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ON STRICT INCLUSIONS OF WEIGHTED DIRICHLET SPACES OF MONOGENIC FUNCTIONS

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We consider a scale of weighted spaces of quaternion-valued functions of three real variables. This scale generalises the idea of $\mathbf{Q_p}$-spaces in complex function theory. The goal of this paper is to prove that the inclusions of spaces from the scale are strict inclusions. As a tool we prove some properties of special monogenic polynomials which have an importance in their own right independently of their use in the scale of $\mathbf{Q_p}$-spaces.

1. INTRODUCTION

In 1995 a new class of holomorphic functions, the scale of so-called $\mathbf{Q_p}$-spaces, was introduced (see [**6**]) and subsequently studied intensively by several authors (see for example, [**3, 5**]). Let $\Delta = \{z : |z| < 1\}$ be the complex unit disk. The Bloch space is then defined by

$$\mathbf{B} = \Big\{f : f \text{ analytic in } \Delta \text{ and } B(f) = \sup_{z\in\Delta}\big(1-|z|^2\big)\big|f'(z)\big| < \infty\Big\}$$

while the Dirichlet space is given by

$$\mathbf{D} = \Big\{f : f \text{ analytic in } \Delta \text{ and } \int_\Delta \big|f'(z)\big|^2\, dxdy < \infty\Big\}.$$

The weight function $g(z,a) = \ln\big|(1-\bar{a}z)/(a-z)\big|$ is defined as the composition of the Möbius transformation $\varphi_a(z) = (a-z)/(1-\bar{a}z)$ and the fundamental solution of the two-dimensional real Laplacian. Then the spaces

$$\mathbf{Q_p} = \Big\{f : f \text{ analytic in } \Delta \text{ and } \sup_{a\in\Delta}\int_\Delta \big|f'(z)\big|^2 g^p(z,a)\, dxdy < \infty\Big\}, \qquad 0 < p < \infty$$

are called $\mathbf{Q_p}$-spaces. This approach was motivated by attempts to substitute a general parameter p for the parameter 2 in the definition of the Bloch space and to define spaces which are invariant under Möbius transformations. Another idea which leads to these

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$\mathbf{Q_p}$-spaces is to find spaces with $\mathbf{D}$ and $\mathbf{B}$, respectively, "at both the end points" of the range. Already before the idea of $\mathbf{Q_p}$-spaces, spaces of analytic functions between the Hardy and the Dirichlet space were studied in [**1**], based on the similar weight function $(1-|x|)^\alpha$. A lot of basic results are already known, such as for instance

$$\begin{aligned} \mathbf{D} \subset \mathbf{Q_p} \subset \mathbf{Q_q} \subset \mathrm{BMOA} \quad & 0<p<q<1 \quad (\text{see } [\mathbf{6}]) \\ \mathbf{Q_1} = \mathrm{BMOA} \quad & \text{see } [\mathbf{6}] \\ \mathbf{Q_p} = \mathbf{B} \quad & \forall p > 1 \quad (\text{see } [\mathbf{3}]). \end{aligned} \tag{1}$$

This means that the spaces $\mathbf{Q_p}$ form a scale as desired and for special values of the scale parameter p these spaces are connected with other known and important spaces of analytic functions. Surveys about special results, boundary values of $\mathbf{Q_p}$-functions, equivalent definitions, applications, and open problems are given in [**7, 13**] (see therein also for a detailed bibliography). There are several attempts to generalise these ideas and the corresponding approaches to higher dimensions [**26, 27, 9, 29**] or to other classes of functions. One idea is to treat the case of the unit ball in $\mathbb{C}^n$. Basic ideas are to replace the derivative f' by the complex gradient of f and the measure $dxdy$ by a weighted measure $d\lambda(z) = (dv)/(1-|z|^2)^{n+1}$, where dv stands for the usual Lebesgue measure. By using an invariant Green function, some results similar to the complex one-dimensional case were proved. The most important results are that

$$\mathbf{Q_p} = \mathbf{B} \quad \text{for } 1 < p < n/(n-1) \quad \text{and} \quad \mathbf{Q_1} = \mathrm{BMOA}(\partial B),$$

where ∂B is the surface of the unit ball in $\mathbb{C}^n$. But, for $p \notin \big((n-1)/n, n/(n-1)\big)$ all $\mathbf{Q_p}$-spaces are trivial, that is, only constant functions belong to $\mathbf{Q_p}$. Other generalisations of $\mathbf{Q_p}$-spaces to higher real dimensions have been published for instance in [**21, 10, 14**]. These approaches are related to harmonic analysis and try to transfer the Möbius invariance of the spaces. Because the differentiability is lost, the complex derivative is replaced by partial derivatives, the gradient, or by finite differences. Nevertheless, a lot of properties analogous to the complex case have already been proved. Our aim is to look for other generalisations of the complex (one-dimensional) ideas, where on the one hand most of the advantages of holomorphic functions (including the differentiability) are preserved and on the other hand the $\mathbb{C}^n$-problems do not appear. In this paper we study hypercomplex generalisations of $\mathbf{Q_p}$-spaces. Instead of holomorphic functions in the unit disk we study regular functions $f : \mathbb{R}^n \mapsto Cl_{0,n-1}$ (that is, solutions of generalised Cauchy-Riemann systems). These functions can be considered in all real space dimensions. Applying the generalised Cauchy-Riemann operator D, its adjoint $\overline{D}$, the hypercomplex Möbius transformation $\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$, and a modified fundamental solution g of the real Laplacian, we consider generalised $\mathbf{Q_p}$-spaces defined by

$$\mathbf{Q_p} = \left\{ f \in \ker D : \sup_{a \in B_1(0)} \int_B |\overline{D} f(x)|^2 \Big(g\big(\varphi_a(x)\big)\Big)^p dx < \infty \right\}.$$

