar{k}^2$$
$$+\frac{0.0099657671\bar{k}\mu^4}{\sigma^8}-0.0019877749\bar{k}^3-\frac{0.0010472117\bar{k}^2\mu}{\sigma^2}$$
$$+\frac{0.0509730414\bar{k}\mu^2}{\sigma^4}-\frac{0.0111900224\bar{k}^2\mu^3}{\sigma^6}+\frac{0.0015732503\bar{k}^4\mu}{\sigma^2}$$
$$+\frac{0.0018363830\bar{k}^3\mu^2}{\sigma^4}-0.0004698285\bar{k}^5,$$

$$\alpha_4[6]=\frac{0.0002491442\bar{k}\mu^4}{\sigma^8}+0.00001558\bar{k}^5+0.0000689336\bar{k}^3-0.0000208494\bar{k}^4$$
$$-0.0002203\bar{k}^2-\frac{0.0004982884\bar{k}^2\mu^3}{\sigma^6}+\frac{0.0003737163\bar{k}^3\mu^2}{\sigma^4}$$
$$-\frac{0.0001245721\bar{k}^4\mu}{\sigma^2}-\frac{0.00173520955\bar{k}^2\mu^2}{\sigma^4}+\frac{0.0004059285\bar{k}^3\mu}{\sigma^2}$$
$$-\frac{0.0010099735\bar{k}^2\mu}{\sigma^2}+\frac{0.0012901233\bar{k}\mu^3}{\sigma^6}+\frac{0.0020706611\bar{k}\mu^2}{\sigma^4}$$
$$+\frac{0.0014428132\bar{k}\mu}{\sigma^2}+0.0004618079\bar{k},$$

$$\beta_4[1]=-0.0002298344\bar{k}+\frac{0.4955894885\bar{k}\mu^3}{\sigma^6}+\frac{0.0078444912\bar{k}\mu}{\sigma^2}$$
$$-\frac{0.0034775004\bar{k}^3\mu^2}{\sigma^2}+\frac{0.247392072\bar{k}^2\mu^2}{\sigma^4}-0.0036039911\bar{k}^4$$
$$-0.0013653983\bar{k}^2+\frac{0.1847055375\bar{k}\mu^4}{\sigma^8}-0.003193344\bar{k}^3$$
$$+\frac{0.02727344243\bar{k}^2\mu}{\sigma^2}+\frac{0.2214043139\bar{k}\mu 2}{\sigma^4}+\frac{0.0938766774\bar{k}^2\mu^3}{\sigma^6}$$
$$-\frac{0.0229629693\bar{k}^4\mu}{\sigma^2}-\frac{0.0453194658\bar{k}^3\mu^2}{\sigma^4}-0.0005151432\bar{k}^5,$$

$$\beta_4[3]=0.000725063\bar{k}+\frac{0.3252643751\bar{k}\mu^3}{\sigma^6}+\frac{0.0637984387\bar{k}\mu}{\sigma^2}-\frac{0.03509022609\bar{k}^3\mu}{\sigma^2}$$
$$+\frac{0.0366159756\bar{k}^2\mu^2}{\sigma^4}-0.0013934668\bar{k}^4+0.0019326258\bar{k}^2$$
$$+\frac{0.0092735277\bar{k}\mu^4}{\sigma^8}-0.00176423111\bar{k}^3+\frac{0.0552426103\bar{k}^2\mu}{\sigma^2}$$
$$+\frac{0.2684284616\bar{k}\mu^2}{\sigma^4}-\frac{0.0305312921\bar{k}^2\mu^3}{\sigma^6}+\frac{0.0075532443\bar{k}^4\mu}{\sigma^2}$$

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$$- \frac{0.0310224137\bar{k}^3\mu^2}{\sigma^4} + 0.001999438\bar{k}^5,$$

$$\beta_4[5] = 0.0035798425\bar{k} + \frac{0.0402734006\bar{k}\mu^3}{\sigma^6} + \frac{0.0225693355\bar{k}\mu}{\sigma^2} - \frac{0.003157655\bar{k}^3\mu}{\sigma^2}$$
$$- \frac{0.0175031212\bar{k}^2\mu^2}{\sigma^4} + 0.0009204328\bar{k}^4 + 0.0016125491\bar{k}^2$$
$$+ \frac{0.0092735277\bar{k}\mu^4}{\sigma^8} - 0.0016394380157\bar{k}^3 - \frac{0.0027181986\bar{k}^2\mu}{\sigma^2}$$
$$+ \frac{0.048843833\bar{k}\mu^2}{\sigma^4} - \frac{0.010800093\bar{k}^2\mu^3}{\sigma^6} + \frac{0.0011734585\bar{k}^4\mu}{\sigma^2}$$
$$+ \frac{0.0022898469\bar{k}^3\mu^2}{\sigma^4} - 0.0003877491\bar{k}^5,$$