where $B_1(0)$ stands for the unit ball in $\mathbb{R}^n$. This definition generalises the complex (one-dimensional) case because for $n = 2$ the definitions coincide. Furthermore, it is known that the function theory of monogenic functions has a very similar structure to the complex (one-variable) function theory (see [**8, 18, 12, 20**]). The first paper following this way was [**16**]. Later on it was proved in [**17**] that $\overline{D}f$ can be considered to be the derivative of a monogenic function f for all dimensions n. For the case of functions $f : \mathbb{R}^4 \mapsto \mathbb{H}$ it is already known from [**30, 25**] that $\overline{D}$ may be interpreted as the derivative of a quaternion-valued regular function. That is why the definition of $\mathbf{Q_p}$ in [**16**] seems to generalise the complex one-dimensional case. In [**10**] another generalisation of the complex case is considered which is also related to the approaches in [**21, 1**]. Instead of the derivative of the function the partial derivatives were used weighted by a more general weight function. The main goal in [**10**] was to construct generalisations of the complex $\mathbf{Q_p}$-spaces conserving the Möbius invariance and at the same time the principal idea of a weight function with some kind of mass concentration around the singularity. The result were $\mathbf{Q_p}$-spaces defined with the help of a weighted gradient norm. A summary and a comparison of different approaches for the case of monogenic functions can be found in [**11**]. Continuing this comparison, it is shown in [**15**] that the norms from [**10**] are equivalent to weighted Sobolev norms and that the norms defined by the gradient in [**10**] and the derivative in [**16**], respectively, are not equivalent in higher dimensions. It should be mentioned that in the complex case all the above approaches define equivalent norms. For better understanding we shall comment in Section 3 of this paper on the essential results of [**16**]. It was there left unsolved whether the inclusions of $\mathbf{Q_p}$-spaces with different parameter p are strict inclusions or not. The positive answer to this question will be proved in Section 4. Additionally, we prove a characterisation of $\mathbf{Q_p}$-functions by characterising their Taylor coefficients.

2. Preliminaries

Let $\mathbf{e}_1, \ldots, \mathbf{e}_m$ be an orthonormal basis in $\mathbb{R}^m$. Consider the 2^m-dimensional Clifford algebra $Cl_{0,m}$ generated from $\mathbb{R}^m$ equipped with a negative inner product. Then we have the anti-commutation relationship $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\mathbf{e}_0$, $i, j = 1, \ldots, m$, where δ_{ij} is the Kronecker delta symbol and $\mathbf{e}_0 = 1$ is the identity of $Cl_{0,m}$. It may be observed that each element of the algebra can be represented in the form

$$a = \sum_A a_A \mathbf{e}_A,$$

where a_A are real numbers and $\mathbf{e}_A, A \subseteq \{1, \ldots, m\}$, with $\mathbf{e}_A = \mathbf{e}_{i_1} \ldots \mathbf{e}_{i_k}$, $\mathbf{e}_{\{i\}} = \mathbf{e}_i$, $i = 1, \ldots, m$, and $\mathbf{e}_\emptyset = \mathbf{e}_0$, are the basis elements of $Cl_{0,m}$.

In what follows identify each element $\mathbf{x} = (x_1, \ldots, x_m)$ of $\mathbb{R}^m$ with the element

$$x = \sum_{k=1}^{m} x_k \mathbf{e}_k$$

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of the Clifford algebra. In this way the vector space $\mathbb{R}^m$ is embedded in $Cl_{0,m}$ and we shall call these elements x of $Cl_{0,m}$ vectors. By

$$\overline{a} = \sum_A a_A \overline{\mathbf{e}}_A,$$

where $\overline{\mathbf{e}}_A = \overline{\mathbf{e}}_{i_k} \dots \overline{\mathbf{e}}_{i_1}$, $\overline{\mathbf{e}}_j = -\mathbf{e}_j$, $j = 1, \dots, m$, we define a conjugate element.

For C^1-functions defined on a domain $\Omega \subset \mathbb{R}^s$ we introduce a generalised Cauchy-Riemann operator by

$$D = \sum_{k=0}^{s} \mathbf{e}_k \frac{\partial}{\partial x_k}, \quad s \leqslant m.$$

Note that $D\overline{D} = \Delta$, where Δ is the Laplacian in $\mathbb{R}^s$.

A function $f : \Omega \mapsto Cl_{0,m}$ is said to be *left-monogenic* if it satisfies the equation $(Df)(x) = 0$ for each $x \in \Omega$.

In Section 3 we shall work in $\mathbb{H}$, the skew field of quaternions. As usual we identify $\mathbb{H}$ with $Cl_{0,2}$ and write $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ instead of $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\}$. Points of $\mathbb{R}^3$ have coordinates (x_0, x_1, x_2), and we use the Cauchy-Riemann operator

$$D = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}.$$

$\overline{D} = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2}$ is the conjugate Cauchy-Riemann operator.

Another important tool in the context of $\mathbf{Q_p}$-spaces is the appropriate explicit form of the Taylor series of a monogenic function. An elementary explanation of this question for Euclidean spaces of arbitrary dimension can be found in [**22**]. To be short we shall only refer to the main facts corresponding to $\mathbb{R}^3$. The major difference to power series in the complex case consists in the absence of regularity of the basic variable $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j}$ and of all of its natural powers x^n, $n = 2, \dots$. This means that we should expect other types of terms which could be designated as generalised powers. Indeed, following [**22**] we use a pair $\underline{z} = (z_1, z_2)$ of two regular variables (see [**18**]) given by

$$z_1 = x_1 - \mathbf{i}x_0 \quad \text{and} \quad z_2 = x_2 - \mathbf{j}x_0$$

and a multi-index $\nu = (\nu_1, \nu_2)$, $|\nu| = (\nu_1 + \nu_2)$ to define the ν-power of $\underline{z}$ by a $|\nu|$-ary product.

DEFINITION 2.1: Let ν_1 elements of the set $a_1, \dots, a_{|\nu|}$ be equal to z_1 and ν_2 elements be equal to z_2. Then the ν-power of $\underline{z}$ is defined by

$$\underline{z}^\nu := \frac{1}{|\nu|!} \sum_{(i_1, \dots, i_{|\nu|}) \in \pi(1, \dots |\nu|)} a_{i_1} a_{i_2} \cdots a_{i_{|\nu|}} \tag{2}$$

where the sum runs over *all* permutations of $(1, \dots, |\nu|)$.

REMARK 2.1. It is evident that for a fixed value of $|\nu| = d$ there exist exactly $(d+1)$ different ν-powers of $\underline{z}$. To distiguish between them we sometimes also use the notation $\underline{z}^\nu = {z_1}^{\nu_1} \times {z_2}^{\nu_2} = {z_2}^{\nu_2} \times {z_1}^{\nu_1}$ but the meaning of the last expressions is slightly different from the usual one in commutative rings and should be understood in the sense of formula (2). We shall use parentheses if the separated powers of z_1 or z_2 have to be understood in the ordinary way. Notice that the algebraic fundamentals for such a definition of generalised powers lie in the application of the symmetric product between d elements of a non-commutative ring as discussed in **[22]**. In this sense the variables z_k, $k = 1, 2$, themselves are symmetric products of $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j}$ with $(-\mathbf{i})$, respectively $(-\mathbf{j})$, in the form

$$z_1 = x_1 - \mathbf{i}x_0 = -\frac{1}{2}(\mathbf{i}x + x\mathbf{i}) \quad \text{and} \quad z_2 = x_2 - \mathbf{j}x_0 = -\frac{1}{2}(\mathbf{j}x + x\mathbf{j}).$$

With this the definition of the ν-power of $\underline{z}$, **[22**, Theorem 2] implies that all polynomials in z_k, $k = 1, 2$, homogeneous of degree $|\nu|$ and of the form

$$f_\nu(z_1, z_2) = \underline{z}^\nu$$

with $\nu = (\nu_1, \nu_2)$ an arbitrary multi-index, are both left and right monogenic and $\mathbb{H}$−linearly independent. Therefore they can serve as basis for generalised power series. In particular, we are interested in *left* power series with centre at the origin and ordered by such homogeneous polynomials. It was shown in **[22]** that the general form of the Taylor series of left monogenic functions in the neighbourhood of the origin is given by

$$P(x) = \sum_{n=0}^{\infty} \Big(\sum_{|\nu|=n} \underline{z}^\nu c_\nu \Big), \quad \text{with} \quad c_\nu \in \mathbb{H}. \tag{3}$$

In Section 4 we need the following estimate.

THEOREM 2.1. *Let $g(x)$ be left monogenic in a neighbourhood of the origin with the Taylor series given in the form* (3). *Then*

$$\left|\frac{1}{2}\overline{D}g(x)\right| \leqslant \sum_{n=1}^{\infty} n \Big(\sum_{|\nu|=n} |c_\nu| \Big) |x|^{n-1}. \tag{4}$$

PROOF: By applying the general formula for the monogenic derivative of generalised powers in **[17]** to our case, we get

$$-\frac{1}{2}\overline{D}\underline{z}^\nu = \nu_1 z_1^{\nu_1 - 1} \times z_2^{\nu_2}\mathbf{i} + \nu_2 z_1^{\nu_1} \times z_2^{\nu_2 - 1}\mathbf{j}, \tag{5}$$

where all the powers have to be understood in the sense of formula (2). Together with

$$|\underline{z}^\nu| \leqslant |x|^{|\nu|}$$

as a consequence of (2) and of the relation of z_k, $k = 1, 2$, to x in the form of symmetric products with $(-\mathbf{i})$, respectively $(-\mathbf{j})$, we obtain the desired result (4) after direct estimation of the modulus of the sums in the monogenic derivative of $\sum_{|\nu|=n} \underline{z}^\nu c_\nu$. It follows that

$$\begin{aligned}
-\frac{1}{2}\overline{D}(g(x)) &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(n-k)z_1^{n-(k+1)}z_2^k\mathbf{i}c_{n-k,k} + \sum_{k=0}^{n} k z_1^{n-k}z_2^{k-1}\mathbf{j}c_{n-k,k}\right]\\
&= \sum_{n=1}^{\infty}\left[\sum_{k=0}^{n-1}(n-k)z_1^{n-(k+1)}z_2^k\mathbf{i}c_{n-k,k} + \sum_{k=1}^{n} k z_1^{n-k}z_2^{k-1}\mathbf{j}c_{n-k,k}\right]\\
&= \sum_{n=1}^{\infty} z_1^{n-(k+1)}z_2^k\left[\sum_{k=0}^{n-1}\left[(n-k)\mathbf{i}c_{n-k,k} + (k+1)\mathbf{j}c_{n-(k+1),k+1}\right]\right].
\end{aligned}$$

Then we get for the modulus

$$\begin{aligned}
&\left|\frac{1}{2}\overline{D}(g(x))\right|\\
&\leqslant \sum_{n=1}^{\infty}|\underline{z}|^{n-1}\left[\sum_{k=0}^{n-1}\left[(n-k)|c_{n-k,k}| + (k+1)\left|c_{n-(k+1),k+1}\right|\right]\right]\\
&= \sum_{n=1}^{\infty}\left[n|c_{n,0}| + \sum_{k=1}^{n-1}(n-k)|c_{n-k,k}| + \sum_{k=0}^{n-2}(k+1)\left|c_{n-(k+1),k+1}\right| + n|c_{0,n}|\right]|x|^{n-1}\\
&= \sum_{n=1}^{\infty}\left[n|c_{n,0}| + \sum_{k=0}^{n-2}(n-(k+1))\left|c_{n-(k+1),k+1}\right|\right.\\
&\qquad\qquad\left. + \sum_{k=0}^{n-2}(k+1)\left|c_{n-(k+1),k+1}\right| + n|c_{0,n}|\right]|x|^{n-1}\\
&= \sum_{n=1}^{\infty} n\left(\sum_{k=0}^{n}|c_{n-k,k}|\right)|x|^{n-1}\\
&= \sum_{n=1}^{\infty} n\left(\sum_{|\nu|=n}|c_\nu|\right)|x|^{n-1}.
\end{aligned}$$

□

In what now follows we shall work in $B_1(0) \subset \mathbb{R}^3$, the unit ball in real three-dimensional space. Moreover, we shall consider functions f defined on $B_1(0)$ with values in $\mathbb{H}$. The contents of Section 3 give an outline of **[16]** and its basic results.

3. Definition of $\mathbf{Q_p}$-spaces in $\mathbb{R}^3$

For $|a| < 1$ we shall denote by

$$\varphi_a(x) = (a-x)(1-\bar{a}x)^{-1}$$

the Möbius transform, which maps the unit ball onto itself. Furthermore, let

$$g(x,a) = \frac{1}{4\pi}\left(\frac{1}{|\varphi_a(x)|} - 1\right)$$

be the modified fundamental solution of the Laplacian in $\mathbb{R}^3$ composed with the Möbius transform $\varphi_a(x)$. We denote for all $p > 0$

$$g^p(x,a) = \frac{1}{4^p\pi^p}\left(\frac{1}{|\varphi_a(x)|} - 1\right)^p.$$

Let $f : B_1(0) \mapsto \mathbb{H}$ be a monogenic function. We shall use, as in **[16]**, the seminorms

- $B(f) = \sup_{x\in B_1(0)} (1-|x|^2)^{3/2}|\overline{D}f(x)|$,
- $Q_p(f) = \sup_{a\in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 g^p(x,a)\, dB_x$,

which lead to the following definitions:

DEFINITION 3.1: The spatial (or three-dimensional) Bloch space $\mathbf{B}$ is the right $\mathbb{H}$-module of all monogenic functions $f : B_1(0) \mapsto \mathbb{H}$ with $B(f) < \infty$.