$$\beta_4[7] = 0.0004092666\bar{k} + \frac{0.001143342\bar{k}\mu^3}{\sigma^6} + \frac{0.0012786599\bar{k}\mu}{\sigma^2} + \frac{0.0003597448\bar{k}^3\mu}{\sigma^2}$$
$$- \frac{0.0012172513\bar{k}^2\mu^2}{\sigma^4} - 0.0000184773\bar{k}^4 - 0.0001952367\bar{k}^2$$
$$+ \frac{0.0002207983\bar{k}\mu^4}{\sigma^8} + 0.0000610908\bar{k}^3 - \frac{0.0008950657\bar{k}^2\mu}{\sigma^2}$$
$$+ \frac{0.0018350756\bar{k}\mu^2}{\sigma^4} - \frac{0.0004415966\bar{k}^2\mu^3}{\sigma^6} - \frac{0.0001103991\bar{k}^4\mu}{\sigma^2}$$
$$+ \frac{0.0003311974\bar{k}^3\mu^2}{\sigma^4} + 0.0000137999\bar{k}^5,$$

$$\gamma_4[1] = 0.0002298344\bar{k} - \frac{0.4955894885\bar{k}\mu^3}{\sigma^6} - \frac{0.007844492\bar{k}\mu}{\sigma^2} + \frac{0.0034775004\bar{k}^3\mu}{\sigma^2}$$
$$- \frac{0.247392072\bar{k}^2\mu^2}{\sigma^4} + 0.0036039911\bar{k}^4 + 0.0013653983\bar{k}^2$$
$$- \frac{0.1854705538\bar{k}\mu^4}{\sigma^8} + 0.003193344\bar{k}^3 - \frac{0.0272734424\bar{k}^2\mu}{\sigma^2}$$
$$+ \frac{0.4452623527\bar{k}\mu^2}{\sigma^4} - \frac{0.0938766774\bar{k}^2\mu^3}{\sigma^6} + \frac{0.0229629693\bar{k}^4\mu}{\sigma^2}$$
$$+ \frac{0.0453194658\bar{k}^3\mu^2}{\sigma^4} + 0.0005151432\bar{k}^5,$$

$$\gamma_4[3] = -0.000725063\bar{k} - \frac{0.3252643751\bar{k}\mu^3}{\sigma^6} + \frac{0.1028682279\bar{k}\mu}{\sigma^2} + \frac{0.0350902261\bar{k}^3\mu}{\sigma^2}$$
$$- \frac{0.0366159756\bar{k}^2\mu^2}{\sigma^4} + 0.0013934668\bar{k}^4 - 0.0019326258\bar{k}^2$$
$$- \frac{0.0927352727\bar{k}\mu^4}{\sigma^8} + 0.0017642311\bar{k}^3 - \frac{0.0552426103\bar{k}^2\mu}{\sigma^2}$$
$$- \frac{0.2684284616\bar{k}\mu^2}{\sigma^4} + \frac{0.0305312921\bar{k}^2\mu^3}{\sigma^6} - \frac{0.0075532443\bar{k}^4\mu}{\sigma^2}$$

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$$+\frac{0.0310224137\bar{k}^3\mu^2}{\sigma^4} - 0.001999438\bar{k}^5,$$

$$\gamma_4[5] = 0.0047534909\bar{k} - \frac{0.0402734006\bar{k}\mu^3}{\sigma^6} - \frac{0.0225693355\bar{k}\mu}{\sigma^2} + \frac{0.0031576551\bar{k}^3\mu}{\sigma^2}$$
$$+\frac{0.0175031212\bar{k}^2\mu^2}{\sigma^4} - 0.0009204328\bar{k}^4 - 0.0016125491\bar{k}^2$$
$$-\frac{0.0092735277\bar{k}\mu^4}{\sigma^8} + 0.0016394380\bar{k}^3 + \frac{0.0027181986\bar{k}^2\mu}{\sigma^2}$$
$$-\frac{0.048843833\bar{k}\mu^2}{\sigma^4} + \frac{0.0108000923\bar{k}^2\mu^3}{\sigma^6} - \frac{0.0011734585\bar{k}^4\mu}{\sigma^2}$$
$$-\frac{0.0022898466\bar{k}^3\mu^2}{\sigma^4} + 0.000388775\bar{k}^5,$$

$$\gamma_4[7] = -\frac{0.0002207983\bar{k}\mu^4}{\sigma^8} - 0.0000137999\bar{k}^5 - 0.00006109082\bar{k}^3$$
$$+0.0000184773\bar{k}^4 + 0.0001952367\bar{k}^2 + \frac{0.0004415966\bar{k}^2\mu^3}{\sigma^6}$$
$$-\frac{0.0003311974\bar{k}^3\mu^2}{\sigma^4} + \frac{0.0001103991\bar{k}^4\mu}{\sigma^2} + \frac{0.0012172513\bar{k}^2\mu^2}{\sigma^4}$$
$$-\frac{0.0003597448\bar{k}^3\mu}{\sigma^2} + \frac{0.0008950657\bar{k}^2\mu}{\sigma^2} - \frac{0.001143342\bar{k}\mu^3}{\sigma^6}$$
$$-\frac{0.0018350756\bar{k}\mu^2}{\sigma^4} - \frac{0.0012786599\bar{k}\mu}{\sigma^2} - 0.0004092666\bar{k}.$$

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