DEFINITION 3.2: The right $\mathbb{H}$-module of all quaternion-valued functions f defined on the unit ball, which are monogenic and satisfy $Q_p(f) < \infty$, is called the $\mathbf{Q_p}$-space.

REMARK 3.1. Because of the special structure of $g(x,a)$, the seminorms $Q_p(f)$ make sense only for $p < 3$ and for increasing dimension s of the space the possible range for p will become smaller and smaller (the same problem as in the C^n-approach). We shall at first in this section consider $\mathbf{Q_p}$-spaces for $p < 3$. In Subsection 3.2 we shall describe another characterisation of $\mathbf{Q_p}$-spaces which is equivalent to the definition above for $p < 3$ and which makes sense for $p \geqslant 3$ and for higher dimensions also.

Obviously, these spaces are not Banach spaces. Nevertheless, if we consider a small neighbourhood of the origin U_ε, with an arbitrary but fixed $\varepsilon > 0$, then we can add the L_1-norm of f over U_ε to our seminorms and $\mathbf{B}$ as well as $\mathbf{Q_p}$ will become Banach spaces. Because this additional term is independent of p we shall consider in the following only the spaces with the corresponding seminorm, but we have to keep in mind that all our results are also true in the case of the norm.

DEFINITION 3.3: The right $\mathbb{H}$-module of monogenic functions $f : B_1(0) \mapsto \mathbb{H}$ with

$$\int_{B_1(0)} |\overline{D}f(x)|^2 dB_x < \infty,$$

is called the spatial (or three-dimensional) Dirichlet space $\mathbf{D}$.

REMARK 3.2. Using the special properties of $g(x,a)$ in $B_1(0)$ one can prove that

$$\mathbf{D} \subset \mathbf{Q_p}, \qquad 0 < p < 3.$$

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3.1. Properties of $\mathbf{Q_p}$-spaces. First we refer to the main steps (see [**16**]) to show that the $\mathbf{Q_p}$-spaces form a range of Banach $\mathbb{H}$-modules (with our additional term added to the seminorm), connecting the spatial Dirichlet space with the spatial Bloch space. In order to do this several lemmas are needed. Although some of these lemmas are only of a technical nature we shall at least state these results to show that the approach to $\mathbf{Q_p}$-spaces in higher dimensions which is sketched in this section is strongly based on properties of monogenic functions.

Proposition 3.1. *Let f be monogenic and $0 < p < 3$, then we have*

$$(1-|a|^2)^3\left|\overline{D}f(a)\right|^2 \leqslant C_1 \int_{B_1(0)} \left|\overline{D}f(x)\right|^2 \left(\frac{1}{|\varphi_a(x)|} - 1\right)^p dB_x, \tag{6}$$

where the constant C_1 does not depend on a and f.

The inequality has the same structure as in the complex one-dimensional case (see for example, [**6**]). Only the exponent 3 on the left hand side shows how the real dimension of the space influences the estimate. To prove this proposition we need a mean value formula coming from properties of the hypercomplex Cauchy integral (see [**19**]), some geometrical properties of the Möbius transformation and the equality

$$\frac{1-|\varphi_a(x)|^2}{1-|x|^2} = \frac{1-|a|^2}{|1-\bar{a}x|^2} \tag{7}$$

which links properties of the (special) Möbius transformation φ_a with the weight function $1-|x|^2$. This equality generalises in a direct way the corresponding property from the complex one-dimensional case. By considering the supremum on both sides of (6) we obtain the following corollary.

Corollary 3.1. *For $0 < p < 3$ we have $\mathbf{Q_p} \subset \mathbf{B}$.*

This corollary means that all $\mathbf{Q_p}$-spaces are subspaces of the Bloch space. We recall that in the complex one-dimensional case all $\mathbf{Q_p}$-spaces with $p > 1$ are equal and coincide with the Bloch space. This leads to a corresponding question in the three-dimensional case considered here. In [**16**] the following theorem is proved.

Theorem 3.1. *Let f be monogenic in the unit ball. Then the following conditions are equivalent:*

1. $f \in \mathbf{B}$;
2. $Q_p(f) < \infty$ *for all* $2 < p < 3$;
3. $Q_p(f) < \infty$ *for some* $p > 2$.

Theorem 3.1 means that all $\mathbf{Q_p}$-spaces for $p > 2$ coincide and are identical with the Bloch space.

3.2. Another characterisation of $\mathbf{Q_p}$-spaces. The one-dimensional analogue of Definition 3.2 was the first definition of $\mathbf{Q_p}$-spaces. This was motivated by the idea to have a range of spaces "approaching" the space BMOA and the Bloch space. Comparing the original definition and one of the equivalent characterisations of BMOA in [**7**] it is obvious that $\mathbf{Q_1}$ = BMOA. Another motivation is given by some invariance properties of the Green function used in the definition. Recent papers (see for example, [**2**]) show that the ideas of these weighted spaces can be generalised in a very direct way to the case of Riemannian manifolds. Caused by the singularity of the Green function, difficulties arise in proving some properties of the scale. One of these properties is the inclusion property with respect to the index p. Considering ideas from [**1**] also the use of polynomial weights seems to be natural and more convenient in the case of increasing space dimension. The idea to relate the Green function with more general weight functions of the type $(1-|x|^2)^p$ is not new. For the complex case it has already been mentioned in [**6, 4**]. Another idea is to prove also a relation of $g^p(x,a)$ with $(1-|\varphi_a|^2)^p$. This way saves on the one hand the advantages of the simple term $(1-|x|^2)^p$ and preserves on the other hand a special behaviour of the weight function under Möbius transforms. As hint we refer to equation (7).

In this subsection we relate these possibilities to characterise $\mathbf{Q_p}$-spaces. Among others, this new (in our case equivalent) characterisation implies the proof of the fact that the $\mathbf{Q_p}$-spaces are a scale of function, spaces with the Dirichlet space at one extreme point and the Bloch space at the other.

THEOREM 3.2. [**16**] *Let f be monogenic in $B_1(0)$. Then, for $1 \leqslant p < 2.99$,*

$$f \in \mathbf{Q_p} \Leftrightarrow \sup_{a\in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 \left(1-|\varphi_a(x)|^2\right)^p dB_x < \infty.$$

At first glance, the condition $p < 2.99$ looks strange. But we have to keep in mind that Theorem 3.1 means that all $\mathbf{Q_p}$-spaces for $p > 2$ are the same, so in fact this condition is only of technical nature caused by the singularity of $g^p(x,a)$ for $p = 3$.

Especially for the proof of this theorem we need the properties of monogenic functions and of the Möbius transformation. The main idea is a change of variables $w = \varphi_a(x)$. (The Jacobian determinant $\left((1-|a|^2)/(|1-\bar{a}w|^2)\right)^3$ has no singularities.) The problem here is that, while $\overline{D}_x f(x)$ is monogenic, after the change of variables $\overline{D}_x f\left(\varphi_a(w)\right)$ is not monogenic. But we know from [**28**] that $\left((1-\overline{w}a)/(|1-\bar{a}w|^3)\right)\overline{D}_x f\left(\varphi_a(w)\right)$ is again monogenic. We also refer to Sudbery [**30**] who studied this problem for the four-dimensional case already in 1979.

The same characterisation can be shown by a different proof [**16**] also in the case that $p < 1$.

PROPOSITION 3.2. *Let f be monogenic in $B_1(0)$. Then, for $0 < p \leqslant 1$,*

$$f \in \mathbf{Q_p} \Leftrightarrow \sup_{a\in B_1(0)} \int_{B_1(0)} |\overline{D}f(x)|^2 \left(1-|\varphi_a(x)|^2\right)^p dB_x < \infty.$$

Using the alternative definition of $\mathbf{Q_p}$-spaces it can be shown that the $\mathbf{Q_p}$-spaces form a scale of Banach spaces. This is a consequence of using the weight function $\left(1-\left|\varphi_a(x)\right|^2\right)$.

PROPOSITION 3.3. *For* $0 < p < q < 2$ *we have:* $\mathbf{Q_p} \subset \mathbf{Q_q}$.

4. Strict inclusions of $\mathbf{Q_p}$-spaces

To prove the strict inclusions of the $\mathbf{Q_p}$ - spaces we need more special tools. One of the basic inequalities is a result on weighted L_p-norms of real analytic functions.

LEMMA 4.1. *Let* $\alpha > 0$, $p > 0$, $n \geqslant 0$, $a_n \geqslant 0$, $I_n = \{k : 2^n \leqslant k < 2^{n+1},\ k \in N\}$, $t_n = \sum\limits_{k \in I_n} a_k$ *and* $f(x) = \sum\limits_{n=1}^{\infty} a_n x^n$. *Then there exists a constant* K *depending only on* p *and* α *such that*

$$\frac{1}{K}\sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \leqslant \int_0^1 (1-x)^{\alpha-1} f(x)^p\, dx \leqslant K \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p. \tag{8}$$

For the proof and a lot more information on weighted $\mathbf{Q_p}$-spaces, Hardy spaces, Bergman spaces, and their connections we refer to [**23**].

For technical reasons it is necessary to evaluate some special integrals.

LEMMA 4.2.

$$\iint\limits_{\partial B_1(0)} \frac{1}{|1-\bar{a}x|^4}\, d\Gamma = \frac{4\pi}{\left(1-|a|^2\right)^2}.$$

PROOF: We begin with the case $a = a_0 > 0$ and use spherical coordinates. Then we have to evaluate

$$\int_0^{\pi}\int_0^{2\pi} \frac{1}{(1+a_0^2 - 2a_0 \sin\varphi_2 \cos\varphi_1)^2}\, d\varphi_1 \sin\varphi_2\, d\varphi_2$$

$$= \int_0^{\pi} \frac{2(1+a_0^2)\pi \sin\varphi_2}{(1-2a_0^2+a_0^4+4a_0^2\cos^2\varphi_2)^{3/2}}\, d\varphi_2 = \frac{4\pi}{(1-|a_0|^2)^2}.$$

To prove the general case we use the orthogonal transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ with $R\begin{pmatrix}|a|\\0\\0\end{pmatrix} = \begin{pmatrix}a_0\\a_1\\a_2\end{pmatrix}$ and let $y = R^T x$. Then we have

$$\iint\limits_{\partial B_1(0)} \frac{1}{|1-\bar{a}x|^4}\, d\Gamma_x = \iint\limits_{\partial B_1(0)} \frac{d\Gamma_x}{\left(1-2\langle a,x\rangle + |a|^2|x|^2\right)^2} = \iint\limits_{\partial B_1(0)} \frac{d\Gamma_y}{\left(1-2\langle a,Ry\rangle + |a|^2|Ry|^2\right)^2}$$

$$= \iint\limits_{\partial B_1(0)} \frac{d\Gamma_y}{\left(1-2\langle R^T a,y\rangle + |a|^2|y|^2\right)^2} = \iint\limits_{\partial B_1(0)} \frac{d\Gamma_y}{(1-|a|y)^4} = \frac{4\pi}{(1-|a|^2)^2}.$$

Here $\langle\cdot,\cdot\rangle$ stands for the usual scalar product in $\mathbb{R}^3$. ∎

Lemma 4.3.

$$\iint\limits_{\partial B_1(0)} \frac{1}{|1-\bar{a}x|^2}\, d\Gamma = \frac{2\pi}{|a|} \ln\left(\frac{1+|a|}{1-|a|}\right).$$

The proof is based on the same ideas as the proof of Lemma 4.2.

Lemma 4.4. *Let* $0 < p \leqslant 2$, $|a| < 1$, $r \leqslant 1$. *Then*

$$\iint\limits_{\partial B_1} \frac{1}{|1-\bar{a}ry|^{2p}}\, d\Gamma_y \leqslant C \frac{1}{(1-|a|r)^p}.$$

Proof: For $1 < p \leqslant 2$ we conclude as follows

$$\iint\limits_{\partial B_1} \frac{1}{|1-\bar{a}rz|^{2p}}\, d\Gamma \leqslant \left(\iint\limits_{\partial B_1} \left(\frac{1}{|1-\bar{a}rz|^{2p}}\right)^{2/p} d\Gamma\right)^{p/2} \left(\iint\limits_{\partial B_1} 1\, d\Gamma\right)^{1-(p/2)}$$
$$= (4\pi)^{1-(p/2)} \left(\iint\limits_{\partial B_1} \frac{1}{|1-\bar{a}rz|^4}\, d\Gamma\right)^{p/2} \leqslant C\left(\frac{1}{\left(1-|a|^2r^2\right)^2}\right)^{p/2}.$$

For $0 < p \leqslant 1$ we estimate

$$\iint\limits_{\partial B_1} \frac{1}{|1-\bar{a}ry|^{2p}}\, d\Gamma_y \leqslant \left[\iint\limits_{\partial B_1} \left(\frac{1}{|1-\bar{a}ry|^{2p}}\right)^{1/p} d\Gamma_y\right]^p$$
$$\left[\iint\limits_{\partial B_1} d\Gamma\right]^{1-p} = C\left(\iint\limits_{\partial B_1} \frac{1}{|1-\bar{a}ry|^2} d\Gamma_y\right)^p.$$

Using

$$\frac{2\pi}{|a|r} \ln \frac{1+|a|r}{1-|a|r} \leqslant \frac{4\pi}{1-|a|r}$$

we obtain the desired result.

From [24] we shall now use the idea of characterising $\mathbf{Q_p}$-functions with the help of the coefficients of their Taylor series expansions.

According to formula (3) of Section 2 the Taylor series expansion of a left monogenic function has the form

$$g(x) = \sum_{n=0}^{\infty} \left(\sum_{|\nu|=n} \underline{z}^{\nu} c_{\nu}\right).$$

In order to formulate the next theorem we introduce the abbreviated notation $H_n(x) := \sum_{|\nu|=n} \underline{z}^{\nu} c_{\nu}$ for such a homogeneous monogenic polynomial of degree n and consider monogenic functions composed by $H_n(x)$ in the following form:

$$f(x) = \sum_{n=0}^{\infty} H_n(x) b_n, \quad b_n \in \mathbb{H}.$$

Taking into account formula (4) we see that in this way we get for this type of function

$$(9)\qquad \left|\frac{1}{2}\overline{D}f(x)\right| \leqslant \sum_{n=0}^{\infty} n\Big(\sum_{|\nu|=n} |c_\nu|\Big)\,|b_n|\,|x|^{n-1}.$$

This is the motivation for another shorthand notation, namely $a_n := \Big(\sum_{|\nu|=n} |c_\nu|\Big)|b_n|$, $(a_n \geqslant 0)$ and we get finally

$$(10)\qquad \left|\frac{1}{2}\overline{D}f(x)\right| \leqslant \sum_{n=0}^{\infty} n a_n |x|^{n-1}.$$

Theorem 4.1. *Let $I_n = \langle k : 2^n \leqslant k < 2^{n+1},\ k \in N\rangle$, $f(x) = \sum_{n=0}^{\infty} H_n(x) b_n$, $b_n \in \mathbb{H}$, H_n be a homogeneous monogenic polynomial of degree n of the aforementioned type, and a_n be defined as before, $0 < p \leqslant 2$. Then*

$$\sum_{n=0}^{\infty} 2^{n(1-p)}\Big(\sum_{k\in I_n} a_k\Big)^2 < \infty \Rightarrow f \in \mathbf{Q_p}.$$

Proof: Assume that $\sum_{n=0}^{\infty} 2^{n(1-p)}\Big(\sum_{k\in I_n} a_k\Big)^2 < \infty$.

$$\iint_{\partial B_1}\left|\frac{1}{2}\overline{D}f\right|^2\left(1-|\varphi_a(x)|^2\right)^p dB_x = \iint_{\partial B_1}\left|\frac{1}{2}\overline{D}\Big(\sum_{n=0}^{\infty}H_n(x)b_n\Big)\right|^2\frac{(1-|x|^2)^p(1-|a|^2)^p}{|1-\bar a x|^{2p}}\,dB_x$$

$$\leqslant \iint_{\partial B_1}\Big(\sum_{n=1}^{\infty} n\,a_n\,|x|^{n-1}\Big)^2\frac{(1-|x|^2)^p(1-|a|^2)^p}{|1-\bar a x|^{2p}}\,dB_x$$

$$= \int_0^1\Big(\sum_{n=1}^{\infty} n\,a_n\,r^{n-1}\Big)^2(1-r^2)^p(1-|a|^2)^p\int_{\partial B_1}\frac{1}{|1-\bar a r z|^{2p}}\,d\Gamma_z\,r^2\,dr$$

$$\leqslant C\int_0^1\Big(\sum_{n=1}^{\infty} n\,a_n\,r^{n-1}\Big)^2(1-r^2)^p(1-|a|^2)^p\frac{1}{(1-|a|r)^p}\,r^2\,dr$$

$$\leqslant 2^pC\int_0^1\Big(\sum_{n=1}^{\infty} n\,a_n\,r^{n-1}\Big)^2(1-r)^p\,\frac{(1+r)^p(1-|a|r)^p}{(1-|a|r)^p}\,r^2\,dr$$

$$\leqslant C2^{2p}\int_0^1\Big(\sum_{n=1}^{\infty} n\,a_n\,r^{n-1}\Big)^2(1-r)^p r^2\,dr = C2^{2p}\int_0^1\Big(\sum_{n=1}^{\infty} n\,a_n\,r^{n}\Big)^2(1-r)^p\,dr$$

$$\leqslant CK2^{2p}\Big(\sum_{n=0}^{\infty}2^{-n(p+1)}\Big(\sum_{k\in I_n} k a_k\Big)^2\Big) \leqslant CK2^{2p}\Big(\sum_{n=0}^{\infty}2^{-n(p+1)}(2^{n+1})^2\Big(\sum_{k\in I_n} a_k\Big)^2\Big)$$

$$= CK\sum_{n=0}^{\infty}2^{-np-n+2p+2n+2}\Big(\sum_{k\in I_n} a_k\Big)^2 = CK2^{2p+2}\sum_{n=0}^{\infty}2^{n(1-p)}\Big(\sum_{k\in I_n} a_k\Big)^2.$$

Therefore,

$$\|f\|_{Q_p} = \sup_{a\in B_1(0)} \iint_{B_1(0)} \left|\frac{1}{2}\overline{D}f\right|^2 (1-|\varphi_a|^2)^p \, dG < \infty.$$

□

To prove a reverse theorem which connects $\mathbf{Q_p}$-functions with weighted norms of the sequence of Taylor coefficients it is necessary to investigate norm relations between a monogenic homogeneous polynomial and its derivative. In the complex (one-dimensional) case all proofs are based on the equality $\overline{\partial} z^n = nz^{n-1}$ and corresponding norm equalities.

One way to achieve the goal in the three dimensional case is the application of monogenic homogeneous polynomials of the form

$$(11)\quad H_{n,\alpha}(x) = (z_1\alpha_1 + z_2\alpha_2)^n = \sum_{k=0}^{n} {z_1}^{n-k} \times {z_2}^k \binom{n}{k} {\alpha_1}^{n-k}{\alpha_2}^k, \quad \alpha_i \in \mathbb{R},\ i=1,2,$$

with the hypercomplex derivative given by

$$(12)\qquad \Big(-\frac{1}{2}\overline{D}\Big) H_{n,\alpha}(x) = nH_{n-1,\alpha}(x)(\alpha_1\mathbf{i} + \alpha_2\mathbf{j}).$$

The $L_2(\partial B_1)$-norm of such a special monogenic homogeneous polynomial of type (11) can be determined explicitly. We get

Lemma 4.5. *Let $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i \in \mathbb{R}$, $i = 1,2$, be the vector of real coefficients defining the monogenic homogeneous polynomial*

$$H_{n,\alpha}(x) = (z_1\alpha_1 + z_2\alpha_2)^n.$$

Suppose that $|\alpha|^2 = \alpha_1^2 + \alpha_2^2 \neq 0$. Then the L_2-norm of this special monogenic homogeneous polynomial is equal to

$$(13)\qquad \|H_{n,\alpha}\|^2_{L_2(\partial B_1)} = \frac{2^{n+2} n! \pi |\alpha|^{2n}}{(2n+1)!!}.$$

Proof:

$$\|H_{n,\alpha}\|^2_{L_2(\partial B_1)}$$
$$= \int_0^{2\pi}\int_0^{\pi} \left(\sin^2\varphi_1(\alpha_1\cos\varphi_2 + \alpha_2\sin\varphi_2)^2 + (\alpha_1^2+\alpha_2^2)\cos^2\varphi_1\right)^n \sin\varphi_1 \, d\varphi_1\, d\varphi_2$$
$$= \int_0^{2\pi}\int_0^{\pi} \Big(|\alpha|^2 + \sin^2\varphi_1\big[(\alpha_1\cos\varphi_2 + \alpha_2\sin\varphi_2)^2 - |\alpha|^2\big]\Big)^n \sin\varphi_1\, d\varphi_1\, d\varphi_2$$
$$= \int_0^{2\pi}\int_0^{\pi} \left(|\alpha|^2 + |\alpha|^2\sin^2\varphi_1\left[\left[\frac{\alpha_1\cos\varphi_2 + \alpha_2\sin\varphi_2}{\sqrt{\alpha_1^2+\alpha_2^2}}\right]^2 - 1\right]\right)^n \sin\varphi_1\, d\varphi_1\, d\varphi_2$$

$$= \int_0^{2\pi}\int_0^{\pi}\Big(|\alpha|^2 + |\alpha|^2\sin^2\varphi_1\big[\sin^2(\omega+\varphi_2)-1\big]\Big)^n \sin\varphi_1\,d\varphi_1\,d\varphi_2$$

$$= \int_0^{2\pi}\int_0^{\pi}\big(|\alpha|^2 - |\alpha|^2\sin^2\varphi_1\cos^2(\omega+\varphi_2)\big)^n \sin\varphi_1\,d\varphi_1\,d\varphi_2$$

$$(14)\qquad = |\alpha|^{2n}\int_0^{2\pi}\int_0^{\pi}\big(1 - \sin^2\varphi_1\cos^2(\omega+\varphi_2)\big)^n \sin\varphi_1\,d\varphi_1\,d\varphi_2$$

where ω is a fixed angle defined uniquely by

$$\sin\omega := \frac{\alpha_1}{\sqrt{\alpha_1^2+\alpha_2^2}} \quad\text{and}\quad \cos\omega := \frac{\alpha_2}{\sqrt{\alpha_1^2+\alpha_2^2}} \quad\text{as usual.}$$

Then we get from formula (14)

$$\|H_{n,\alpha}\|^2_{L_2(\partial B_1)} = |\alpha|^{2n}\int_0^{2\pi}\int_0^{\pi}\big(1 - \sin^2\varphi_1\cos^2(\omega+\varphi_2)\big)^n \sin\varphi_1\,d\varphi_1\,d\varphi_2$$

$$(15)\qquad = |\alpha|^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\int_0^{2\pi}\cos^{2k}(\omega+\varphi_2)\,d\varphi_2\right)\left(\int_0^{\pi}\sin^{2k+1}\varphi_1\,d\varphi_1\right).$$

Both integrals in the last formula (15) can be evaluated with integration by parts. We have

$$I_{1k} := \int_0^{\pi}\sin^{2k+1}\varphi_1\,d\varphi_1 = \frac{k!2^{k+1}}{(2k+1)!!}$$

and similarly

$$I_{2k} := \int_0^{2\pi}\cos^{2k}(\omega+\varphi_2)\,d\varphi_2 = \pi\frac{(2k-1)!!}{2^{k-1}k!}.$$

These values and formulae (14) and (15) together lead to

$$\|H_{n,\alpha}\|^2_{L_2(\partial B_1)} = |\alpha|^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\int_0^{2\pi}\cos^{2k}(\omega+\varphi_2)\,d\varphi_2\right)\left(\int_0^{\pi}\sin^{2k+1}\varphi_1\,d\varphi_1\right)$$

$$= \pi|\alpha|^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot\frac{(2k-1)!!}{2^{k-1}k!}\cdot\frac{k!2^{k+1}}{(2k+1)!!}$$

$$(16)\qquad = 2^2\pi|\alpha|^{2n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot\frac{1}{(2k+1)}.$$

The last step in proving the lemma is the application of the relation

$$(17)\qquad s_n := \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(2k+1)} = \frac{2^n n!}{(2n+1)!!}.$$

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which results from a recursive formula of the form

$$s_n = \frac{2n}{2n+1} \cdot s_{n-1} \quad \text{with} \quad s_0 = 1.$$

To obtain these relations in an elementary way one has only to use well known properties of the binomial coefficients. Inserting (17) in (16) we obtain immediately

$$\|H_{n,\alpha}\|^2_{L_2(\partial B_1)} = \frac{2^{n+2} n! \pi |\alpha|^{2n}}{(2n+1)!!} \tag{18}$$

and the lemma is proved. □

Corollary 4.1. *With the help of* (12) *we get immediately the quotient of the* $L_2(\partial B_1)$ *norms of* $H_{n,\alpha}$ *and its hypercomplex derivative* $-1/2(\overline{D}H_{n,\alpha})$ *in the simple form:*

$$\frac{\left\| \left(-\frac{1}{2}\overline{D}H_{n,\alpha}\right) \right\|_{L_2(\partial B_1)}}{\|H_{n,\alpha}\|_{L_2(\partial B_1)}} = \sqrt{n\left(n+\frac{1}{2}\right)}\,.$$

Notice that the quotient does not depend on the choice of α *but only on the degree of the homogeneous monogenic polynomial.*

Theorem 4.2. *Let* $0 < p \leqslant 2$,

$$f(x) = \left(\sum_{k=0}^{\infty} \frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} a_k \right) \in \mathbf{Q_p}.$$

Then $\sum\limits_{k=0}^{\infty} 2^{k(1-p)} |a_k|^2 < \infty$.

Proof:

$$\|f\|^2_{Q_p} \geqslant \iint\limits_{B_1(0)} \left| \frac{1}{2}\overline{D}f \right|^2 (1-|x|^2)^p dG$$

$$= \iint\limits_{B_1(0)} \left| \sum_{k=1}^{\infty} \left(-\frac{1}{2}\overline{D}\right) \left(\frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} \right) a_k \right|^2 (1-|x|^2)^p dG = (*)$$

$-1/2\overline{D}\Big((H_{2^k,\alpha})/(\|H_{2^k,\alpha}\|_{L_2(\partial B_1)})\Big)$ is a homogeneous monogenic polynomial of degree $2^k - 1$ and will be written in the form

$$\left(-\frac{1}{2}\overline{D}\right) \left(\frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} \right) = r^{2^k-1} \Phi_k(\varphi_1, \varphi_2)$$

with

$$\Phi_k(\varphi_1, \varphi_2) := \left(\left(-\frac{1}{2}\overline{D}\right) \left(\frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} \right) \right) \Big|_{\partial B_1}.$$

Then we continue

$$(*) = \int_0^1 \int_{\partial B_1(0)} \left| \sum_{k=1}^{\infty} r^{2^k-1} \Phi_k(\varphi_1,\varphi_2) a_k \right|^2 r^2(1-r^2)^p d\Gamma\, dr$$

$$= \int_0^1 \int_{\partial B_1(0)} \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \overline{a}_k r^{2(2^k-1)} \overline{\Phi}_k(\varphi_1,\varphi_2)\Phi_j(\varphi_1,\varphi_2) a_j r^2(1-r^2)^p d\Gamma\, dr$$

$$= \int_0^1 \sum_{k=1}^{\infty} |a_k|^2 r^{2(2^k-1)} \|\Phi_k\|^2_{L_2(\partial B_1)} r^2(1-r^2)^p dr$$

$$= \int_0^1 \sum_{k=1}^{\infty} |a_k|^2 r^{2(2^k-1)} 2^k\Big(2^k+\frac{1}{2}\Big) r^2(1-r^2)^p dr$$

$$\geqslant \int_0^1 \sum_{k=1}^{\infty} |a_k|^2 r^{2(2^k-1)} (2^k-1)^2 r^3(1-r^2)^p dr$$

$$= \frac{1}{2}\int_0^1 \sum_{k=1}^{\infty} |a_k|^2 x^{2^k-1}(2^k-1)^2 x(1-x)^p dx$$

$$= \frac{1}{2}\sum_{k=1}^{\infty} |a_k|^2 (2^k-1)^2 \int_0^1 x^{2^k}(1-x)^p dx \geqslant \frac{1}{8}\sum_{k=1}^{\infty}|a_k|^2 2^{2k}\int_0^1 x^{2^k}(1-x)^p dx$$

$$\geqslant \frac{1}{96}\sum_{k=1}^{\infty}|a_k|^2 2^{2k} 2^{-k(p+1)} = \frac{1}{96}\sum_{k=1}^{\infty}|a_k|^2 2^{k(1-p)}$$

Here we used the quaternion-valued inner product

$$(f,g)_{L(\partial B_1)} = \int_{\partial B_1} \overline{f}(x) g(x) d\Gamma$$

and the orthogonality of the spherical monogenics Φ_k (see [8]). ∎

REMARK. Theorem 4.1 and Theorem 4.2 prove that

$$f = \sum_{k=0}^{\infty} \frac{H_{2^k,\alpha}}{\|H_{2^k,\alpha}\|_{L_2(\partial B_1)}} a_k \in Q_p \Longleftrightarrow \sum_{k=0}^{\infty} 2^{k(1-p)}|a_k|^2 < \infty. \tag{19}$$

THEOREM 4.3. *The inclusions $\mathbf{Q}_{p_1} \subset \mathbf{Q}_p$ are strict for all $0 < p_1 < p \leqslant 2$.*

PROOF: Let

$$f(x) = \sum_{n=0}^{\infty} \frac{H_{2^n,\alpha}}{\|H_{2^n,\alpha}\|_{L_2(\partial B_1)}} a_n, \quad H_{n,\alpha}(x) = (z_1\alpha_1 + z_2\alpha_2)^n,$$

$$|\alpha|^2 = \alpha_1^2 + \alpha_2^2 \neq 0, \; a_n = \frac{1}{2^{n(1-p_1)/2}}.$$

Then,

$$\sum_{n=0}^{\infty} 2^{n(1-p)}|a_n|^2 = \sum_{n=0}^{\infty} \frac{1}{2^{n(p-p_1)}} < \infty \qquad \forall p > p_1$$

and

$$\sum_{n=0}^{\infty} 2^{n(1-p_1)}|a_n|^2 = \sum_{n=0}^{\infty} 1 = \infty.$$

By Theorem 4.1 and Theorem 4.2 we have that $f \in \mathbf{Q_p}$ but $f \notin \mathbf{Q_{p_1}}$. The idea of the proof is completely analogous to the ideas used in [**23**]. ∎

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